Introduction to Evolutionary Game Theory

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What is a game

- It is a situation in which two or more players compete (Ferguson and Gould, 1975).

- It is any situation in which individuals must make strategic decisions and in which the final result depends on what each one decides to do (Nicholson, 1997).

- Any decision-making problem, where the payoff (that a person obtains) depends not only on his/her own decisions but also on the decisions of the other people participating in the game (Maddala and Miller, 1991).

What is game theory

- It is a formal way of analyzing the interactions between agents that behave strategically.
- It is the mathematics of decision making in conflict situations
- It is applied in economics, military affairs, politics, ethology, sociology, ecology, and evolutionary biology.

Goal of Game Theory

Game

- Confronting players
- Strategies, decision

Theory

- Explicanation
- Prediction

GOAL OF GAME THEORY

Find the patterns of rational behavior in which the results depend on the actions of the interacting players.

Game Theory is the study of rational interactions in games i.e. it is the study of the **logic of interaction in games.** It is the study of how agents can do as well as possible in games.

The origin

- First formalized by Von Neumann and Morgenstern in 1944

  The Theory of Games and Economic Behaviour.

- Has found application in economics, politics, biology, computer science, psychology, sociology, etc.

Elements of a game

- **Agents** — Individuals, companies, groups of people, countries, etc. They are players who make decisions. They can choose from a set of possible alternatives

- **Strategies** — Are action plans: anticipated decisions with regarding the future. One strategy corresponds to each course of action a player can choose.

- **Payoffs** — Earnings Earnings correspond to the returns that each player gets when the game ends, represented by “payoff matrix” or benefits and losses.

Rational Choice

Decisions are made by individuals trying to maximize their benefits and minimize their costs.

Players make decisions about how to act (adopt a strategy) by comparing costs and benefits (payoff) of different actions.

"It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own interest." Adam Smith (1776)

"[Political economy] does not treat the whole of man's nature as modified by the social state, nor of the whole conduct of man in society. It is concerned with him solely as a being who desires to possess wealth, and who is capable of judging the comparative efficacy of means for obtaining that end." John Stuart Mill (1836)

Homo Economicus

Homo Economicus acts to obtain the greatest welfare for himself using all the available information on opportunities and constraints. **In a way, it suggests a rational, selfish and effort-averse individual.**

20th century: Lionel Robbins' rational choice theory
Individuals always tend to maximize their utility or benefit and tends to reduce costs or risks.

Homo sociologicus

Ralf Dahrendorf (1958) to refer to the image of human nature with which many sociological models try to limit the social forces that determine individual tastes and social values. This concept suggests that man is a tabula rasa on which societies and cultures write their values and goals so that their only goal is to fulfill the social role.

Cooperative and non cooperative

COOPERATIVE GAMES

Players can negotiate binding contracts.

"They choose strategies together."

NON COOPERATIVE GAMES

Players can NOT negotiate binding contracts.

"Each one chooses his optimal strategy independently".

- Understand the point of view of a "rational" adversary.
- Deduce own response to other actions.

Cooperative and non cooperative

Non-cooperative games:

It is not possible to negotiate and enforce a contract binding between players.

Example: Two rival firms take each other's likely behavior into account as they independently set their prices and advertising strategies to capture more market share.

Classes of games

Zero Sum	What one player wins is what the other player loses. The social, economic or political actors must understand the nature of this type of game.
Simultaneous vs sequential	Simultaneous Games: Players play simultaneously or are unaware of other players' previous moves. Sequential (or dynamic) games: Last players have some knowledge of previous actions.
Single vs repeated	One round is played Repeated a known or unknown number of times (repeated infinite times)
Complete vs Incomplete information	Players may have all or part of their opponent's move information

Game representation

The representation of a game can be done in two ways

Extensive form - game tree

It is a graphic representation of a strategic situation. Each node represents the possible courses of action for each player, at the end of the tree the profits obtained by each player are presented.

Normal Form - payoff matrix.

It is a representation of a strategic situation through a matrix. The strategies of each player are presented to the left and at the top of the table. The winnings obtained by each of the players at the end of the game are presented in the inner part of the table.

Example: Prisoners' Dilemma I

The Prisoners' Dilemma is one of the most famous games studied in game theory. We will get back to it later

Two suspects are arrested for a crime

The DA wants to extract a confession so he offers each a deal

Example: Prisoners' Dilemma

The Deal

"if you fink on your companion, but your companion doesn't fink on you, you will be freebut your companion gets a three-year sentence"

"if you both fink on each other, you will each get a two-year sentence"

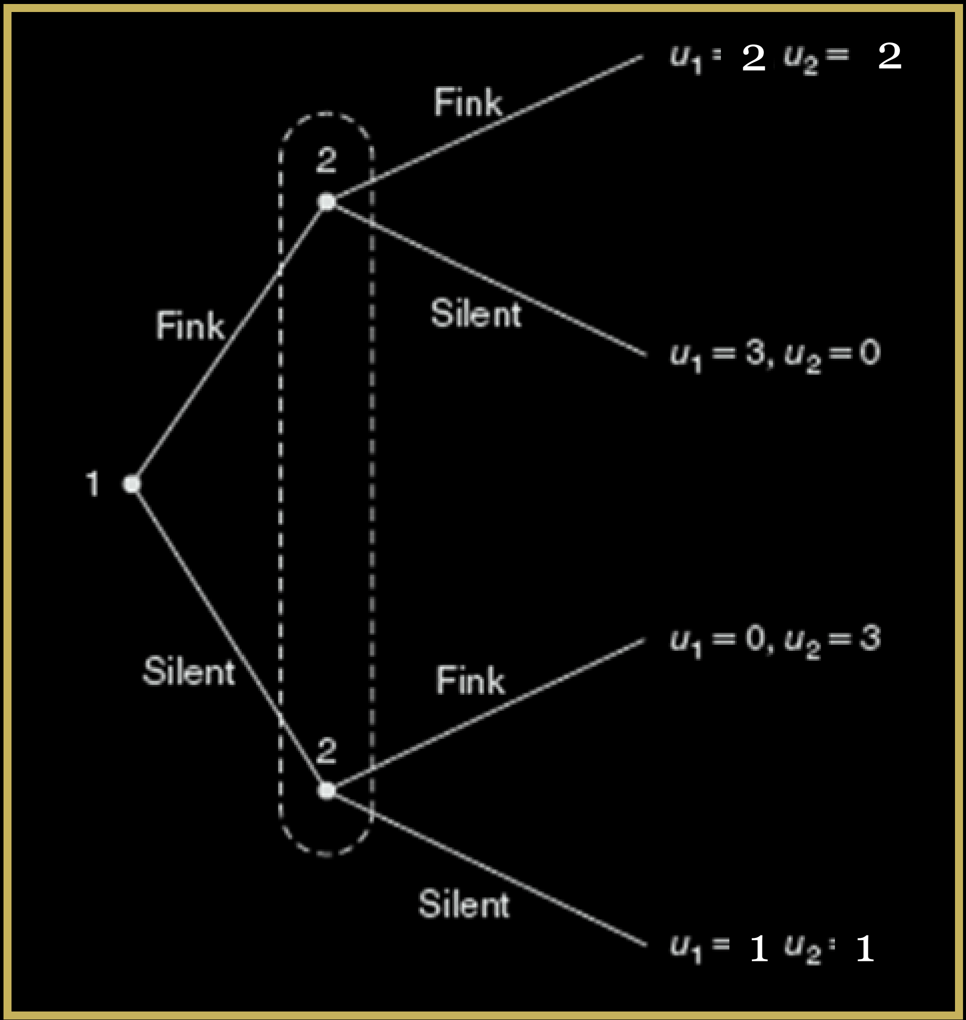
"if neither finks, we will get tried for a lesser crime and each get a one-year sentence"

Example: Prisoners' Dilemma

There are 4 combinations of strategies and two payoffs for each combination

- useful to use a game tree or a matrix to show the payoffs
  - a game tree is called the <u>extensive form</u>
  - a matrix is called the <u>normal form</u>

Example: Prisoners' Dilemma Extensive Form



- Each node represents a decision point
- The dotted oval means that the nodes for player 2 are in the same information set
  - player 2 doesn't know player 1's move

Example: Prisoners' Dilemma Normal Form

Sometimes it is more convenient to represent games in a matrix

	Fink	Silent
Fink	2	0
Silent	3	1

Dominant Strategies

A normal form game is defined by a triplet $\langle I, (S_i)_{i \in I}, (u_i)_{i \in I} \rangle$

Where

I is a finite set of palyers
S_i is the set of possible action for i
$s_i \in S_i$ is thes strategy adopted by i
$u_i : S \to \Re$ is the payoff $of\ i$, S is the profile of all the possible actions

$s_{-i} = [s_j]_{j \neq i}$: others´ actions
$(s_i, s_{-i}) \in S$ strategies profile

Dominant strategy: A strategy $s_i \in S_i$ is dominant for player *i* if

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})\ \forall s'_i \in S_i \wedge \forall s_{-i} \in S_{-i}$$

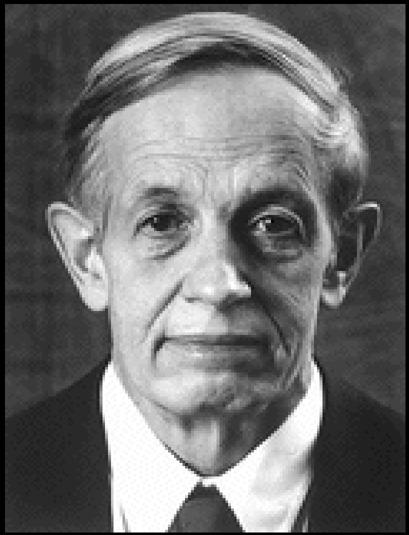
Dominant strategies equilibria: A strategy profile s∗ is the dominant strategies equilibrium if for every player *i* , $s^i{*}$ is dominant.

Group photo

- Wednesday after coffee break

Nash equilibrium

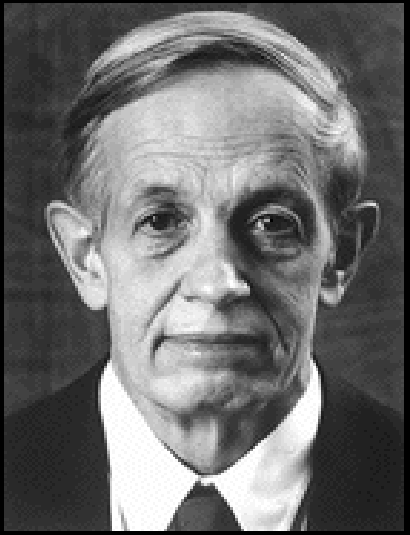
A Nash equilibrium is reached when each player chooses a strategy that is best option, given what their opponents are doing. It is not convenient for any of the players to deviate from the chosen strategy



Nash equilibrium involves strategic choices that, once made, provide no incentives for players to alter their behavior

It is best choice for each player given the other players’ equilibrium strategies

Nash equilibrium



Any game in which the players have a finite number of possible strategies has an equilibrium in terms of mixed strategies.

In mixed strategies, the Nash equilibrium is one in which each agent chooses the optimal frequency with which it will follow its strategies, given the frequency chosen by the other.

Finding the Nash equilibrium can be tricky

Hands On

- Generaliza PD to multiple player
- Consider a game on a network
- Compare individual vs colective game

Eliminating Dominated Strategies

We can use this technique in normal-form games to find the equilibrium:

	f	s
f	2,2	0,3
s	3,0	1,1

Eliminating Dominated Strategies

	f	s
f	2,2	0,3
~~s~~	~~3,0~~	~~1,1~~

Fink is a dominant strategy for row

Eliminating Dominated Strategies

	f	s
f	2.2	~~0 3~~
s	~~3.0~~	~~1,1~~

Fink is a dominant strategy for row

(f,f) is the unique Nash equilibrium.

Prisoner’s Dilemma

Even though both players would be better off cooperating, mutual defection is the dominant strategy.

What drives this?

- One-shot game
- Inability to trust your opponent
- Perfect rationality

How do players escape this dilemma?

Play repeatedly

Find a way to ‘guarantee’ cooperation

Change payment structure

Dominated Strategies

Strict dominated strategy: A strategy $s_i \in S_i$ is strictly dominated for player *i if* there is another strategy $s'_i \in S_i$ that verifies

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \ \forall s_{-i} \in S_{-i}$$

Weakly dominated strategy: A strategy $s_i \in S_i$ s weakly dominated for player *i* if there is another strategy $s'_i \in S_i$ such that

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \ \text{ for some } \ s_{-i} \in S_{-i}$$

Dominant strategies must never be chosen and can be iterartively eliminated

La información sobre los payoff y la racionalidad resulta en una eliminación iterada

Dominated strategies

Example 3x3:

Column

File

	C1	C2	C3
F1	70,20	55,40	65,30
F2	80,21	35,10	30,50
F3	30,22	60,30	55,25

Dominated strategies

Column

File		C1	C2	C3
	F1	70,20	55,40	65,30
	F2	80,21	35,10	30,50
	F3	30,22	60,30	55,25

C3 dominates **C1** for player column

Dominated strategies

Column

File

	C1	C2	C3
F1	70,20	55,40	65,30
F2	~~80,21~~	~~35,10~~	~~30,50~~
F3	30,22	60,30	55,25

F2 is dominated by **F1** and **F3** for player file.

Dominated strategies

Column

File

	C1	C2	C3
F1	70,20	55,40	65,30
F2	~~80,21~~	~~35,10~~	~~30,50~~
F3	30,22	60,30	55,25

C2 dominates **C3** for player column, that will always choose **C2**

Dominated strategies

Column

File

	C1	C2	C3
F1	~~70,20~~	~~55,40~~	~~65,30~~
F2	~~80,21~~	~~35,10~~	~~30,50~~
F3	~~30,22~~	60,30	~~55,25~~

So file chooses **F3**.

Best response – Nash equlibrium

We have n players.

Each player *i* adopts one strategy among a set of strategies S_i

A given strategy $s_i \in S_i$ is a best response to the strategy of the oponent s_o if no other strategy in S_i gets a higher payoff

$$u_i(s_i^*, s_o) \geq u_i(s_i, s_o)\ \forall s_i \in S_i$$

A Nash equilibrium is a strategy that is a best response to itself or a set of strategies that are mutually bets responses one to the others,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)\ \forall s_i \in S_i$$

Nash equilibrium

Neither player can profitably deviate given the strategies of the other players. Therefore, in Nash equilibrium, the best responses intersect

In other words, the players' guesses are consistent: each player i chooses s*i expecting the rest of the players to choose s*-I

The player's guess is verified in a Nash equilibrium.

This has a "steady state" feel to it. In fact, the two ways to justify Nash Equilibrium are based on this concept

Nash equilibrium

Sometimes can not trivially found

Column

File		St	Pa	Sc
	St	0, 0	-1,1	1,-1
	Pa	1,-1	0,0	-1,1
	Sci	-1,1	1,-1	0,0

Nash equilibrium

Sometimes there is more than one Nash equilibrium

Battle of sexes

A couple plans their vacation. The woman prefer to go to the beach, the man to the mountains. Both prefer to spend their vacations together than apart

	Mountain	Beach
Mountain	2,1	0,0
Beach	0,0	1,2

Mixed strategies

In the cases discussed above, the player chooses a specific course of action (strategy) and sticks with it.

However, in some games there is no Nash equilibrium of pure strategies, so it is essential to extend the concept of Nash equilibrium by incorporating the concept of mixed strategies.
A mixed strategy is one in which the player randomly chooses between two or more possible options, based on a pre-established distribution of probabilities.

Mixed strategies

"Matching pennies"

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

	p H	(1-p) T
q H	1,-1	-1,1
(1-q) T	-1,1	1,-1

If there is a Nash equilibrium, the players should be able to choose an optimal response frequency based on what the other does.

Mixed strategies

	p H	(1-p) T
q H	1,-1	-1,1
(1-q) T	-1,1	1,-1

Each player plays each strategy with a certain probability, but wants his/her profit to be independent of what the other player does.

If A plays Heads; B wins –p+(1-p) if A plays Tails B wins p-(1-p)

$p-(1-p)=-p+(1-p) \Rightarrow p=1/2$

$q-(1-q)=-q+(1-q) \Rightarrow q=1/2$

Formalizing game theory

Each player has at his/her availability the same (or not) finite set of pure strategies. $R = \{R_1, R_2, \ldots, R_N\}$.

For this strategies there is a payoof matrix that contains information about payoffs

Payoff Matrix

$$\boxed{A = (a_{ij})}$$

a_{ij} is the payoff that R_i gets when confronting R_j

We can also define a mixed strategy. A player can adopt any of the pure strategies or a mixed strategy *p*, which is defined by the probability of adopting each one of the pure strategies

$$p=\{p_1, p_2, \ldots, p_n\}, \quad 1 \geq p_i \geq 0.$$

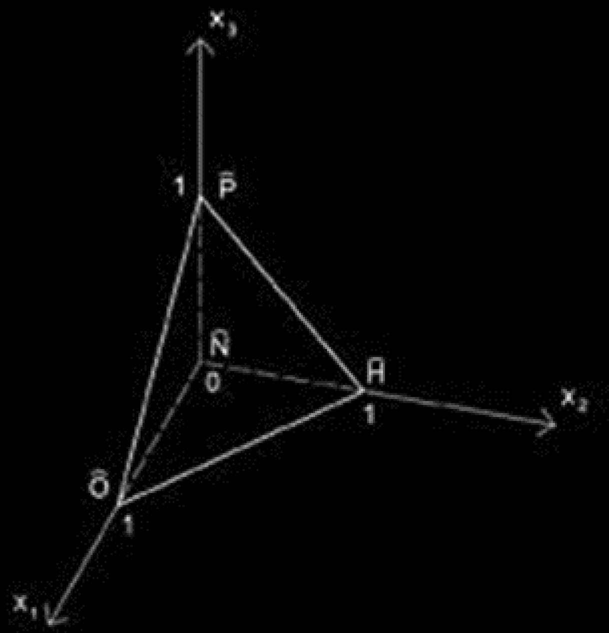
Formalizing game theory

As the values p_i represent probabilities they must satsfy

$$\sum p_i = 1 \quad \text{and} \quad 1 \geq p_i \geq 0$$

That defines a simplex in the R^n vectorial space

The vertices of the simplex are pure strategies



Formalizing game theory

To calculate the payoff of any strategy p_1 against another p_2 we calculate

$$p^1 A p^2 = \sum_{i,j} p_i^1 p_j^2 a_{ij}$$

If we have a pure strategy R_i, its payoff when confronting p^1 is

$$e_i A p^1 = \sum_j p_j^1 a_{ij} \equiv A(p^1\ )_i$$

with $\quad \overline{e_i} = \{0,0, \dots, 1, .. 0\}$

Formalizing game theory

A best response to a strategy $\boldsymbol{p}^1$ is a strategy $\boldsymbol{p}^2$ such that the value $\boldsymbol{p}^2 A \ \boldsymbol{p}^1$ is maximum.

If all players have the same choice of strategies, a Nash equilibrium is a strategy that is the best response to itself.

$$p^j A p^i \leq p^i A p^i \ \forall j$$

And it is called strict if it is the only best answer, that is

$$p^j A p^i < p^i A p^i \ \forall j \neq i$$

The central question of evolutionary game theory is whether there is a profile of strategies, in a population, that is stable against perturbations. That is, if it is invaded by mutants, can they take advantage ?

Formalizing game theory

The set of best responses to a strategy *t* is *β(t)*. Suppose p belongs to *β(t)*.
The *pAt* is maximum

$$\sum_i \sum_j a_{ij} p_i t_j = \sum_i p_i \sum_j a_{ij} t_j$$

Suppose there exists a pair *k,h* such that p_h and p_k are >0 and

$$\sum_j a_{hj} t_j > \sum_j a_{kj} t_j \qquad (1)$$

I can replace strategy *p* by *p** so that

$$p^*_h = p_h + p_k,\ p^*_k = 0$$

And now p*At > pAt, but this is absurd. So (1) cannot hold. That means that the payoffs of all the pure strategies that participate in *p* when faced with *t* is the same. Therefore, if a mixed strategy is in *β(t)* all the pure ones that form it will be. The only strict Nash equilibria are pure strategies.

Evolutionarily Stable Strategy

Suppose a population adopts a unique strategy

By mutation or invasion, an individual (mutant) in the population adopts a different strategy.

If the mutant is doing better than the rest of the population, the population will imitate it and change the oruginal strategy.

If it goes worse, the rest of the population will ignore it

If the population adopted a strategy such that no mutant strategy can take advantage of the situation, that strategy is called

Evolutionarily Stable Strategy (ESS).

Evolutionarily Stable Strategy

Suppose that the population has strategy r_i and that there is a small invasion of individuals with strategy r_j. We need

$$r_j A((1-\varepsilon)r_i + \varepsilon r_j) < r_i A((1-\varepsilon)r_i + \varepsilon r_j) \quad \forall j \neq i$$

$$(1-\varepsilon)\boxed{(r_i A r_i - r_j A r_i)} + \varepsilon \boxed{(r_i A r_j - r_j A r_j)} > 0$$

1) $r_i A r_i > r_j A r_i$ Strict Nash Equilibrium

2) $r_i A r_i = r_j A r_i \Rightarrow r_i A r_j > r_j A r_j$ Stability

An ESS should be a Strict Nash Equilibrium or be a Stable Nash Equilibrium

Hawks and Doves

Two individuals must compete for a valuable resource

Hawk:
Always fight for the resource. If you win you receive the benefit
If he loses, he may be injured and must pay the cost of the assault.

Dove:
Never fight for the resource.
He retreats if he is confronted by a Hawk.
Split the resource if you meet another Dove
There is no cost of aggression

Hawks and Doves

Hawks and Doves

Payoff **G**ain for obtaining the resource > 0
Cost per aggression > 0 and > G

	Hawk	Dove
Hawk	*(G-C)/2, B-C/2*	*B*,0
Dove	0, *B*	*G/2,G/2*

Let us call P(A,B) the payoff received by a strategy A in a population (with strategy) B

P(H, H) > P(D,H) then H is ESS

P(H, H) > P(D,H) if (G-C)/2 > 0

C < G, H is ESS

Hawks and Doves

Suppose the entire population behaves like Hawk. Is it an Evolutionarily Stable Strategy (ESS)?

If $P(H, H) > P(D,H) \Rightarrow$ H is ESS

$P(H, H) > P(D,H) \Rightarrow (G-C)/2 > 0$

C < **G**, **Hawk** is ESS, if **C** > **G**, it is not

Suppose the entire population behaves like Dove. Is it an Evolutionarily Stable Strategy (ESS)?

If $P(D, D) > P(H,D) \Rightarrow$ D is ESS

$P(P, P) > P(H,P) \Rightarrow G/2 > B$, false

Dove can not be ESS

Hawks and Doves

If C > G, neither H nor P is evolutionarily stable

We can look for an EES among the mixed strategies

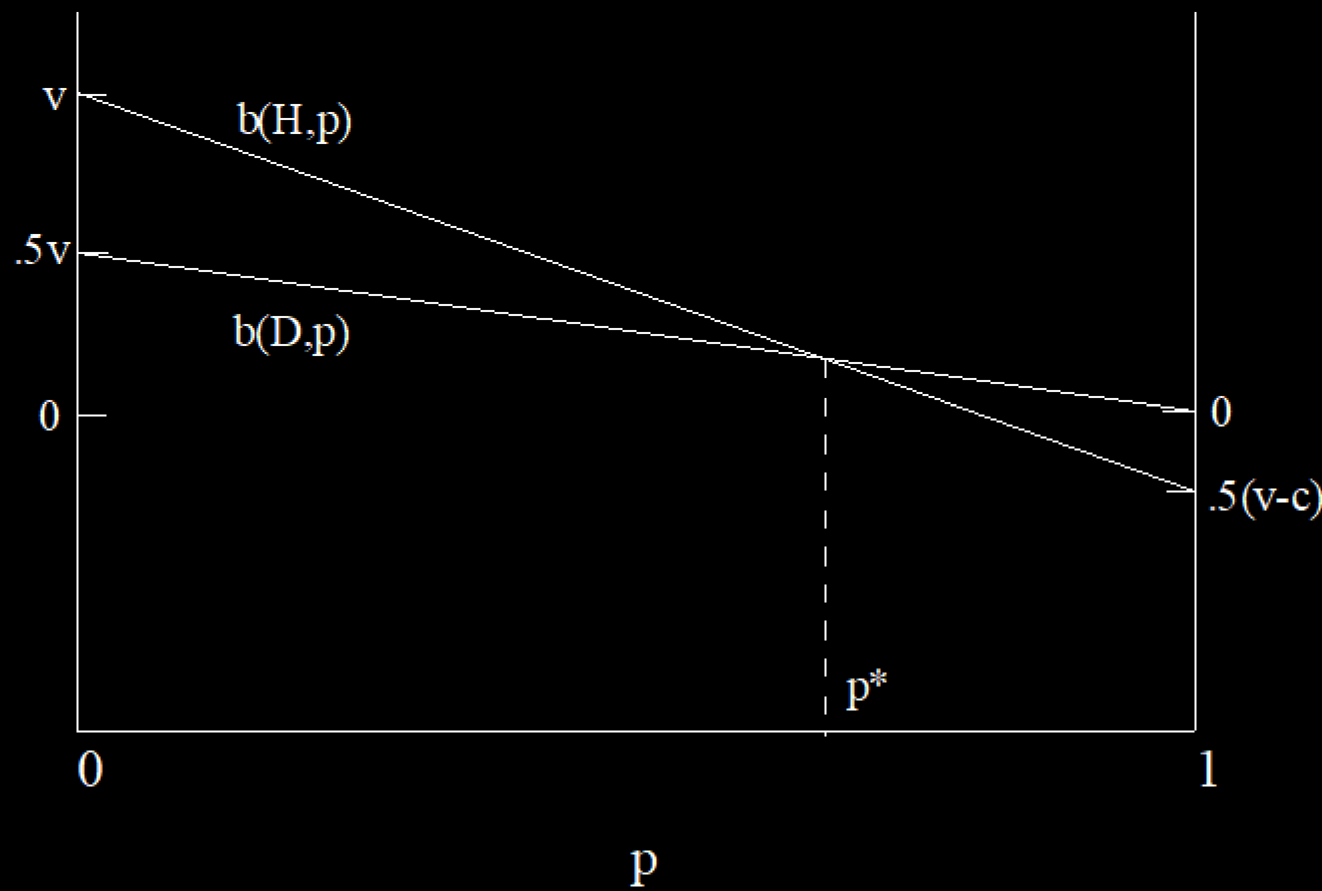
Suppose then that H is adopted with probability ρ and D with probability (1- ρ)

The payoff for choosing H = ρ E(H,H) + (1 - ρ) E(H,P) = ρ[(G-C)/2] + (1 – ρ) G

The payoff for choosing D = ρ E(P,H) + (1 - ρ) E(P,P) = ρ[0] + (1 – ρ) B/2

If I want neither H nor P to take advantage ρ[(G-C)/2] + (1 – ρ) B = (1 – ρ) B/2 $\Rightarrow$ ρ=G/C

Hawks and Doves



Hawks and Doves

Suppose now that the population adopts the mixed strategy with *ρ=B/C*

And that an individual enters with a strategy *(α H, (1- α)P)*

The payment of this individual is

$$\alpha \frac{B}{C} P(H,H) + \alpha(1-\frac{B}{C})P(H,P) + (1-\alpha)\frac{B}{C}P(P,H) + (1-\alpha)(1-\frac{B}{C})P(P,P)$$

$$\alpha \frac{B}{C}\frac{(B-C)}{2} + \alpha(1-\frac{B}{C})B + (1-\alpha)\frac{B}{C}0 + (1-\alpha)(1-\frac{B}{C})\frac{B}{2}$$

$$\alpha\frac{B^2}{2C} - \alpha\frac{B}{2} + \alpha B - \alpha\frac{B^2}{C} + \frac{B}{2} - \alpha\frac{B}{2} + \alpha\frac{B^2}{2C} - \frac{B^2}{2C} \rightarrow \boxed{\frac{B}{2} - \frac{B^2}{2C}}$$

Independent of α

Hawks and Doves

The previous calculation shows us that although the chosen strategy is in the set of its best answers, it is not the only one, since they all are.

Let's see if the payoff of *(B/C H, (1-B/C)P)* against *(α H, (1- α)P)* is greater than that of the latter against itself

$$\frac{B}{C}[\alpha\frac{B-C}{2}+(1-\alpha)B]+\frac{C-B}{C}(1-\alpha)\frac{B}{2}=-\alpha B+\frac{B}{2}+\frac{B^2}{2C}$$

$$\alpha[\alpha\frac{(B-C)}{2}+(1-\alpha)B]+(1-\alpha)\frac{B}{2}(1-\alpha)=\frac{B}{2}-\frac{C}{2}\alpha^2$$

$$\left(-\alpha B+\frac{B}{2}+\frac{B^2}{2C}\right)-\left(\frac{B}{2}-\frac{C}{2}\alpha^2\right)=\frac{1}{2C}(B-\alpha C)^2>0$$

Hawks and Doves

Diversity : 2 Variants

1. Coexistence of individuals with two different behaviors. The population is composed of ρ*Hawks and , (1 – ρ*) Doves

2. Adoption of mixed strategies. Each individual has the same mixed strategy, which makes him choose Hawk with frequency ρ

Evolutionary games

Behavior can be defined by trial and error. Adaptation and learning are key factors

Rational behavior constraint can be relaxed

Games are played in a population, where each individual receives a score

Strategies that perform better than average spread while the others disappear

The initial assignment of strategies is random Each player plays with all his neighbors (mean field vs spatial distribution) and his payoff from him is the sum of the payoffs

The success of each player determines the number of followers or descendants in the next step (Selection) The descendants or imitators inherit or copy the strategy with some error (Mutation) If the (global) Nash equilibrium is reached, no other strategy can invade

Allowed dynamics

Any dynamics should lead to a possible strategy. It must leave the simplex invariant. The dynamics will give us equations for $\dot{x}_i$.
Zero must be a fix point of $\dot{x}_i$

If we call $S = \sum_i x_i$
Then 1 must be a fix point of $\dot{S} = \sum_i \dot{x}_i$

Replicator dynamics

The replicator equation describes the evolution of phenotype frequencies in a population with selection proportional to fitness.

If we call x_i the frequency of the phenotype *i*, and f_i to its fitness the equation is

$$\dot{x}_i = x_i(f_i(\bar{x}) - \bar{f}(\bar{x}))$$

where

$$\bar{f}(\bar{x}) = \sum_i x_i f_i(\bar{x})$$

Replicator dynamics

Zero must be a fix point of $\dot{x}_i$

$$\dot{x}_i = x_i(f_i(\bar{x}) - \bar{f}(\bar{x}))$$

1 must be a fix point of $\dot{S} = \sum_i \dot{x}_i = \sum_i x_i(f_i(\bar{x}) - \bar{f}(\bar{x}))$

$$\dot{S} = \sum_i x_i f_i(\bar{x}) - \bar{f}(\bar{x}) \sum_i x_i = \bar{f}(\bar{x})\,(1 - S)$$

Replicator dynamics

Payoff matrix $A = \{a_{ij}\}$

$$\bar{x} = \{x_1, x_2, \dots., x_N\}$$

$$\sum_i x_i = 1$$

$$\overline{e_i} = \{0,0, \dots, 1, ..0\}$$

$$f_i(\bar{x}) = e_i A \bar{x}$$

$$\bar{f}(\bar{x}) = \bar{x} A \bar{x}$$

Simple evolutionary game simulations

- Everyone starts with a random strategy
- Everyone population plays game against everyone else
- The payoffs are added up
- The total payoff determines the number of offspring (**Selection**)
- Offspring inherit ***approximately*** the strategy of their parents (**Mutation**)
- [Note similarity to genetic algorithms.]
- [Nash equilibrium in a population setting- no other strategy can invade]

Hawks and Doves

Payoff Hawk

$$f_H = (1{,}0)A\begin{pmatrix} x \\ 1-x \end{pmatrix}$$

Payoff Dove

$$f_P = (0{,}1)A\begin{pmatrix} x \\ 1-x \end{pmatrix}$$

Mean payoff

$$\overline{f} = (x, 1-x)\ A\ \begin{pmatrix} x \\ 1-x \end{pmatrix}$$

Replicator equation

$$\frac{dx}{dt} = x\left(f_H - \overline{f}\,\right)$$

Hawks and Doves

Replicator equation

$$\frac{dx}{dt} = x(f_H - \overline{f})$$

$$\overline{f} = xf_H + (1-x)f_D$$

$$\frac{dx}{dt} = x(1-x)(f_H - f_D)$$

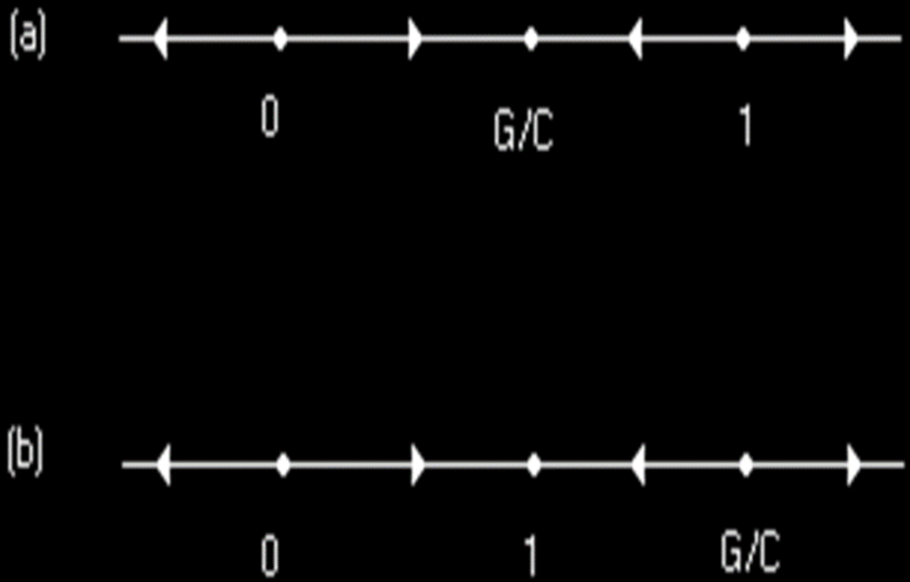
$$\frac{dx}{dt} = \frac{1}{2}x(1-x)(G - Cx)$$

$$\frac{dx}{dt} = 0 \Rightarrow x = 0 \lor x = 1 \lor x = \frac{G}{C}$$

Hawks and Doves

G<C, dimorphic equilibrium

G>C, pure hawk equilibrium



Stone, Paper, Scissors

	Stone	Paper	Scissors
Stone	0, 0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

$$\downarrow$$

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Stone, Paper, Scissors

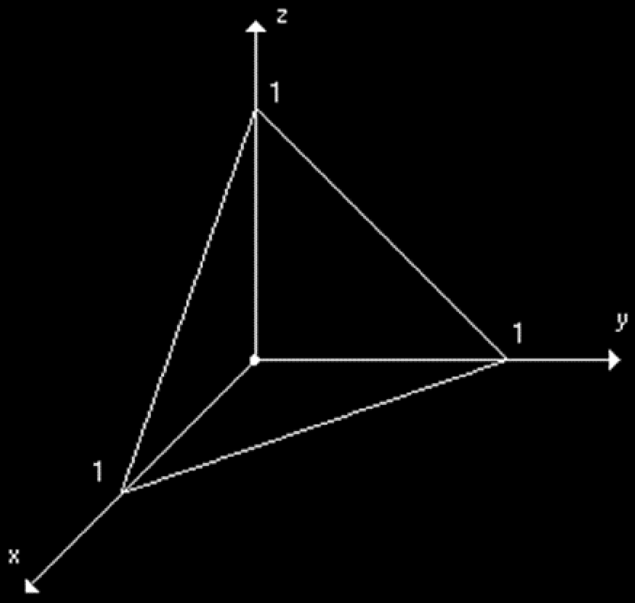
$$\frac{dx}{dt} = x(y - z)$$

$$\frac{dy}{dt} = y(z - x)$$

$$\frac{dz}{dt} = z(x - y)$$

$$\frac{dx}{dt} = -x + x^2 + 2xy$$

$$\frac{dy}{dt} = y - y^2 - 2xy$$



Stone, Paper, Scissors

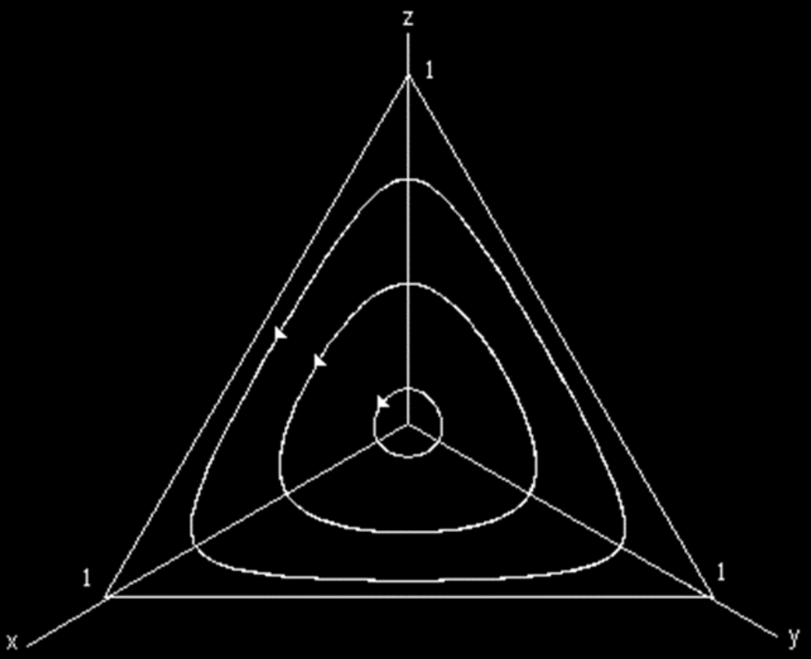
Equilibria

(1,0,0)
(0,1,0)
(0,0,1)

Saddle

$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

Center



Stone, Paper, Scissors

$$\frac{dx}{dt} = x(-1 + x + 2y)$$
$$\frac{dy}{dt} = y(1 - y - 2x)$$

$$= x^* + \varepsilon$$
$$y = y^* + \delta$$

$$\frac{d(x^* + \varepsilon)}{dt} = \frac{d\varepsilon}{dt} = (x^* + \varepsilon)(-1 + x^* + \varepsilon + 2(y^* + \delta))$$
$$\frac{d(y^* + \delta)}{dt} = \frac{d\delta}{dt} = (y^* + \delta)(1 - y^* - \delta - 2(x^* + \varepsilon))$$

$$\frac{d\varepsilon}{dt} = x^*(-1 + x^* + 2y^*) + x^*(\varepsilon + 2\delta) + \varepsilon(-1 + x^* + 2y^*)$$
$$\frac{d\delta}{dt} = y^*(1 - y^* - 2x^*) - y^*(2\varepsilon + \delta) + \delta(1 - y^* - 2x^*)$$

$$\frac{d\varepsilon}{dt}, \frac{d\delta}{dt}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix}$$

$$= 2(x^* + y^*) - 1$$
$$b = 2x^*$$
$$c = -2y^*$$
$$d = 1 - 2(x^* + y^*)$$

Stone, Paper, Scissors

$x^* = 0$
$y^* = 0$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = -1$

$x^* = 0$
$y^* = 1$

$$\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = -1$

$x^* = 1$
$y^* = 0$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = -1$

$x^* = 1/3$
$y^* = 1/3$

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

$\lambda_1 = i\sqrt{3}$
$\lambda_2 = -i\sqrt{3}$

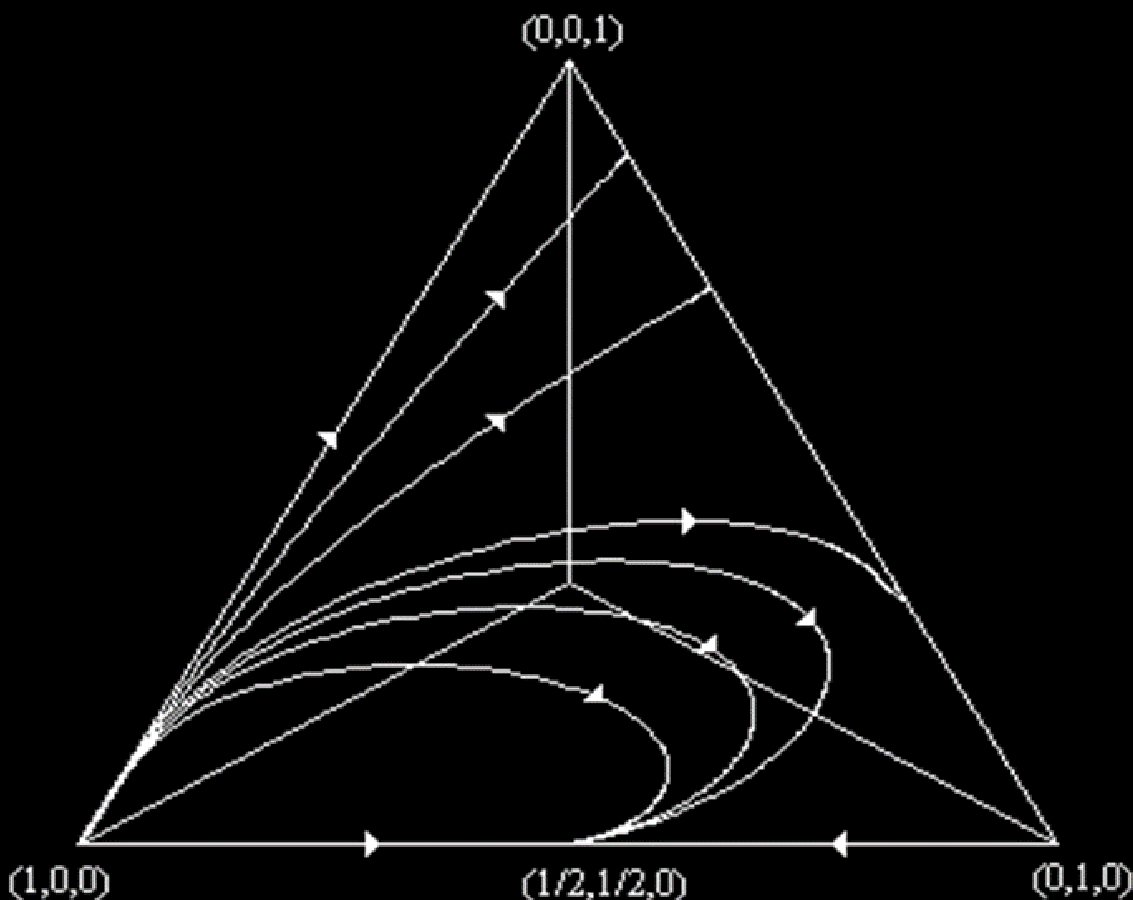
Extended hawks and doves

Retaliator: behaves like a dove in front of a dove and like a hawk in front of a hawk. Fight if brought to fight

	Hawk	Dove	Retaliator
Hawk	(G-C)/2, (G-C)/2	G,0	(G-C)/2, (G-C)/2
Dove	0, G	G/2, G/2	G/2, G/2
Retaliator	(G-C)/2, (G-C)/2	G/2, G/2	G/2, G/2

$$A = \begin{pmatrix} \frac{G-C}{2} & G & \frac{G-C}{2} \\ 0 & \frac{G}{2} & \frac{G}{2} \\ \frac{G-C}{2} & \frac{G}{2} & \frac{G}{2} \end{pmatrix}$$

Extended hawks and doves



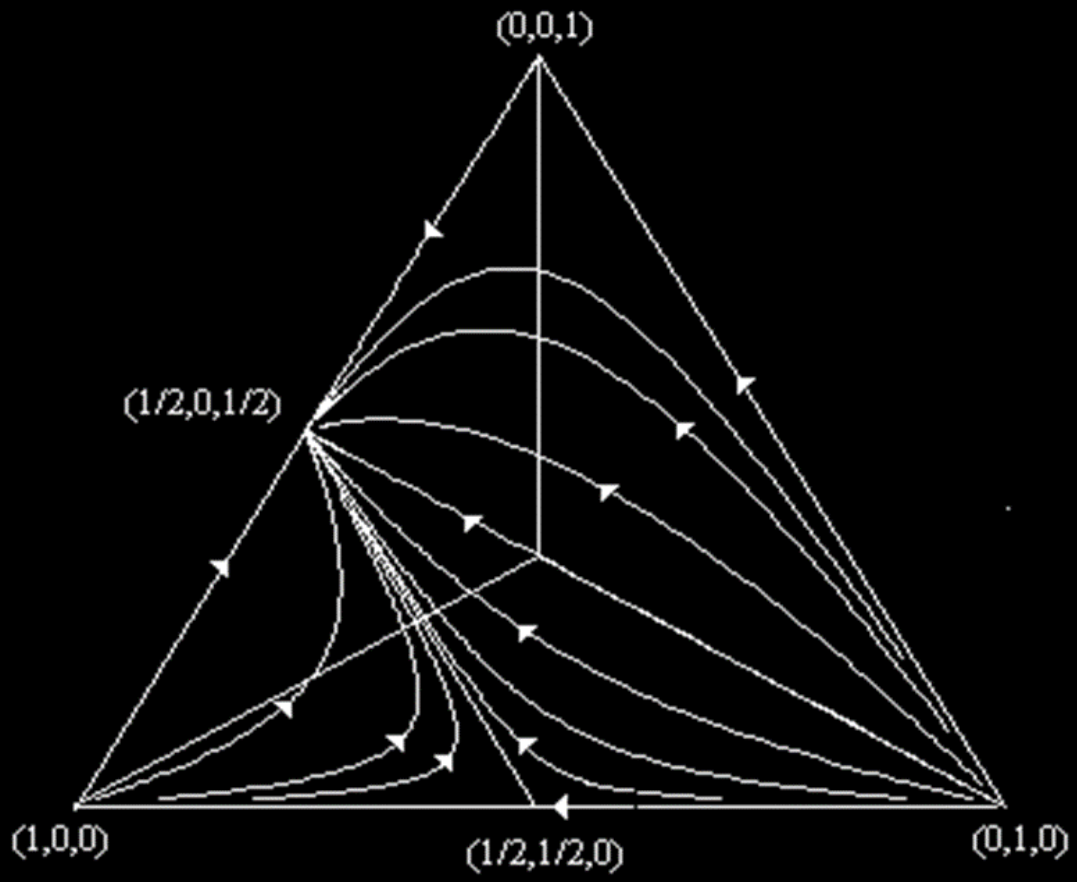
Extended hawks and doves II

Bully - Behaves like a hawk in front of dove, but cowers in front of a hawk. Avoid the fight if he is brought to fight

	Hawk	Dove	Bully
Hawk	(G-C)/2, (G-C)/2	G,0	G, 0
Dove	0, G	G/2, G/2	0, G
Bully	0, G	G, 0	G/2, G/2

$$A = \begin{pmatrix} \frac{G-C}{2} & G & G \\ 0 & \frac{G}{2} & 0 \\ 0 & G & \frac{G}{2} \end{pmatrix}$$

Extended hawks and doves II



2x2 Symmetric Games

Each player can choose between two available strategies

	E1	E2
E1	a,a	b,c
E2	c,b	d,d

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Consider a population that plays E1 with prob. x and E2 with prob. (1-x)

$$f_1 = xa + (1-x)b$$

$$f_2 = xc + (1-x)d$$

$$\overline{f} = x[xa + (1-x)b] + (1-x)[xc + (1-x)d]$$

$$\frac{dx}{dt} = x(f_1 - \overline{f}) = x(1-x)(x(a-b-c+d) + (b-d))$$

2x2 Symmetric Games

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a-c & 0 \\ 0 & d-b \end{pmatrix}$$

$$\overline{f} = x(a-c) + (1-x)(d-b)$$

$$\frac{dx}{dt} = x(f_1 - \overline{f}) = x(x(a-c) - [x^2(a-c) + (1-x)^2(d-b)]$$

$$\frac{dx}{dt} = x(1-x)[x(a-c) - (1-x)(d-b)] = x(1-x)[x(a-b-c+d) + (b-d)]$$

2x2 Symmetric Games

Each player can choose between two available strategies

	E1	E2
E1	a_1, a_1	0,0
E2	0,0	a_2,a_2

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

Consider a population that plays E1 with prob. x and E2 with prob. (1-x)

$$f_1 = xa + (1-x)b$$

$$f_2 = xc + (1-x)d$$

$$\frac{dx}{dt} = x(f_1(\vec{x}) - \overline{f}(\vec{x})) \ = \ x(1-x)(xa_1 - (1-x)a_2$$

$$\frac{dx}{dt} = 0 \quad \Rightarrow x = 1, \quad x = 0, \quad x = \frac{a_2}{a_1 + a_2}$$

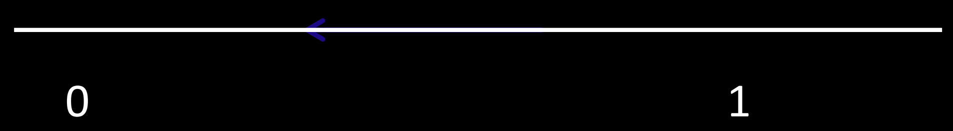
2x2 Symmetric Games

$$x_1(1-x_1)(x_1 a_1 - (1-x_1)a_2)$$

$$x_1^* = \frac{a_2}{a_1 + a_2}$$

If $a_1 > 0$ and $a_2 < 0$, the flux is always

If $a_1 < 0$ and $a_2 > 0$, the flux is always



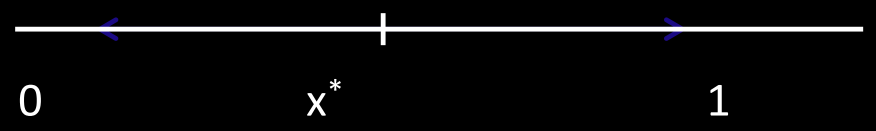
$$|x_1^*| > 1$$

2x2 Symmetric Games

$$\dot{x}_1 = x_1(1 - x_1)(x_1 a_1 - (1 - x_1)a_2)$$

$$x_1^* = \frac{a_2}{a_1 + a_2}$$

If $a_1 > 0$ and $a_2 > 0$, the flux close to x* , where the derivative is null



If $a_1 < 0$ and $a_2 < 0$, the flux close to x* , where the derivative is null

2x2 Symmetric Games

There are three (four) types of 2x2 Symmetric Games

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

Games type I (IV) : $a_1 < 0$ *y* $a_2 > 0$ ($a_1 > 0$ *y* $a_2 < 0$) — Prisoner's dilemma

Games type II: $a_1 > 0$ *y* $a_2 > 0$ — Coordination Games

Games type III: $a_1 < 0$ *y* $a_2 < 0$ — Hawks and doves

Replicator equation with mutation

Suppose there is a replication error.

The probability that a type j mutates to a type i is considered.

The replicator equation with mutation is

$$\dot{x}_i = \sum_j x_j f_j(\vec{x}) q_{ji} - \overline{f}(\underline{\vec{x}}) x_i$$

Adaptive dynamics

The game contemplates a continuous space of strategies

The population is homogeneous, everyone adopts the same strategy

Mutation generates strategic variants close to that of the population

If the mutant is better than the resident strategy, it is adopted by the population, if not it is rejected

It is used to find evolutionarily stable strategies

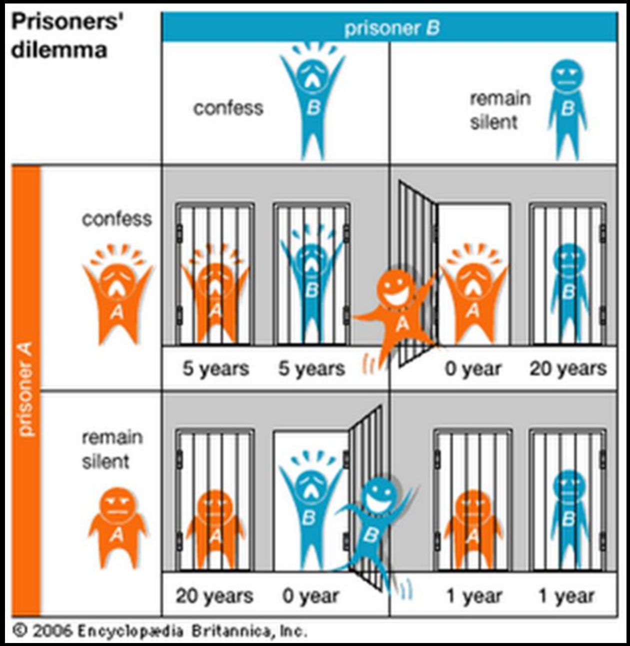
Adaptive dynamics

- Strategies are described by continuous parameters :
- Expected score of mutant $S'(p_1', p_2', \ldots., p_n')$against S is given by E(S',S)
- The adaptive dynamics flow in the direction which maximises the score:

$$\dot{p}_i = \left. \frac{\partial E(S', S)}{\partial p_i'} \right|_{S' \to S}, i = 1, \ldots, n$$

Prisoner’s dilemma

Cooperation is much more frequent than suggested by models based on rational behavior



Prisoner's dilemma

	Cooperate	Defect
Cooperate	R, R	S, T
Defect	T, S	P, P

R REWARD for mutual cooperation
S SUCKER's payoff
T TEMPTATION to defect
P PENALTY for mutual defection

With $T>R>P>S$ and $R > (T+S)/2$

Prisoner’s dilemma

DP Iterative PD vs. P.S. simple

In simple instances, rational decision prevails. Always desert. However, in the iterative, defecting is always not optimal since mutual cooperation can cause a net gain for both agents.

While cooperation is collectively rational behavior, from the individual point of view desertion is appropriate.

$T > R > P > S$ and $R > (T+S)/2$. The condition $R > (T+S)/2$ is important when the game is repeated. This ensures that individuals do better cooperating with each other than alternating between cooperating and not cooperating.

The lack of cooperation is the tragedy of the commons. A situation in which several individuals, motivated only by self-interest and acting independently but rationally, end up destroying a limited shared resource—the common—even though it is clearly the case that it is not in their interest—either as individuals or altogether—for such destruction to take place.

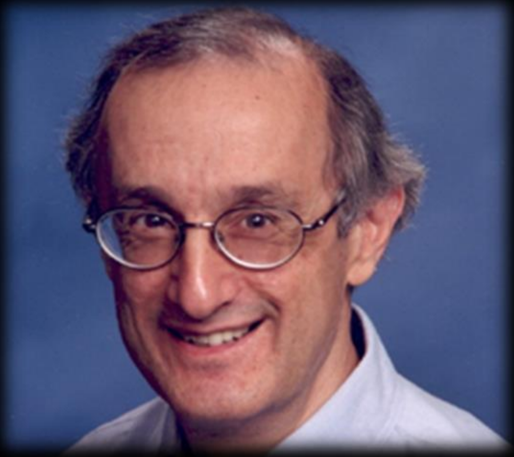
Axelrod' s Tournament

Made in the early 80's

Axelrod invited game theory researchers to propose strategies to play the iterated D.P.

Each strategy had to compete against all the others, including itself and a strategy that randomly cooperated or betrayed

TFT won the first tournament and the second, which was held after having informed the competitors of the result of the first.



Strategies

Tit For Tat - cooperate in the first game, and then repeat the opponent's last choice.

Tit For Tat and Random - Replays the opponent's last choice biased by random adjustment

Tit For Two Tats - Like TFT, except the opponent must make the same choice twice in a row before it is reciprocated

Naive Prober (Tit For Tat with D Random) – Plays TFT, but sometimes plays D randomly

Remorseful Prober (Tit For Tat with Random Defection) – As above, but if the opponent responds D to the taunt, they show remorse and re-cooperate

Naive Peace Maker (Tit For Tat con Random Cooperation) – TFT but from time to time tries to make peace and cooperate

True Peace Maker (Hybrid of TFT and TF2T and Random Cooperation) – Starts TF2T, plays D once, but sometimes cooperates when he must defraud

Random - 50% chance of C or D.

Always Defect

Always Cooperate

Grudger (Cooperate, but only support a D) - Cooperate until the opponent plays D, then always D

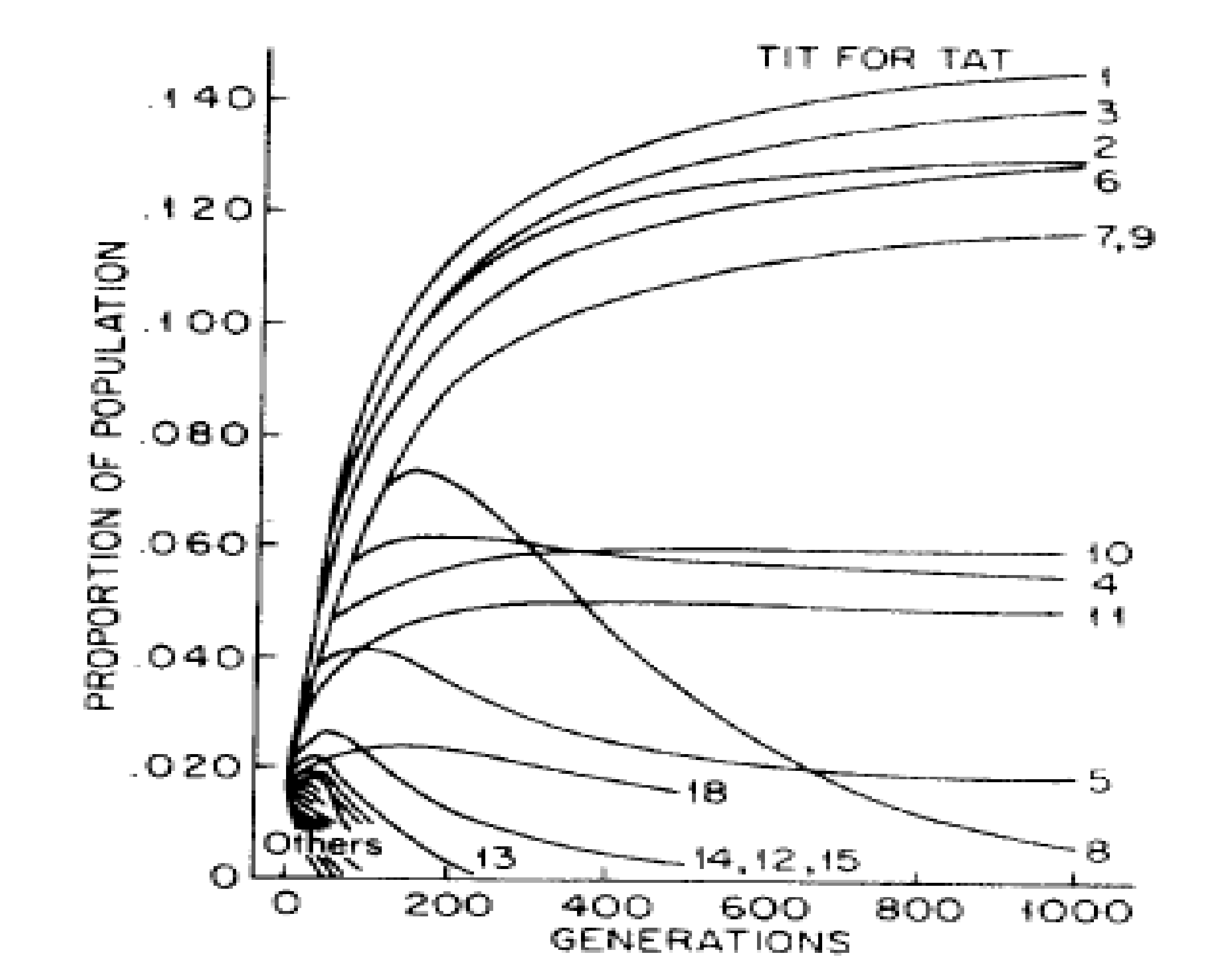
Pavlov- repeat last choice if good

Pavlov and Random

Gradual – Cooperate until the opponent defrauds, and play D the same number of times your opponent did in the game. Cooperate again twice.

Suspicious Tit For Tat - Like TFT but starts with D.

Iterated PD



TIT FOR TAT
1
3
2
6
7,9
10
4
11
5
8
18
14, 12, 15
13
Others
.140
.120
.100
.080
.060
.040
.020
0
PROPORTION OF POPULATION
0
200
400
600
800
1000
GENERATIONS

Altruistic Fish

Until now an exact representation of the conditions of the prisoner's dilemma has not been identified in nature.

Predator inspection in gregarious fish is close, but the stage is debatable

A pair of fish may break away from the group to swim close by and inspect the predator chasing the school

They receive a payment in the form of gaining knowledge about the predator

Two fish can move closer to the predator, so they benefit from cooperation

Also, one can "defect" benefit from the knowledge without risk So T>R>P>S is satisfied, but... Can they recognize previous deserters in order to punish them? Do they really prefer to approach in pairs? Does an inspector share information with the group?

Altruistic Fish

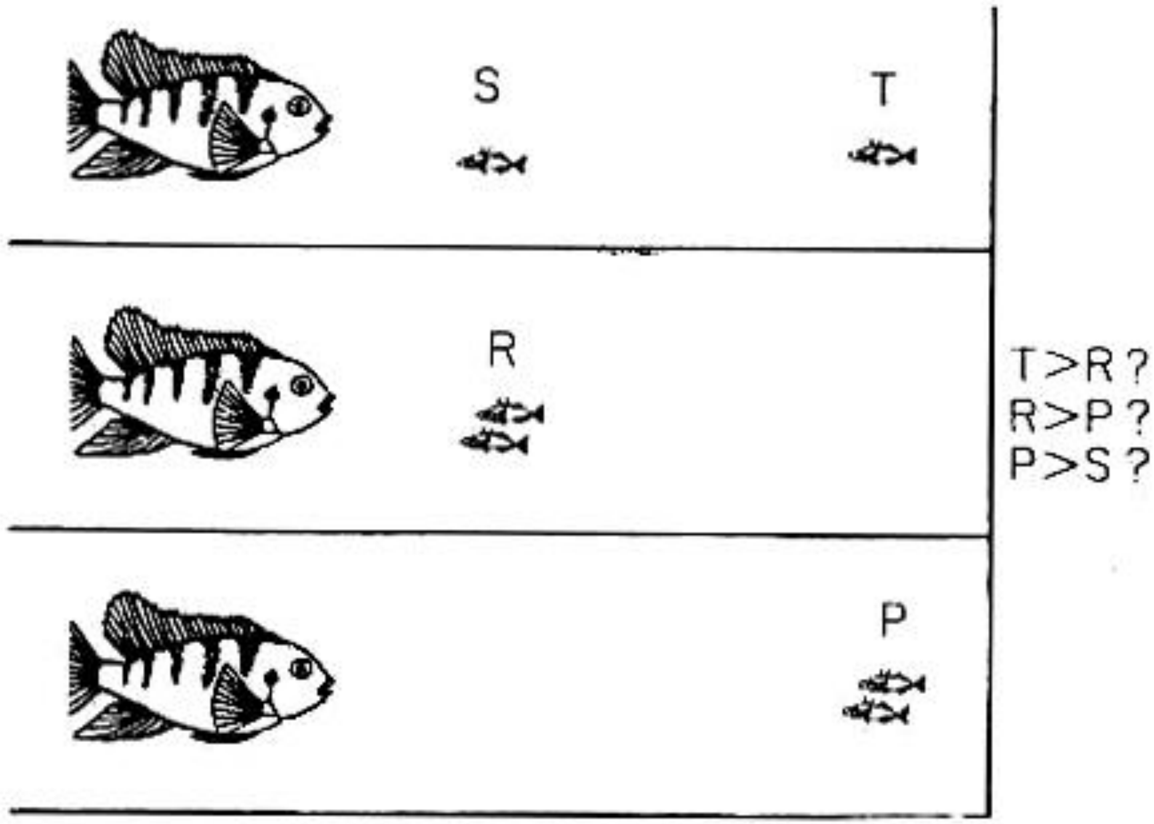


Fig. 3.8. Predator inspection behavior in pairs of fish as a possible prisoner's dilemma game. When one fish alone inspects, it receives the "sucker's" payoff (S), with its partner obtaining the "temptation to cheat" payoff (T). When both inspect, each receive the reward for mutual cooperation (R), and when neither inspects, they receive the punishment for mutual defection (P) (after Milinski, 1990).

Altruistic Fish

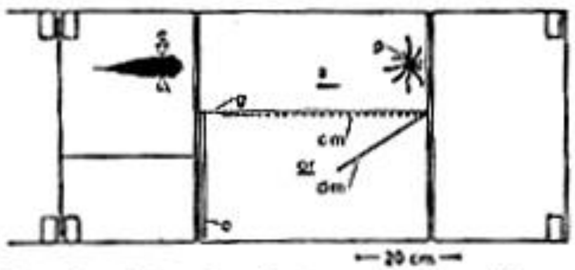


Fig. 1 Plan of the experimental equipment. The glass tanks shown contain the cichlid (c) and the plant (p), respectively. A typical position of a test stickleback (s) is also shown, as are the positions of the cooperating mirror (cm) and the defecting mirror (dm); 0, opaque screen; g, glass partition. Only one of the two mirrors was present in each experiment.

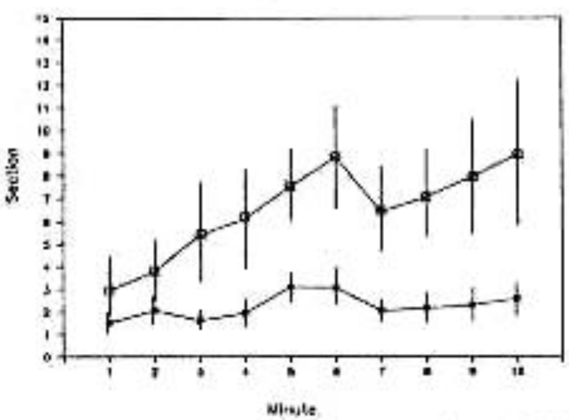


Fig. 1. Mean position (± 1 SE) of a guppy during each minute of a trial

Fish get closer with parallel mirrors than with oblique mirrors

Altruistic Vampire

Vampires: Desmodus rotundus, studied by Gerald Wilkinson in Trinidad

They live in groups

There are various degrees of kinship, sometimes 0

Sometimes vampires go out and get nothing

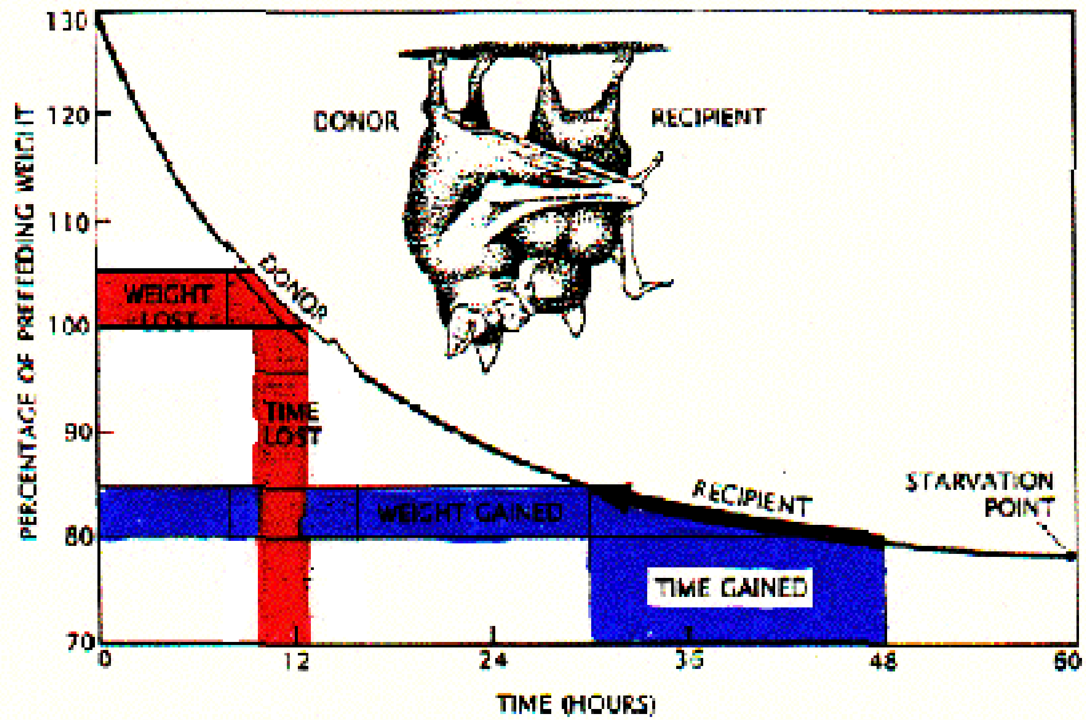
Other times, they get a cow and come back with a stomach full of blood.

They regurgitate blood to their companions

They do it with reciprocity, even in the absence of kinship

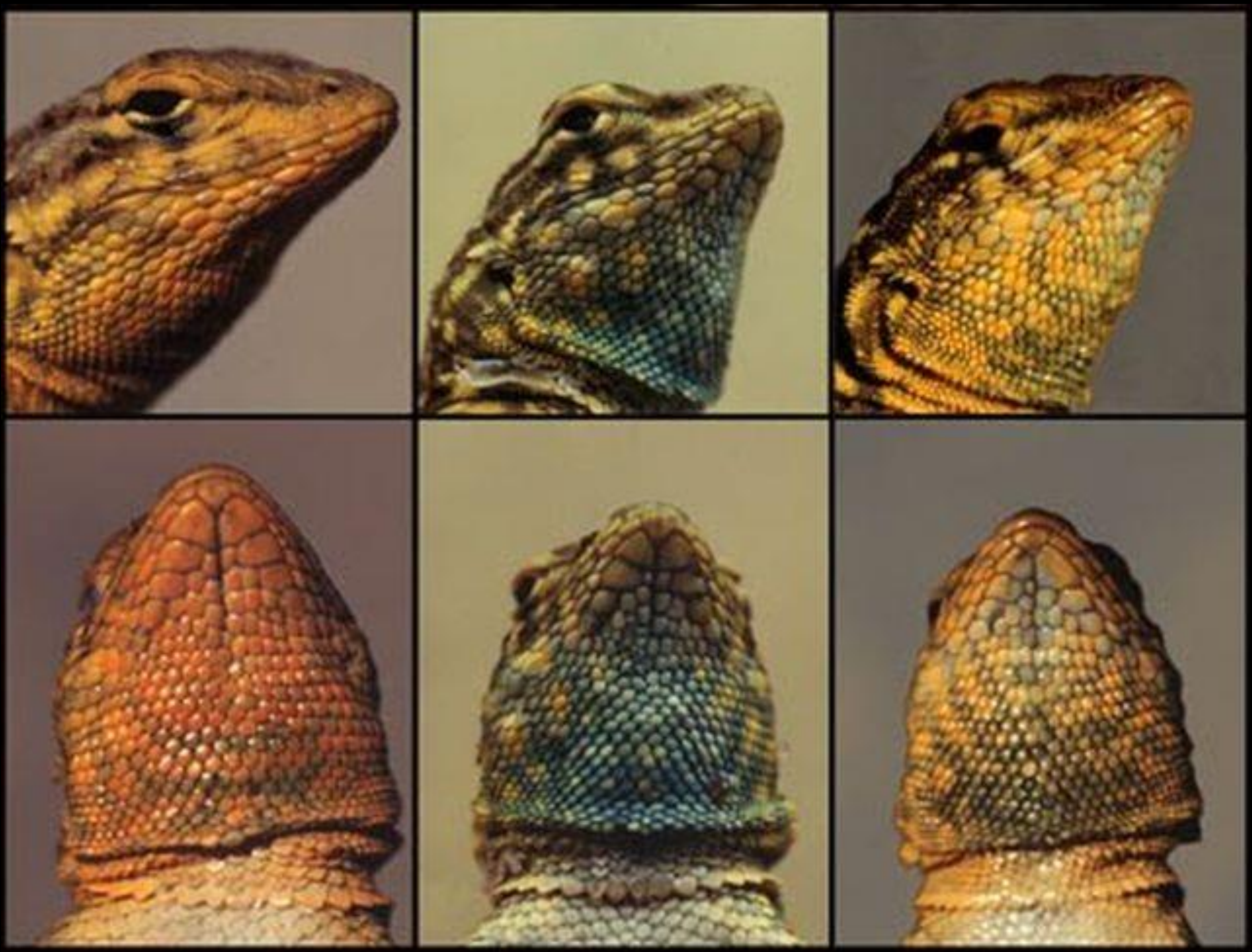
Altruistic Vampire

Altruistic Vampire



COST BENEFIT ANALYSIS of blood sharing among vampire bats indicates that recipients benefit more than donors lose. The author weighed adult females returning to the roost after feeding and then weighed them every hour for the next 24 hours. An individual who had fed might return at 130 percent of its prefeeding weight (half the weight of a blood meal is lost through urination within the first hour after feeding), whereas a bat who failed to feed on two successive nights might return at 80 percent of its earlier weight. By regurgitating five milliliters of condensed blood to a hungry roostmate, the donor bat might drop from 110 to 95 percent of its prefeeding weight but lose only six hours of the time it has remaining until starvation. The recipient bat, however, gains 18 hours and so benefits more than the donor loses.

Stone, Paper, Scissors



Stone, Paper, Scissors

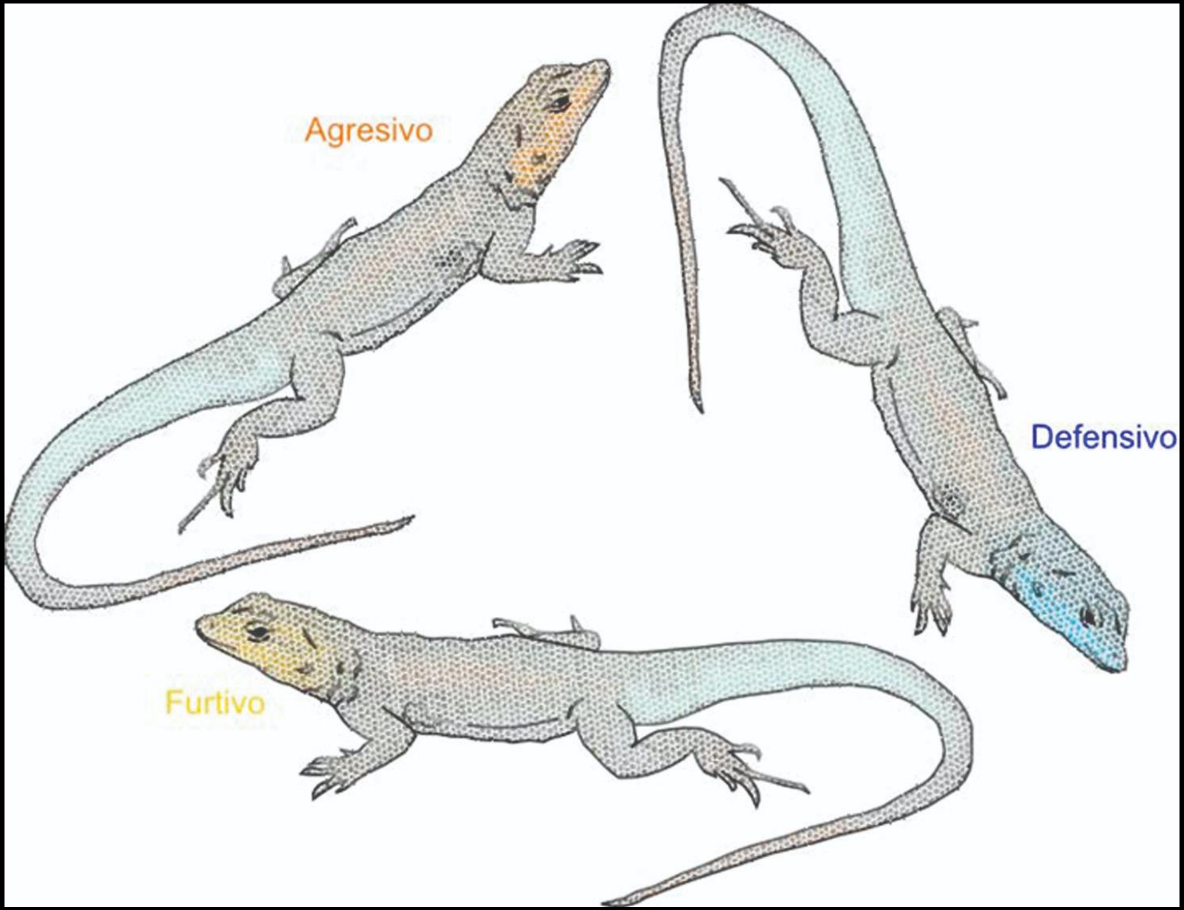
Male spotted flank lizards, Uta stansburiana, can exhibit one of three different reproductive strategies.
It lives in the arid zones of the southern USA and northern Mexico.

The three strategies in question are associated with males whose bodies have throats colored orange, blue or yellow

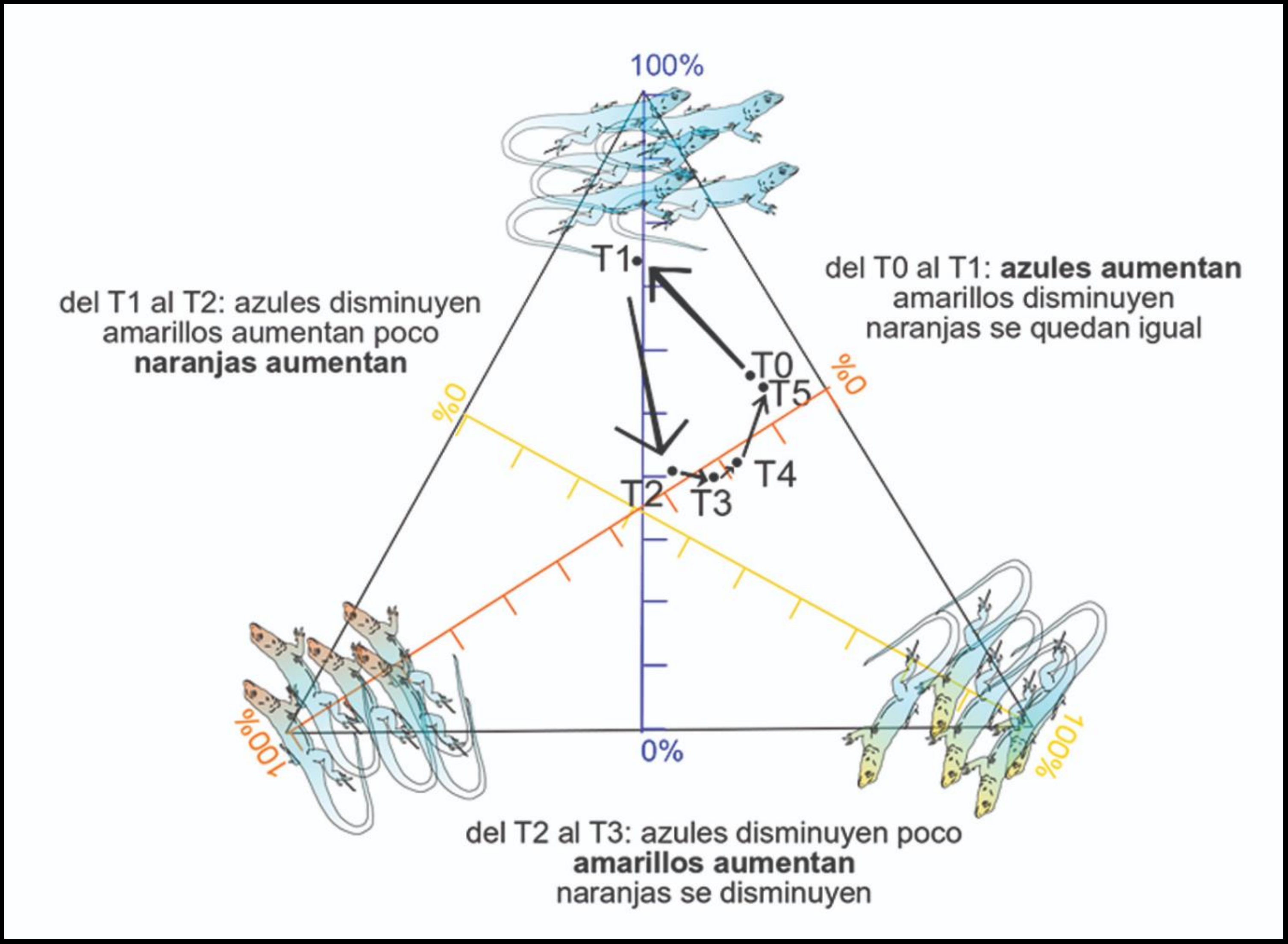
Orange-throated males have more testosterone and are the most aggressive, defending large territories and invading territories of blue-throated males.
In turn, the blue ones are less aggressive and defend smaller territories, they are less polygamous than the orange ones and they particularly defend certain females.

On the other hand, the males with yellow lines on their throats are furtive because they are confused with the females whose throat also has yellow lines; they even behave like them, and thus manage to go unnoticed by the territorial males and copulate with the females.

Stone, Paper, Scissors



Stone, Paper, Scissors



Mechanisms for the Evolution of Cooperation

Kin selection: I cooperate with genetic relatives.

Direct reciprocity: I help you, you help me.

Indirect Reciprocity: I help you, someone will help me.

Spatial Reciprocity: Neighbors help each other.

Group Selection: Groups of cooperators have better performance

Mechanisms for the Evolution of Cooperation

		Payoff matrix C	Payoff matrix D	
Kin selection	C	$(b-c)(1+r)$	$br-c$	r...genetic relatedness
	D	$b-rc$	0	
Direct reciprocity	C	$(b-c)/(1-w)$	$-c$	w...probability of next round
	D	b	0	
Indirect reciprocity	C	$b-c$	$-c(1-q)$	q...social acquaintanceship
	D	$b(1-q)$	0	
Network reciprocity	C	$b-c$	$H-c$	k...number of neighbors
	D	$b-H$	0	
Group selection	C	$(b-c)(m+n)$	$(b-c)m-cn$	n...group size m...number of groups
	D	bn	0	

b = benefit c= cost