Building Quantum Machines with Light

LMCAL

Marcelo Martinelli
Laboratório de Manipulação Coerente de Átomos e Luz

INCT-IQ

Representations of the field state

Marcelo Martinelli
Laboratório de Manipulação Coerente de Átomos e Luz

Quantum Optics – Density Operators

Pure X Mixed States

$$\boxed{|\psi\rangle = \sum c_n |a_n\rangle}$$

$$\boxed{\sum |a_m\rangle\langle a_m| = 1}$$

$$\boxed{c_n = \langle a_n|\psi\rangle}$$

$$\boxed{\langle a_m|a_n\rangle = \delta_{mn}}$$

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

$$\langle a_m|A|a_n\rangle = A_{mn}$$

Introducing the density operator (von Neumann – 1927)

$$c_n c_m^* = \rho_{nm}$$

$$\rho = |\psi\rangle\langle\psi|$$

Quantum Optics – Density Operators

$$\boxed{\langle A \rangle = \sum \langle a_n | \rho | a_m \rangle \langle a_m | A | a_n \rangle}$$

$$\boxed{= \sum \langle a_n | \rho A | a_n \rangle = Tr\{\rho A\}}$$

Now we can represent a statistical mixture of pure states!

$$\rho = \sum p_k \rho_k \qquad \sum p_k = 1$$

$$\boxed{\langle A \rangle = Tr\{\rho A\}} \qquad \boxed{Tr\rho = 1}$$

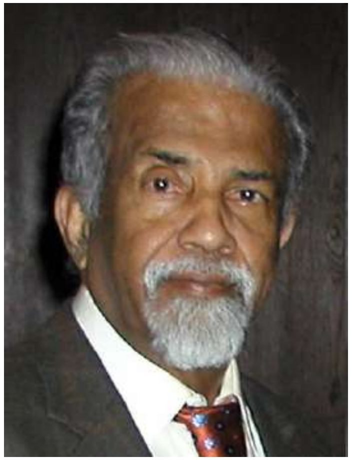
$$\boxed{Tr\rho \geq Tr\rho^2}$$

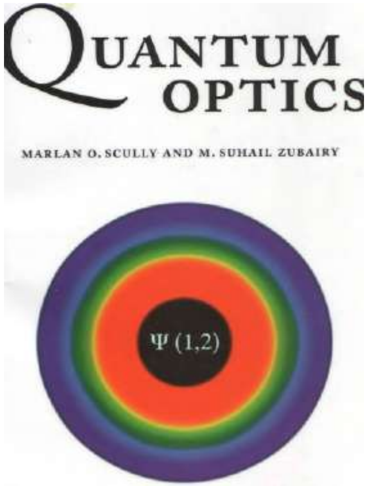
Quantum Optics – Density Operators

Coherent States $|\alpha\rangle$ $\qquad$ $\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\, d^2\alpha$

P(α) : representation of the density operator:

Glauber and Sudarshan





PRL, v. 10, p. 84-87, Feb/1st/1963,

PHOTON CORRELATIONS*

Roy J. Glauber
Lyman Laboratory, Harvard University, Cambridge, Massachusetts
(Received 27 December 1962)

PRL, v. 10, p. 277-279, Apr/1st/1963,

EQUIVALENCE OF SEMICLASSICAL AND QUANTUM MECHANICAL DESCRIPTIONS OF STATISTICAL LIGHT BEAMS

E. C. G. Sudarshan
Department of Physics and Astronomy, University of Rochester, Rochester, New York
(Received 1 March 1963)

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Coherent and Incoherent States of the Radiation Field*

ROY J. GLAUBER
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
(Received 29 April 1963)

P-Representation

Sudarshan & Glauber (1963)

Given the density operator

$$\hat{\rho} = \sum_{n,m} \rho_{nm} |n\rangle\langle m|$$

We may look for a description on the coherent state basis $|\alpha\rangle$

Immediate attempt: use $1 = \frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha$

with $d^2\alpha = d\{Re(\alpha)\}\, d\{Im(\alpha)\}$

$$\hat{\rho} = \left(\frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha\right) \hat{\rho} \left(\frac{1}{\pi} \int |\beta\rangle\langle\beta| d^2\beta\right)$$

$$= \frac{1}{\pi^2} \int \langle\alpha|\hat{\rho}|\beta\rangle\, |\alpha\rangle\langle\beta|\, d^2\alpha\, d^2\beta$$

$$\hat{\rho} = \int \mathcal{F}(\alpha,\beta)\, |\alpha\rangle\langle\beta|\, d^2\alpha\, d\beta^2 \quad = \sum_{n,m} \rho_{nm} |n\rangle\langle m|$$

$\mathcal{F}(\alpha,\beta) \equiv \frac{\langle\alpha|\hat{\rho}|\beta\rangle}{\pi^2}$ is equivalent to $\rho_{nm} = \langle n|\hat{\rho}|m\rangle$

$$\hat{\rho} = \int \mathcal{F}(\alpha,\beta)\, |\alpha\rangle\langle\beta|\, d^2\alpha\, d\beta^2 = \sum_{n,m} \rho_{nm} |n\rangle\langle m|$$

$\mathcal{F}(\alpha,\beta) \equiv \dfrac{\langle\alpha|\hat{\rho}|\beta\rangle}{\pi^2}$ is equivalent to $\rho_{nm} = \langle n|\hat{\rho}|m\rangle$

$\downarrow$ continuous, 4D

$\downarrow$ discrete, 2D

$\mathcal{F}(\alpha,\beta)$ is a map of the $\hat{\rho}$ matrix.

Nevertheless, in a doubled space. Could we use, at least, a 2D mapping?

Proposal: $\mathcal{F}(\alpha,\beta) \rightarrow P(\alpha)$ (or $P(\alpha,\alpha^*)$)

such as $\underline{\hat{\rho} = \int P(\alpha)\, |\alpha\rangle\langle\alpha|\, d^2\alpha}$

Use: calculation of normally-ordered Operators

$$O_N(\hat{a},\hat{a}^\dagger)=\sum_{nm} c_{nm}\,\hat{a}^{\dagger n}\hat{a}^m$$

$$\langle O_N(\hat{a},\hat{a}^\dagger)\rangle = Tr\left[\hat{\rho}\cdot O_N(\hat{a},\hat{a}^\dagger)\right]=\sum_{nm} c_{nm}\,Tr\left[\rho\,\hat{a}^{\dagger n}a^m\right]$$

$$Tr\left[\hat{A}\hat{B}\right]=\sum_n \langle n|\hat{A}\hat{B}|n\rangle=\sum_n\langle n|\hat{A}\left(\sum_m|m\rangle\langle m|\right)\hat{B}|n\rangle$$

$$=\sum_{nm}\langle n|\hat{A}|m\rangle\langle m|\hat{B}|n\rangle=\sum_{nm}\langle m|\hat{B}|n\rangle\langle n|\hat{A}|m\rangle$$

$$=\sum_m\langle m|\hat{B}\left(\sum_n|n\rangle\langle n|\right)\hat{A}|m\rangle=\sum_m\langle m|\hat{B}\cdot\hat{A}|m\rangle=Tr\left[\hat{B}\hat{A}\right]$$

$$\langle O_N(\hat{a},\hat{a}^\dagger)\rangle=\sum_{nm} c_{nm}\,Tr\left[\hat{a}^m\hat{\rho}\,\hat{a}^{\dagger n}\right]$$

$$\hat{\rho} = \int P(\alpha) |\alpha\rangle\langle\alpha| \, d^2\alpha \Rightarrow$$

$$\langle O_N(\hat{a}, \hat{a}^\dagger)\rangle = \sum_{nm} C_{nm} \, Tr\left[\int P(\alpha) \left(\hat{a}^m |\alpha\rangle\langle\alpha| \hat{a}^{\dagger n}\right) d^2\alpha \right]$$

$$= \sum_{nm} C_{nm} \, Tr\left[\int P(\alpha) \, \alpha^m |\alpha\rangle\langle\alpha| \alpha^{*n} \, d^2\alpha \right]$$

$$= Tr\left[\int P(\alpha) \left(\sum_{nm} C_{nm} \alpha^m \alpha^{*n} \right) |\alpha\rangle\langle\alpha| \, d^2\alpha \right]$$

$$= Tr\left[\int P(\alpha) \, O_N(\alpha, \alpha^*) |\alpha\rangle\langle\alpha| \, d^2\alpha \right]$$

$$= \int P(\alpha) \, O_N(\alpha, \alpha^*) \, Tr\left[|\alpha\rangle\langle\alpha| \right] d^2\alpha$$

but $Tr\left[|\alpha\rangle\langle\alpha| \right] = \sum_n \langle n|\alpha\rangle\langle\alpha|n\rangle = \sum_n |\langle n|\alpha\rangle|^2 = 1$

$$\therefore \langle O^N(\hat{a},\hat{a}^+)\rangle = \int d^2\alpha\, P(\alpha,\alpha^*)\, O_N(\alpha,\alpha^*)$$

→ density function
probabitity function

Other ways of calculating:

Given $\hat{\rho}$

$$\langle -\beta|\hat{\rho}|\beta\rangle = \int P(\alpha)\langle -\beta|\alpha\rangle\langle\alpha|\beta\rangle\, d^2\alpha$$

Since $\langle\alpha|\beta\rangle = \exp\left[-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^*\beta\right]$

$$\langle -\beta|\hat{\rho}|\beta\rangle = e^{-|\beta|^2}\int P(\alpha)e^{-|\alpha|^2}\cdot\exp[\beta\alpha^* - \beta^*\alpha]\, d^2\alpha$$

but $\beta\alpha^* - \beta^*\alpha = -2i(x_\beta y_\alpha - x_\alpha y_\beta)$

$$\therefore \langle -\beta|\rho|\beta\rangle e^{|\beta|^2} = \iint P(x,y)\, e^{-(x^2+y^2)} e^{2i(y_\beta x - x_\beta y)}\, dx\, dy$$

$$\therefore \langle -\beta|\rho|\beta\rangle e^{|\beta|^2} = \iint P(x,y)\, e^{-(x^2+y^2)} e^{2i(y_\beta x - x_\beta y)}\, dx\, dy$$

2D Fourier transform

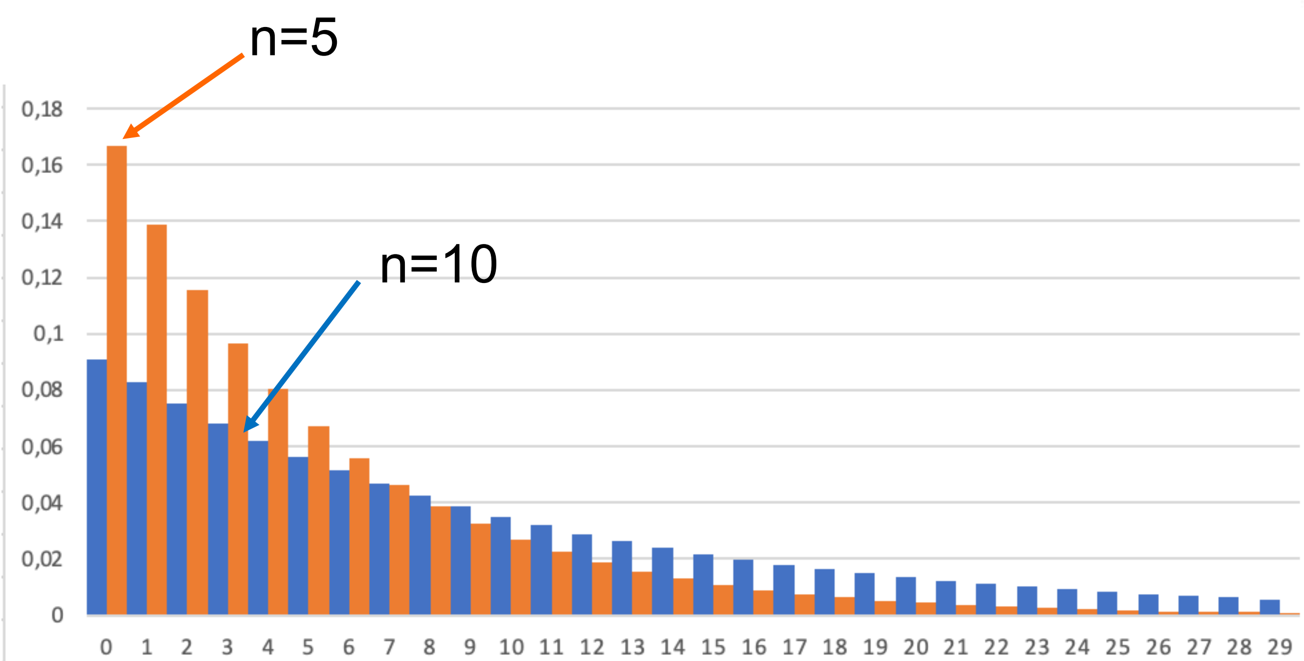
$\Rightarrow$ taking the inverse

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2} \int \langle -\beta|\hat{\rho}|\beta\rangle e^{|\beta|^2} \exp[-\beta\alpha^* + \beta^*\alpha]\, d^2\beta$$

States in the P representation

Thermal state: $$\rho = \sum_n \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|$$

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_B T}-1}$$



States in the P representation

Thermal state: $\rho = \sum_n \frac{1}{1+\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}} \right)^n |n\rangle\langle n|$

$$\langle -\beta|\rho|\beta\rangle = \sum_n \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \langle -\beta|n\rangle\langle n|\beta\rangle$$

$$= \frac{e^{-|\beta|^2}}{1+\bar{n}} \sum_n \frac{-|\beta|^{2n}}{n!} \frac{\bar{n}^n}{(1+\bar{n})^n}$$

$$= \frac{e^{-|\beta|^2}}{1+\bar{n}} \cdot \exp\left[- \frac{|\beta|^2 \cdot \bar{n}}{1+\bar{n}} \right]$$

$$\Rightarrow P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2(1+\bar{n})} \int \exp\left[-\frac{|\beta|^2 \bar{n}}{1+\bar{n}}\right] e^{-\beta\alpha^* + \alpha\beta^*} d^2\beta$$

$$\boxed{\int_{-\infty}^{\infty} e^{-ax^2} e^{-i 2\pi k x} dx = \sqrt{\frac{\pi}{a}}\, e^{-\pi^2 k^2/a}}$$

$\pi k \to x_\alpha$

$x \to y_\beta$

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2(1+\bar{n})} \cdot \left(\frac{1+\bar{n}}{\bar{n}}\right) \cdot \pi \cdot e^{-|\alpha|^2\left(\frac{1}{\bar{n}}+1\right)}$$

$$= \frac{1}{\pi \bar{n}} e^{-|\alpha|^2/\bar{n}} \quad \longrightarrow \quad \text{Gaussian distribution}$$

Coherent state $|\beta\rangle$

$$P(\alpha) = \mathrm{Tr}\left[|\beta\rangle\langle\beta|\delta(\alpha^* - \hat{a}^\dagger)\delta(\alpha - \hat{a})\right]$$

$$= \mathrm{Tr}\left[\delta(\alpha - \hat{a})|\beta\rangle\langle\beta|\delta(\alpha^* - \hat{a}^\dagger)\right]$$

$$= \mathrm{Tr}\left[\delta(\alpha - \beta)\delta(\alpha^* - \beta^*)\right]$$

$$= \delta^{(2)}(\alpha - \beta)$$

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$$

$$= \int \delta^{(2)}(\alpha - \beta)|\alpha\rangle\langle\alpha| d^2\alpha = |\beta\rangle\langle\beta|$$

Number state: $|n\rangle$

$$\langle -\beta | n\rangle \langle n | \beta\rangle = \exp -|\beta|^2 \frac{(-1)^n |\beta|^{2n}}{n!}$$

$$P(\alpha) = \frac{(-1)^n e^{|\alpha|^2}}{\pi^2 n!} \underbrace{\int |\beta|^{2n} e^{-\beta\alpha^* + \beta^*\alpha} d^2\beta}_{\frac{\partial^{2n}}{\partial\alpha^n \partial\alpha^{*n}} \int e^{-\beta\alpha^* + \beta^*\alpha} d^2\beta}$$

$$= \frac{e^{|\alpha|^2}}{n!} \frac{\partial^n}{\partial\alpha^n} \cdot \frac{\partial^n}{\partial\alpha^{*n}} \delta^2(\alpha)$$

Awful!

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Relation between Quantum and Semiclassical Description of Optical Coherence*

C. L. MEHTA
Department of Physics and Astronomy, University of Rochester, Rochester, New York

AND

E. C. G. SUDARSHAN†
Institute for Theoretical Physics, University of Bern, Bern, Switzerland
(Received 12 October 1964)

Proc. Phys. Math. Soc. Jpn. 22: 264-314 (1940)

Some Formal Properties of the Density Matrix.

by Kôdi HUSIMI.

(Read Sept. 23, 1939 and Jan. 27, 1940)

Q-Representation

P- Glauber & Sudarshan (1963)

$$P(\alpha) = \mathrm{Tr}\left[\rho\, \delta(\alpha^* - \hat{a}^\dagger)\, \delta(\alpha - \hat{a})\right]$$

$$\langle O^N(\hat{a}, \hat{a}^\dagger)\rangle = \int P(\alpha)\, O_N(\alpha, \alpha^*)\, d^2\alpha$$

Q - Husimi (1940)

$$Q(\alpha) = \mathrm{Tr}\left[\rho\, \delta(\alpha - \hat{a})\, \delta(\alpha^* - \hat{a}^\dagger)\right]$$

$$Q(\alpha) = \mathrm{Tr}\left[\rho\, \delta(\alpha - \hat{a})\, \delta(\alpha^* - \hat{a}^\dagger)\right]$$

<u>Implications:</u>

$$Q(\alpha) = \mathrm{Tr}\left[\rho\, \delta(\alpha - \hat{a}) \left(\int \frac{|\alpha'\rangle\langle\alpha'|}{\pi}\, d\alpha'^2\right) \delta(\alpha^* - a^\dagger)\right]$$

$$= \frac{1}{\pi}\, \mathrm{Tr}\left[\rho \left(\int \delta(\alpha - \alpha')\, \delta(\alpha^* - \alpha'^*)\, |\alpha'\rangle\langle\alpha'|\, d^2\alpha'\right)\right]$$

$$= \frac{1}{\pi}\, \mathrm{Tr}\left[\rho\, |\alpha\rangle\langle\alpha|\right] =$$

$$= \frac{1}{\pi} \sum_n \langle n|\rho|\alpha\rangle\langle\alpha|n\rangle = \frac{1}{\pi} \sum_n \langle\alpha|n\rangle\langle n|\rho|\alpha\rangle$$

$$Q(\alpha) = \frac{\langle \alpha | \rho | \alpha \rangle}{\pi}$$

$$\Rightarrow \int Q(\alpha)\, d^2\alpha = 1 \; ; \quad Q(\alpha) \geq 0$$

And since $\rho = \sum p_k |\psi_k\rangle\langle\psi_k|$

$$Q(\alpha) = \frac{1}{\pi} \sum_k p_k |\langle \psi_k | \alpha \rangle|^2$$

$$|\langle \psi_k | \alpha \rangle|^2 \leq 1$$

$$p_k \leq 1$$

$$\Rightarrow Q(\alpha) \leq \frac{1}{\pi}$$

Operator evaluation

$$O^{A}(\hat{a},\hat{a}^{\dagger}) = \sum_{nm} d_{nm}\,\hat{a}^{n}\,\hat{a}^{\dagger m}$$

$$\rightarrow \langle O^{A}(\hat{a},\hat{a}^{\dagger})\rangle = \mathrm{Tr}\left[\, O^{A}(a,a^{\dagger})\rho \,\right]$$

$$= \mathrm{Tr}\left[\sum_{nm} d_{nm}\,\hat{a}^{n}\,\hat{a}^{\dagger m}\rho\right]$$

$$= \mathrm{Tr}\left[\sum_{nm} d_{nm}\,\hat{a}^{n}\left(\int d^{2}\alpha\,\frac{|\alpha\rangle\langle\alpha|}{\pi}\right)\hat{a}^{\dagger m}\rho\right]$$

$$= \mathrm{Tr}\left[\sum_{nm} d_{nm}\, \hat{a}^{n} \left(\int d^2\alpha\, \frac{|\alpha\rangle\langle\alpha|}{\pi}\right) \hat{a}^{+m} \rho\right]$$

$$= \frac{1}{\pi} \mathrm{Tr}\left[\sum_{nm} d_{nm} \left(\int d^2\alpha\, \alpha^{n} \alpha^{*m}\, |\alpha\rangle\langle\alpha|\right) \rho\right]$$

$$= \int d^2\alpha \cdot O^{A}(\alpha, \alpha^*)\; \mathrm{Tr}\left[|\alpha\rangle\langle\alpha|\rho\right] \frac{1}{\pi}$$

$$\therefore \langle O^{A}(\hat{a}, \hat{a}^{+})\rangle = \int d^2\alpha\, O^{A}(\alpha, \alpha^*)\, Q(\alpha)$$

Given: $\hat{O}(\hat{a},\hat{a}^\dagger) = \hat{O}_N(\hat{a},\hat{a}^\dagger) = \sum_{nm} c_{nm}\, \hat{a}^{\dagger n} \hat{a}^m$

$= \hat{O}_A(\hat{a},\hat{a}^\dagger) = \sum_{k\cdot l} d_{kl}\, \hat{a}^k \hat{a}^{\dagger l}$

$[\hat{a},\hat{a}^\dagger] = 1$

$$\langle \hat{O}(\hat{a},\hat{a}^\dagger) \rangle = \mathrm{Tr}\left[\hat{\rho} \cdot \hat{O}(\hat{a},\hat{a}^\dagger)\right]$$

$$= \int d^2\alpha \; P(\alpha,\alpha^*)\; O_N(\alpha,\alpha^*)$$

$$= \int d^2\alpha \; Q(\alpha,\alpha^*)\; O_A(\alpha,\alpha^*)$$

$P(\alpha,\alpha^*) =$

$= \mathrm{Tr}\left[\hat{\rho}\, \delta(\alpha^* - \hat{a}^\dagger)\, \delta(\alpha - \hat{a})\right]$

$Q(\alpha,\alpha^*) =$

$= \mathrm{Tr}\left[\hat{\rho}\, \delta(\alpha - \hat{a})\, \delta(\alpha^* - \hat{a}^\dagger)\right]$

$$|\beta\rangle \Rightarrow \quad \boxed{Q(\alpha) = \frac{e^{-|\alpha-\beta|^2}}{\pi}}$$

$$Q(\alpha) = \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\pi} = \frac{|\langle\beta|\alpha\rangle|^2}{\pi}$$

$$Q(\beta) = \frac{1}{\pi} \quad ; \quad P(\beta) = \delta^{(2)}(\alpha-\beta)$$

<u>Thermal State:</u> $\rho = \sum_n \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |n\rangle\langle n|$

$$Q(\alpha) = \sum_n \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |\langle n|\alpha\rangle|^2 \quad \frac{1}{\pi}$$

$$= \frac{e^{-|\alpha|^2}}{\pi(1+\bar{n})} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \frac{|\alpha|^{2n}}{n!}$$

$$= \frac{e^{-|\alpha|^2}}{\pi(1+\bar{n})} \exp\left[\frac{|\alpha|^2 \bar{n}}{1+\bar{n}}\right]$$

$$= \frac{1}{\pi} \exp\left[|\alpha|^2\left(\frac{\bar{n} - 1 - \bar{n}}{1+\bar{n}}\right)\right] \cdot \frac{1}{1+\bar{n}}$$

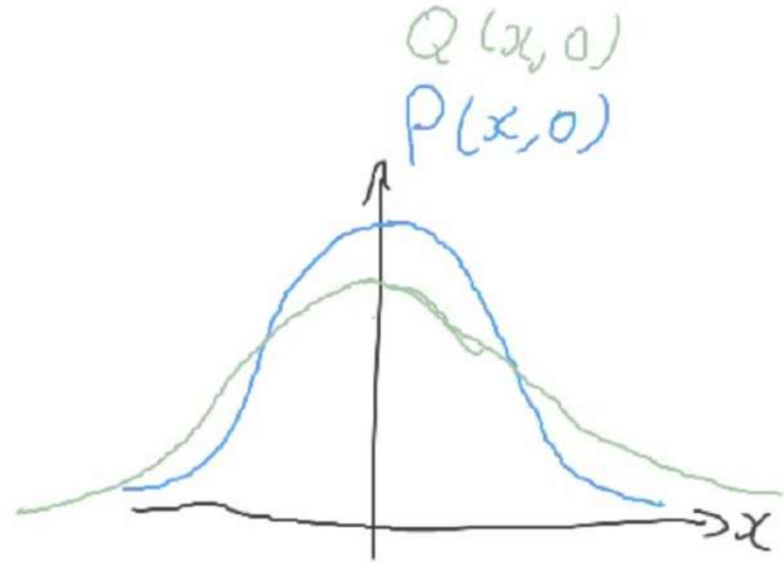
$$= \frac{1}{\pi} \exp\left[|\alpha|^2 \left(\frac{\bar{n} - 1 - \bar{n}}{1+\bar{n}} \right) \right] \cdot \frac{1}{1+\bar{n}}$$

$$\boxed{Q(\alpha) = \frac{e^{-|\alpha|^2/1+\bar{n}}}{\pi(1+\bar{n})}}$$

$$= \frac{1}{\pi} \exp\left[|\alpha|^2 \left(\frac{\bar{n} - 1 - \bar{n}}{1+\bar{n}} \right) \right] \cdot \frac{1}{1+\bar{n}}$$

$$\boxed{Q(\alpha) = \frac{e^{-|\alpha|^2/1+\bar{n}}}{\pi(1+\bar{n})}}$$

$$P(\alpha) = \frac{e^{-|\alpha|^2/\bar{n}}}{\pi\,\bar{n}}$$



$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1 = \hat{n} + 1$$

$$= \frac{1}{\pi} \exp\left[|\alpha|^2 \left(\frac{\bar{n} - 1 - \bar{n}}{1+\bar{n}} \right) \right] \cdot \frac{1}{1+\bar{n}}$$

$$\boxed{Q(\alpha) = \frac{e^{-|\alpha|^2 / 1+\bar{n}}}{\pi(1+\bar{n})}}$$

<u>Number state</u>: $Q(\alpha) = \frac{\langle \alpha | n \rangle \langle n | \alpha \rangle}{\pi}$

$$\boxed{Q(\alpha) = |\alpha|^{2n} \frac{e^{-|\alpha|^2}}{\pi n!}}$$

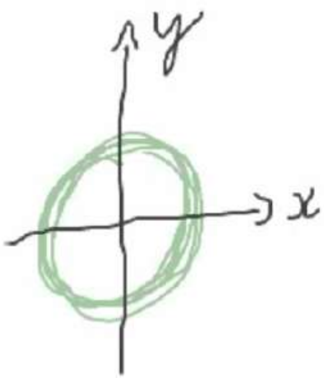
$$= \frac{1}{\pi} \exp\left[|\alpha|^2\left(\frac{\bar{n} - 1 - \bar{n}}{1+\bar{n}}\right)\right] \cdot \frac{1}{1+\bar{n}}$$

$$\boxed{Q(\alpha) = \frac{e^{-|\alpha|^2/1+\bar{n}}}{\pi(1+\bar{n})}}$$

<u>Number state</u>: $Q(\alpha) = \frac{\langle\alpha|n\rangle\langle n|\alpha\rangle}{\pi}$

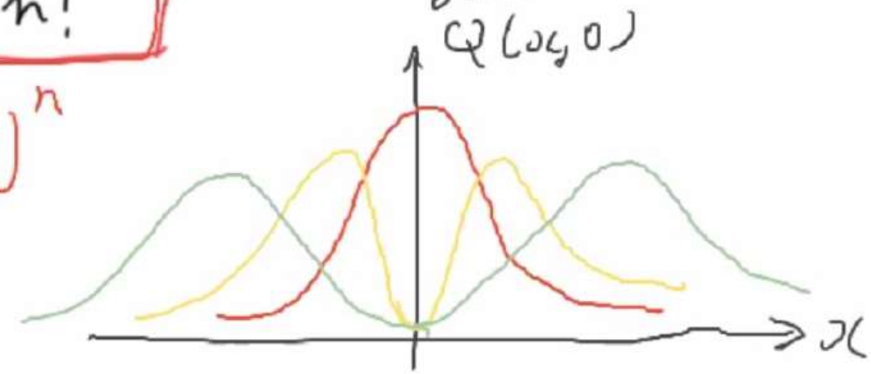
$$Q(\alpha) = |\alpha|^{2n}\,\boxed{\frac{e^{-|\alpha|^2}}{\pi n!}}$$

$$P \propto \frac{\partial^n}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha)$$



Polinomial $(r^2)^n$

$r = |\alpha|$



Quantum Optics – Density Operators

2D Representations of the density operators provide a simple way to describe the state of the field as a function of dimension 2N, where N is the number of modes involved.

P representation is a good way to present “classical” states, like thermal light or coherent states – statistical mixtures

But it is singular for “non classical states” (e.g. Fock and squeezed states).

Husimi-Q is positive, limited, well defined. Looks like a probability distribution.

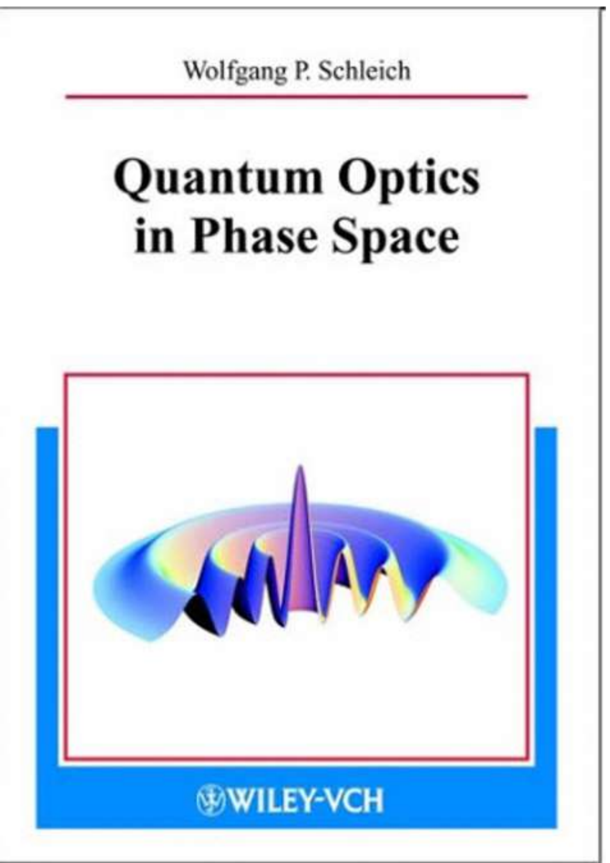
Washes away the quantum features – everything looks like a statistical mixture!

We will see and old trick for an intermediate representation..

Quasi-Probability Representations

P- Glauber – Sudarshan (1962) $\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\, \mathrm{d}^2\alpha$

Q – Husimi (1940) $Q(\alpha) = \dfrac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi}$



Wigner (1932)

$$\bar{W}(\bar{x}, \bar{p}) = \frac{1}{\pi\hbar}\int dy \langle \bar{x} + y|\rho|\bar{x} - y\rangle \exp(-2iy\bar{p}/\hbar)$$

JUNE 1, 1932 *PHYSICAL REVIEW* *VOLUME 40*

On the Quantum Correction For Thermodynamic Equilibrium

By E. Wigner
Department of Physics, Princeton University
(Received March 14, 1932)

Wigner function

Definition: a 2D representation

$$W(x,p) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \, \exp\left(-\frac{i}{\hbar} p \xi\right) \left\langle x + \frac{\xi}{2} \middle| \hat{\rho} \middle| x - \frac{\xi}{2} \right\rangle$$

1D-Fourier transform of: $\rho(x', x'') = \langle x' | \hat{\rho} | x'' \rangle$

normalized as $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \, W(x,p) = 1$

Meaning: Coupling from state $|x''\rangle$ to $\langle x'|$ by a projector $|\Psi\rangle\langle\Psi| \rightarrow$ quantum jump

$$W(x,p) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \, \exp\left(-\frac{i}{\hbar} p \xi\right) \left\langle x + \frac{\xi}{2} \middle| \hat{\rho} \middle| x - \frac{\xi}{2} \right\rangle$$

1D - Fourier transform of: $\rho(x', x'') = \langle x' | \hat{\rho} | x'' \rangle$

$$\rho(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2} \int \langle -\beta | \hat{\rho} | \beta \rangle e^{|\beta|^2} \exp[-\beta\alpha^* + \beta^*\alpha] \, d^2\beta$$

$$W(x,p) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \exp\left(-\frac{i}{\hbar} p\xi\right) \left\langle x + \frac{\xi}{2} \middle| \hat{\rho} \middle| x - \frac{\xi}{2} \right\rangle$$

1D - Fourier transform of: $\rho(x', x'') = \langle x' | \hat{\rho} | x'' \rangle$

normalized as $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \, W(x,p) = 1$

Meaning: Coupling from state $|x''\rangle$ to $\langle x'|$ by a projector $|\Psi\rangle\langle\Psi|$ $\rightarrow$ quantum jump

Equivalent to a dipole moment in a transition $\langle n' | \hat{\mu} | n'' \rangle$

Arbitrary as it may sound, it carries important properties

Marginals:

$$\int_{-\infty}^{\infty} dp\, W(x,p) = \int_{-\infty}^{\infty} d\xi \langle x+\tfrac{\xi}{2}|\hat{\rho}|x-\tfrac{\xi}{2}\rangle \underbrace{\int_{-\infty}^{\infty} dp \exp\left(-\tfrac{i}{\hbar}p\xi\right)\cdot\frac{1}{2\pi\hbar}}_{\delta(\xi)}$$

$$= \langle x|\hat{\rho}|x\rangle = P(x)$$

↳ probability distribution on x

$$P(q_{\mathbf{k}s}) = \langle n_{\mathbf{k}s}|q_{\mathbf{k}s}\rangle\langle q_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = |\langle q_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle|^2.$$

$$\int_{-\infty}^{\infty} dx\, W(x,p) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\xi\, \exp\left(-i\frac{p}{\hbar}\xi\right) \langle \underbrace{x+\tfrac{1}{2}\xi}_{x''} | \hat{\rho} | \underbrace{x-\tfrac{1}{2}\xi}_{x'} \rangle$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx''\, \exp\left[-\frac{i}{\hbar} p (x''-x')\right] \langle x'' | \hat{\rho} | x' \rangle$$

Relation with the probability distribution in p

$$\mathcal{P}(p) = \langle p | \hat{\rho} | p \rangle = \langle p | \int dx'' |x''\rangle\langle x''|\, \hat{\rho} \int dx' |x'\rangle\langle x'| \,|p\rangle$$

$$= \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dx'\, \langle x'' | \rho | x' \rangle\, \underbrace{\langle p | x'' \rangle}_{\exp\left(-\frac{i}{\hbar} p x''\right)}\, \underbrace{\langle x' | p \rangle}_{\exp\left(\frac{i}{\hbar} p x'\right)}$$

Relation with the probability distribution in p

$$\mathcal{P}(p) = \langle p | \hat{\rho} | p \rangle = \langle p | \int dx'' |x''\rangle\langle x''| \, \hat{\rho} \int dx' |x'\rangle\langle x'| \, |p\rangle$$

$$= \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dx' \, \langle x'' | \rho | x' \rangle \, \underset{\exp\left(-\frac{i}{\hbar} p x''\right)}{\underbrace{\langle p | x'' \rangle}} \, \underset{\exp\left(\frac{i}{\hbar} p x'\right)}{\underbrace{\langle x' | p \rangle}}$$

$$\therefore \quad \mathcal{P}(p) = \int_{-\infty}^{\infty} dx \, W(x,p)$$

Overlap of Quantum States

Given $\hat{\rho}_1, \hat{\rho}_2$; $\quad Tr(\hat{\rho}_1, \hat{\rho}_2) = 2\pi\hbar \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \; W_1(x,p)\, W_2(x,p)$

Consequence: $W(x,p)$ is bounded!

$$\mathrm{Tr}\{\hat{\rho}^2\} = 2\pi\hbar \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \; W^2(x,p) \leq 1$$

$$\underline{\text{"Area"}} \; \left[\int dx \int dp \; W^2(x,p)\right]^{-1} \geq 2\pi\hbar$$

From Cauchy-Schwarz inequality

$$|\langle \phi_1 | \phi_2 \rangle|^2 \leq \langle \phi_1 | \phi_1 \rangle \langle \phi_2 | \phi_2 \rangle = 1$$

$$\Rightarrow \; |W(x,p)| \leq \frac{1}{\pi\hbar}$$

Since $\exists \; \hat{\rho}_1, \hat{\rho}_2 \; / \; \mathrm{Tr}\{\hat{\rho}_1, \hat{\rho}_2\} = 0$

$$\int dx \int dp \;\; W_1(x,p) \, W_2(x,p) = 0 \;\; \Rightarrow \;\; \exists \; W(x,p) < 0$$

Observables: in the Harmonic Oscillator

$$\langle O_s(\hat{a}, \hat{a}^{+}) \rangle = \int dx\, dp\; W(x,p) \cdot O_s(\alpha, \alpha^{*})$$

$$\alpha = x + ip$$

Observables: in the Harmonic Oscillator

$$\langle \hat{O}_s(\hat{a},\hat{a}^\dagger)\rangle = \int dx\,dp\; W(x,p)\cdot O_s(\alpha,\alpha^*)$$

$$\alpha = x + ip$$

$$\hat{n} = \hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - 1$$

$\hat{a}^\dagger\hat{a}$ $\Downarrow$ Normal Ordering

$\hat{a}\hat{a}^\dagger - 1$ $\Downarrow$ anti normal ordering

$$\hat{n}_n = \hat{a}^\dagger\hat{a}$$

$$\hat{n}_{an} = \hat{a}\hat{a}^\dagger - 1$$

$$\hat{n}_s = \frac{\hat{n}_n + \hat{n}_{an}}{2}$$

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

Observables: in the Harmonic Oscillator

$$\langle \hat{O}_s(\hat{a}, \hat{a}^\dagger) \rangle = \int dx\, dp\ W(x,p) \cdot O_s(\alpha, \alpha^*)$$

$$\alpha = x + ip$$

$$\hat{n} = \frac{\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger}{2} - \frac{1}{2}$$

$$\hat{n}_n = \hat{a}^\dagger \hat{a}$$

$$\hat{n}_{an} = \hat{a}\hat{a}^\dagger - 1$$

$$\hat{H}_n = \hbar\omega\left(\hat{a}^\dagger \hat{a} + 1/2\right)$$

$$\hat{n}_s = \frac{\hat{n}_n + \hat{n}_{an}}{2}$$

$$\hat{H}_a = \hbar\omega\left(\hat{a}\hat{a}^\dagger - 1/2\right)$$

$$\hat{H}_s = \hbar\omega\left(\frac{\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger}{2}\right) = \hbar\omega(x^2 + y^2) \qquad \hat{a} = \hat{x} + i\hat{y}$$

Observables: in the Harmonic Oscillator

$$\langle O_s(\hat{a}, \hat{a}^\dagger)\rangle = \int dx\, dp\; W(x,p) \cdot O_s(\alpha, \alpha^*)$$

$\alpha = x + ip$

$\hat{x} = \hat{a} + \hat{a}^\dagger$

Symmetric ordering

$\hat{x}^2 = O(2)$

up to $\underline{\text{power } 2}$

linear
bilinear functions
of $\hat{a}, \hat{a}^\dagger$

$O(3), O(4) \ldots$

Wigner Function

Coherent state : $\hat{\rho} = |\alpha_0\rangle\langle\alpha_0| \Rightarrow$

$$W(x,y) = \frac{2}{\pi} \exp\left[-\frac{1}{2}\left[(x-x_0)^2 + (y-y_0)^2\right]\right]$$

Center @ $\alpha_0 = x_0 + i y_0$

Squeezed state : $|\alpha, \varepsilon\rangle = |\alpha, e^{r}\rangle$ $\qquad \varepsilon \in \mathbb{R}$

$$W(x,y) = \frac{2}{\pi} \exp\left[-\frac{1}{2}\left[(x-x_0)^2 e^{-2r} + (y-y_0) e^{2r}\right]\right]$$

$\hookrightarrow$ Gaussian

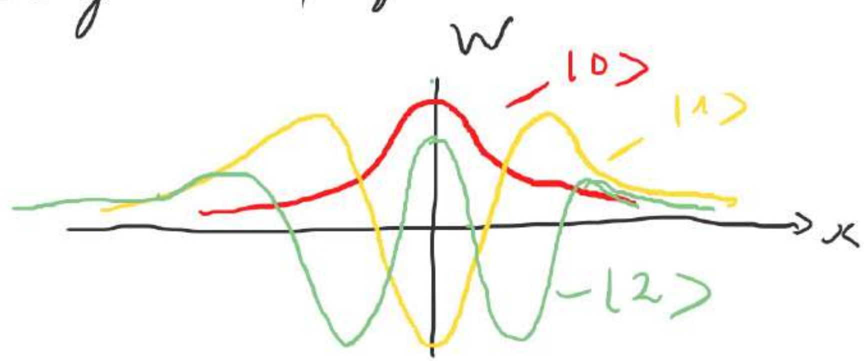
Number state: $\hat{\rho} = |n\rangle\langle n|$

$$W(x,y) = \frac{2}{\pi}(-1)^n L_n(4r^2) e^{-2r^2}$$

$$r = \sqrt{x^2 + y^2}$$

$L_n(x) \to$ Laguerre polynomial

$W(x,y) < 0$



Characteristic Function

$$\chi \leftrightarrow \hat{\rho} \quad \rightarrow \quad \mathcal{F}(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta^*\alpha}\,\chi(\eta)\,d^2\eta$$

Defines quasiprobability as a Fourier Transform of the characteristic function

$$\text{CF:} \quad \chi_N(\eta) = \mathrm{Tr}\{\rho\, \underbrace{e^{\eta\hat{a}^\dagger} e^{-\eta^*\hat{a}}}_{\text{normal ordering}}\}$$

$$P(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta^*\alpha}\,\chi_N(\eta)\,d^2\eta$$

if $\chi_A(\eta) = \mathrm{Tr}\{\hat{\rho}\, \underbrace{e^{-\eta^* a} e^{\eta a^\dagger}}_{\text{anti normal}}\}$

$$Q(\alpha) = \frac{1}{\pi^2}\int e^{\eta^*\alpha - \eta\alpha^*}\, \chi_A(\eta)\, d^2\eta$$

Baker-Haussdorf $e^{A+B} = e^A e^B e^{-[A,B]/2}$;

if $[A,[A,B]] = [B,[A,B]] = 0$

$\chi_S = \mathrm{Tr}\{\hat{\rho}\, \underbrace{e^{\eta a^\dagger - \eta^* a}}_{D(\eta)}\}$; $\quad W(\alpha) = \frac{1}{\pi^2}\int e^{(\eta^*\alpha - \eta\alpha^*)}\chi(\eta)\, d^2\eta$

$$\chi_S = \mathrm{Tr}\{\hat{\rho}\,\hat{D}(\eta)\}$$

$$\chi_A = \mathrm{Tr}\{\hat{\rho}\,\hat{D}(\eta)\}\,e^{-\frac{1}{2}|\eta|^2} \rightarrow \text{Gaussian weight}$$

$$\chi_N = \mathrm{Tr}\{\hat{\rho}\,\hat{D}(\eta)\}\,e^{\frac{1}{2}|\eta|^2} \rightarrow \text{Inverse Gausian}$$

$$\Rightarrow W(\alpha) = \frac{2}{\pi}\int P(\beta)\,e^{-2|\alpha-\beta|^2}\,d^2\beta \rightarrow \text{bounded, continuous}$$

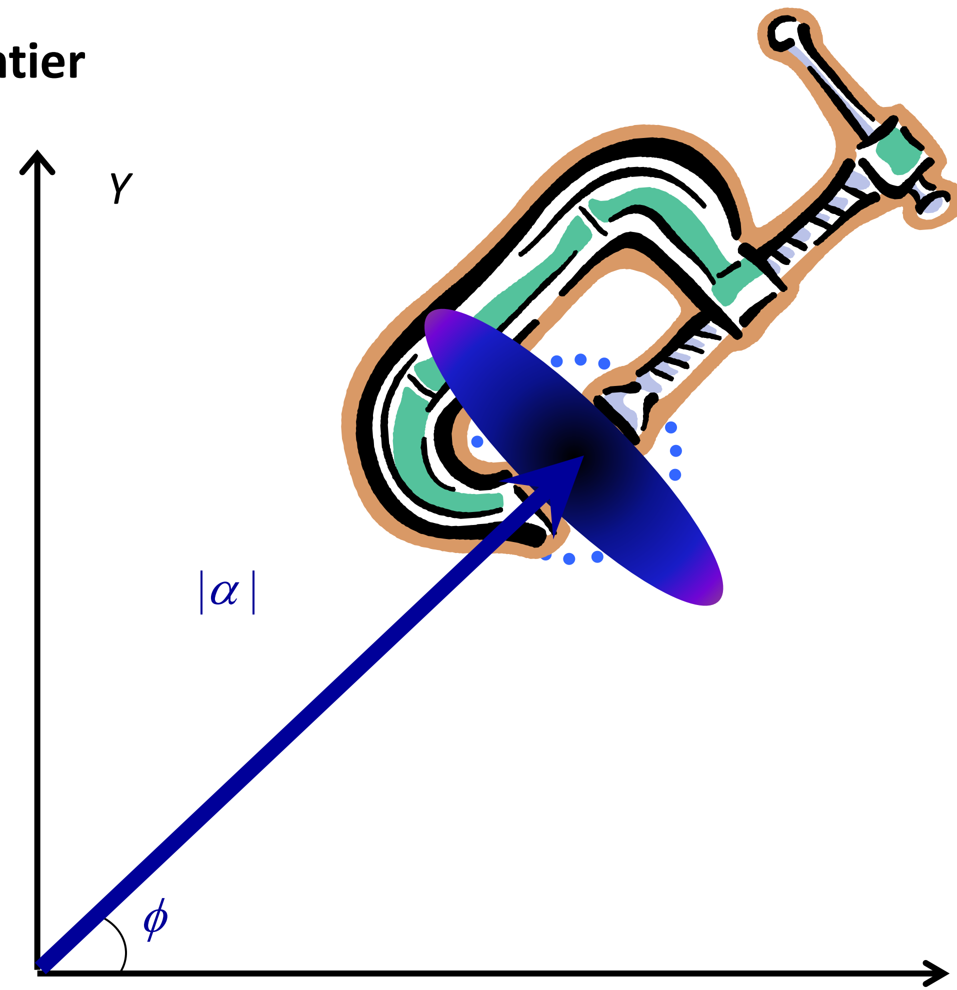
$$Q(\alpha) = \frac{1}{\pi}\int P(\beta)\,e^{-|\alpha-\beta|^2}\,d^2\beta \rightarrow \text{posite, bonded, continuous}$$

$$= \frac{2}{\pi}\int W(\beta)\,e^{-2|\alpha-\beta|^2}\,d^2\beta \quad \text{(check)}$$

Wigner Representation

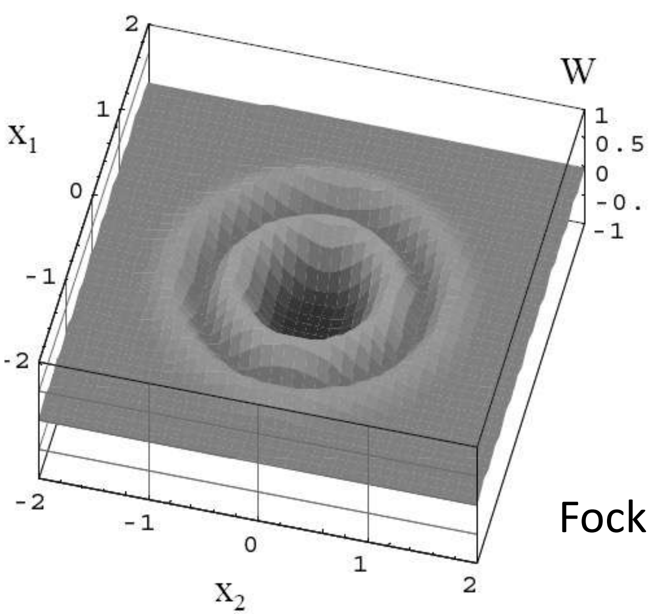
Evident quantum/ classical frontier

Squeezed states



States with W<0

n=3



Fock states $|n\rangle$

Reconstruction of non-classical cavity field states with snapshots of their decoherence

Samuel Deléglise[1], Igor Dotsenko[1,2], Clément Sayrin[1], Julien Bernu[1], Michel Brune[1], Jean-Michel Raimond[1] & Serge Haroche[1,2]

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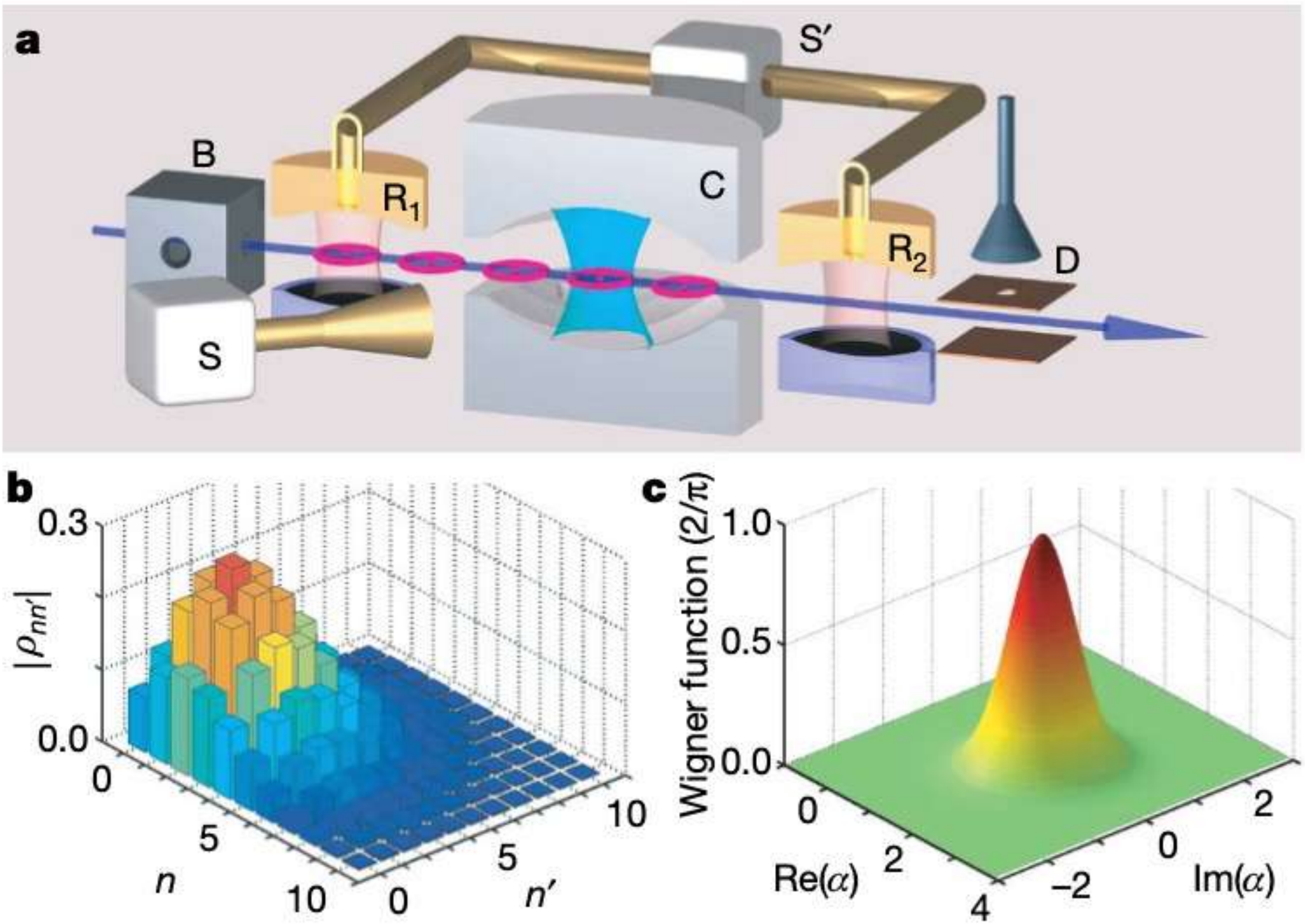
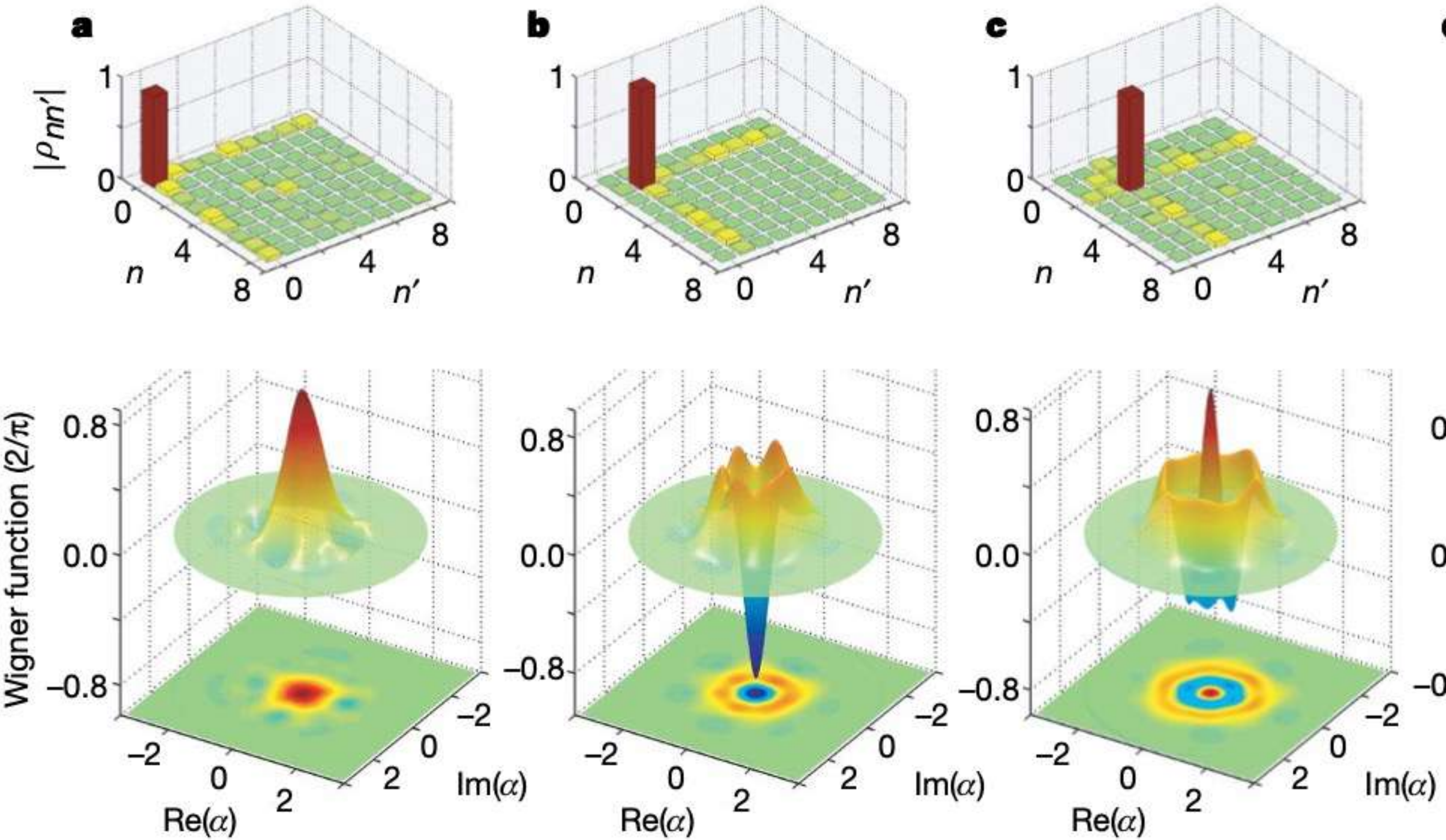
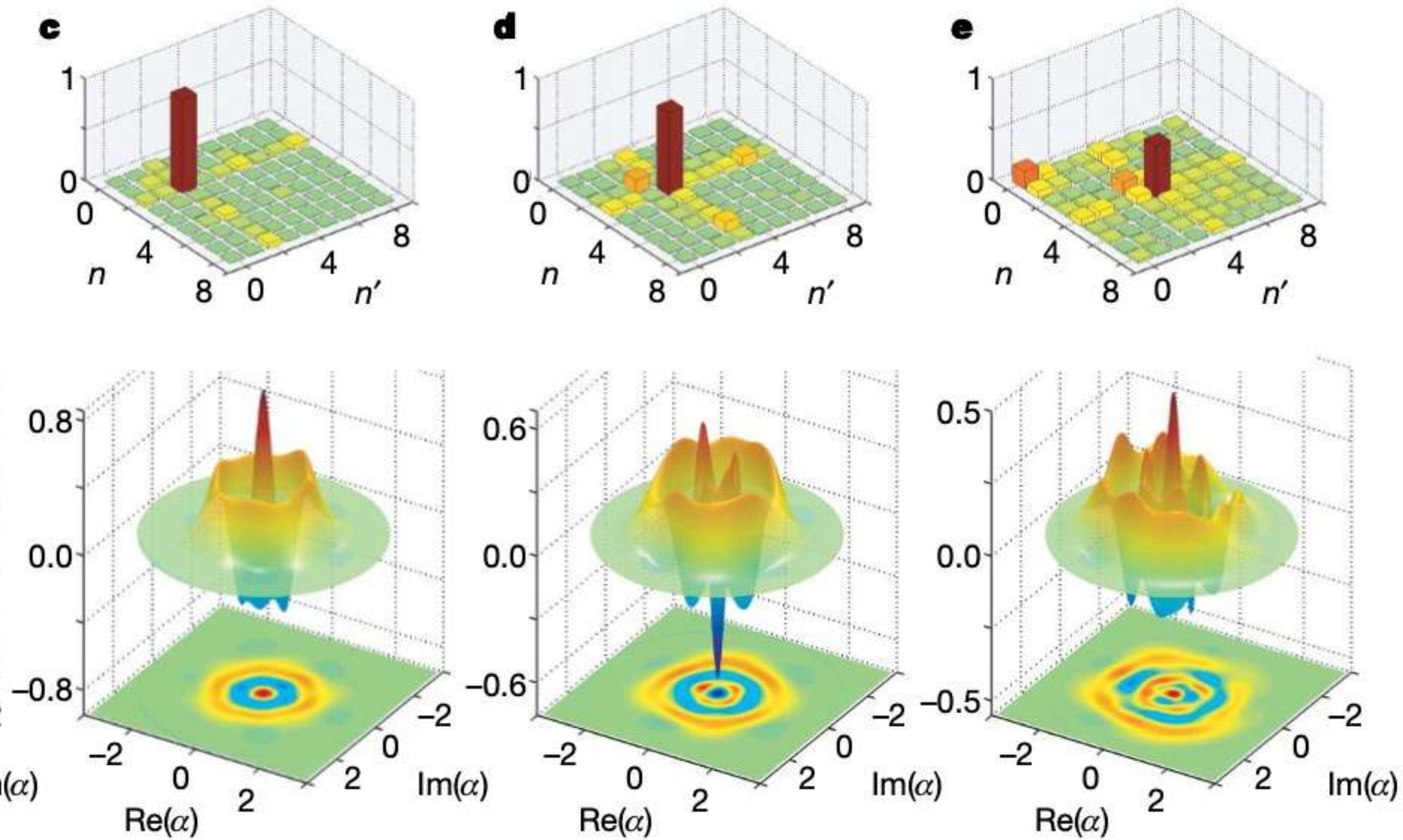
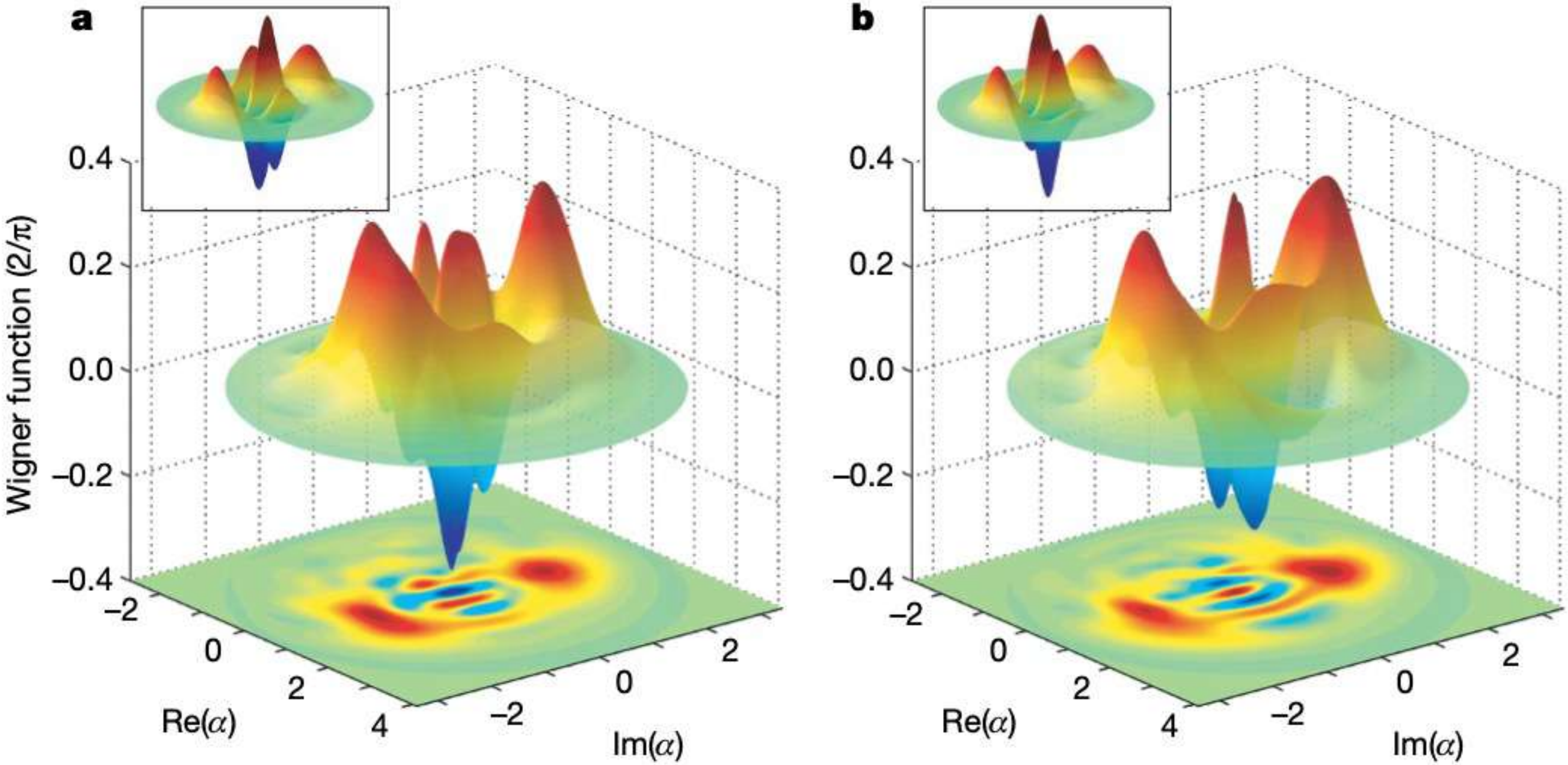




a
b
c
$|\rho_{nn'}|$
1
0
0
4
8
n
n'
Wigner function (2/π)
0.8
0.0
−0.8
−2
0
2
Re(α)
Im(α)


c
d
e
1
0
0
4
8
n
0
4
8
n′
0.8
0.0
−0.8
0.6
−0.6
0.5
−0.5
−2
0
2
Re(α)
Im(α)



$|\alpha\rangle + |-\alpha\rangle$ $\qquad$ $|\alpha\rangle - |-\alpha\rangle$