Chiral Anomaly and the Interpretation with Spacetime Dependent Electromagnetic Fields

Kenji Fukushima

The University of Tokyo

— Workshop on Electromagnetic Effects in Strongly Interacting Matter —

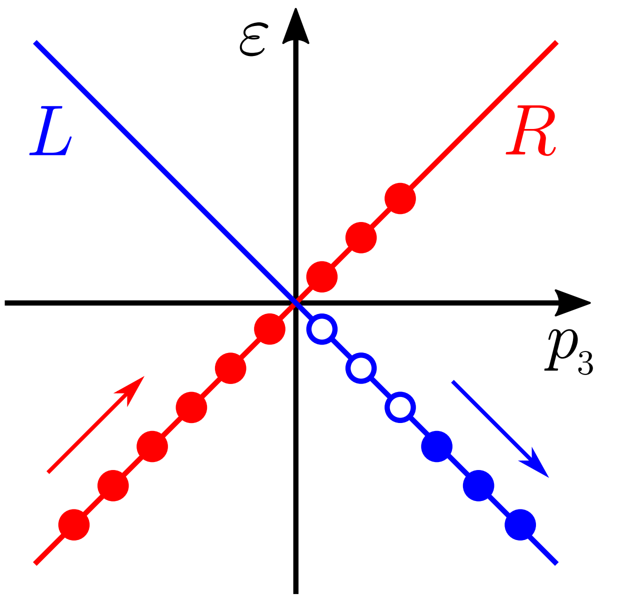
Overview: Constant Fields

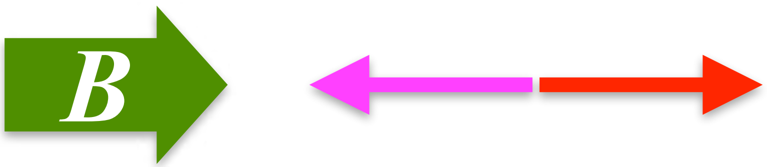
Particle/Chirality Production

Schwinger formula for the particle production

$$\omega = \frac{e^2 EB}{4\pi^2} \coth\left(\frac{B}{E}\pi\right) \exp\left(-\frac{\pi m^2}{eE}\right)$$

identifiable with chirality when $B >> E$ (LLL)

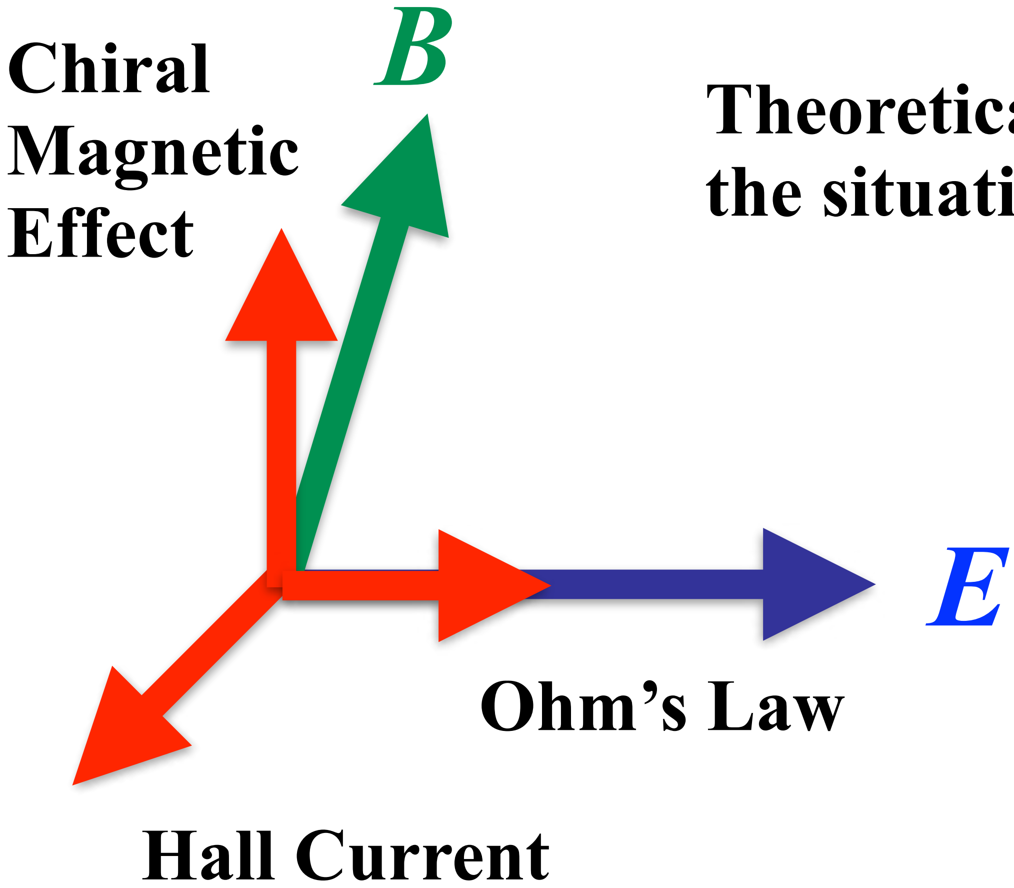




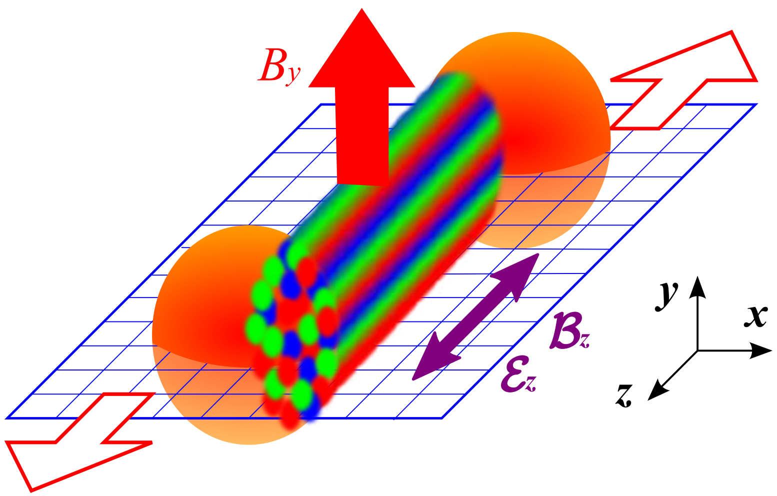
$(\Delta N_5 = 2)$

Strong $B \longrightarrow$ Dimensional Reduction
Momentum direction ~ Chirality

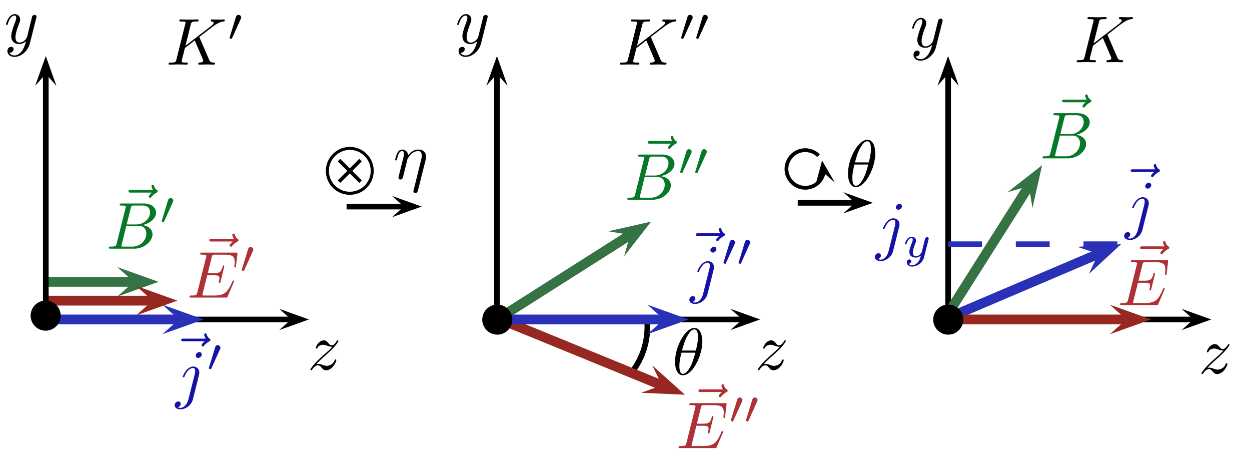
Particle/Chirality Production



Theoretical idealization of the situation in heavy-ion collisions



Particle/Chirality Production



Fukushima-
-Kharzeev-
-Warringa (2010)

Combination of the Lorentz boost + rotation of the axes

(In *K'* frame) $$\Gamma = \frac{q^2 E'_z B'_z}{4\pi^2} \coth\left(\frac{B'_z}{E'_z}\pi\right) \exp\left(-\frac{m^2\pi}{|qE'_z|}\right)$$

(In *K''* frame) $$\partial_t j_y \simeq \frac{q^2 B_y}{2\pi^2} \frac{g\mathcal{E}_z\mathcal{B}_z^2}{\mathcal{B}_z^2+\mathcal{E}_z^2} \coth\left(\frac{\mathcal{B}_z}{\mathcal{E}_z}\pi\right) \exp\left(-\frac{2m^2\pi}{|g\mathcal{E}_z|}\right)$$

Particle/Chirality Production

Schwinger formula for particle production vs. the axial Ward identity for chiral anomaly

$$\omega = \frac{e^2 EB}{4\pi^2} \coth\left(\frac{B}{E}\pi\right) \exp\left(-\frac{\pi m^2}{eE}\right)$$

$$\partial_t n_5 = \frac{e^2 EB}{2\pi^2} + 2m \underline{\langle \bar{\psi} i\gamma_5 \psi \rangle}$$

Mass dependence seemingly look very different?

Parity-odd condensate induced by *E* an *B*

Particle/Chirality Production

How to reconcile two correct formulas?

$$\partial_t n_5 = \frac{e^2 EB}{2\pi^2} + 2m\langle\bar{\psi} i\gamma_5 \psi\rangle$$

$$\langle \mathrm{out}|\bar{\psi} i\gamma_5 \psi|\mathrm{in}\rangle = -\frac{e^2 EB}{4\pi^2 m}$$

No chirality production in equilibrium

$$\langle \mathrm{in}|\bar{\psi} i\gamma_5 \psi|\mathrm{in}\rangle$$

$$= -\frac{e^2 EB}{4\pi^2 m}\left(1 - e^{-\pi m^2/eE}\right)$$

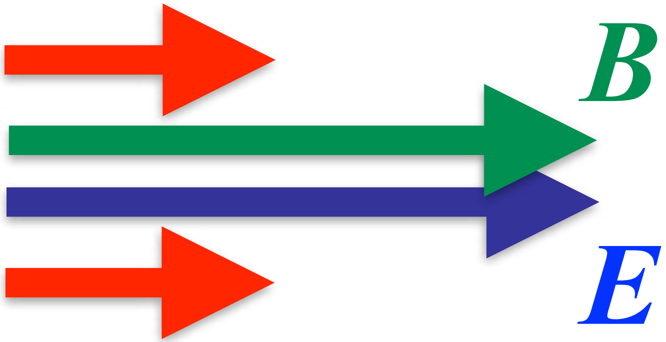
Schwinger formula reproduced!

Copinger-Fukushima-Pu (2018)

Particle/Chirality Production

Common setup for theory and experiment

$$j_{\mathrm{CME}} = \underline{(\boldsymbol{E}\cdot\boldsymbol{B})}\boldsymbol{B} \quad \propto \quad B^2$$



(relaxation time approx. is assumed to make the current finite)

$$j_{\mathrm{Ohm}} = \sigma \boldsymbol{E}$$

$$j = (\sigma_{\mathrm{Ohm}} + \sigma_{\mathrm{CME}})E \qquad \sigma_{\mathrm{CME}} \propto B^2$$

Son-Spivak (2012)

Configuration of “external” fields

Particle/Chirality Production

In theoretical calculations constant fields are always assumed for simplicity.

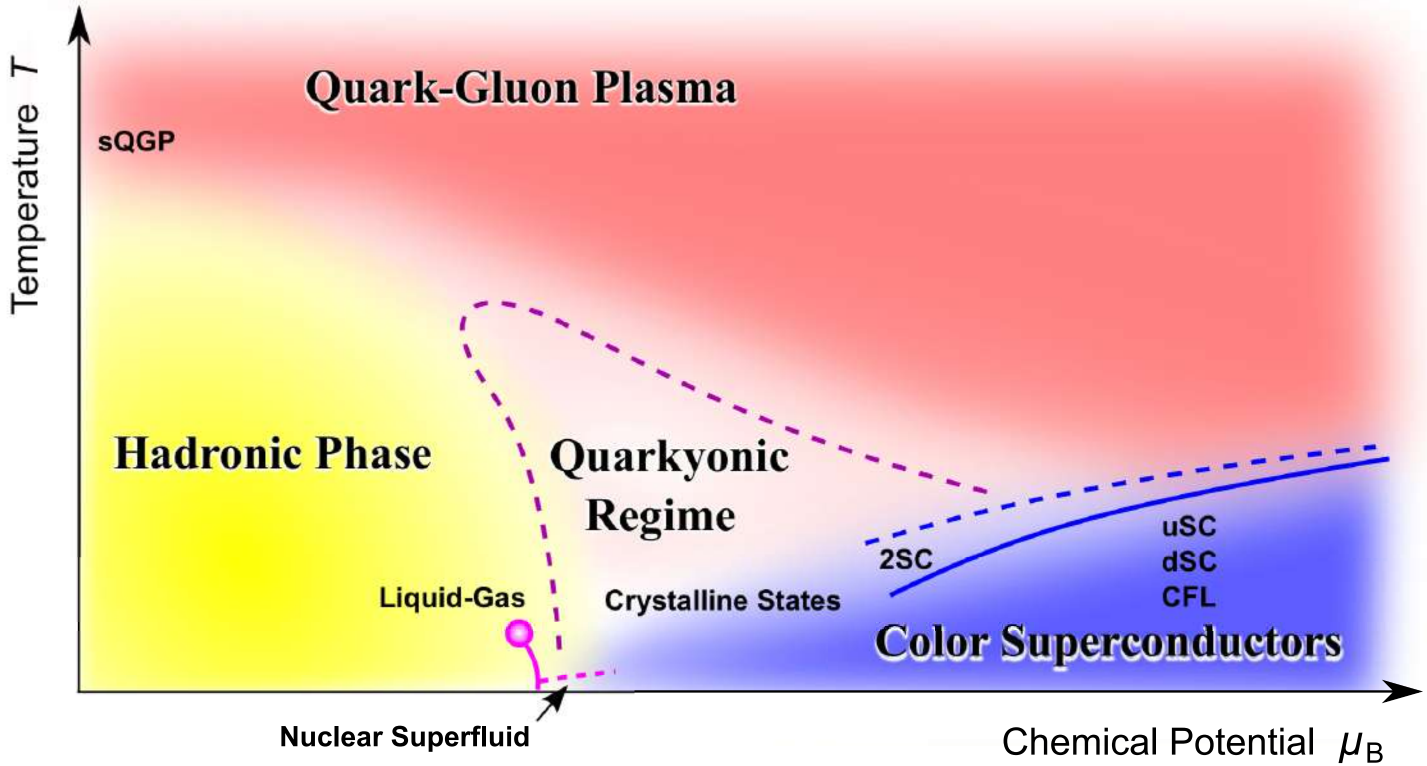
In the real world constant fields cannot be found at all!

Not only EM fields but QCD matter may be also very inhomogeneous.

Frequently asked question but…

Some Nontrivial Examples: Inhomogeneity in QCD

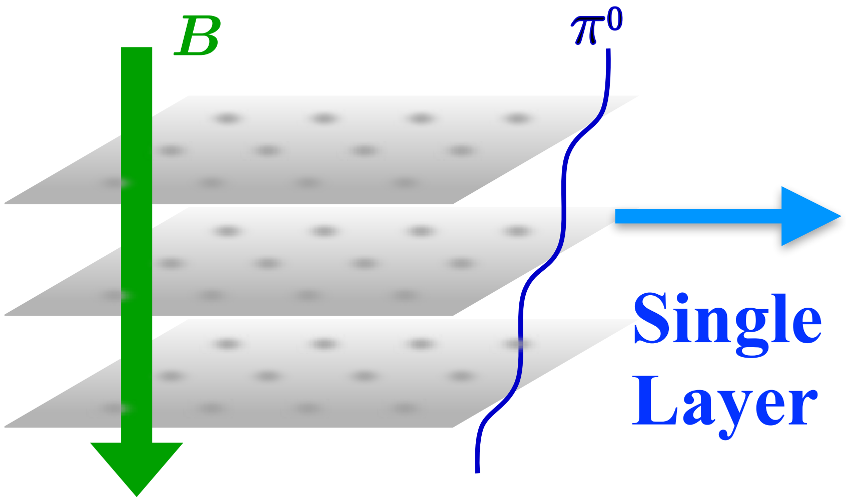
Case I : High-Density QCD



High-density QCD matter (Quarkyonic Matter)
→ Inhomogeneous Skyrme Crystal (in large N_c).

Case I : High-Density QCD

Chiral Soliton Lattice by Brauner-Yamamoto: 1609.05213



One Skyrmion in a "**periodic**" box can realize a Skyrme Crystal.
(Klebanov 1985, Goldhaber-Manton 1987)

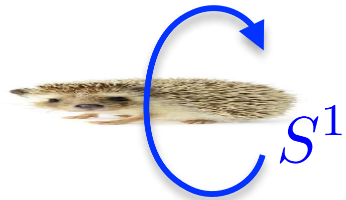
The boundary condition must be consistent with the discrete symmetries and the baryon number (winding).

Only two possibilities: Normal Crystal & Anomalous Crystal

Case I : High-Density QCD

Without B the baryon number is given by

$$\pi_3(\mathrm{SU}(2)) = \mathbb{Z}$$



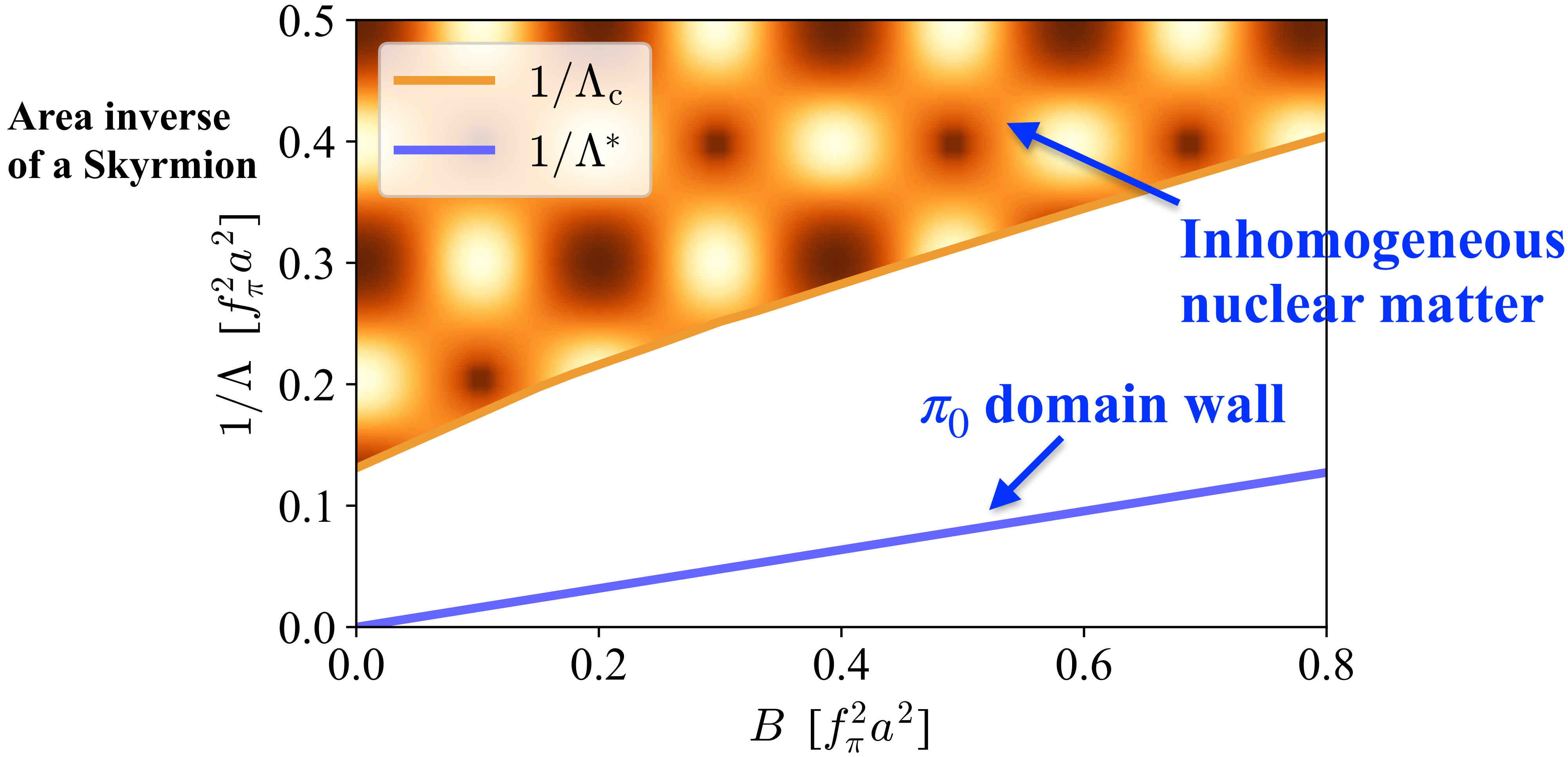
Distinct homotopy connected?

$$\pi_3(\mathrm{SU}(2)) = \mathbb{Z} \longrightarrow \pi_1(\mathrm{U}(1)) = \mathbb{Z}$$

$B = 0$ **Phase Transition** $B \to \infty$

Case I : High-Density QCD

Chen-Fukushima-Qiu (2021)

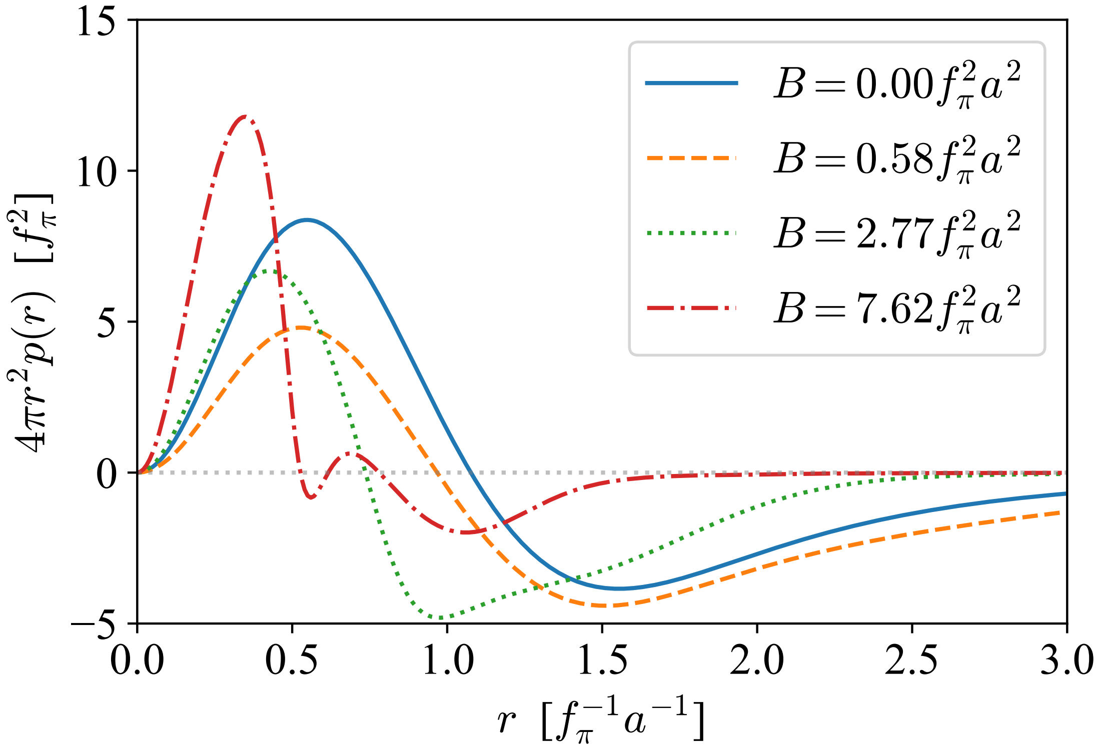


Case I : High-Density QCD

Chen-Fukushima-Qiu (2021)

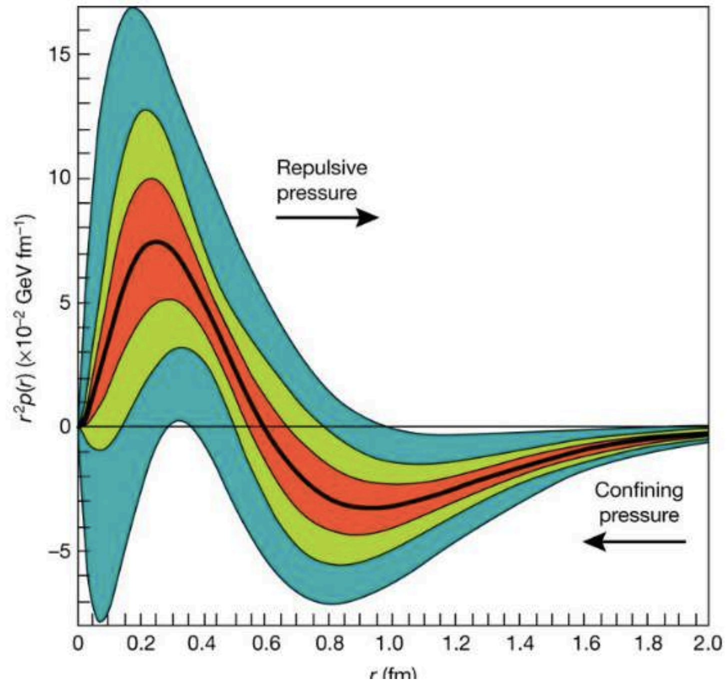
Single Skyrmion (baryon) under strong *B* is as interesting!

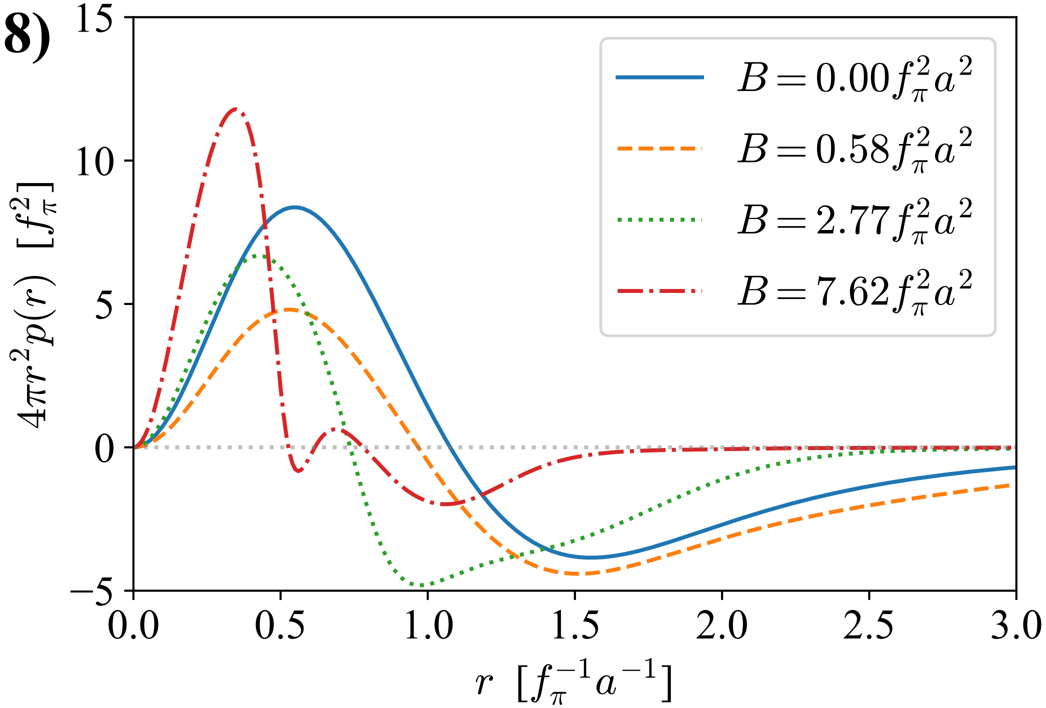
[Pressure Distribution]



Case I : High-Density QCD

Burkert-Elouadrhiri-Girod (2018)





Confinement force is probed in EIC.

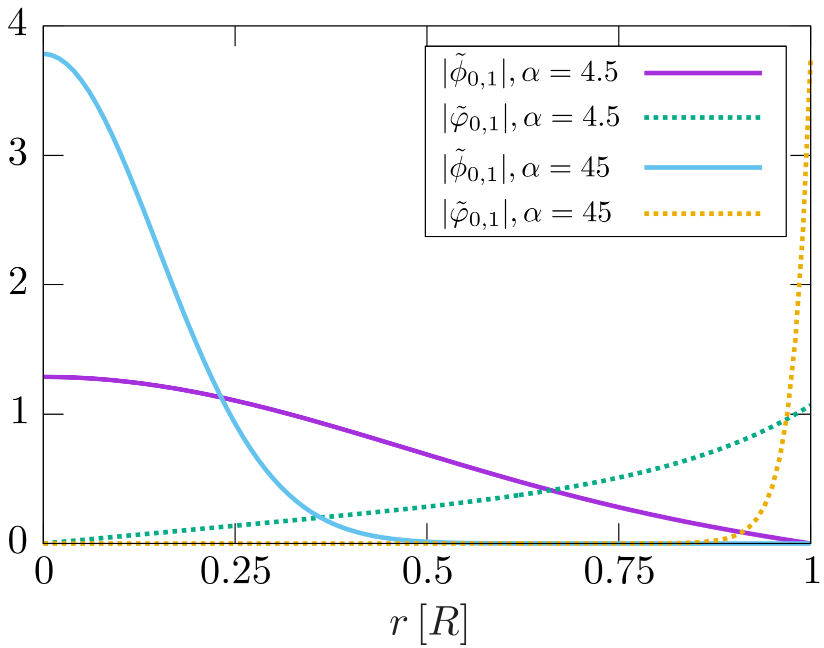
Extra confining force from the external *B*!?

Chen-Fukushima-Qiu (soon)

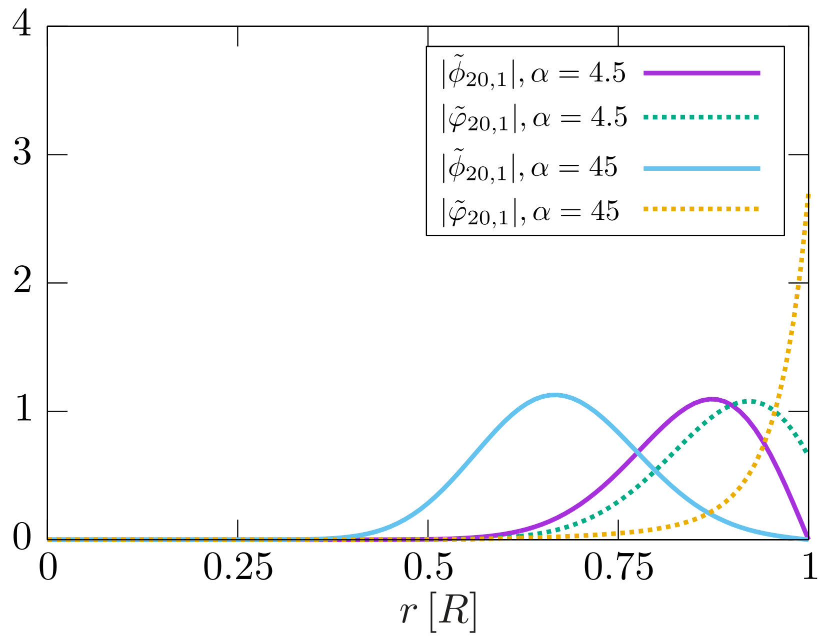
$$\int d^3x P(x) = -\frac{2}{3}\boldsymbol{m}\cdot\boldsymbol{B}$$

Case II : Finite Systems

For another example, the physical system is always finite, and as soon as the boundary condition in a finite box is imposed, you cannot avoid inhomogeneity!



$l = 0$



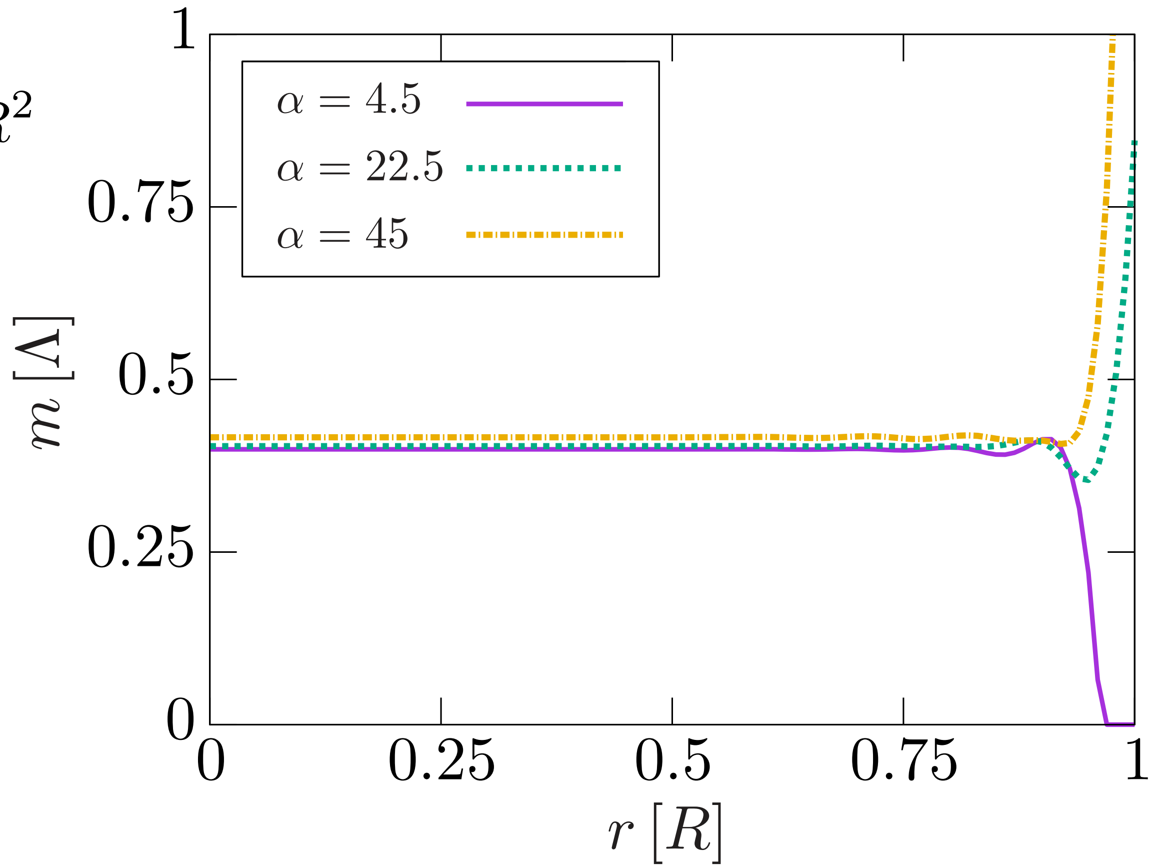
$l = 20$

Centrifugal force pushes the wave-functions farer.

Case II : Finite Systems

Magnetic catalysis is inhomogeneous in a box.

$$\alpha \equiv \frac{1}{2} eBR^2$$



Chen-Fukushima-Huang-Mameda (2017)

Case III : Vortical Matter

Inhomogeneous EM? Of course Yes!

One unknown example, for which we are now working very hard : Fukushima-Yu (soon).

Production of twisted particles in heavy-ion collisions

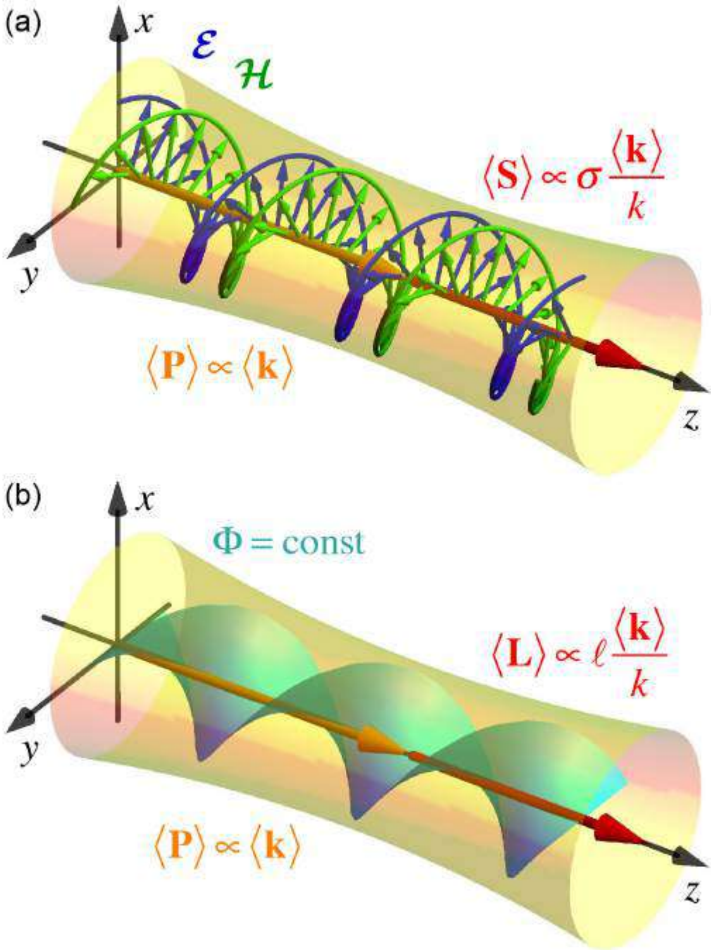
Liping Zou[1],* Pengming Zhang[2],† and Alexander J. Silenko[3,4,5]‡

arXiv:2112.12404 [hep-ph]

Photons can carry not only the helicity but also the twisted wave-function (with orbital angular momentum) ← Paraxial Photons

Case III : Vortical Matter

Bliokh-Nori (2015)

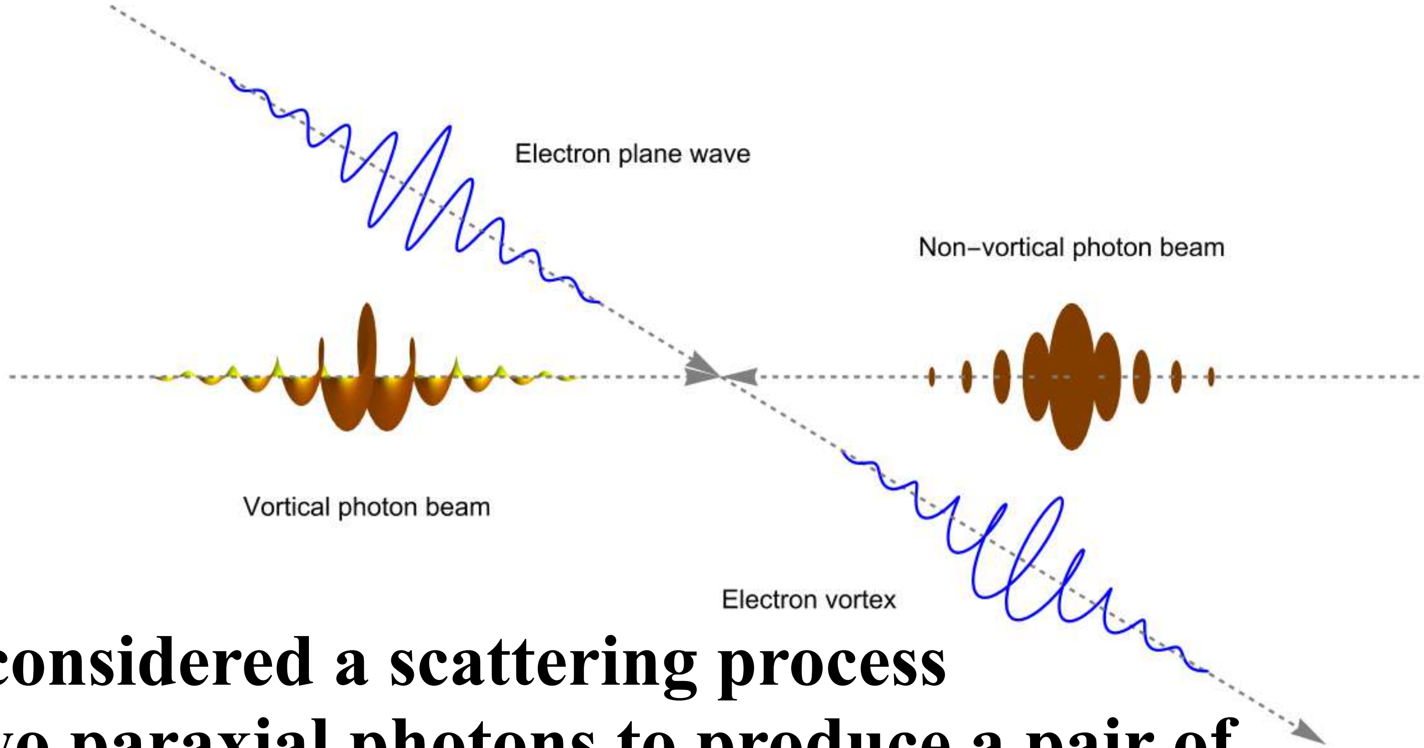


Polarization

Photon vortex beam
with finite orbital angular mom.

Case III : Vortical Matter

Fukushima-Yu (soon)

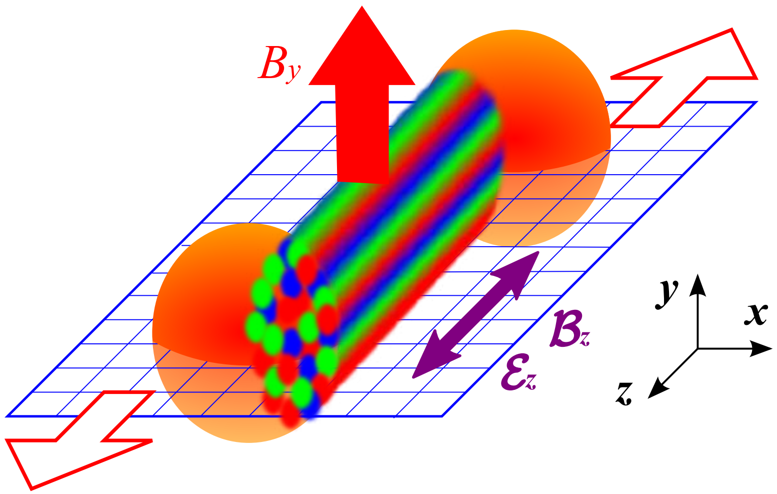


We considered a scattering process of two paraxial photons to produce a pair of fermions, confirming finite chirality production.

New (and more convenient) probe to the chiral anomaly

Simple Modeling of Local Parity Violation

CME requires Parity-odd EM



Possibly, it appears from the Glasma initial condition with parallel chromo-*E*/*B*.

**QCD (QED) is a parity conserving theory,
and the parity is not broken in the whole system.**

Violation is only local $\rightarrow$ **Local Parity Violation**

$$\boldsymbol{E}\cdot\boldsymbol{B}\neq 0 \qquad \int d^3x\,\boldsymbol{E}\cdot\boldsymbol{B}=0$$ **Very inhomogeneous**

LPV from the Maxwell Eq.

Electromagnetic plane waves with parallel electric and magnetic fields E||H in free space

Koichi Shimoda, Toshio Kawai, and Kiyoji Uehara
Department of Physics, Keio University, 3-14-1 Hiyoshi, Kohokuku, Yokohama 223, Japan

(Received 31 October 1988; accepted for publication 7 June 1989)

Maxwell's equations for electromagnetic waves in free space, with parallel electric and magnetic fields, are solved under the condition that the fields are functions of z and t, independent of either x or y. A number of interesting examples of plane waves obtained from the general solution are shown. Their properties and the vanishing Poynting vector are discussed.

Usually, the electromagnetic waves (as solutions of the free/source-less Maxwell equations) have propagating *E* and *B* perpendicular to each other.

$$\boldsymbol{E}\cdot\boldsymbol{B}=0$$

LPV from the Maxwell Eq.

They found a family of solutions:

$$E_x = \cos(F+G)\cos(F-G)$$

F and *G* are arbitrary functions of *z*+*t* and *z*-*t*

$$E_y = \cos(F+G)\sin(F-G)$$

Standing Wave

$$H_x = \sin(F+G)\cos(F-G)$$

These are not general solutions, and other types of solutions do exist.

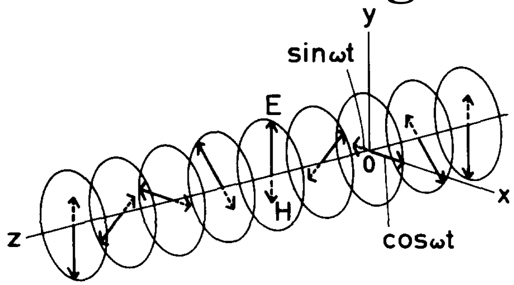
$$H_y = \sin(F+G)\sin(F-G)$$

Parity-odd domains (local parity violation) may occur locally but its spatial average is zero not to break parity.

LPV from the Maxwell Eq.

A particularly interesting example:

Helical Standing Wave



$$F = \frac{k}{2}(z + t) \qquad G = -\frac{k}{2}(z - t)$$

An example of the vector potential (giving *E* and *B*)

$$A_0 = 0 \qquad \boldsymbol{A} = \frac{a}{\omega}(-\cos\omega z \sin\omega t, \cos\omega z \cos\omega t, 0)$$

$$\boxed{\boldsymbol{E} \cdot \boldsymbol{B} = \frac{a^2}{2} \sin 2\omega z} \qquad \boxed{(\boldsymbol{E}^2 - \boldsymbol{B}^2) = a^2 \cos 2\omega z}$$

Twisted-modes: Evtuhov-Siegman (1965) / Chu-Ohkawa (1982)

LPV from the Maxwell Eq.

$$\partial_\mu j_5^\mu = \frac{e^2}{2\pi^2}\boldsymbol{E}\cdot\boldsymbol{B} \neq 0$$ **Particle Production?**

No particle production (no real-time physics)

We can compute the Chern-Simons current:

$$K^\mu = \frac{e^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma$$ so that $$\partial_\mu(j_5^\mu - K^\mu) = 0$$

In particular $\int d^3x\, K^0$ is gauge inv. (magnetic helicity)

$$K^0 = 0 \qquad K^z = \frac{e^2}{4\pi^2}\frac{a^2}{\omega}\cos^2\omega z$$ **Only spatial current!?**

LPV from the Maxwell Eq.

How to proceed…?

We tried to find analytical solutions, but it seems to be impossible (or very difficult).

$$\boldsymbol{A} = \frac{a}{\omega}(-\cos\omega z \sin\omega t, \cos\omega z \cos\omega t, 0)$$

This is a space-time dependent external potential.

Time-periodically driven system!

A very good theoretical approach is known: Floquet

Floquet

Floquet Theorem:

For periodic Hamiltonians: $H(t+T) = H(t)$
The time evolution operator is decomposed into:

$$U(t,t') = \mathrm{e}^{-\mathrm{i}K(t')}\mathrm{e}^{+\mathrm{i}H_{\mathrm{F}}(t-t')}\mathrm{e}^{+\mathrm{i}K(t)}$$

Kick operator
(t-periodic to moving on a "co-rotating" bases)

Floquet Hamiltonian
t-independent physics!

→ **Floquet Engineering**

Floquet

Construction of K and H_F

The existence is guaranteed, but the actual calculation is a different story… a well-developed approach is a systematic expansion in terms of $1/\omega$

Decomposition is not unique and a sort of "gauge fixing" is necessary.

→ Different schemes for the high-freq. (Magnus) expansion. (We adopt the van Vleck expansion.)

Floquet

A question:

Floquet Hamiltonian is static.

Theory drops information along the time axis…
How it is possible to retain the chiral anomaly that can exist in (3+1) dimensions, but NOT in (3+0) dimensions…???

We can understand this by solving everything explicitly for the chiral anomaly in (1+1) dimensions.
(It does not exist in (1+0) dimensions, either.)

Analysis in (1+1) Dimensions

Let us consider a simpler problem in (1+1)D.

Fukushima-Hidaka-Shimazaki-Taya (soon)

Arbitrary background fields satisfying:

$$A^1 = 0 \qquad\qquad A^0(t+T, x) = A^0(t, x)$$

The answer is already known:

$$\partial_\mu J_5^\mu = \frac{eE}{\pi} + 2mP, \qquad P := \langle \hat{\bar{\psi}} \mathrm{i}\gamma_5 \hat{\psi} \rangle$$

Analysis in (1+1) Dimensions

Original Hamiltonian for Weyl fermions

$$H_{\mathrm{R/L}} = \underbrace{\mp \mathrm{i}\partial_x + \sqrt{2}e\tilde{A}_0^{\mp}}_{=\tilde{H}_0} + \underbrace{\sum_{l\neq 0} \mathrm{e}^{+\frac{2\pi \mathrm{i} l}{T}t}\sqrt{2}e\tilde{A}_l^{\mp}}_{=\sum_{l\neq 0}\mathrm{e}^{+\frac{2\pi \mathrm{i} l}{T}t}\tilde{H}_{l\neq 0}}$$

Explicitly constructed Kick opeartor:

$$K(t,x) = \sum_{n=1}^{\infty}(\mp\partial_x)^{n-1}\int^t \mathrm{d}t_n \int^{t_n}\mathrm{d}t_{n-1}\cdots\int^{t_2}\mathrm{d}t_1\sqrt{2}\left(eA^{\mp}(t_1,x) - e\tilde{A}_0^{\mp}(x)\right)$$

Floquet Hamiltonian

$$H_{\mathrm{F}} = \tilde{H}_0$$

Simply only in (1+1)D!

Analysis in (1+1) Dimensions

To make the long story very short, the axial current expectation value involves the point-splitting regularization leading to a derivative term from the Kick operator.

Floquet Hamiltonian does not carry information of the chiral anomaly, as suspected correctly.

Very first time to point out that the Kick operator is so essential for physics.

Analysis in (1+1) Dimensions

One might think that this is only an academic exercise?

No… the Kick operator is a transformation onto the “co-rotating” bases: consider a real rotating system! EM fields look like time-periodically driven potentials!

Time-periodicity can be canceled by the unitary rotation, but the chiral anomaly is retained in this part.

A very interesting subject: Chiral anomaly with EM fields in a rotating frame

Fukushima-Hattori-Mameda-Taya (on going)

Summary

- **Many are known for constant EM fields**
  - □ Relation between the particle production and the axial Ward identity (chiral anomaly).
  - □ In-In and In-Out differences

- **Inhomogeneity (in spacetime) is everywhere**
  - □ QCD phase diagram with strong *B* effect
  - □ Rotating QCD medium producing paraxial photons
  - □ LPV backgrounds modeled with the Maxwell eq.

- **Theoretical approach based on the Floquet theory**