Scaling functions in disordered elastic materials

Lectures for the ICTP-SAIFR School on Disordered Elastic Systems

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P.W. Anderson, in his 1978 Les Houches lectures on Ill Condensed Matter, emphasized that they were talking about a new paradigm -- asking ‘*How do disordered systems differ from regular systems’* rather than ‘*How may they be reduced to them’*. My first two lectures focus on avalanches – abrupt shifts from one locally stable state to another under an external stress, something that is different from behavior of liquids, gases, and perfect crystals. My last two lectures will discuss recent work with Danilo Liarte and Stephen Thornton, on how some aspects of disordered elastic systems may be studied by finding an effective 'ordinary' material describing the emergent properties. Everywhere, I focus on the use and importance of universal scaling functions.

I. Avalanches, the renormalization group, and universal scaling functions.

We introduce avalanches and crackling noise. We show how the renormalization group explains universality and emergent scale invariance. We use the renormalization group to derive universal critical exponents that relate pairs of quantities, and derive the universal scaling functions that arise when three or more variables are involved. We use the avalanche size distribution to illustrate universal scaling functions. We introduce (solvable) mean field theories, explain why their universal scaling functions are important even though they predict the behavior everywhere. We show how to derive the universal scaling function for the Ising model using the Curie-Weiss mean-field model very near to the critical temperature.

II. Universal scaling of avalanche properties in various systems

Avalanches arise in a wide variety of systems, and their universal scaling functions illustrate many important features found in Liarte and Thornton’s scaling theory in lecture IV. We introduce a mean-field model for pandemic outbreaks that also describes the random field Ising model above six dimensions. The mean-field prediction for the average temporal shape agrees well with Brazilian experiments done in Rio de Janeiro. We study the universal scaling function describing *corrections to scaling* from irrelevant variables, as applied to a model of fracture precursors – crackling noise heard in sea shells and bones before they finally break in two. We discuss *dangerous* irrelevant variables, that can’t be set to zero in the scaling function even though they vanish under coarse-graining because they control the behavior of important physical quantities. We discuss *crossover scaling* as applied to a experiment measuring the roughness of crack surfaces, showing a crossover from one theoretical prediction to another. And (postponed perhaps to lecture III) we discuss very different universal scaling functions needed for systems in the lower and upper critical dimension, and their analysis using normal form theory. We apply this normal form theory to the unusual avalanches seen in the 2D random-field Ising avalanche model.

III. Rigidity and linear elastic theory

We introduce two different rigidity transitions – systems where tuning a parameter takes one from a fluid, floppy state to a rigid, elastic state. The jamming transition of frictionless, soft spheres happens when they first all touch one another; it has a bulk modulus that jumps abruptly and a shear modulus that vanishes linearly at the transition. The rigidity percolation transition is modeled by diluting spring networks; it behaves differently, with both bulk and shear modulus vanishing linearly from the elastic side. We describe the sophisticated theory of linear response – correlation functions, susceptibilities, Greens functions, the dynamical matrix, and the relations between them. We argue that the frequency-dependent elastic moduli (calculated in lecture IV) can be used to predict all the linear response properties of jammed materials, and their crossover to rigidity percolation. We touch upon the relation of this mean-field theory to the glass transition.

IV. Universal scaling function for rigidity transitions: linear theory

We review the spring-lattice model for jamming of Liarte, Stenull, Mao, and Lubensky, and their coherent potential calculation of the effective frequency-dependent moduli. We introduce a simplified model – punching spherical holes into an isotropic elastic material, that allows us to solve for the universal scaling properties of this model explicitly, in a way we can explain (hopefully comprehensibly) in this lecture period. (0) We write the (complicated) self-consistent mean-field CPA equations for our isotropic jamming theory. (1) We change variables to the distances to the jamming and rigidity percolation points. (2) We cheat, by plugging in the correct scaling behavior for the control variables and the resulting behavior, and keep the lowest order terms in the distance to jamming. We including the leading correction to scaling, and show that it is a dangerous irrelevant variable, which is needed to describe the low frequency behavior. We end by discussing the predicted correlation function – agreeing both with the ‘boson peak’ of glasses and the constant density of states seen in simulations at the jamming transition.