Building Quantum Machines with Light

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INCT-IQ

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Quantizing the Field

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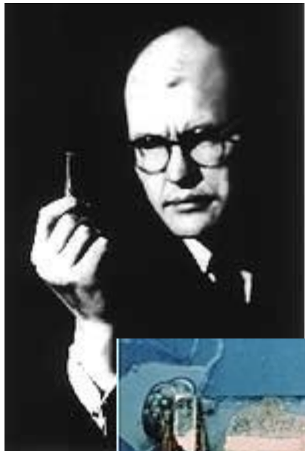
USP

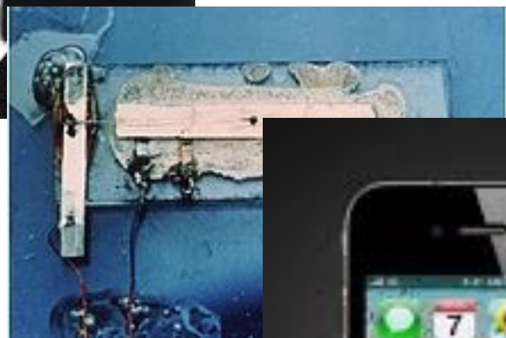
- Quantum Mechanics as an Information Theory.
- Complete knowledge about a system.
- Use of QM for information processing.
- Superposition: a much denser space than allowed by classical mechanics.
- Entanglement: nonlocal correlations at quantum level.
- Which hardware will we use?
- What can we learn about Quatum Mechanics along this process?

First Quantum Revolution

John Bardeen, William Shockley and Walter Brattain at Bell Labs, 1948. Nobel 1956

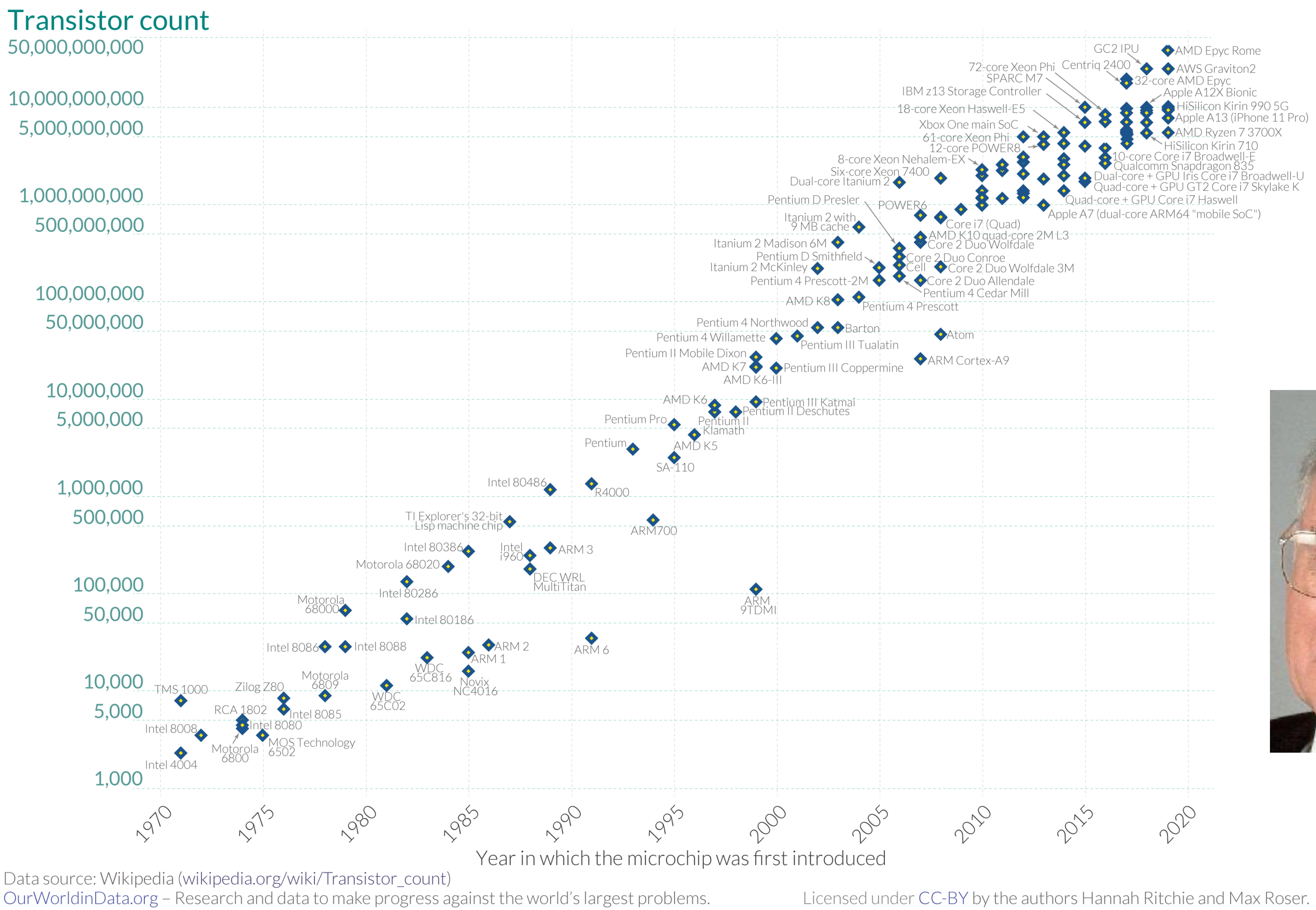
Jack Kilby
(Texas Instrument, 1958),
Robert Noyce (Intel co-founder)
Nobel 2000

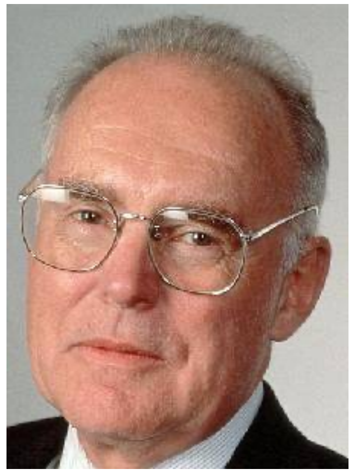




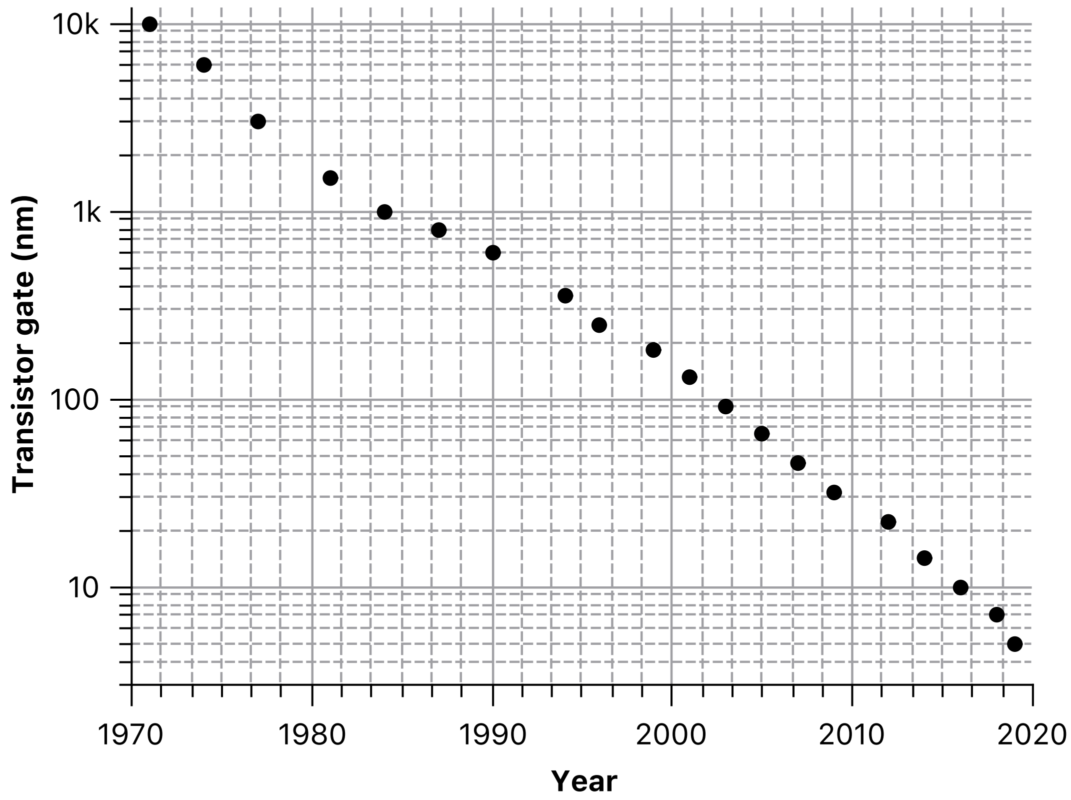
Theodore Maiman @ Hughes Research Lab (1960)

- Living in a world ruled by Information.
- A transformation as big as the Industrial Revolution.
- Built on an exponential increase of processing resources.





- The pattern is reproduced in storage capacity (memories), bit rate in communication …



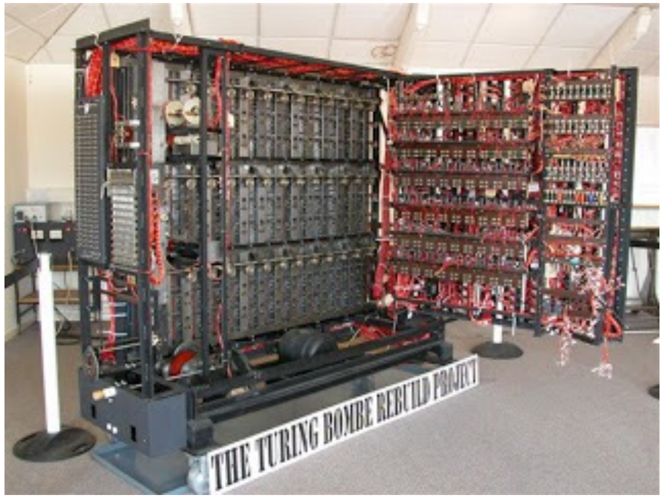
- But for how long can the silicon based CMOS stand the growth? Dissipation, clock frequency, transistor size, switching voltage are imposing limits.

- Expected saturation by 2050.

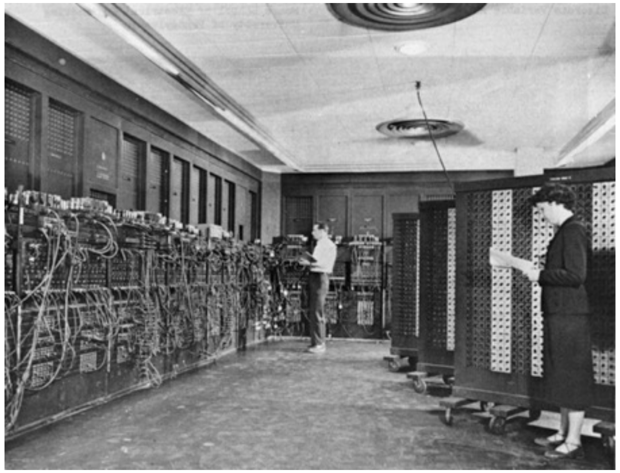
- Electronics and photonics are a proof of success in quantum information.

- But information itself remained treated in a classical way:

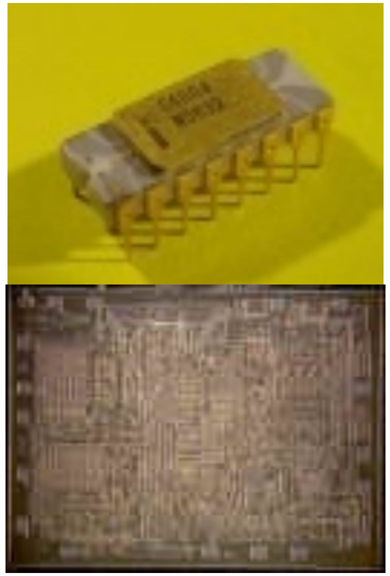
  we have just miniaturized relays and wires.



Bombe →
(bomba kryptologiczna)



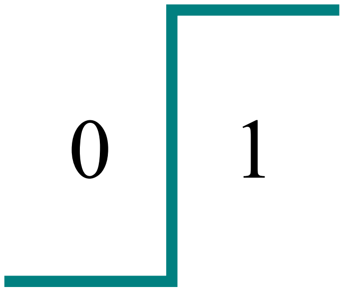
Eniac (1946) →



Intel 4040 (1971)

- What will be the ultimate processing units?
- Will we be forced to change from classical to quantum "bits"?

Bit

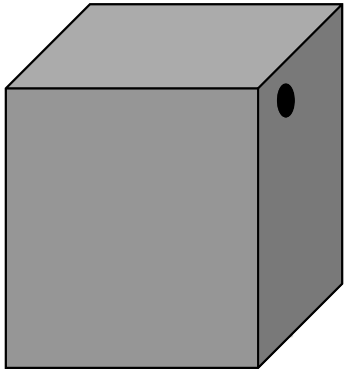


Qubit

Cross the boundary between classical and quantum…

Quantum Mechanics

Birth of a revolution at the dawn of the 20th Century



Introduction of the concept of "quanta"

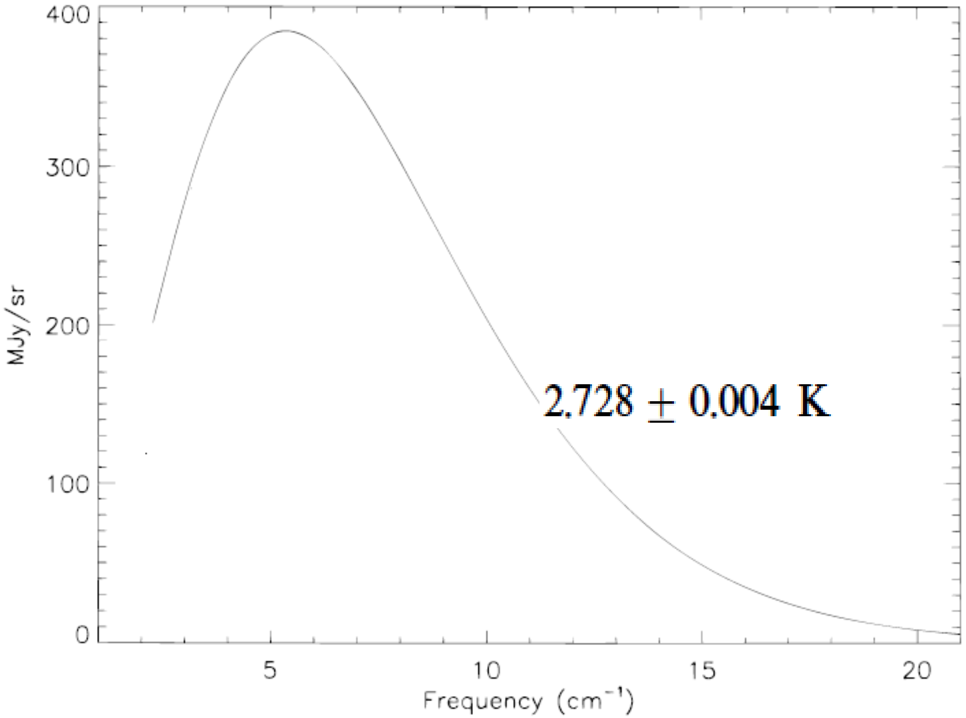


Fig. 4.—Uniform spectrum and fit to Planck blackbody (T). Uncertainties are a small fraction of the line thickness.

Energy per unit volume per unit wavelength
$$S_\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

Energy per unit volume per unit frequency
$$S_\upsilon = \frac{8\pi h}{c^3}\frac{\upsilon^3}{e^{h\upsilon/kT}-1}$$

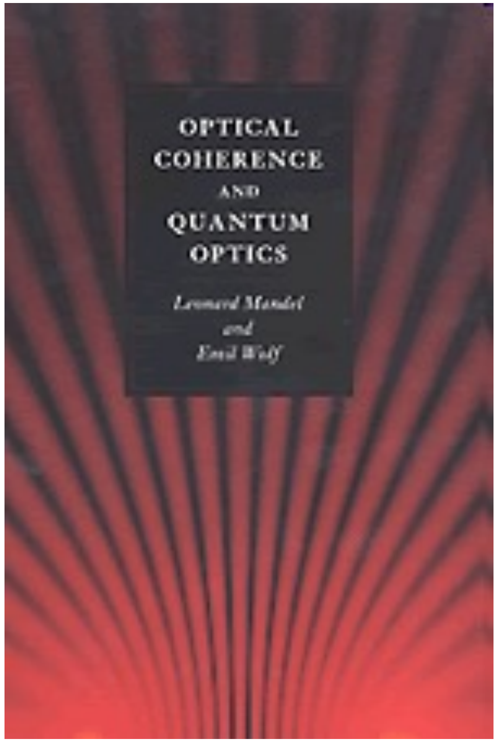
THE COSMIC MICROWAVE BACKGROUND SPECTRUM FROM THE FULL *COBE*[1] FIRAS DATA SET

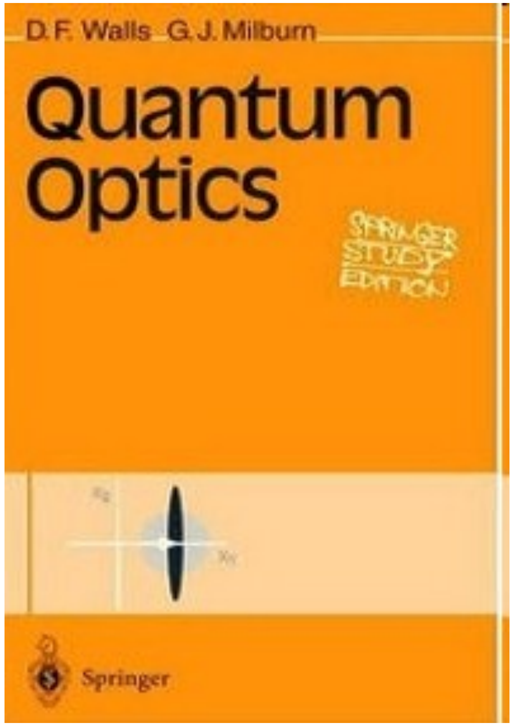
D. J. Fixsen,[2] E. S. Cheng,[3] J. M. Gales,[2] J. C. Mather,[3] R. A. Shafer,[3] and E. L. Wright[4]

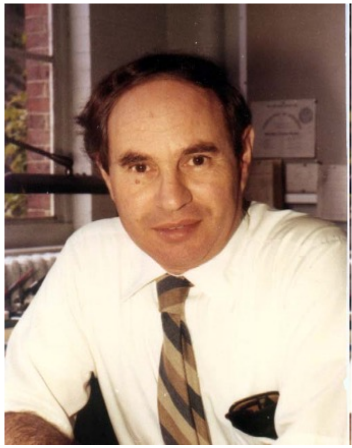
The Astrophysical Journal, 473:576–587, 1996 December 20

Quantum Optics

Quantization of the Electromagnetic Field (on the shoulders...)







Optics

Maxwell Equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Solution in a Box

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{\epsilon_0 L^3}} \sum_{\mathbf{k}} \sum_{s} i\omega \left[u_{\mathbf{k}s}(t) \boldsymbol{\epsilon}_{\mathbf{k}s} e^{i\mathbf{k}\cdot\mathbf{r}} - u^*_{\mathbf{k}s}(t) \boldsymbol{\epsilon}^*_{\mathbf{k}s} e^{-i\mathbf{k}\cdot\mathbf{r}} \right],$$

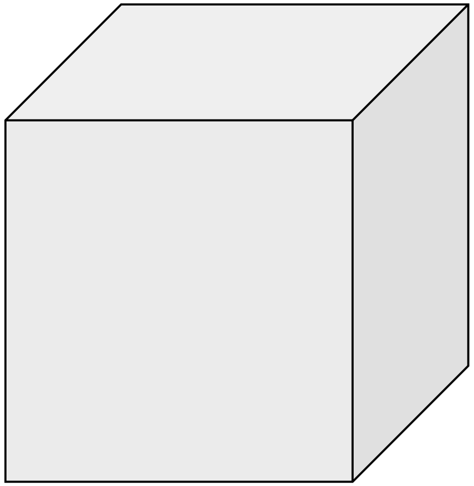
$$\mathbf{B}(\mathbf{r},t) = \frac{i}{\sqrt{\epsilon_0 L^3}} \sum_{\mathbf{k}} \sum_{s} \left[u_{\mathbf{k}s}(t)(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}s}) e^{i\mathbf{k}\cdot\mathbf{r}} - u^*_{\mathbf{k}s}(t)(\mathbf{k} \times \boldsymbol{\epsilon}^*_{\mathbf{k}s}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right]$$

Wavevector

$$k_j = 2\pi n_j / L$$

Angular Frequency

$$\omega = c|\mathbf{k}|$$



Amplitude

$$u_{\mathbf{k}s}(t) = c_{\mathbf{k}s} e^{-i\omega t}$$

Polarization

$$\boldsymbol{\epsilon}^*_{\mathbf{k}s} \cdot \boldsymbol{\epsilon}_{\mathbf{k}s'} = \delta_{ss'}$$

$$\boldsymbol{\epsilon}^*_{\mathbf{k}1} \times \boldsymbol{\epsilon}_{\mathbf{k}2} = \mathbf{k}/k$$

Energy of the EM Field

$$\mathcal{H} = \frac{1}{2}\int_V \left[\epsilon_0 \mathbf{E}^2(\mathbf{r},t) + \frac{\mathbf{B}^2(\mathbf{r},t)}{\mu_0}\right] dv = 2\sum_{\mathbf{k}}\sum_{s}\omega^2 |u_{\mathbf{k}s}(t)|^2$$

Canonical Variables: going into Hamiltonian formalism

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + u^*_{\mathbf{k}s}(t)$$
$$p_{\mathbf{k}s}(t) = -i\omega\left[u_{\mathbf{k}s}(t) - u^*_{\mathbf{k}s}(t)\right]$$

Energy of the EM Field

$$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{s}\left[p_{\mathbf{k}s}^2(t) + \omega^2 q_{\mathbf{k}s}^2(t)\right]$$

Canonical Variables: going into Hamiltonian formalism

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + u_{\mathbf{k}s}^*(t)$$

$$p_{\mathbf{k}s}(t) = -i\omega\left[u_{\mathbf{k}s}(t) - u_{\mathbf{k}s}^*(t)\right]$$

Energy of the EM Field

$$\mathcal{H} = \frac{1}{2} \sum_{\mathbf{k}} \sum_{s} \left[p_{\mathbf{k}s}^2(t) + \omega^2 q_{\mathbf{k}s}^2(t) \right]$$

A very familiar Hamiltonian!

Sum over independent harmonic oscillators

Energy of the EM Field

$$\hat{\mathcal{H}} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{s}\left[\hat{p}^2_{\mathbf{k}s}(t) + \omega^2\hat{q}^2_{\mathbf{k}s}(t)\right]$$

Using creation and annihilation operators, associated with amplitudes u_{ks}

$$\hat{q}_{\mathbf{k}s}(t) = \sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}_{\mathbf{k}s}(t) + \hat{a}^\dagger_{\mathbf{k}s}(t)\right]$$

$$\hat{p}_{\mathbf{k}s}(t) = i\sqrt{\frac{\hbar\omega}{2}}\left[\hat{a}_{\mathbf{k}s}(t) - \hat{a}^\dagger_{\mathbf{k}s}(t)\right]$$

$$\left[\hat{a}_{\mathbf{k}s}(t), \hat{a}^\dagger_{\mathbf{k}'s'(t)}\right] = \delta^3_{\mathbf{k}\mathbf{k}'}\delta_{ss'}$$

$$\left[\hat{a}_{\mathbf{k}s}(t), \hat{a}_{\mathbf{k}'s'(t)}\right] = 0$$

$$\left[\hat{a}^\dagger_{\mathbf{k}s}(t), \hat{a}^\dagger_{\mathbf{k}'s'(t)}\right] = 0.$$

$$\hat{a}_{\mathbf{k}s}(t) = \hat{a}_{\mathbf{k}s}e^{-i\omega t} \qquad \hat{a}^\dagger_{\mathbf{k}s}(t) = \hat{a}^\dagger_{\mathbf{k}s}e^{i\omega t}$$

Energy of the EM Field

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_{\mathbf{k}} \left(\hat{a}^{\dagger}_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} + \frac{1}{2} \right)$$

Amplitudes of Electric and Magnetic Fields

$$\hat{\mathbf{E}}(\mathbf{r},t) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar\omega}{2\epsilon_0}} \left[i\hat{a}_{\mathbf{k}s} \boldsymbol{\epsilon}_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - i\hat{a}^{\dagger}_{\mathbf{k}s} \boldsymbol{\epsilon}^{*}_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right]$$

$$\hat{\mathbf{B}}(\mathbf{r},t) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar}{2\omega\epsilon_0}} \left[i\hat{a}_{\mathbf{k}s} (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}s}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - i\hat{a}^{\dagger}_{\mathbf{k}s} (\mathbf{k} \times \boldsymbol{\epsilon}^{*}_{\mathbf{k}s}) e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right] .$$

Field Quadratures – Classical Description

- Classical Description of the Electromagnetic Field:

Fresnel Representation of a single mode

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{\sqrt{\epsilon_0 L^3}}\sum_{\mathbf{k}}\sum_{s} i\omega\left[u_{\mathbf{k}s}(t)\boldsymbol{\epsilon}_{\mathbf{k}s}e^{i\mathbf{k}\cdot\mathbf{r}}-u^*_{\mathbf{k}s}(t)\boldsymbol{\epsilon}^*_{\mathbf{k}s}e^{-i\mathbf{k}\cdot\mathbf{r}}\right],$$

$$\mathbf{B}(\mathbf{r},t)=\frac{i}{\sqrt{\epsilon_0 L^3}}\sum_{\mathbf{k}}\sum_{s}\left[u_{\mathbf{k}s}(t)(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}s})e^{i\mathbf{k}\cdot\mathbf{r}}-u^*_{\mathbf{k}s}(t)(\mathbf{k}\times\boldsymbol{\epsilon}^*_{\mathbf{k}s})e^{-i\mathbf{k}\cdot\mathbf{r}}\right]$$

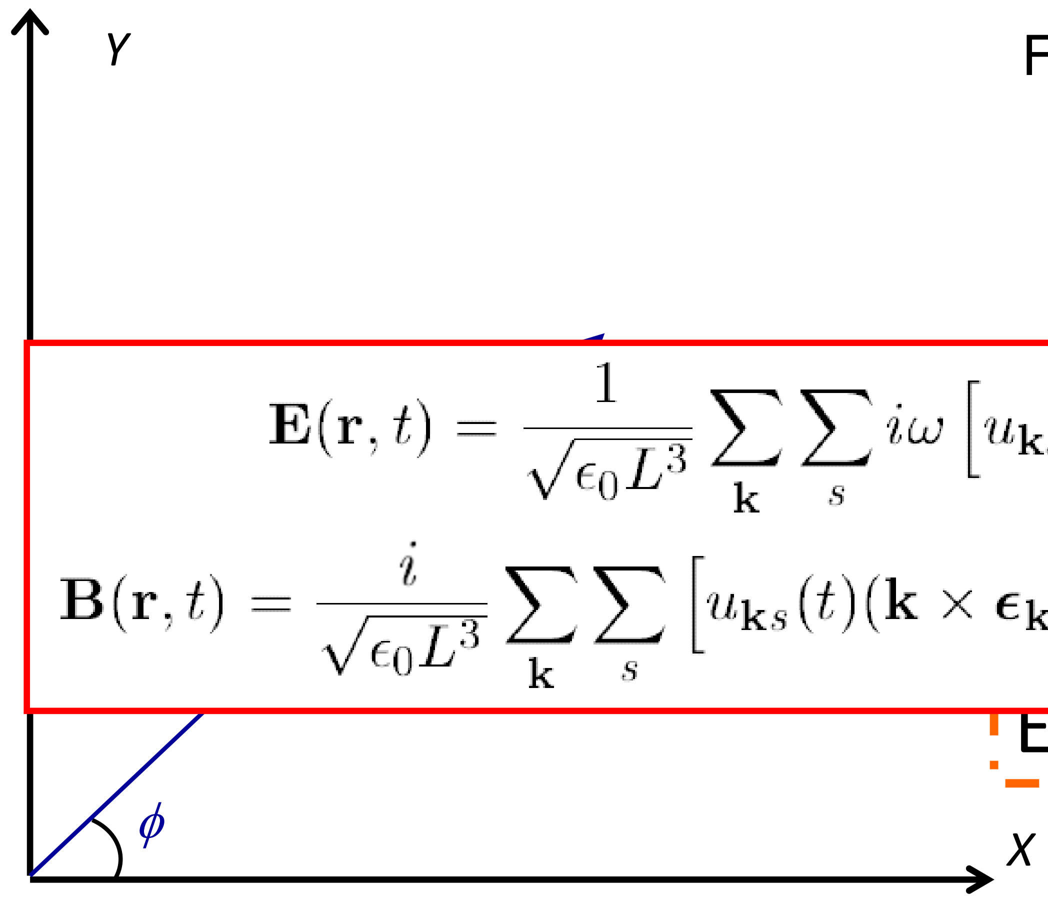
Field Quadratures – Classical Description

- Classical Description of the Electromagnectic Field:

Fresnel Representation of a single mode

For a fixed position

E(t)=Re[α exp(iωt)]



$$\mathbf{E}(\mathbf{r},t)=\frac{1}{\sqrt{\epsilon_0 L^3}}\sum_{\mathbf{k}}\sum_{s} i\omega\left[u_{\mathbf{k}s}(t)\boldsymbol{\epsilon}_{\mathbf{k}s}e^{i\mathbf{k}\cdot\mathbf{r}}-u^*_{\mathbf{k}s}(t)\boldsymbol{\epsilon}^*_{\mathbf{k}s}e^{-i\mathbf{k}\cdot\mathbf{r}}\right],$$

$$\mathbf{B}(\mathbf{r},t)=\frac{i}{\sqrt{\epsilon_0 L^3}}\sum_{\mathbf{k}}\sum_{s}\left[u_{\mathbf{k}s}(t)(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}s})e^{i\mathbf{k}\cdot\mathbf{r}}-u^*_{\mathbf{k}s}(t)(\mathbf{k}\times\boldsymbol{\epsilon}^*_{\mathbf{k}s})e^{-i\mathbf{k}\cdot\mathbf{r}}\right]$$

E(t)=X cos(ωt)+ Y sen(ωt)

Field Quadratures – Quantum Optics

The electric field can be decomposed as

$$\hat{\mathbf{E}}^{(+)} = \frac{i}{L^{3/2}} \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar\omega}{2\epsilon_0}} \left[\hat{a}_{\mathbf{k}s} \mathbf{u}_{\mathbf{k}s}(\mathbf{r}) e^{-i\omega t}\right] \quad ; \quad \hat{\mathbf{E}}^{(-)} = \left[\hat{\mathbf{E}}^{(+)}\right]^{\dagger}$$

And also as

$$\hat{\mathrm{E}} = \frac{2i}{L^{3/2}} \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar\omega}{2\epsilon_0}} \epsilon \left[\hat{X} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) + \hat{Y} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)\right]$$

X and Y are the field quadrature operators, satisfying

$$\hat{X}_\theta(t) = \mathrm{e}^{-i\theta}\, \hat{a}(t) + \mathrm{e}^{i\theta}\, \hat{a}^\dagger(t)\ , \qquad \hat{Y}_\theta(t) = -i \left[\, \mathrm{e}^{-i\theta}\, \hat{a}(t) - \mathrm{e}^{i\theta}\, \hat{a}^\dagger(t)\, \right]$$

$$\left[\hat{X}(\theta), \hat{X}\left(\theta + \frac{\pi}{2}\right)\right] = 2i$$

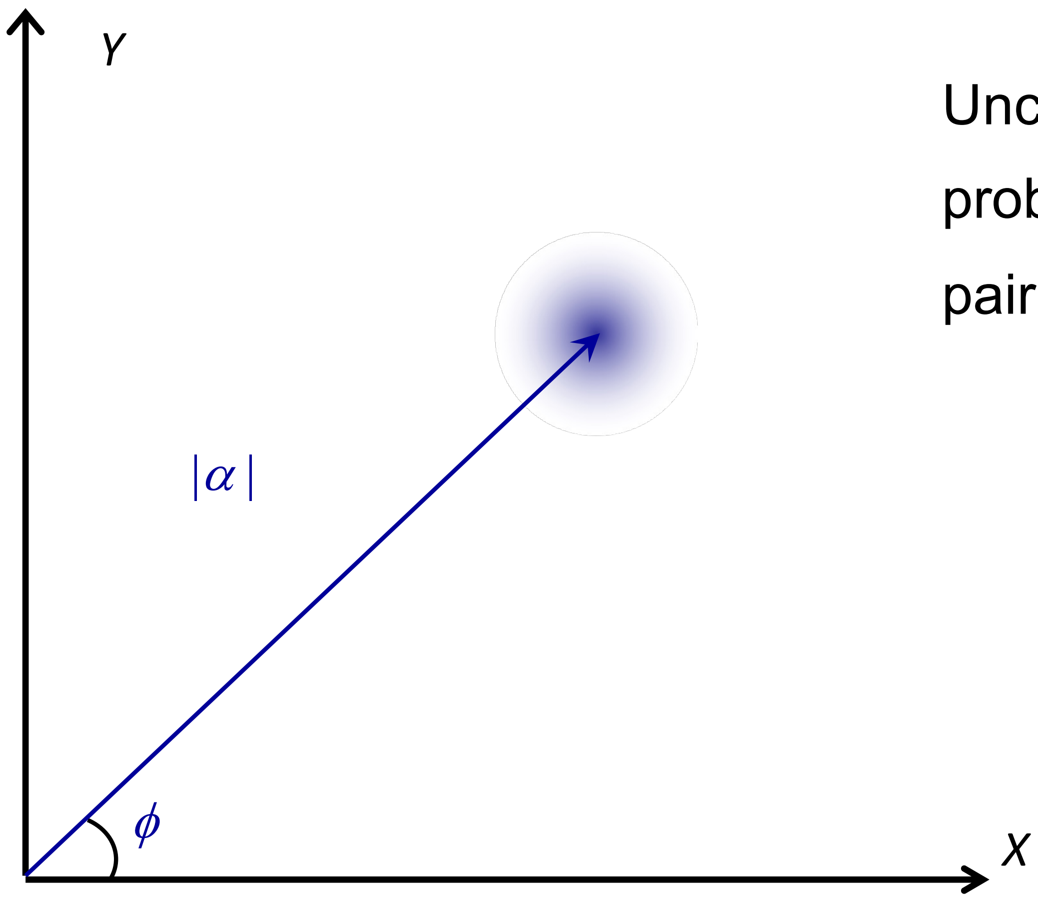
Thus,

$$\Delta X \Delta Y \geq 1$$

Field Quadratures – Quantum Optics

$$\left[\hat{X}(\theta), \hat{X}\left(\theta + \frac{\pi}{2}\right)\right] = 2i$$

Thus, $\Delta X \Delta Y \geq 1$



Uncertainty relation implies in a probability distribution for a given pair of quadrature measurements

Field quadratures behave just as position and momentum operators!

Now we know that:

- the description of the EM field follows that of a set of harmonic oscillators,
- the quadratures of the electric field are observables, and
- they must satisfy an uncertainty relation.

But how to describe different states of the EM field?

Can we find appropriate basis for the description of the field?

Or alternatively, can we describe it using density operators?

And how to characterize these states?

Quantum Optics – Number States

Eigenstates of the number operator

$$\hat{n}_{\mathbf{k}s} = \hat{a}^{\dagger}_{\mathbf{k}s}\hat{a}_{\mathbf{k}s} \qquad \hat{n}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = n_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle$$

Number of excitations in a given harmonic oscillator →

number of excitations in a given mode of the field →

number of photons in a given mode!

Annihilation and creation operators:

$$\hat{a}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \sqrt{n_{\mathbf{k}s}}|n_{\mathbf{k}s} - 1\rangle,$$

$$\hat{a}^{\dagger}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \sqrt{n_{\mathbf{k}s} + 1}|n_{\mathbf{k}s} + 1\rangle,$$

$$\hat{a}_{\mathbf{k}s}|0\rangle = 0.$$

Fock States:
Eigenvectors of the Hamiltonian

$$|\{n\}\rangle = \prod_{\mathbf{k}s} |n_{\mathbf{k}s}\rangle$$

$$\hat{\mathcal{H}}|\{n\}\rangle = \left[\sum_{\mathbf{k}s}(n_{\mathbf{k}s} + 1/2)\hbar\omega\right]|\{n\}\rangle$$

$$\mathcal{E} = \sum_{\mathbf{k}s}\left[\hbar\omega_{\mathbf{k}}\left(\hat{n}_{\mathbf{k}} + \frac{1}{2}\right)\right]$$

Quantum Optics – Number States

Complete, orthonormal, discrete basis

$$\langle n_{\mathbf{k}s} | m_{\mathbf{k}s} \rangle = \delta_{n_{\mathbf{k}s} m_{\mathbf{k}s}} \Rightarrow \langle \{n\} | \{m\} \rangle = \prod_{\mathbf{k}s} \delta_{n_{\mathbf{k}s} m_{\mathbf{k}s}},$$

$$\sum_{n_{\mathbf{k}s}=0}^{\infty} |n_{\mathbf{k}s}\rangle\langle n_{\mathbf{k}s}| = 1 \Rightarrow \sum_{\{n\}} |\{n\}\rangle\langle\{n\}| = 1.$$

Disadvantage: except for the vacuum mode it is quite an unusual state of the field.

Can we find something better?

Quantum Optics – q - States

q-states in Fock space:
Describing the usual textbook solution of the harmonic oscillator Hamiltonian with energy eigenstates.

$$P(q_{\mathbf{k}s}) = \langle n_{\mathbf{k}s}|q_{\mathbf{k}s}\rangle\langle q_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = |\langle q_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle|^2.$$

$$\begin{aligned}\langle q_{\mathbf{k}s}|\hat{a}_{\mathbf{k}s}|0_{\mathbf{k}s}\rangle = 0 &= (\omega/2\hbar)^{1/2}\langle q_{\mathbf{k}s}|[\hat{q}_{\mathbf{k}s} + (\mathrm{i}/\omega)\hat{p}_{\mathbf{k}s}]|0_{\mathbf{k}s}\rangle \\ &= (\hbar/2\omega)^{1/2}[(\omega/\hbar)q_{\mathbf{k}s} + \partial/\partial q_{\mathbf{k}s}]\langle q_{\mathbf{k}s}|0_{\mathbf{k}s}\rangle,\end{aligned}$$

$$\langle q_{\mathbf{k}s}|\hat{p}_{\mathbf{k}s}|\psi\rangle = (\hbar/\mathrm{i})(\partial/\partial q_{\mathbf{k}s})\langle q_{\mathbf{k}s}|\psi\rangle.$$

Quantum Optics – q - States

$$0 = (\hbar/2\omega)^{1/2}[(\omega/\hbar)q_{\mathbf{k}s} + \partial/\partial q_{\mathbf{k}s}]\langle q_{\mathbf{k}s}|0_{\mathbf{k}s}\rangle$$

$$\langle q_{\mathbf{k}s}|0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\tfrac{1}{2}q_{\mathbf{k}s}^2\omega/\hbar\right) \qquad \int_{-\infty}^{\infty} |\langle q_{\mathbf{k}s}|0\rangle|^2 \, \mathrm{d}q_{\mathbf{k}s} = 1$$

So far, we have the ground state.

$$\langle q_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \frac{1}{\sqrt{(n_{\mathbf{k}s}!)}}\langle q_{\mathbf{k}s}|(\hat{a}^{\dagger}_{\mathbf{k}s})^{n_{\mathbf{k}s}}|0\rangle$$

Quantum Optics – q - States

Recursive operations on the ground state give us

$$\langle q_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \frac{1}{\sqrt{(n_{\mathbf{k}s}!)}}\langle q_{\mathbf{k}s}|(\hat{a}^{\dagger}_{\mathbf{k}s})^{n_{\mathbf{k}s}}|0\rangle$$

$$= \frac{1}{\sqrt{(n_{\mathbf{k}s}!)}}\left(\frac{\hbar}{2\omega}\right)^{n_{\mathbf{k}s}/2}\left[\frac{\omega}{\hbar}q_{\mathbf{k}s} - \frac{\partial}{\partial q_{\mathbf{k}s}}\right]^{n_{\mathbf{k}s}}\langle q_{\mathbf{k}s}|0\rangle$$

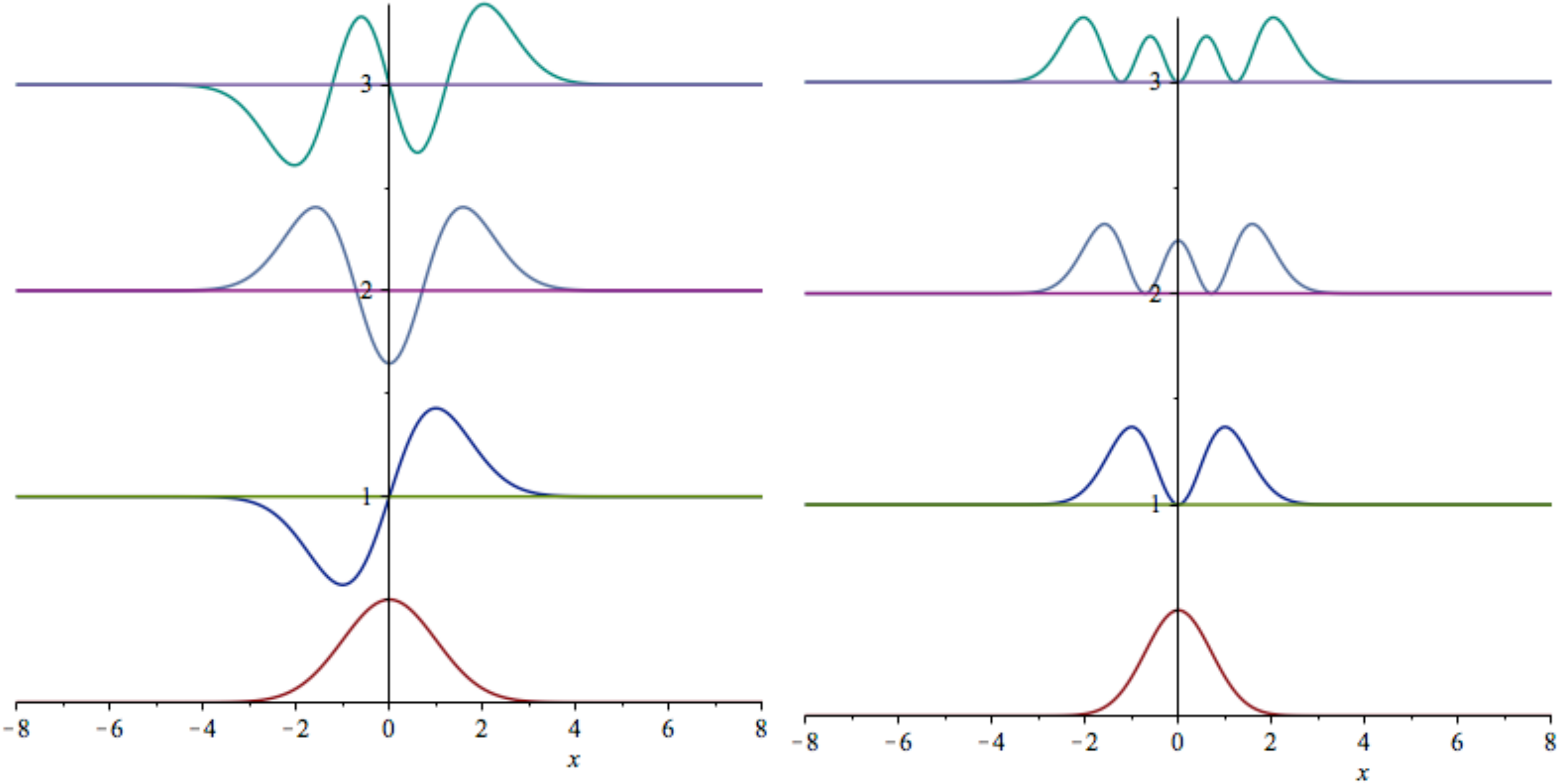
$$= \frac{1}{\sqrt{(n_{\mathbf{k}s}!)}}\left(\frac{\hbar}{2\omega}\right)^{n_{\mathbf{k}s}/2}\left[\frac{\omega}{\hbar}q_{\mathbf{k}s} - \frac{\partial}{\partial q_{\mathbf{k}s}}\right]^{n_{\mathbf{k}s}}\left(\frac{\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\tfrac{1}{2}q^2_{\mathbf{k}s}\omega/\hbar\right)$$

Leading to the n^{th}-order Hermitian polynomial

$$\langle q_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \frac{1}{(2^{n_{\mathbf{k}s}}n_{\mathbf{k}s}!)^{1/2}}\left(\frac{\omega}{\pi\hbar}\right)^{1/4}H_{n_{\mathbf{k}s}}[\sqrt{(\omega/\hbar)}q_{\mathbf{k}s}]\exp\left(-\tfrac{1}{2}q^2_{\mathbf{k}s}\omega/\hbar\right)$$

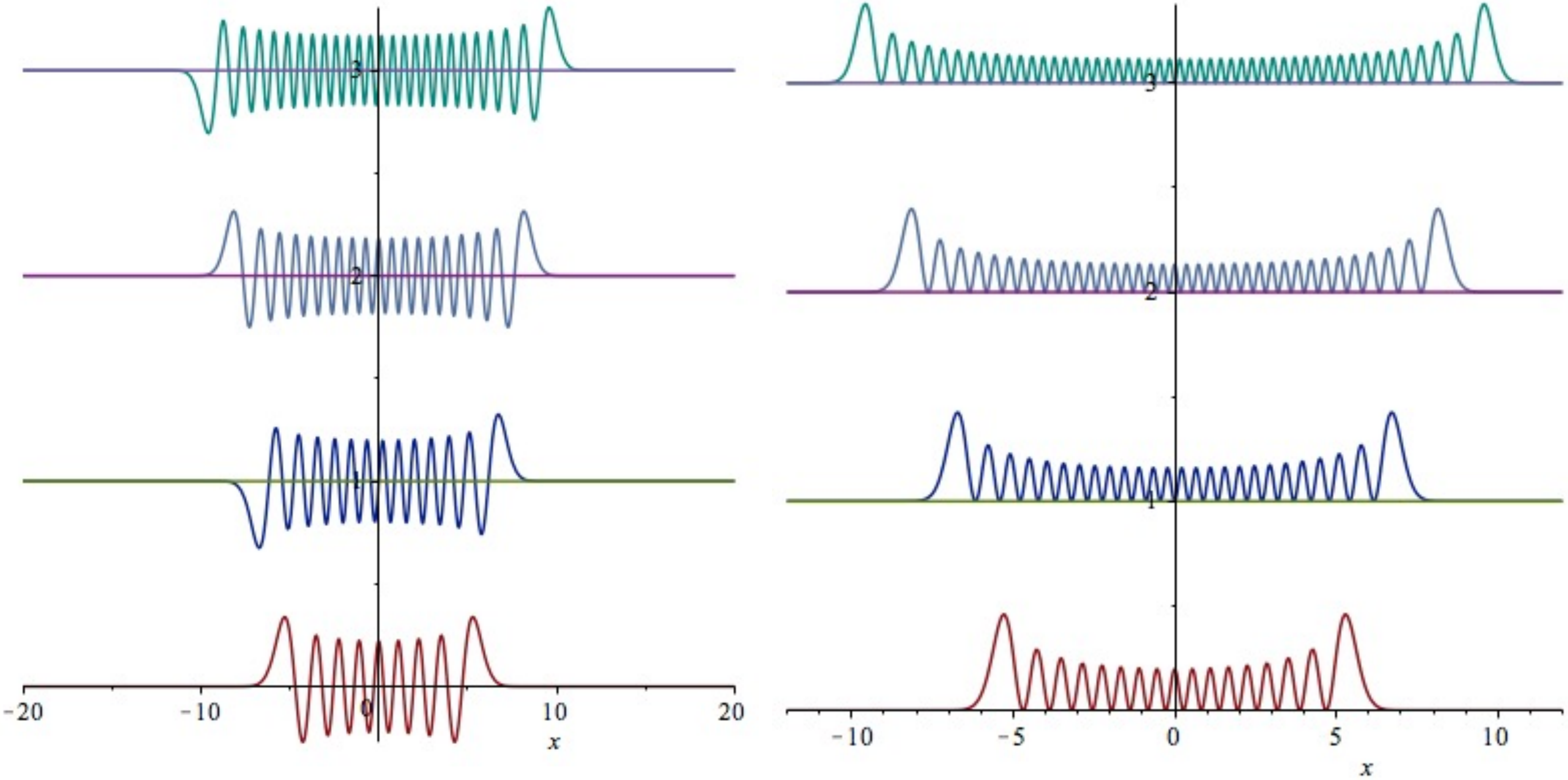
Quantum Optics – q - States

$$\langle q_{\mathbf{k}s} | n_{\mathbf{k}s} \rangle \qquad P(q_{\mathbf{k}s}) = \langle n_{\mathbf{k}s} | q_{\mathbf{k}s} \rangle \langle q_{\mathbf{k}s} | n_{\mathbf{k}s} \rangle$$



Quantum Optics – q - States

$$\langle q_{\mathbf{k}s} | n_{\mathbf{k}s} \rangle \qquad P(q_{\mathbf{k}s}) = \langle n_{\mathbf{k}s} | q_{\mathbf{k}s} \rangle \langle q_{\mathbf{k}s} | n_{\mathbf{k}s} \rangle$$



Quantum Optics – Number States

Disadvantage: except for the vacuum mode it is quite an unusual state of the field.

Can we find something better?

Quantum Optics – q - States

Eigenfunctions of the position operador: $\hat{q}|q\rangle = q|q\rangle$

Continuous spectra.

Nice, but limited for representing non-pure (mixed) states.

We can do something better using the Fresnel plane

Quantum Optics – Coherent States

Eigenvalues of the annihilation operator: $a_{\mathbf{k}s}|\alpha_{\mathbf{k}s}\rangle = \alpha_{\mathbf{k}s}|\alpha_{\mathbf{k}s}\rangle$

In the Fock State Basis:

$$|\alpha_{\mathbf{k}s}\rangle = e^{-|\alpha_{\mathbf{k}s}|^2/2} \sum_{n_{\mathbf{k}s}=0}^{\infty} \frac{\alpha_{\mathbf{k}s}^{n_{\mathbf{k}s}}}{\sqrt{n_{\mathbf{k}s}!}} |n_{\mathbf{k}s}\rangle$$

Completeness:

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha = 1$$

but is not orthonormal

$$\langle\alpha|\alpha'\rangle = exp\left(-\frac{1}{2}|\alpha|^2 + \alpha'\alpha^* - \frac{1}{2}|\alpha'|^2\right)$$

Over-complete!

Moreover:

- corresponds to the state generated by a classical current,
- reasonably describes a monomode laser well above threshold,
- it is the closest description of a “classical” state.

Quantum Optics – Number States

Precise number of photons

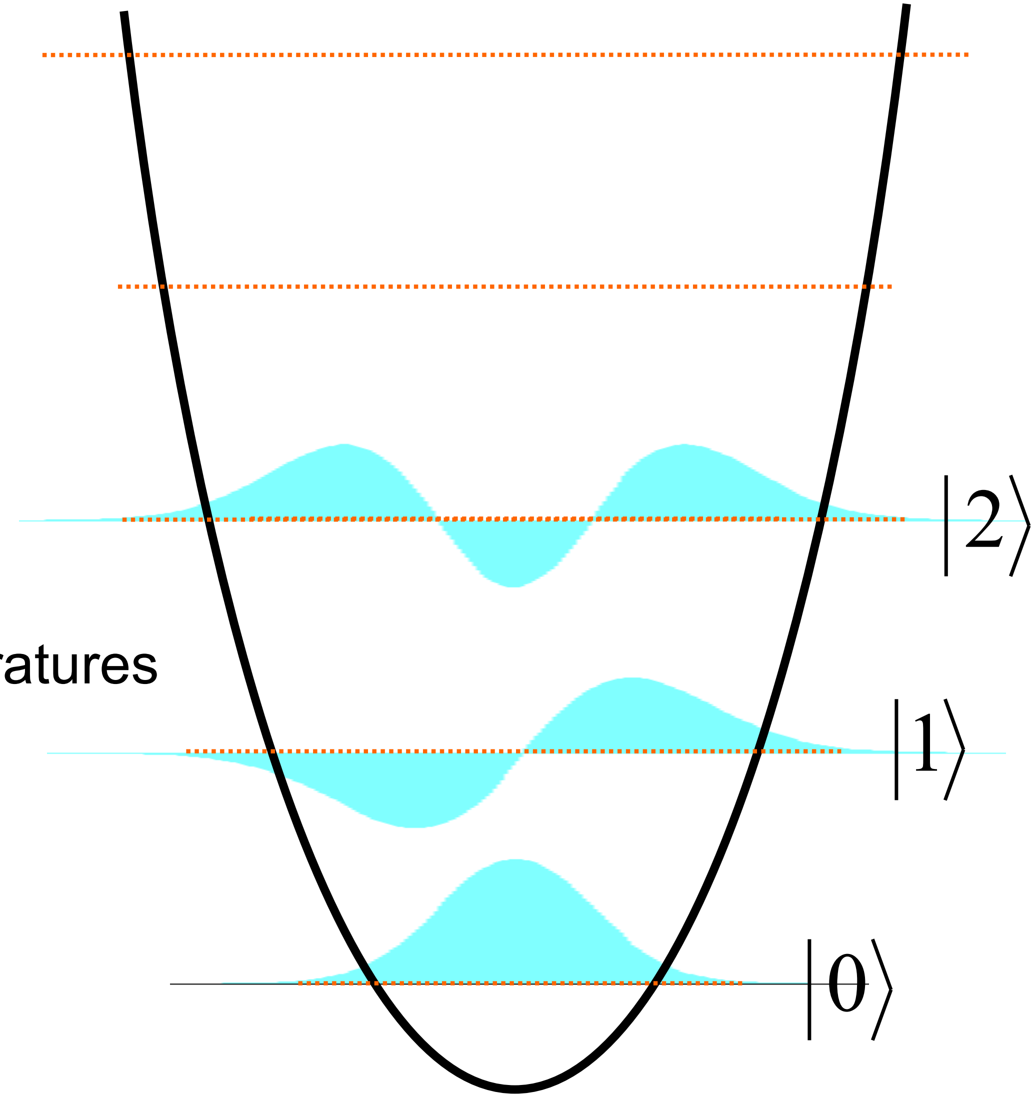
$$\langle \hat{n} \rangle = n$$

$$\Delta \hat{n}^2 = 0$$

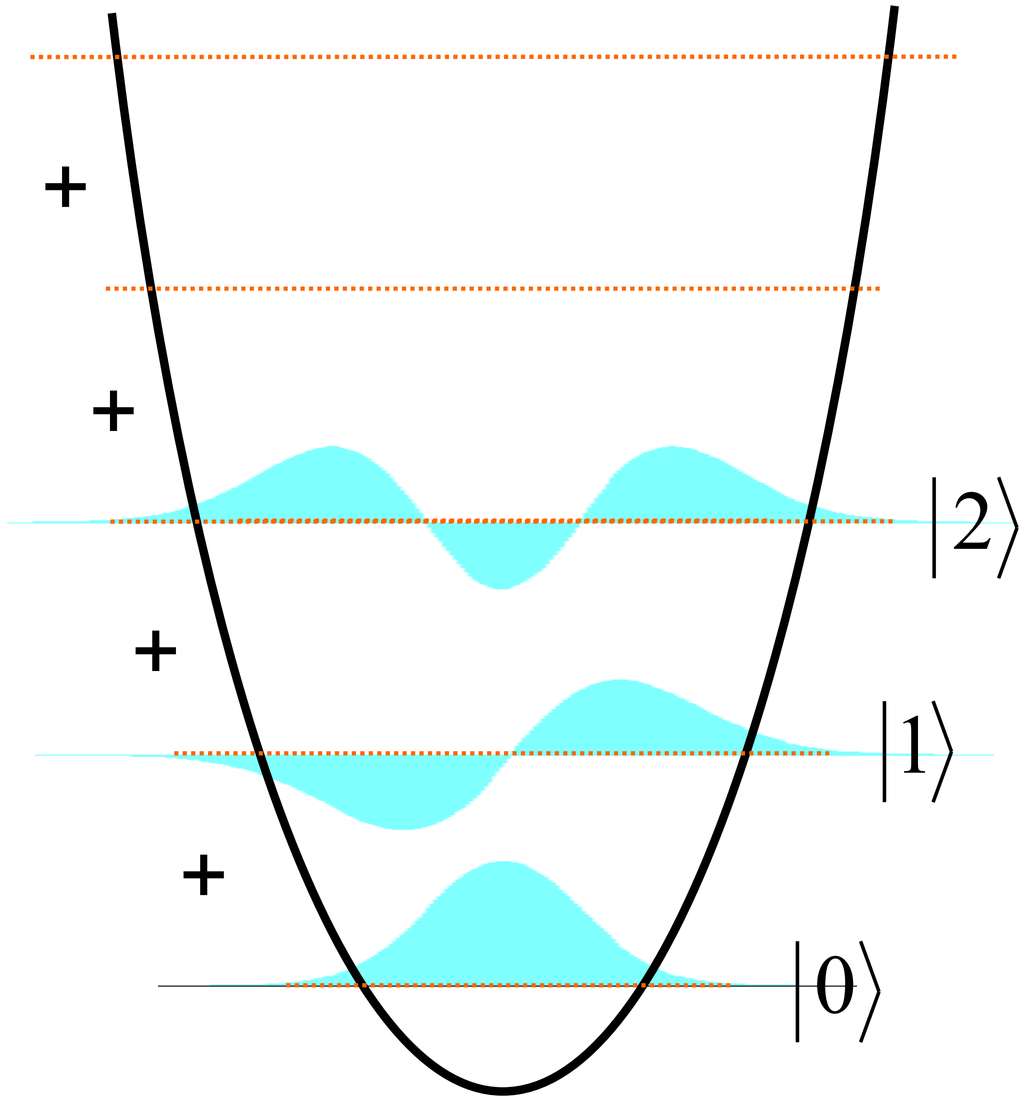
Growing dispersion of the quadratures

$$\langle \hat{X} \rangle = \langle \hat{Y} \rangle = 0$$

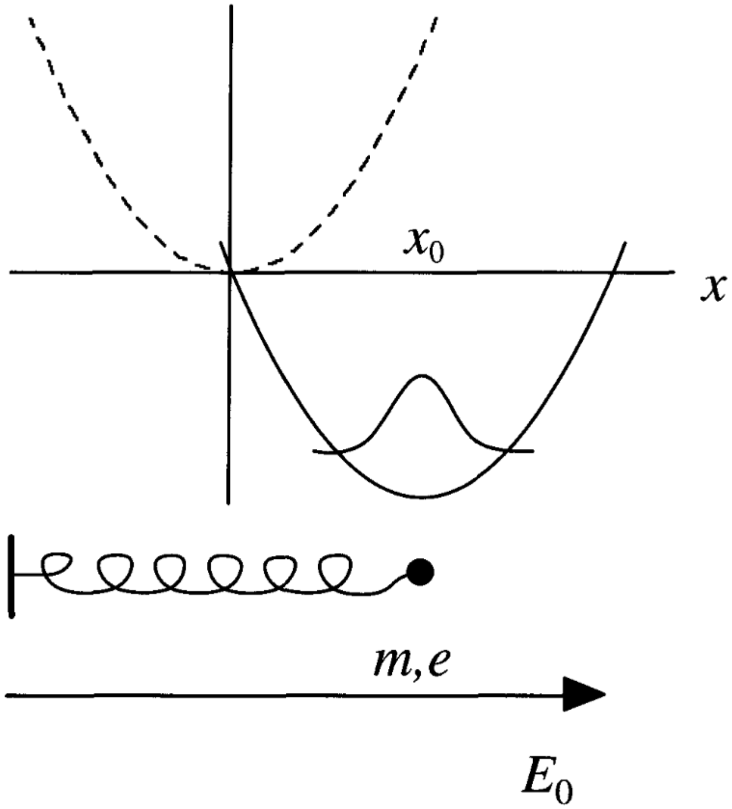
$$\langle \hat{X}^2 \rangle = \langle \hat{Y}^2 \rangle = 2n + 1$$



Quantum Optics – Coherent State



$$|\alpha_{\mathbf{k}s}\rangle = e^{-|\alpha_{\mathbf{k}s}|^2/2} \sum_{n_{\mathbf{k}s}=0}^{\infty} \frac{\alpha_{\mathbf{k}s}^{n_{\mathbf{k}s}}}{\sqrt{n_{\mathbf{k}s}!}} |n_{\mathbf{k}s}\rangle$$



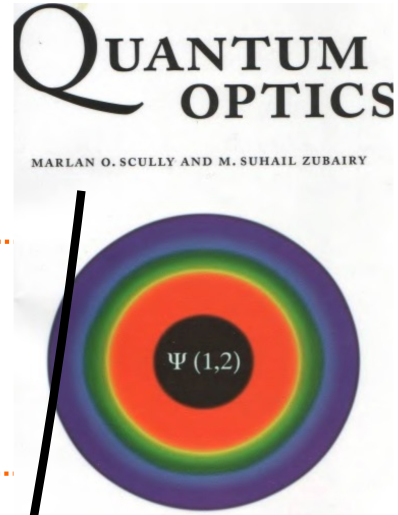
$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - eE_0x,$$

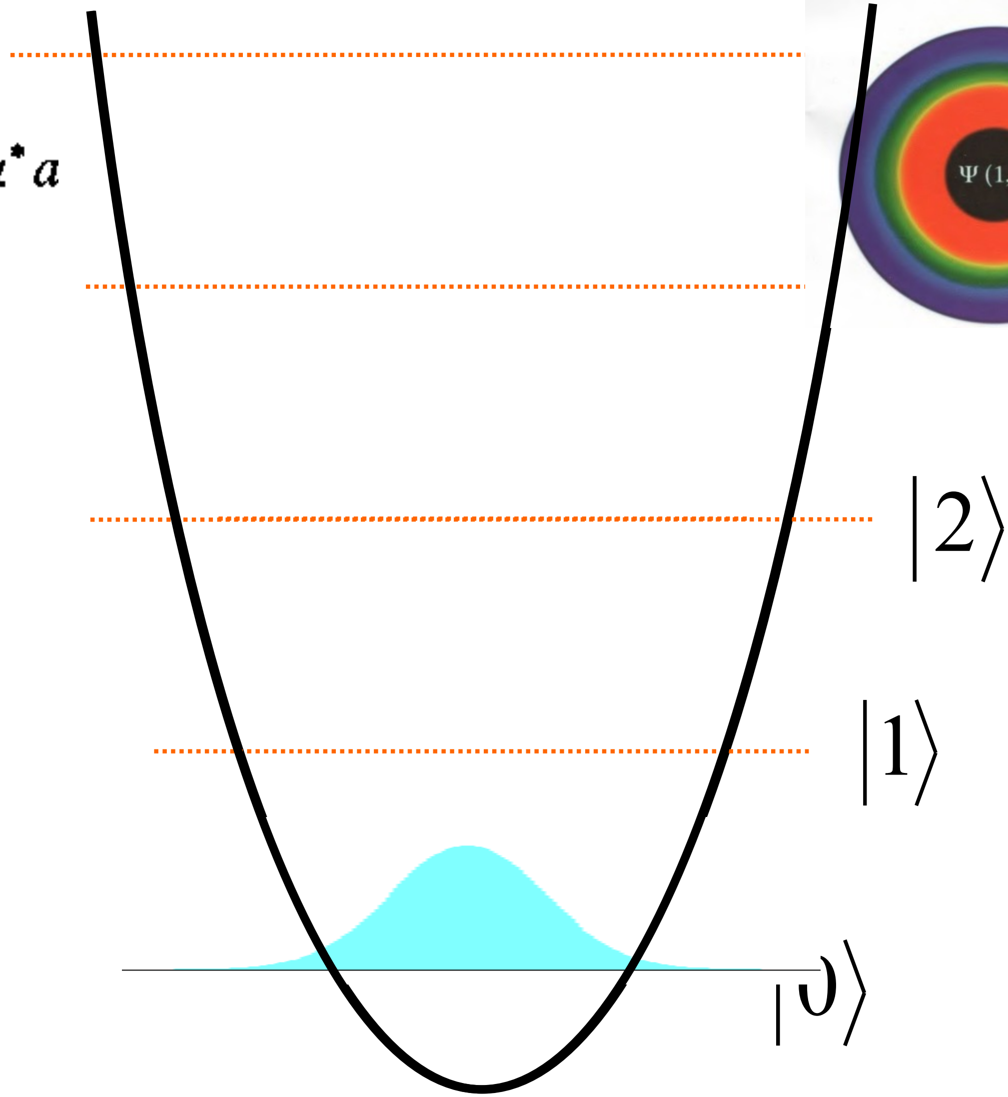
$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}k\left(x - \frac{eE_0}{k}\right)^2 - \frac{1}{2}k\left(\frac{eE_0}{k}\right)^2$$

Quantum Optics – Coherent State

$$|\alpha\rangle = D(\alpha)|0\rangle$$

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$



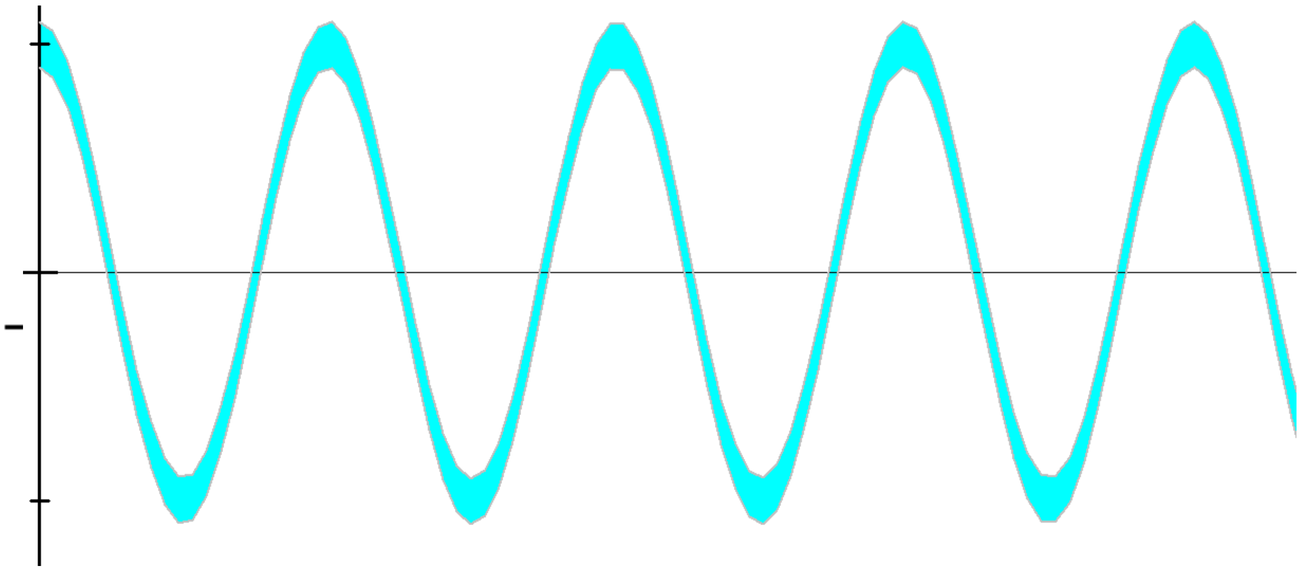


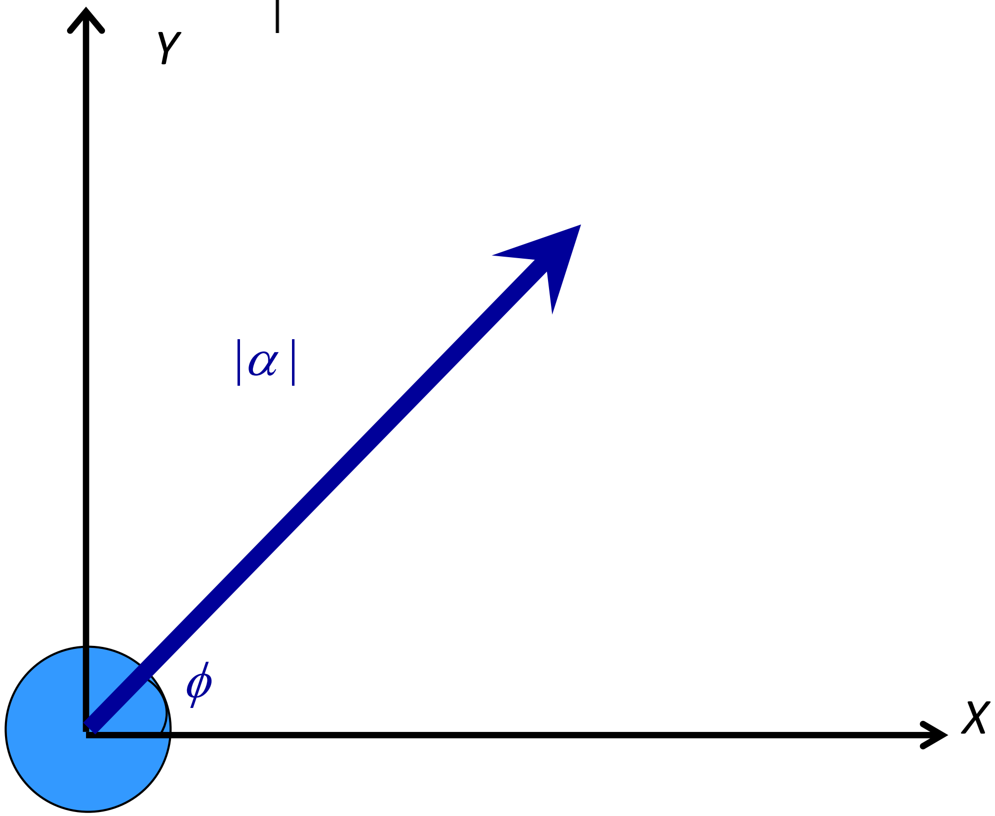
$$D^\dagger(\alpha) = D(-\alpha) = [D(\alpha)]^{-1}.$$

Quantum Optics – Coherent State

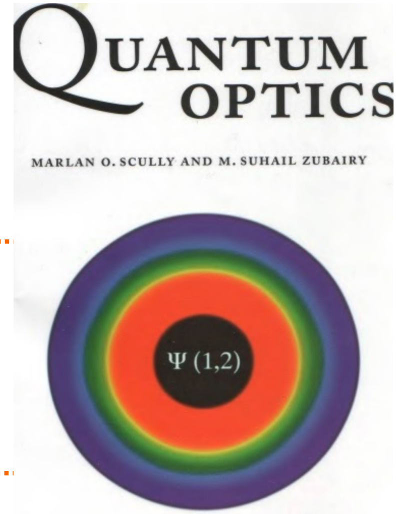
$$|\alpha\rangle = D(\alpha)|0\rangle$$

$$D(\alpha) = \exp(\alpha a^{\dagger} -$$



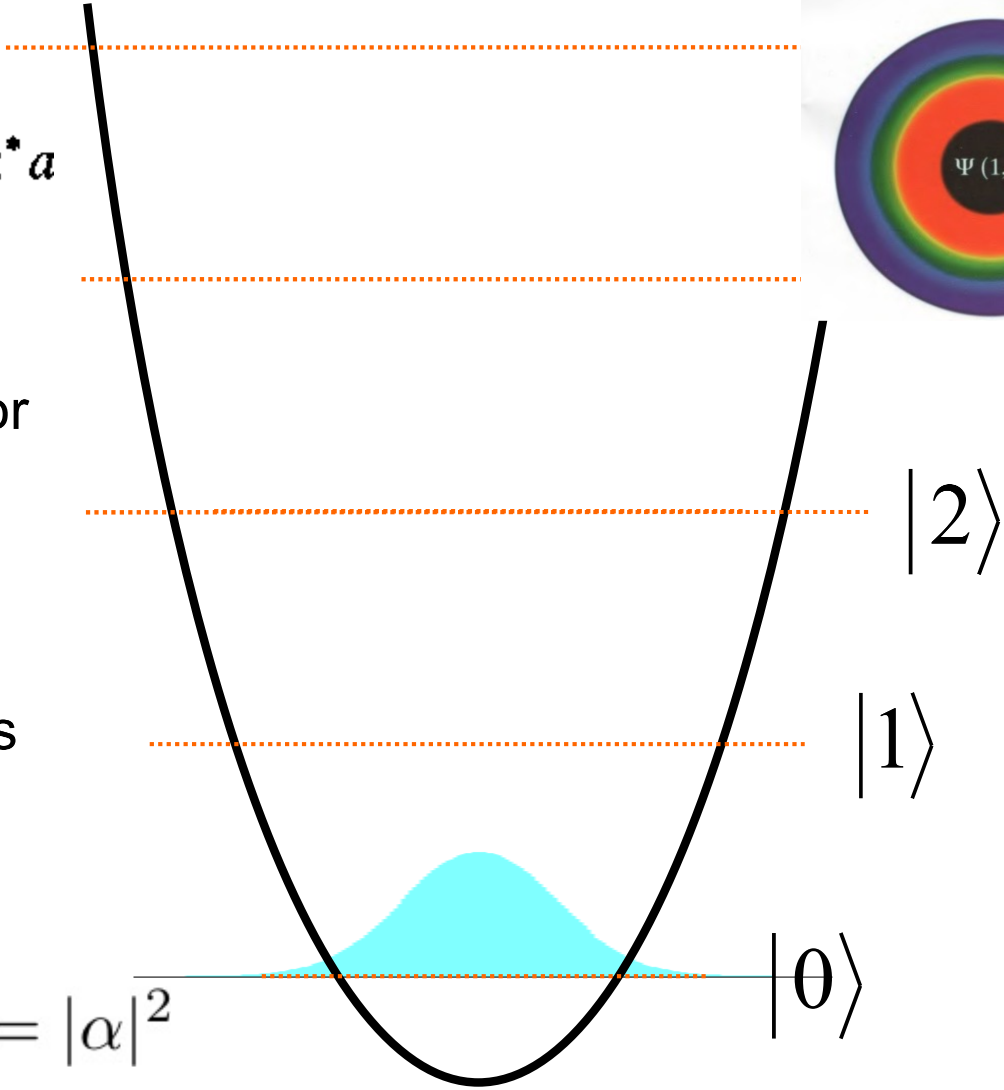


Quantum Optics – Coherent State



$$|\alpha\rangle = D(\alpha)|0\rangle$$

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$



Mean value of number operator

$$\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$$

Poissonian distribution of photons

$$p(n) = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!}$$

$$\langle n\rangle = |\alpha|^2$$

Therefore, variance of photon number is equal to the mean number!

$$\Delta^2\hat{n} = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 = |\alpha|^2$$

Quantum Optics – Coherent Squeezed States

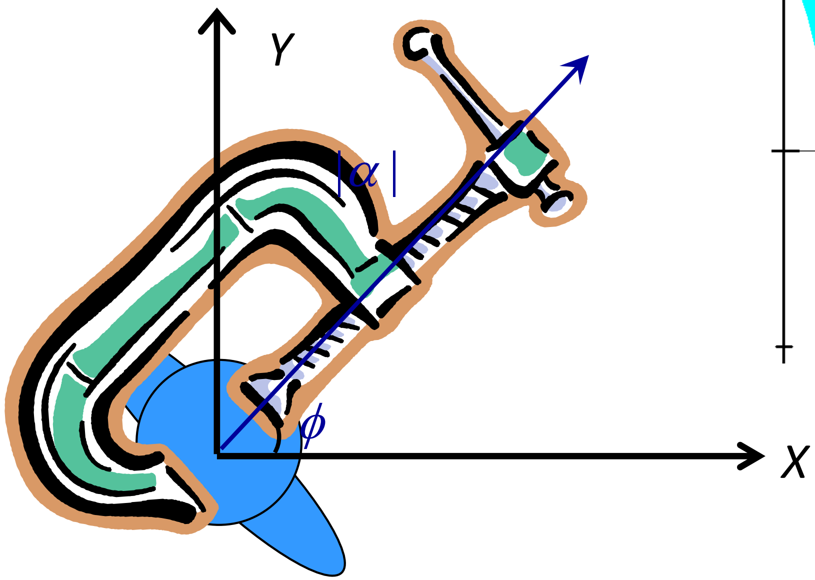
$$|\alpha\rangle = D(\alpha)|0\rangle$$

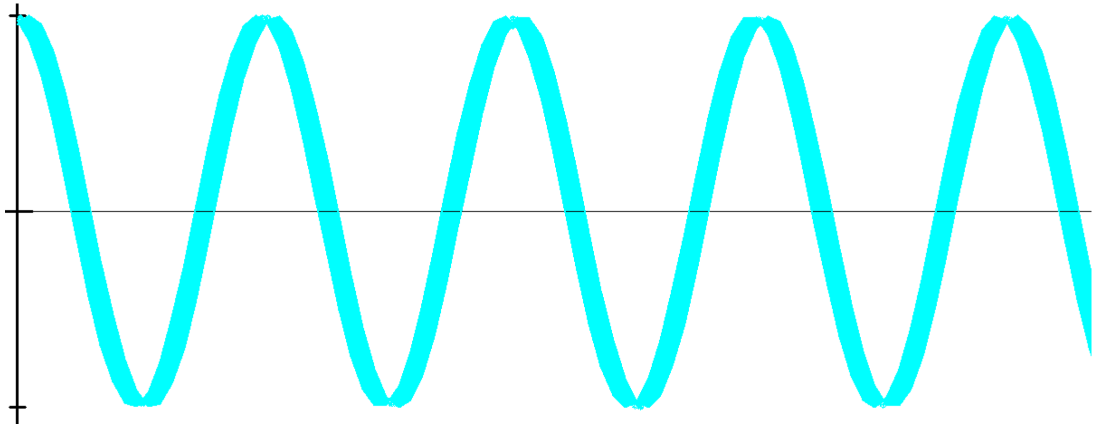
$$D(\alpha) = \exp(\alpha a^{\dagger} - \alpha^{*} a)$$

$$S(\varepsilon) = \exp(1/2\,\varepsilon^{*} a^{2} - 1/2\,\varepsilon a^{\dagger 2})$$

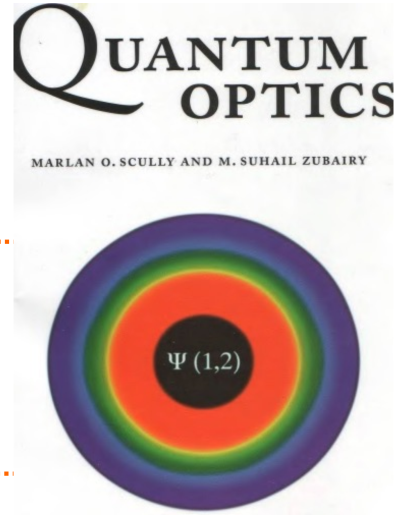
$$\varepsilon = r e^{2i\phi}$$

$$|\alpha, \varepsilon\rangle = D(\alpha)S(\varepsilon)|0\rangle$$





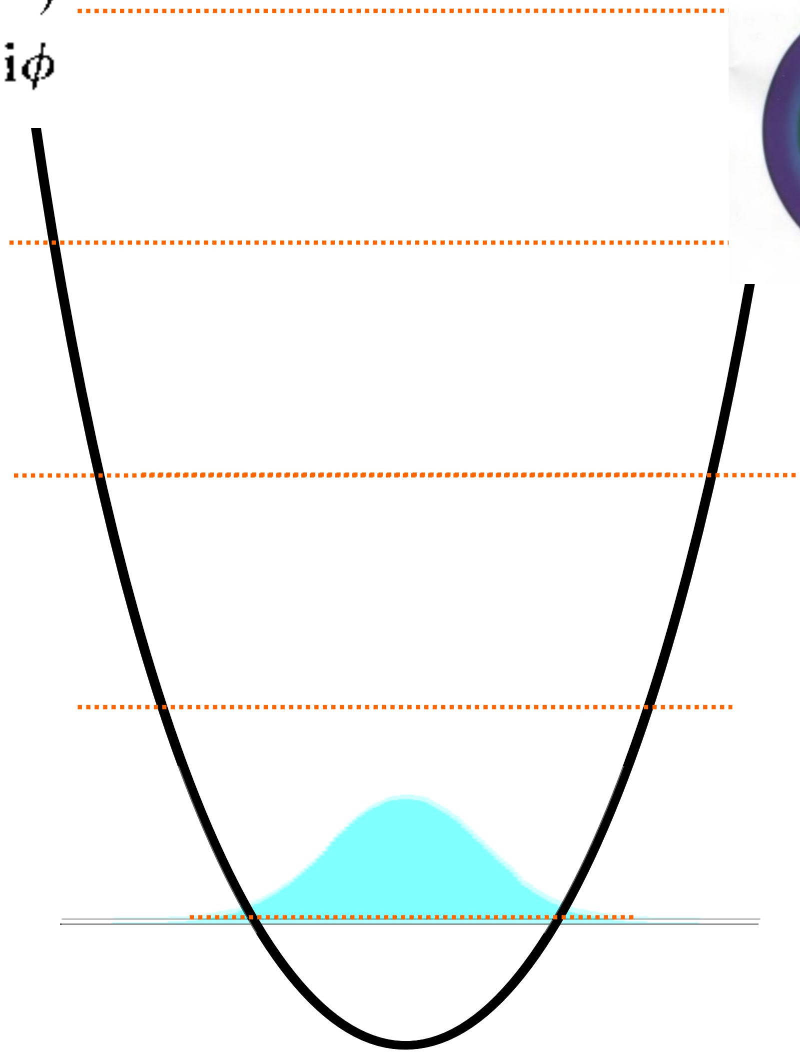
Quantum Optics – Squeezed State



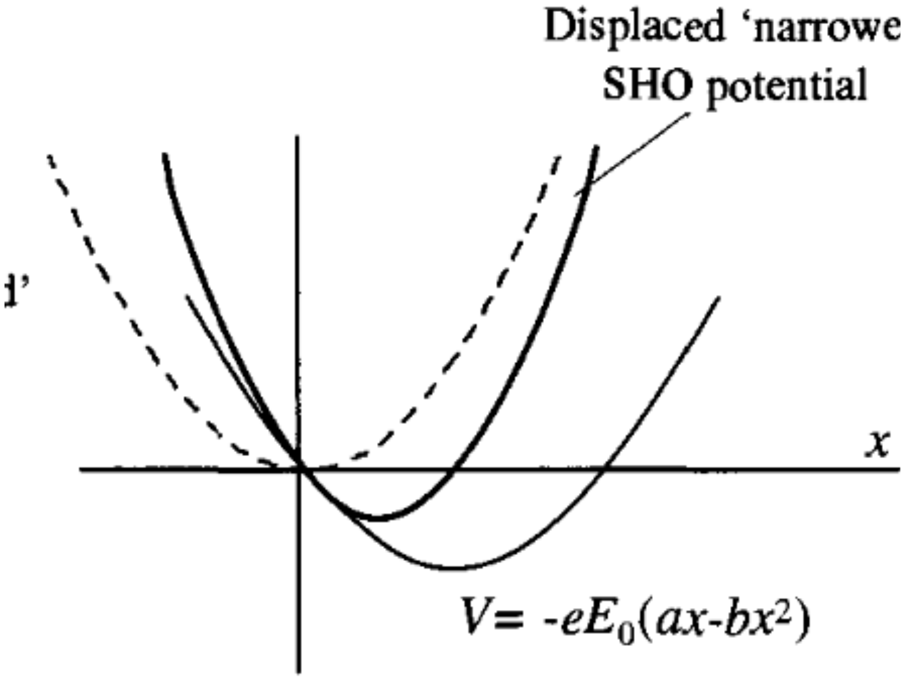
$$S(\varepsilon) = \exp(1/2\varepsilon^* a^2 - 1/2\varepsilon a^{\dagger 2})$$

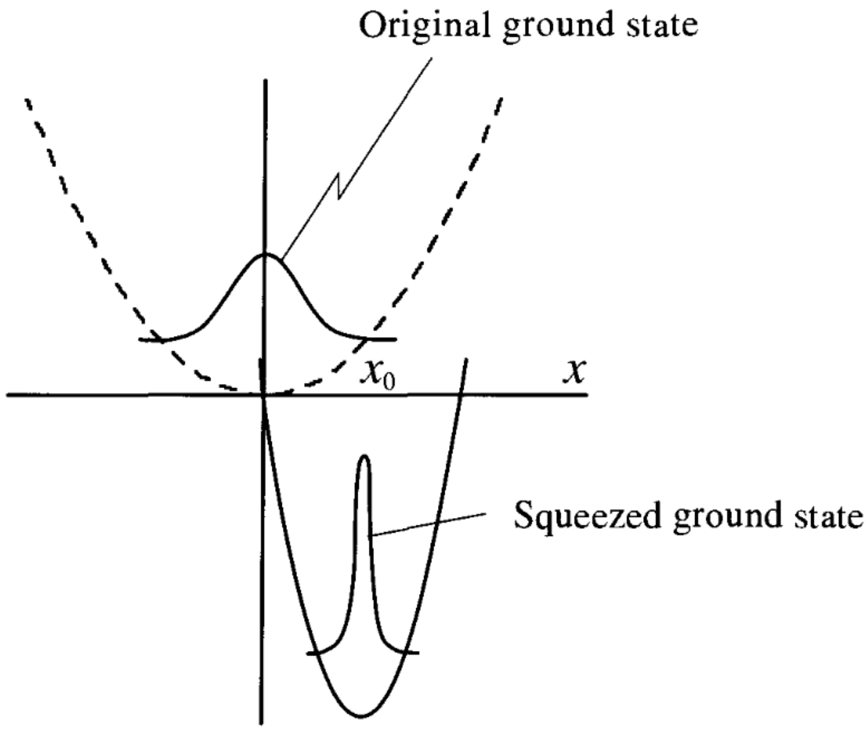
$$\varepsilon = re^{2i\phi}$$

$$S(\epsilon)|0\rangle = |0,\epsilon\rangle$$



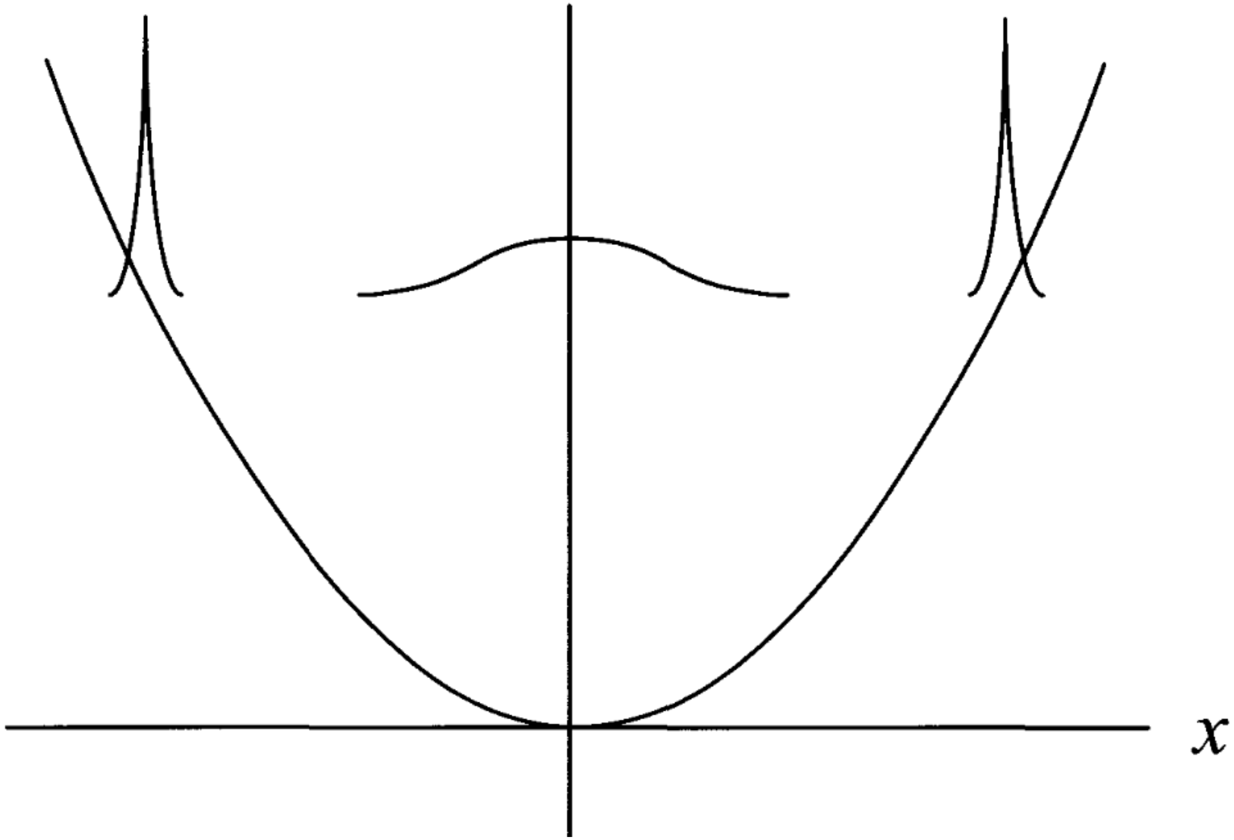
$$S^\dagger(\xi) = S^{-1}(\xi) = S(-\xi).$$





$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - eE_0(ax - bx^2),$$

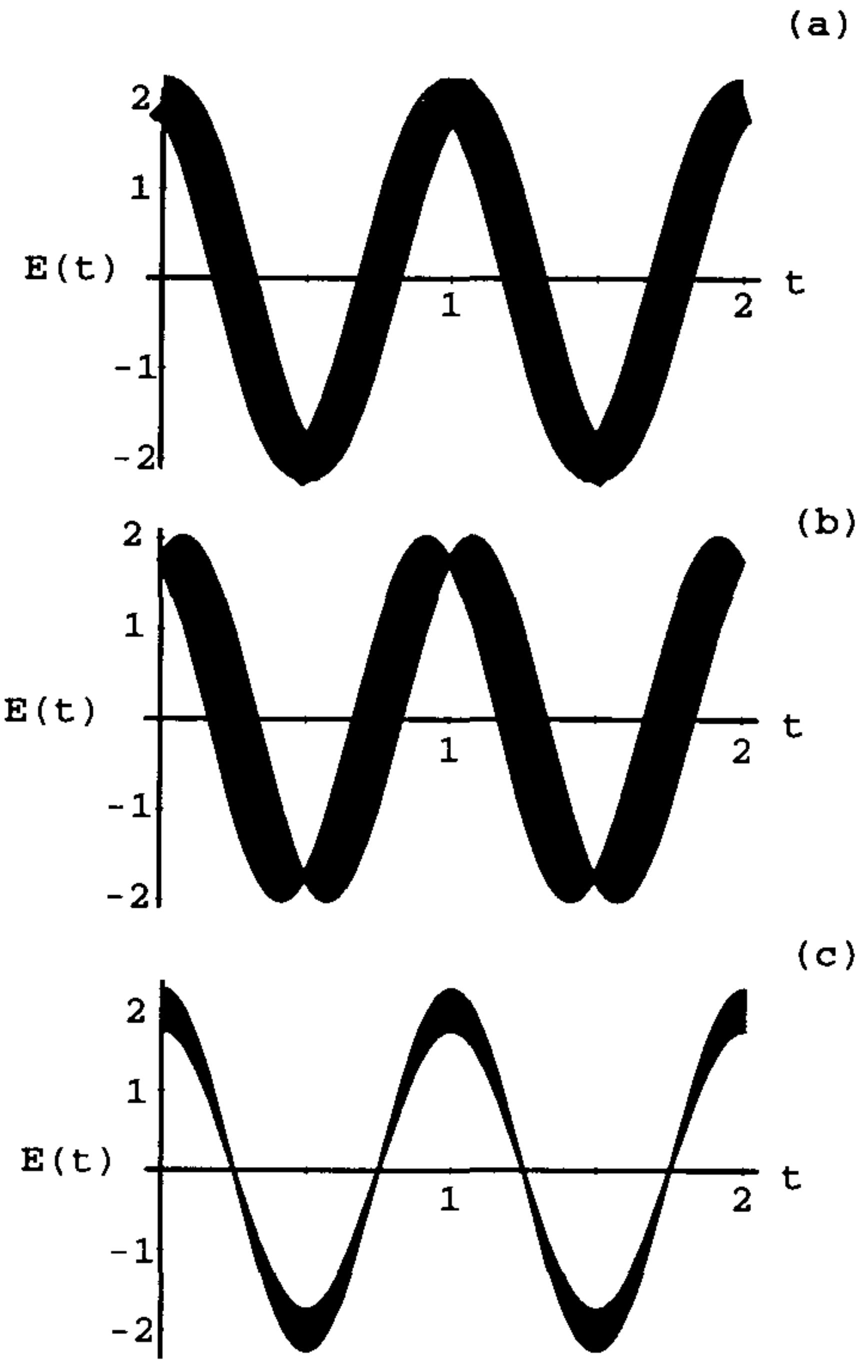
$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}(k + 2ebE_0)x^2 - eaE_0x.$$

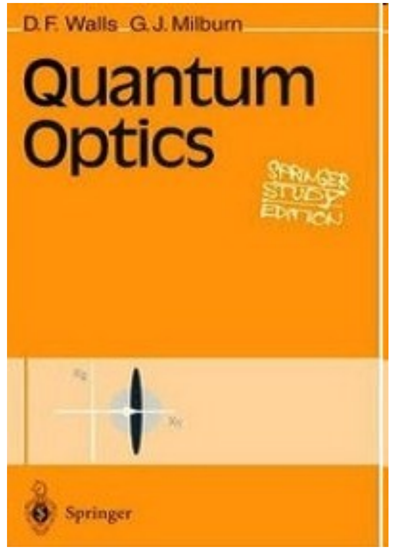


$$\psi(x, t = 0) = \delta(x - x_0),$$

$$\psi(x, t = \pi/2\nu) = \sqrt{\frac{m\nu}{2\pi\hbar}} \exp\left[i\left(\frac{m\nu x_0}{\hbar}\right)x\right],$$

$$\psi(x, t = \pi/\nu) = \delta(x + x_0).$$


(a)
(b)
(c)
E(t)
t
2
1
-1
-2
1
2


D.F. Walls G.J. Milburn
Quantum Optics
Springer

Quantum Optics – Density Operators

We have states defined by

The Hamiltonian operator: $\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$

The position operator" $\hat{q}|q\rangle = q|q\rangle$

The annihilation operator: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

How can we represent a mixed state (e. g. a thermal state)?

Quantum Optics – Density Operators

Pure X Mixed States

$$|\psi\rangle = \sum c_n |a_n\rangle$$

$$\sum |a_m\rangle\langle a_m| = 1$$

$$c_n = \langle a_n|\psi\rangle$$

$$\langle a_m|a_n\rangle = \delta_{mn}$$

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

$$\langle a_m|A|a_n\rangle = A_{mn}$$

<u>Introducing the density operator (von Neumann – 1927)</u>

$$c_n c_m^* = \rho_{nm}$$

$$\rho = |\psi\rangle\langle\psi|$$

Quantum Optics – Density Operators

$$\langle A\rangle = \sum \langle a_n|\rho|a_m\rangle\langle a_m|A|a_n\rangle$$

$$= \sum \langle a_n|\rho A|a_n\rangle = Tr\{\rho A\}$$

Now we can represent a statistical mixture of pure states!

$$\rho = \sum p_k \rho_k \qquad \sum p_k = 1$$

$$\langle A\rangle = Tr\{\rho A\} \qquad Tr\rho = 1$$

$$Tr\rho \geq Tr\rho^2$$

Calculate the mean value and the variance of the number operator $\hat{n}$, and the quadrature operators $\hat{X} = (\hat{a} + \hat{a}^\dagger)$ and $\hat{Y} = -i(\hat{a} - \hat{a}^\dagger)$), for the following states:

1. Fock state $|n\rangle$
2. Coherent state $|\alpha\rangle$
3. Squeezed state $|\alpha, \eta\rangle = D(\alpha)S(\eta)|0\rangle$, with $\eta \in \mathbb{R}$.
4. Squeezed state $|\beta, \eta\rangle = S(\eta)D(\beta)|0\rangle$, with $\eta \in \mathbb{R}$.

Displacement operator : $D(\alpha)$

$|\alpha\rangle = D(\alpha)|0\rangle$

$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a}$

$|0\rangle$ → Fock number basis
→ Coherent state basis

Normal ordering of $a^\dagger, a$

Exercise: Baker-Hausdorff equation

if. $[[A,B],A] = [[A,B],B] = 0$

then $e^{A+B} = e^{-[A,B]/2} e^{A} e^{B}$

$$\begin{cases} D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} & \rightarrow \text{Normal Ordering} \\ D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} & \rightarrow \text{Symmetric Ordering} \\ D(\alpha) = e^{|\alpha|^2/2} e^{-\alpha^* a} e^{\alpha a^\dagger} & \rightarrow \text{Anti normal Ordering} \end{cases}$$

Unitary operator

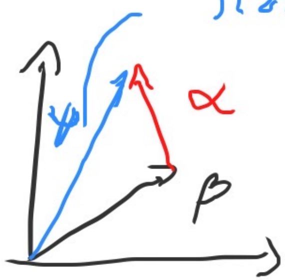
$$D^\dagger(\alpha) = D(-\alpha) = [D(\alpha)]^{-1}$$

$$D^{-1}(\alpha)\, a\, D(\alpha) = a + \alpha \quad / \quad D^{-1}(\alpha)\, a^{+} D(\alpha) = a^{+} + \alpha^{*}$$

α → Complex amplitude of the EM field

Example: state → $|\beta\rangle$ → "Amplitude" of the field

$$\mathcal{E} = \langle\beta|a|\beta\rangle = \langle\beta|\beta|\beta\rangle = \beta$$



Displaced state: $D(\alpha)|\beta\rangle = |\psi\rangle = |\alpha+\beta\rangle$

$$\mathcal{E} = \langle\psi|a|\psi\rangle = \langle\beta|D^{+}(\alpha)\cdot a\cdot D(\alpha)|\beta\rangle$$

$$\langle\beta|a+\alpha|\beta\rangle = \langle\beta|a|\beta\rangle + \langle\beta|\beta\rangle\alpha$$

$$= \beta + \alpha$$

Displacement Operator: $D(\alpha)|0\rangle = |\alpha\rangle$

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} = e^{-|\alpha|^2/2} \sum_{j=0}^{\infty} \frac{(\alpha \hat{a}^\dagger)^j}{j!} \sum_{i=0}^{\infty} \frac{(-\alpha^* \hat{a})^i}{i!}$$

$$= e^{|\alpha|^2/2} e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} = e^{|\alpha|^2/2} \sum_{j=0}^{\infty} \frac{(-\alpha^* \hat{a})^j}{j!} \sum_{i=0}^{\infty} \frac{(\alpha \hat{a}^\dagger)^i}{i!}$$

$$= e^{\alpha a^\dagger - \alpha^* a} = \sum_{j=0}^{\infty} \frac{(\alpha \hat{a}^\dagger - \alpha^* \hat{a})^j}{j!}$$

All powers of $(\hat{a}^\dagger, \hat{a})$ are present

Squeezing operator

$$\hat{H}_I = i\hbar\left(g\hat{a}^{+2} - g^*\hat{a}^2\right) \rightarrow |\psi(t)\rangle = e^{(g\hat{a}^{+2} - g^*\hat{a}^2)t}|0\rangle$$

$$|\psi(t)\rangle = S(\varepsilon)|0\rangle$$

$$\Rightarrow S(\varepsilon) = \exp\left(\frac{1}{2}\varepsilon^*\hat{a}^2 - \frac{1}{2}\varepsilon\hat{a}^{+2}\right)$$

$$\varepsilon = r\exp(i\theta) \quad (r, \theta \in \mathbb{R})$$

$$S^+(\varepsilon) = S^{-1}(\varepsilon) = S(-\varepsilon)$$

Why squeezed?

Since $e^{A} B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \dots$

$$\cosh r = \frac{e^{r} + e^{-r}}{2}$$

$$\sinh r = \frac{e^{r} - e^{-r}}{2}$$

$$e^{r} = \cosh r + \sinh r$$

$$S^{\dagger}(\varepsilon)\, \hat{a}\, S(\varepsilon) = \hat{a} \cosh r - \hat{a}^{\dagger} e^{i\theta} \sinh r$$

$$S^{\dagger}(\varepsilon)\, \hat{a}^{\dagger} S(\varepsilon) = \hat{a}^{\dagger} \cosh r - \hat{a} e^{-i\theta} \sinh r$$

If we rotate the frame

$$X_2 + i Y_2 = (X_1 + i Y_1)\, e^{-i\theta/2}$$ (I'm using a distinct notation here)

$$\hat{b} = \hat{a}\, e^{-i\theta/2}$$

$$S^{\dagger}(\varepsilon)\,(X_2 + i Y_2)\, S(\varepsilon) = X_2 e^{-r} + i Y_2 e^{r}$$

Squeezed States

compress   stretch

$$S^\dagger(\varepsilon)\,(X_2 + iY_2)\,S(\varepsilon) = X_2 e^{-r} + i Y_2 e^{r}$$

Acting on vacuum states: $S(\varepsilon)|0\rangle = |0,\varepsilon\rangle$

$$\langle 0,\varepsilon| X_2 |0,\varepsilon\rangle = 0 = \langle 0,\varepsilon| Y_2 |0,\varepsilon\rangle$$

$$\langle 0,\varepsilon| X_2^2 |0,\varepsilon\rangle = \langle 0,\varepsilon| X_2 \cdot X_2 |0,\varepsilon\rangle = \langle 0| S^\dagger(\varepsilon) X_2 \cdot X_2 S(\varepsilon) |0\rangle$$

$$\langle 0| S^\dagger(\varepsilon) X_2 \underbrace{S(\varepsilon) \cdot S^\dagger(\varepsilon)}_{I} X_2 S(\varepsilon) |0\rangle = \langle 0| X_2^2 |0\rangle e^{-2r}$$

$$\boxed{\Delta^2 X_2 = e^{-2r}\,;\ \Delta^2 Y_2 = e^{2r}}$$

→ Noise compression

Squeezing operator: Quadratic, rescale the axis

Displacement operator; Linear, shift the origin

Combining both: $|\beta, \varepsilon\rangle = D(\beta) S(\varepsilon)|0\rangle$ (Walls & Milburn)

$|\alpha, \varepsilon\rangle = S(\varepsilon) D(\alpha)|0\rangle$ (Scully & Zubairy)

Distinct, yet equivalent...