2) Consider the equation $(\Box - m^2)\phi = 0$ in the Poincaré patch of AdS_{d+1}. Check that near the boundary $x_0 = 0$, the two independent solutions go like $x_0^{2h_\pm}$, with

$$2h_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2R^2}. \tag{11.70}$$

(so that $2h_+ = \Delta$, the conformal dimension of the operator dual to ϕ).

3) Check that near $x_0 = 0$, the massless scalar field $\phi = \int K_B\phi_0$, with

$$K_B(\vec{x}, x_0; \vec{x}') = c\left(\frac{x_0}{x_0^2 + |\vec{x} - \vec{x}'|^2}\right)^d, \tag{11.71}$$

goes to a constant, ϕ_0. Then check that for the massive scalar case, replacing in K_B the power d by $2h_+$, we have $\phi \to x_0^{2h_-}\phi_0$ near the boundary.

4) Check that the (1-loop) anomaly of R-currents is proportional to N^2 at leading order, by doing the trace over indices in the diagram.

5) Write down the classical equations of motion for the 5 dimensional Chern-Simons action for A_μ^a.

6) Consider a scalar field ϕ in AdS_5 supergravity, with action

$$S = \int d^5x\sqrt{-g}\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\frac{\phi^3}{3}\right]. \tag{11.72}$$

Is the 4-point function of operators $\mathcal{O}$ sourced by ϕ, $\langle\mathcal{O}(x_1)...\mathcal{O}(x_4)\rangle$, zero or nonzero, and why?

- The classical on-shell supergravity action in AdS space has divergences near the boundary that correspond to UV divergences in the field theory, so need to be regularized and renormalized by adding counterterms to the action.

- The one-point function is given by the normalizable mode $\phi_{(2\Delta-d)}$ from the expansion near the boundary, which is linearly independent in the near-boundary expansion, but is *dependent* on the non-normalizable mode $\phi_{(0)}$ in an exact solution (there exists a unique regular solution in Euclidean AdS).

- Higher n-point functions can be derived from further differentiations of the *exact* mode $\phi_{(2\Delta-d)}$, viewed as a function of $\phi_{(0)}$.

- In the near-boundary expansion, we generically have an expansion in z^2, and the coefficients $\Phi_{(2k)}$ with $2k < 2\Delta - d$ are algebraically defined in terms of $\Phi_{(0)}$.

- In the holographic renormalization method, it is crucial to perform all calculations at a finite distance ϵ from the boundary. We integrate only down to ϵ, and write counterterms in terms of fields on a boundary at ϵ.

References and further reading

For more details on the method of holographic renormalization, see the review [93]. The method was first used in [94] where the holographic Weyl anomaly was calculated (we will mention this anomaly next chapter).

Exercises, Chapter 22.

1) Calculate the asymptotic expansion (perturbative solution) for a gauge field (with action $\int \sqrt{g} F_{\mu\nu}^2/4$) in AdS_{d+1}, for $d > 3$, as a function of boundary values.

2) Do the same for AdS_4. What changes?

3) Calculate the one-point function (operator VEV) for a scalar with $\Delta = d/2 + 2$.

4) Calculate the exact solution for a scalar with $\Delta = d/2 + 2$, and from it, find $\phi_{(4)}$ as a function of $\phi_{(0)}$.

5) How are the 3-point function calculations affected by the addition of the counterterm action (22.30)?

6) Do the Fourier transform from (22.41) to (22.42) and find the explicit form for $\mathcal{R}1/x^6$.

- Jet quenching, the energy loss of a heavy quark ("jet") passing through the sQGP medium, can be modelled by the energy loss of a string moving with constant v at the boundary, and trailing behind all the way to the horizon. One finds a drag coefficient $\eta_D \propto \sqrt{\lambda} T^2$.

- One can calculate the jet quenching parameter $\hat{q}$ from a Wilson loop with two long lightlike sides of length L_- and two short spacelike sides of length L, and AdS/CFT gives $\hat{q}_{SYM} \propto \sqrt{\lambda} T^3$.

- A chemical potential for the R-charge of $\mathcal{N}=4$ SYM corresponds to a constant source for A_0 on the boundary of AdS_5, i.e. $A \to \mu dt$ as $z \to 0$, and gives an electrically charged solution in AdS, namely the Reissner-Nordstrom AdS black hole.

- The grand-canonical ensemble of constant μ is found from the usual AdS action for supergravity, whereas the canonical ensemble for constant charge density is found by adding an extra boundary term to the AdS action for the gauge field.

- Adding an external magnetic field in $\mathcal{N}=4$ SYM with respect to the R symmetry group is done by adding a magnetic field in AdS. In AdS_4, one simply adds magnetic charge to the AdS black hole.

References and further reading

The prescription for AdS/CFT at finite temperature was done by Witten in [43]. The calculation of the jet quenching parameter from a (partially) lightlike Wilson loop was originally done in [80]. More details about the drag on heavy quarks, jet quenching and $\mathcal{N}=4$ SYM plasmas in general and how they apply to heavy ion collisions can be found in [79]. The way to add magnetic field in AdS_4 and more details on adding chemical potential can be found in [76, 78]. In [77] it is described how to add magnetic field in AdS_5.

Exercises, Chapter 15.

1) Parallel the calculation of the Schwarzschild black hole to show that the extremal ($Q=M$) black hole has zero temperature.

2) Check that the rescaling plus the limit given in (15.36) gives the Witten background for finite temperature AdS/CFT.

3) Take a near-horizon nonextremal D3-brane metric,

$$ds^2 = \alpha' \left\{ \frac{U^2}{R^2}[-f(U)dt^2 + d\vec{y}^2] + R^2 \frac{dU^2}{U^2 f(U)} + R^2 d\Omega_5^2 \right\}$$
$$f(U) = 1 - \frac{U_0^4}{U^4}\,, \tag{15.89}$$

where U_0 is fixed, $U_0 = \pi T R^2$ (T=temperature). Note that for $f(U) = 1$ we get the near-horizon extremal D3 brane, i.e. $AdS_5 \times S^4$. Check that a light ray travelling between the boundary at $U = \infty$ and the horizon at $U = U_0$ takes a finite time (for $U_0 = 0$, it takes an infinite time to reach $U = 0$).

4) Check that the rescaling

$$U = \rho \cdot \frac{U_0}{R}; \quad t = \frac{\tau R}{U_0}; \quad \vec{y} = \vec{x}\frac{R^2}{U_0} , \tag{15.90}$$

where R =AdS radius, takes the above near-horizon nonextremal D3-brane metric to the Witten finite T AdS/CFT metric.

5) Check that the temperature of the AdS-Reissner-Nordstrom solution (15.75) is given by (15.77).

6) Calculate the grand-canonical thermodynamic potential (15.79) by calculating the regularized on-shell action, subtracting the contribution of pure AdS space.

7) Check that for the magnetic solution with (15.82), the temperature is given by (15.86), and the grad-canonical potential by (15.87).