PDFs, GPDs and TMDPDFs from lattice QCD

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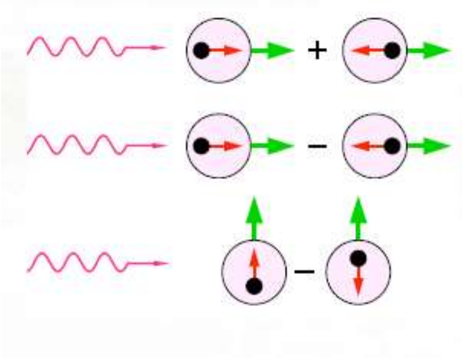
With thanks to my collaborators for the works presented here:

C. Alexandrou, S. Bhattacharya, K. Cichy, M. Constantinou, J. Dodson, X. Gao,
K. Hadjiyiannakou, K. Jansen, A. Metz, S. Mukherjee, A. Scapellato, A. Sen, Y. Zhao

Twist-2 Parton Distribution Functions (PDFs)

Complete set of twist-2 parton distribution functions



Unpolarized PDF γ^+: $f_1(x)$

Helicity PDF $\gamma^+\gamma_5$: $g_1(x)$

Transversity PDF σ^{+j}: $h_1(x)$

Light-cone coordinates

$$\gamma^+ = \frac{\gamma^0+\gamma^3}{\sqrt{2}}$$

$$x = x_{Bj} = \frac{-q^2}{2P.q}$$

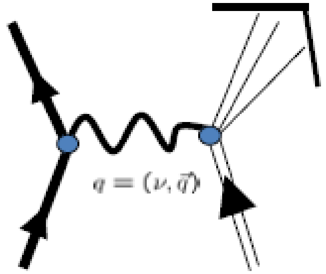
is the momentum fraction carried by a given parton

Example:

$$f_1(x) = \frac{1}{4\pi}\int dz^- e^{-iP^+z^-}\langle P|\bar{\psi}(z^-)\gamma^+\psi(0)|P\rangle$$

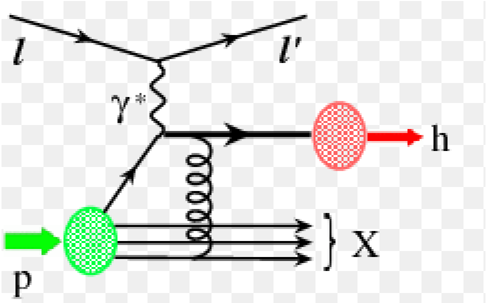
Cross sections are measured:

Totally inclusive



Have access to the chiral-even distributions $f_1(x)$ (unpolarized) and $g_1(x)$ (helicity)

Semi-inclusive



Have access to the chiral-odd distribution $h_1(x)$ (transversity). Naturally more difficult to obtain data on transversity

Twist-3 PDFs

Twist expansion: $f_i(x) = f_i^{(0)} + \frac{f_i^{(1)}}{Q} + \frac{f_i^{(2)}}{Q^2} + \cdots$

Twist-2 + Twist-3 +Twist-4

Twist-3:

$$\left.\begin{array}{lcl} \hat{1} & : & e(x) \\ \gamma^j\gamma_5 & : & g_T(x) \\ \sigma^{jk} & : & h_L(x) \end{array}\right\}$$

No density interpretation;

Contain information of quark-gluon-quark correlations;

Possible zero mode contribution;

Hard do determine experimentally.

Examples: $g_T(x) \equiv g_1(x) + g_2(x)$

Can be interpreted as a transverse force acting on the quark being scattered **Burkardt PRD88 (2013) 114502**

$$g_T^{ww}(x) = \int_x^{+1} dy\, g_1(y)$$

Wandzura-Wilczek approximation
Experimentally: Possible 15-40% violation

Accardi et al., JHEP 11, 093 (2009)

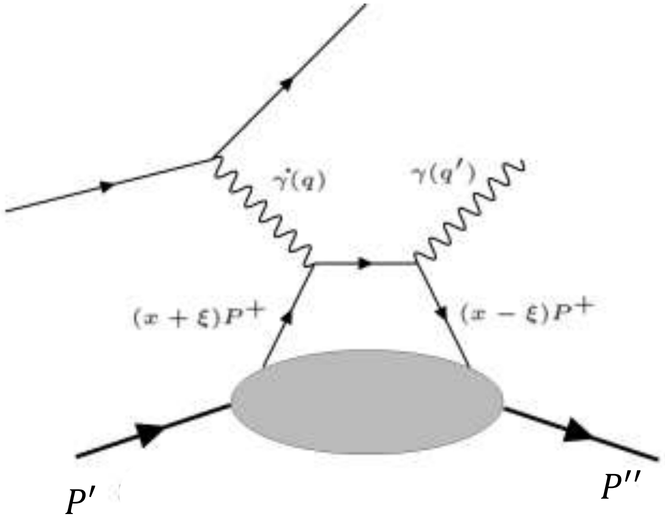
$$\int_{-1}^{+1} dx\, g_T(x) = \int_{-1}^{+1} dx\, g_1(x)$$

Burkhardt-Cottingham sum rule

$$\int_{-1}^{+1} dx(e^u(x) + e^d(x)) = \frac{\sigma_{\pi N}}{m}$$

The $e(x)$ PDF is related to the pion-nucleon sigma term

Generalised PDFs (GPDs)



A virtual photon is exchanged, with a real photon measured in the final state

Momentum transfer: $\Delta \equiv P'' - P', \qquad t \equiv \Delta^2,$

Fraction of the momentum transfer: $\xi \equiv -\dfrac{P''^+ - P'^+}{P''^+ + P'^+} = -\dfrac{2\Delta^+}{P^+}$, ξ is called skewness

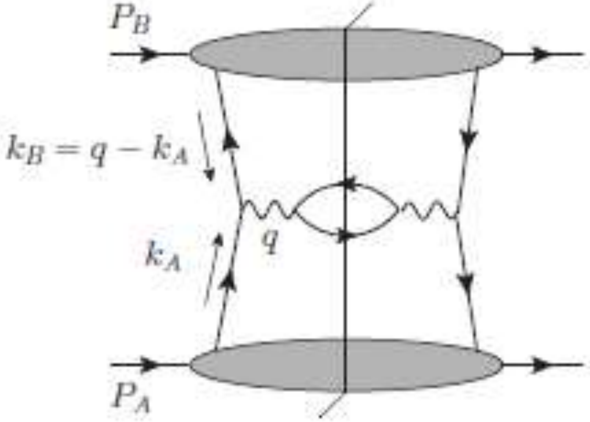
GPDs are multidimensional objects, depending on x, t, ξ

Transverse momentum dependent PDFs (TMDPDFs)

Why TMDPDFs?

If we measure only the invariant mass of the final lepton pair:

$$\frac{d\sigma}{dQ^2} = \sum_{i,j} \int dx_1 dx_2 f_i(x_1) f_j(x_2) H(x_1, x_2) \left(1 + \mathcal{O}\left(\frac{m^2}{Q^2}\right)\right)$$



If we measure the transverse momentum $\vec{q}_T$ of the lepton pair, we have access to the transverse momentum of the quarks!

$$\frac{d^2\sigma}{dQ^2 dq_T^2} = \sum_{i,j} \int dx_1 dx_2 \int d^2 b_T e^{i\vec{b}_T \cdot \vec{q}_T} f_i(x_1, \vec{b}_T) f_j(x_2, \vec{b}_T) H(x_1, x_2) \left(1 + \mathcal{O}\left(\frac{\Lambda_{QCD}}{Q^2}, \frac{q_T^2}{Q^2}\right)\right), \qquad q_T \ll Q$$

Transverse momentum dependent PDFs

Last 5 years witnessed enormous progress on first principles computations of both PDFs and GPDs

Theoretical papers

X. Ji, PRL 110, 262002 (2013) - Quasi
A.V. Radyushkin, PRD 96, 034025 (2017) - Pseudo
A. J. Chambers et al., PRL 118, 242001 (2017) - OPE without OPE
Yan-Qing Ma and Jian-Wei Qiu, PRL 120, 022003 (2018) - Good lattice cross sections

Exploratory studies

LP3, PRD 91, 054510 (2015)
ETMC, PRD 92, 014502 (2015)

Nucleon PDFs at physical pion mass using Quasi

ETMC, NPB 923, 394 (2017) - Nonperturbative renormalization
ETMC, PRL 121, 112001 (2018) - Unpolarized and helicity
LP3, PRL 121, 242003 (2018) - Helicity

Nucleon PDFs at physical pion mass using Pseudo

ETMC, PRD 103, 034510 (2021)
HadStruc., PRL 125, 232003 (2020) - Extrapolated to physical pion mass

Nucleon GPDs

ETMC, PRL 125, 262001 (2020) - Unpolarized and helicity
ETMC+Temple+BNL+ANL, PRD 106 125, 115412 (2022) Symmetric and asymmetric frames

Twist-3

ETMC/Temple, PRD 102, 111501 (2020)
ETMC/Temple, PRD 104 115410 (2021)

List restricted to physical pion mass results or exploratory studies. There are many more works on the subject and I apologize to authors of works not listed

TMDPDs just starting

Theoretical papers

M. A. Ebert, I. W. Stewart, Y. Zhao, PRD 99, 034505 (2019)
M. A. Ebert et al., JHEP 37, 2019 (2019)
X. Ji, Y. Liu, Yu-Sheng Liu, Phys. Lett. B 811, 135956 (2020)
X. Ji, Y. Liu, Yu-Sheng Liu, Nucl. Phys. B 955, 115054 (2020)
P. Shanahan, M. Wagman, Y. Zhao, PRD 102, 014511 (2020)
M. A. Ebert et al., arXiv:2201.08401

Exploratory studies – Soft function

LPC, PRL 125, 192001 (2020)
ETMC, PRL 128, 062002 (2022)

Exploratory studies – Collins-Sopper kernel

ETMC, PRL 128, 062002 (2022)
LPC, arXiv: 2204.00200

Exploratory studies – Beam functions and TMDPDFs

ETMC+PKU, *PoS* Lattice2022 (2023) 123
ETMC+PKU, *PoS* Lattice2022 (2023) 733
LPC, arXiv: 2211.02340

Light-cone PDFs and quasi PDFs

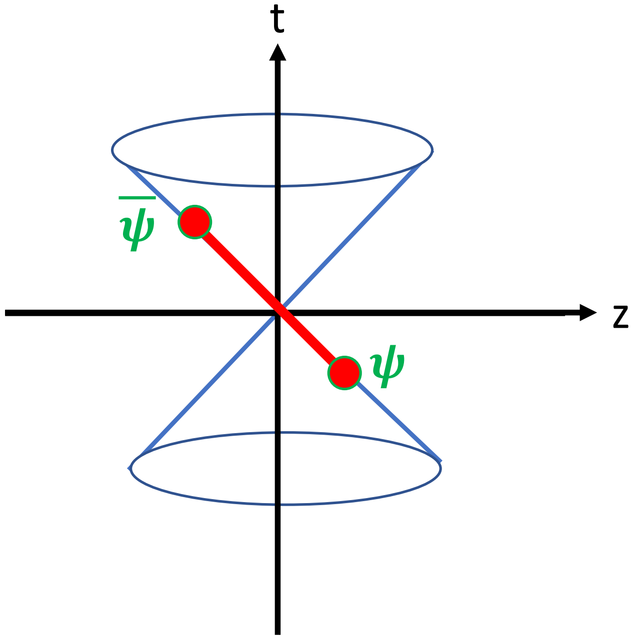
$$q(x) = \frac{1}{4\pi}\int dz^- e^{-iP^+z^-}\langle P|\bar{\psi}(z^-)\gamma^+ W\,(z^-,0)\psi(0)|P\rangle$$

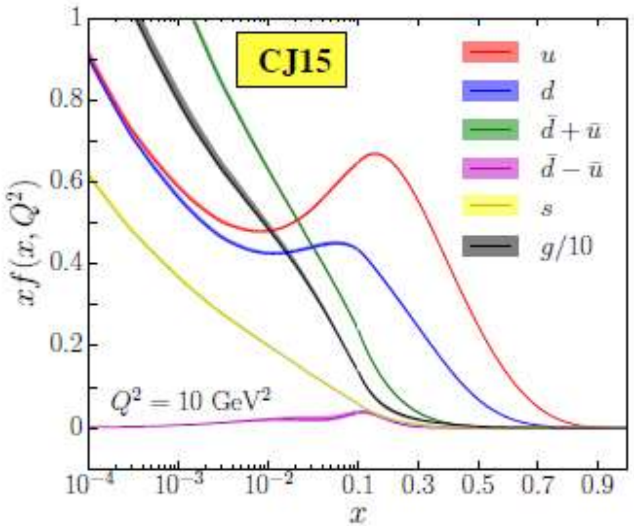
Dirac Structure

Wilson line

Quark distribution is given by a light-front correlation

$$z^- = \frac{t-z}{\sqrt{2}}, P^+ = \frac{E+P^z}{\sqrt{2}}$$

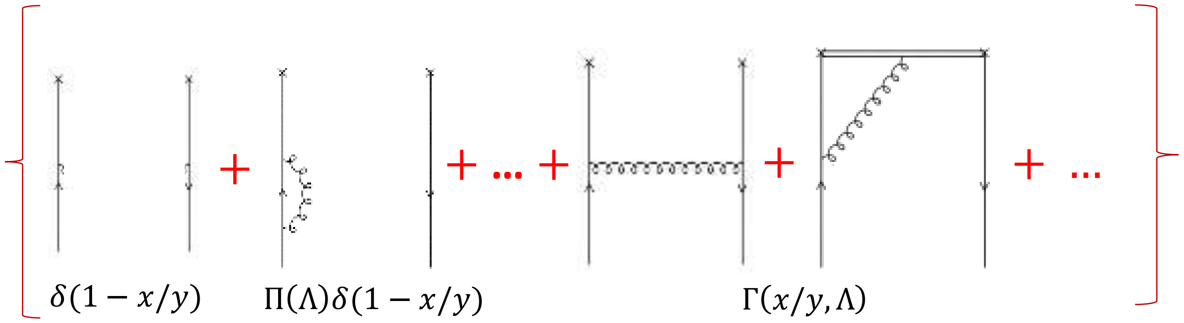




Our focus: isovector quark distributions, $q(x) \equiv u(x) - d(x)$

Perturbative correction to isovector quark distributions :

$q(x, \Lambda)=$

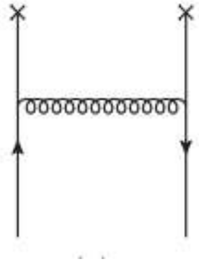


$\otimes q_{bare}(y) + \mathcal{O}(\alpha_s^2)$

Regulator of IR and UV divergences

$$q(x, \Lambda) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi}\Pi(\Lambda)\right\} + \frac{\alpha_s}{2\pi}\int_x^1 \Gamma\left(\frac{x}{y}, \Lambda\right) q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

Simplest diagram



$$= -ig^2 C_F \int \frac{dk^+ dk^- d^2k_\perp}{(2\pi)^4} \frac{\bar{u}(p)\gamma^\mu k\cdot\gamma\gamma^+ k\cdot\gamma\gamma_\mu u(p)}{(k^2)^2(p-k)^2}\,\delta\left(y - \frac{k^+}{p^+}\right)$$

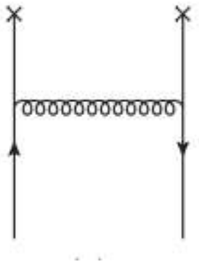
$$p = (\xi P^+, 0,0,0);\;\; \xi = \frac{p^+}{P^+}$$

$$k^2 + i\epsilon = 2yp^+\left(k^- - \frac{k_\perp^2}{2yp^+} + i\epsilon\right)$$

$$(p-k)^2 + i\epsilon = -2p^+(1-y)\left(k^- + \frac{k_\perp^2}{2p^+(1-y)} - i\epsilon\right)$$

For $0 < y < 1$, one pole in the upper half and other in the lower half of the complex plane

For $y > 1$ or $y < 0$, the poles are either on the lower half or on the upper half of the complex plane

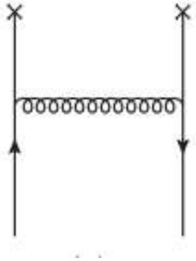


$$= 2\alpha_s C_F (1-y)\int \frac{d^2k_\perp}{(2\pi)^2}\frac{\bar{u}(p)\gamma^+ u(p)}{k_\perp^2} = \frac{\alpha_s}{2\pi}4p^+(1-y)\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} + \ln\left(\frac{\mu^2}{\mu_F^2}\right)\right)$$

With support only in the physical region, $0 < y < 1$

DR used for IR and UV divergences

Infinite momentum frame (IMF)

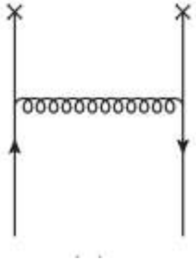


$$= -ig^2 C_F \int \frac{dk^0 dk^3 d^2k_\perp}{(2\pi)^4} \frac{\bar{u}(p)\gamma^\mu k\cdot\gamma\gamma^3 k\cdot\gamma\gamma_\mu u(p)}{(k^2)^2(p-k)^2}\,\delta\left(y-\frac{k^3}{p^3}\right)$$

$$k^2 + i\epsilon = \left(k^0 - \sqrt{k_\perp^2 + y^2(p^3)^2} + i\epsilon\right)\left(k^0 + \sqrt{k_\perp^2 + y^2(p^3)^2} - i\epsilon\right)$$

$$(p-k)^2 + i\epsilon = \left(k^0 - p^3 - \sqrt{k_\perp^2 + (1-y)^2(p^3)^2} + i\epsilon\right)\left(k^0 - p^3 + \sqrt{k_\perp^2 + (1-y)^2(p^3)^2} - i\epsilon\right)$$

Integrating in k^0 and taking the $p^3 \to \infty$ limit:



$$= 2\alpha_s C_F (1-y)\int \frac{d^2k_\perp}{(2\pi)^2}\frac{\bar{u}(p)\gamma^3 u(p)}{k_\perp^2} \quad = \frac{\alpha_s}{2\pi} 4p_3(1-y)\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} + \ln\left(\frac{\mu^2}{\mu_F^2}\right)\right)$$

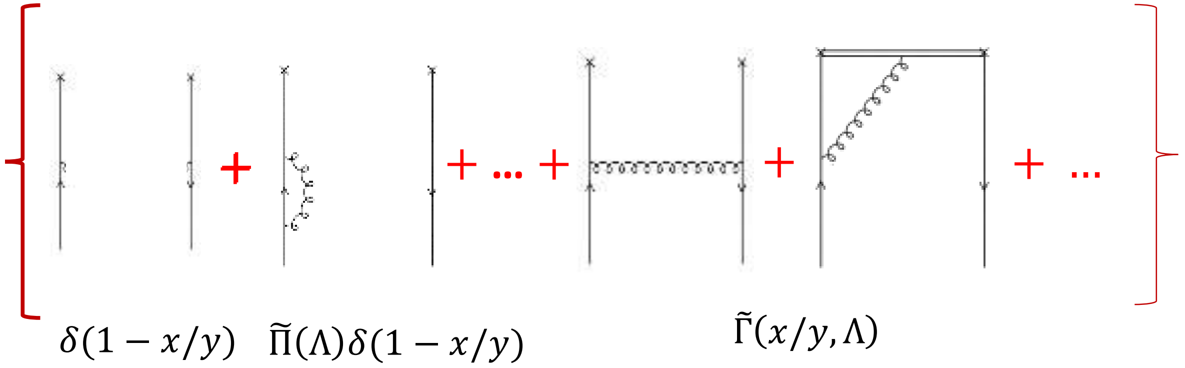
with $0 < y < 1$

LC and IMF have the same IR and UV behaviour and are equivalente

Unfortunately, they can not be computed within LQCD

What if p_3 is kept finite?

$\tilde{q}(x,\Lambda)=$

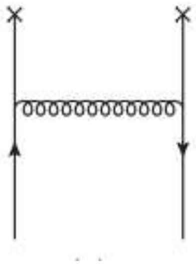


$\otimes q_{bare}(y)+\mathcal{O}(\alpha_s^2)$

Regulator of IR and UV divergences

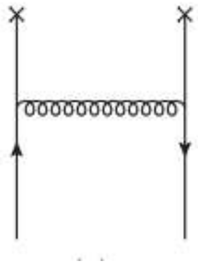
$$\tilde{q}(x,\Lambda)=q_{bare}(x)\left\{1+\frac{\alpha_s}{2\pi}\tilde{\Pi}(\Lambda)\right\}+\frac{\alpha_s}{2\pi}\int_x^1\tilde{\Gamma}\left(\frac{x}{y},\Lambda\right)q_{bare}(y)\frac{dy}{y}+\mathcal{O}(\alpha_s^2)$$

Keeping p_3 finite



$$= -ig^2 C_F \int \frac{dk^0 dk^3 d^2k_\perp}{(2\pi)^4} \frac{\bar{u}(p)\gamma^\mu k\cdot\gamma\gamma^3 k\cdot\gamma\gamma_\mu u(p)}{(k^2)^2(p-k)^2}\,\delta\left(y-\frac{k^3}{p^3}\right)$$

Integrating in k^0 and keeping p_3 finite, we have an integral over k_T which is UV finite! But has an IR divergence. Using Dimensional Regularization:



$$= \frac{\alpha_s}{2\pi}4p_3\left((1-y)\left(-\frac{1}{\epsilon_{IR}}+ln\left(\frac{p_3^2}{\mu_F^2}\right)+\ln(4y(1-y))\right)+1\right), \qquad 0<y<1$$

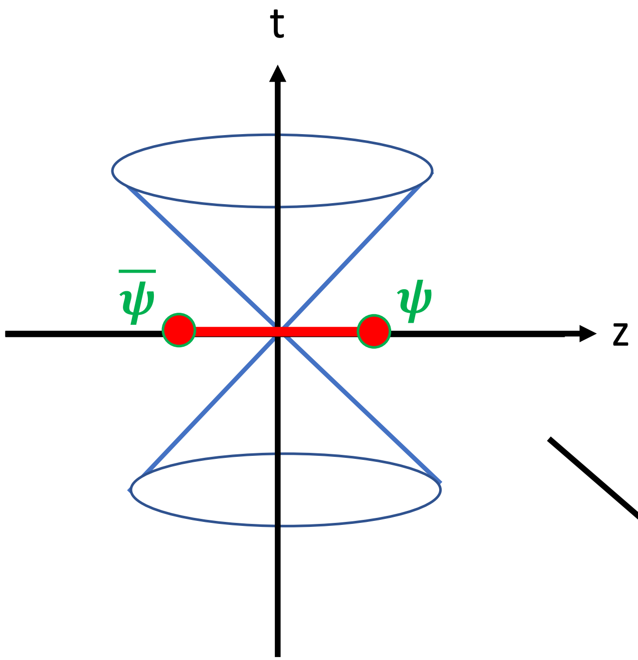
$$+\frac{\alpha_s}{2\pi}4p_3\left((1-y)\ln\left(\frac{x}{x-1}\right)+1\right), \qquad y>1$$

$$+\frac{\alpha_s}{2\pi}4p_3\left((1-y)\ln\left(\frac{x-1}{x}\right)-1\right), \qquad y<0$$

Support outside the physical region!

Same IR pole as in the LC and IMF cases

UV divergence appears only when integrating over all parton momentum fraction y

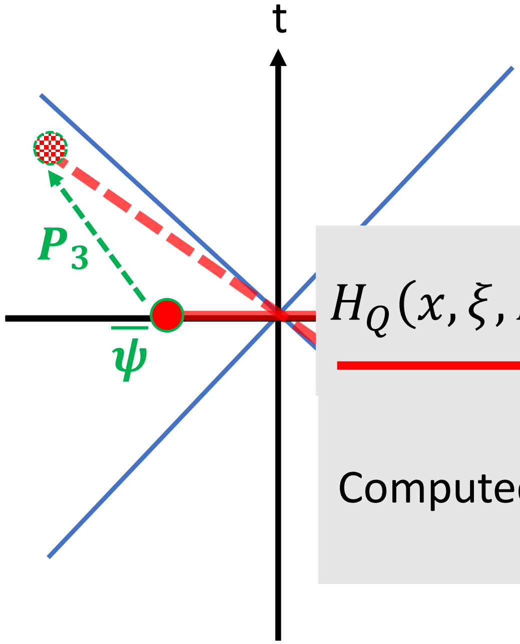


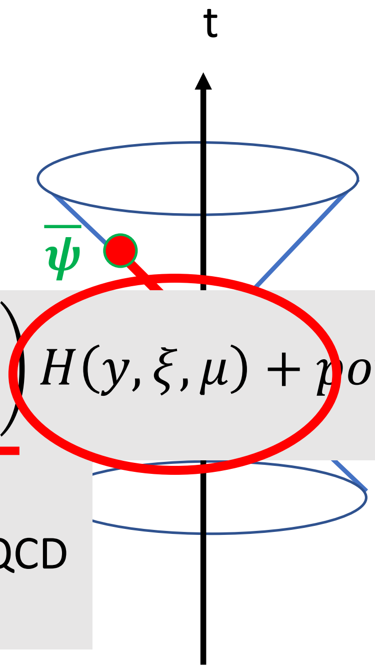
$$\frac{1}{2}\int\frac{dz^3}{2\pi}e^{ik^3z^3}\left\langle P';\lambda'\middle|\bar{\psi}\left(-\frac{z^3}{2}\right)\Gamma\mathcal{W}\left(-\frac{z^3}{2},\frac{z^3}{2}\right)\psi\left(\frac{z^3}{2}\right)\middle|P;\lambda\right\rangle$$

Purely spatial correlation

X. Ji, PRL 110 (2013) 262002.

We want to go from a purely spatial correlation to a light-front correlation





$$\underline{H_Q(x,\xi,P^3,\mu)} = \int\frac{dy}{|y|}\underline{C\left(\frac{x}{y},\frac{\xi}{y},\frac{\mu}{yP^3}\right)}H(y,\xi,\mu) + power\ corrections$$

Computed in LQCD

Computed in pQCD

Results for Twist-2

ETMC

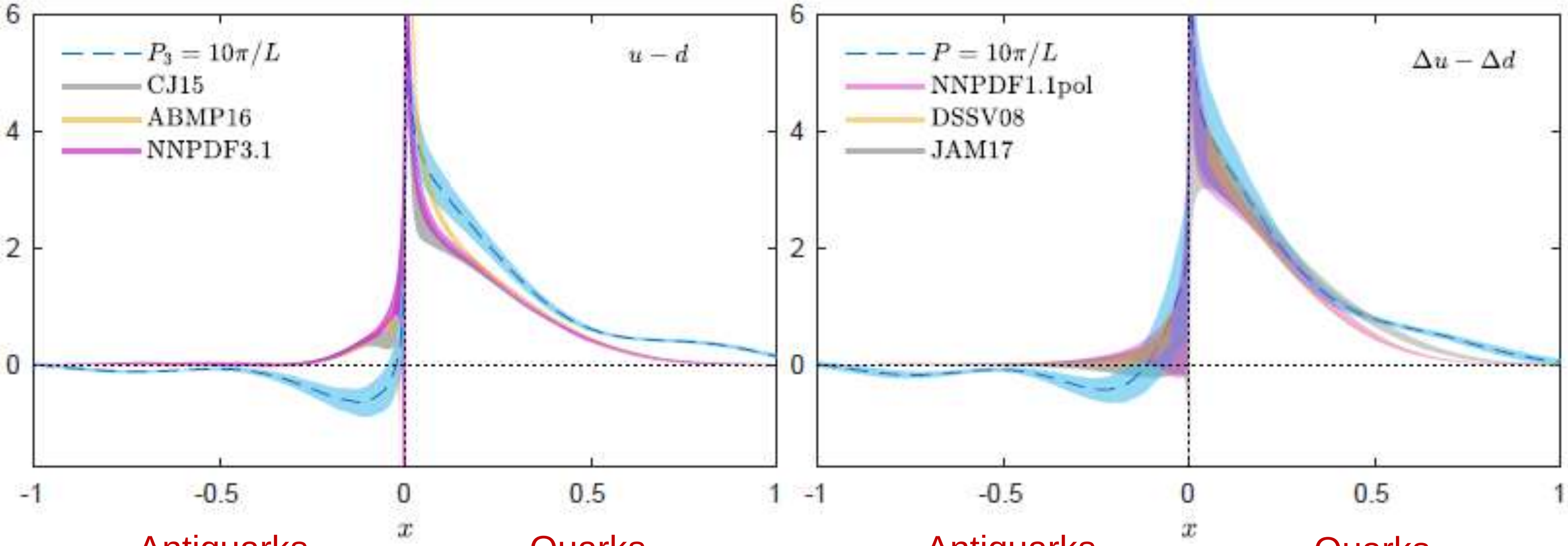
ETMC, NPB 923, 394 (2017) - Nonperturbative renormalization

ETMC, PRL 121, 112001 (2018) - Unpolarized and helicity quasi

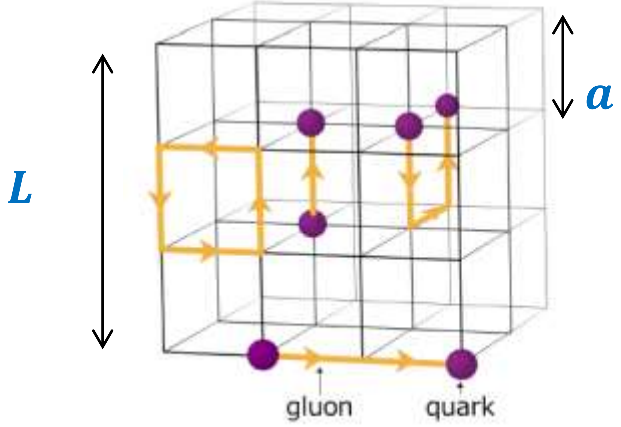
ETMC, PRD 103, 034510 (2021) - Unpolarized and helicity pseudo

Quasi-PDF approach

Unpolarized $P_3 = 1.38\ GeV$ **Helicity**



Antiquarks Quarks Antiquarks Quarks



Source: JICFuS, Tsukuba

C. Alexandrou, K. Cichy, M. Constantinou, K. Hadjiyiannakou, K. Jansen, A. Scapellato and F. Steffens, PRL 121, 112001 (2018)

$m_\pi \cong 130$ MeV

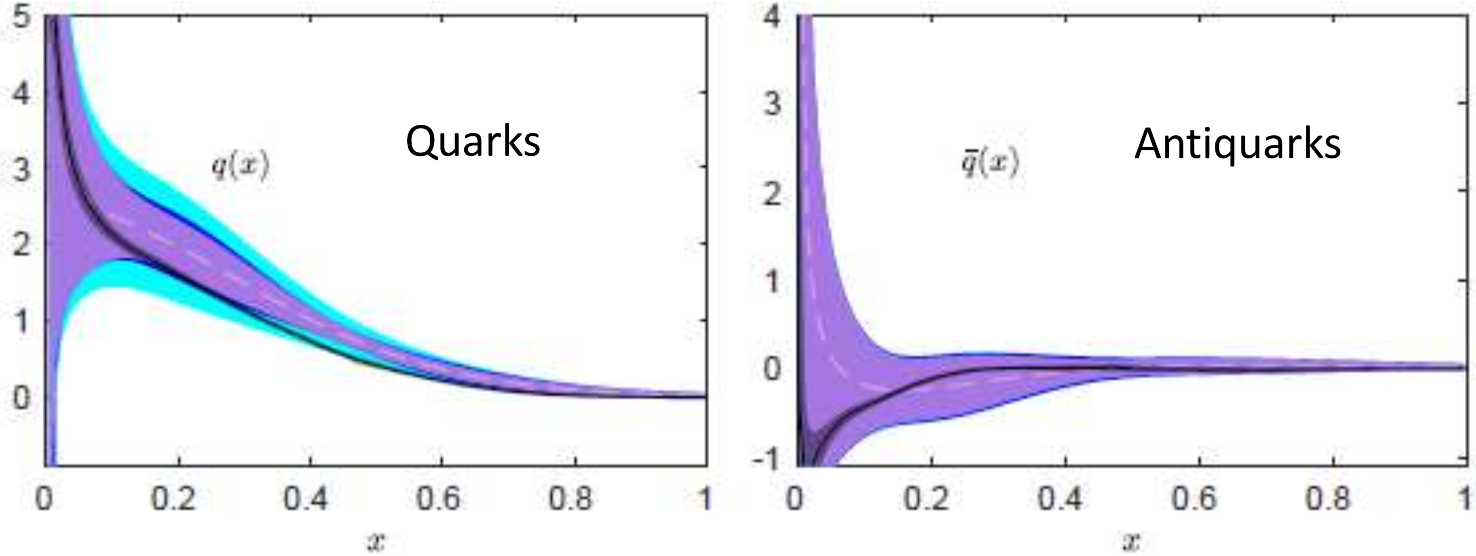
$48^3 \times 96$ lattice

$a \cong 0.093$ fm

Pseudo-PDF approach

Uses same ensamble as the quasi approach

Different from quasi case, here a pheno inspired ansatz is used to reconstruct he x dependence



Purple: Statistical error
Blue: quantified systematics
Cyan: estimated systemantics

Black: NNPD3.1 parametrization

ETMC, PRD 103, 034510 (2021)

Results for Twist-3

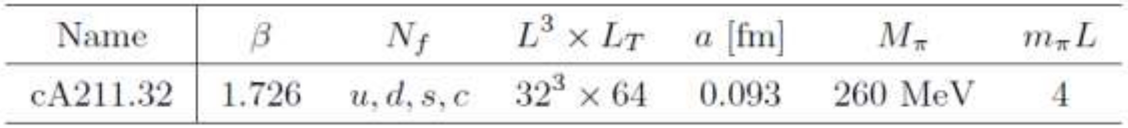
ETMC + Temple

PRD 102 (2020) 11, 111501 - Lattice $g_T(x)$

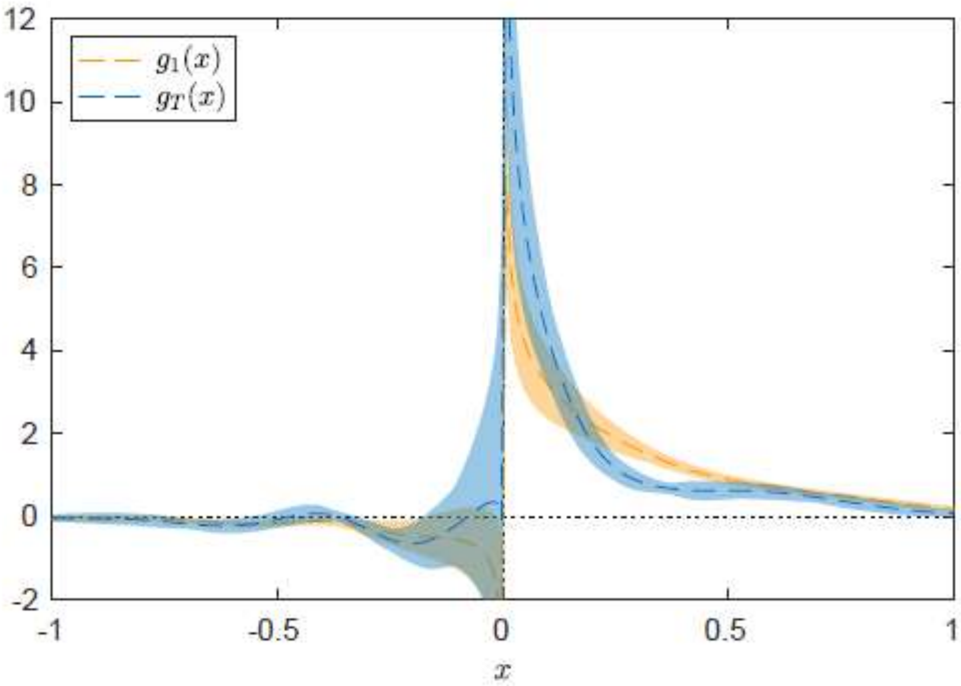
PRD 102 (2020) 3, 034005 - Matching $g_T(x)$

PRD 102 (2020) 224025 - Matching $e(x)$ and $h_L(x)$

arXiv: 2107.02574 - Lattice $h_L(x)$



Name	β	N_f	$L^3 \times L_T$	a [fm]	M_π	$m_\pi L$
cA211.32	1.726	u, d, s, c	$32^3 \times 64$	0.093	260 MeV	4

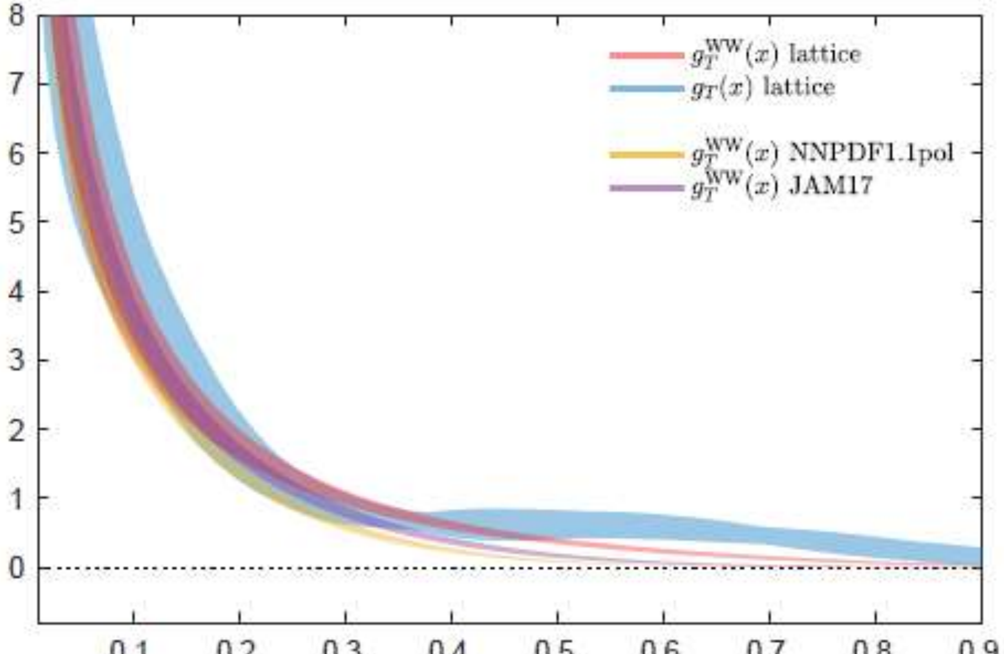


The $g_T(x) = g_1(x) + g_2(x)$ distribution

$P_3 = 1.67\ GeV$

The BC sum rule is verified:

$$\int_{-1}^{+1} dx\ g_T(x) - \int_{-1}^{+1} dx\ g_1(x) = 0.01(20)$$



The WW approximation

$$g_T^{ww}(x) = \int_x^{+1} dy\ g_1(y)$$

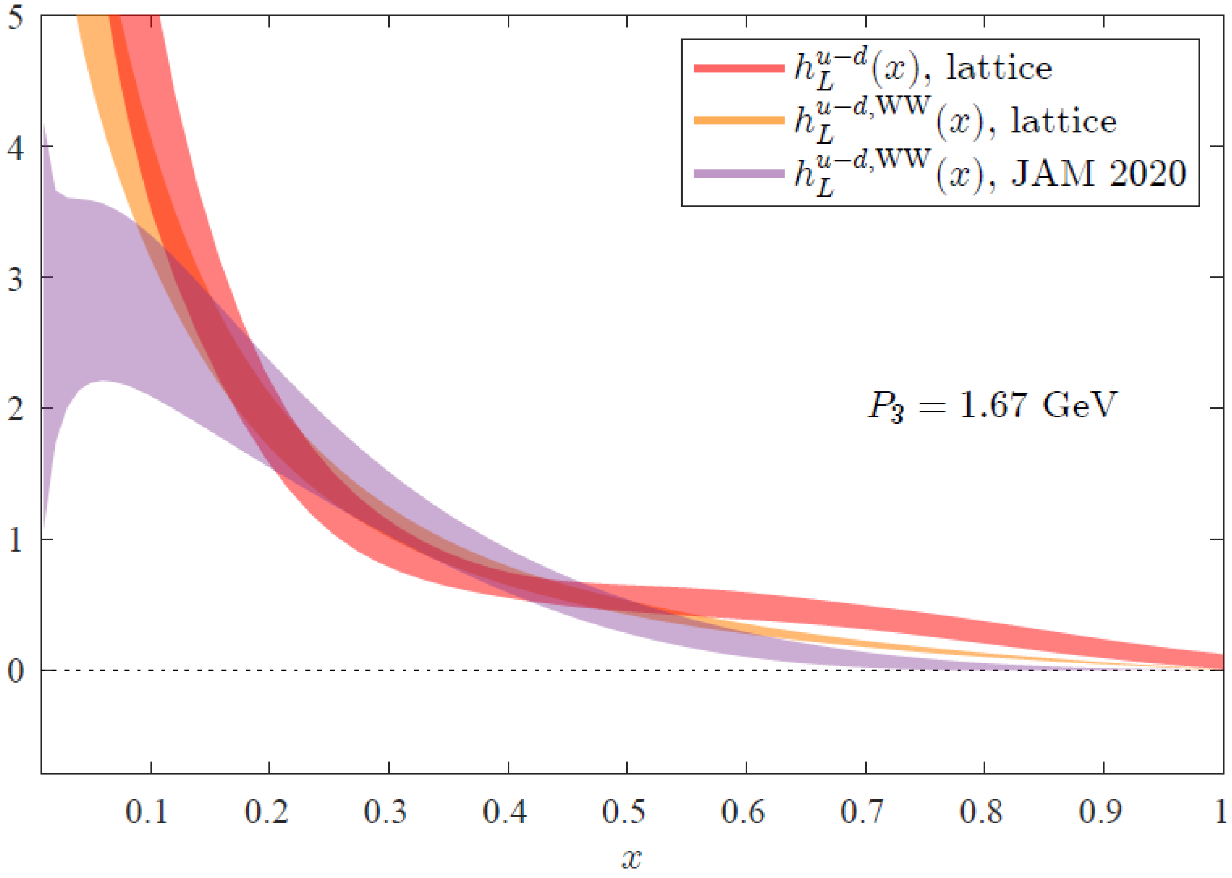
Up to $x < 0.5$, $g_T(x)$ agrees with $g_T^{ww}(x)$

Violations of 30-40% possible

The chiral-odd twist-3 distribution $h_L(x)$

The WW approximation relates $h_L(x)$ to its twist-2 counterpart $h_1(x)$

$$h_L^{ww}(x) = 2x\int_x^{+1}\frac{dy}{y^2}h_1(x)$$



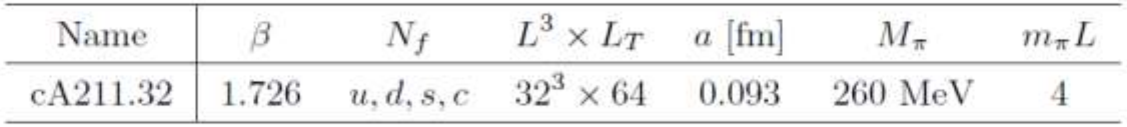
Suggests that the twist-3 distribution can be determined From its twis-2 counterpart

Twist-2 GPDs

ETMC

PRL 125, 262001 (2020) - Unpolarized ane helicity

PRD 105, 034501 (2022) - Transversity



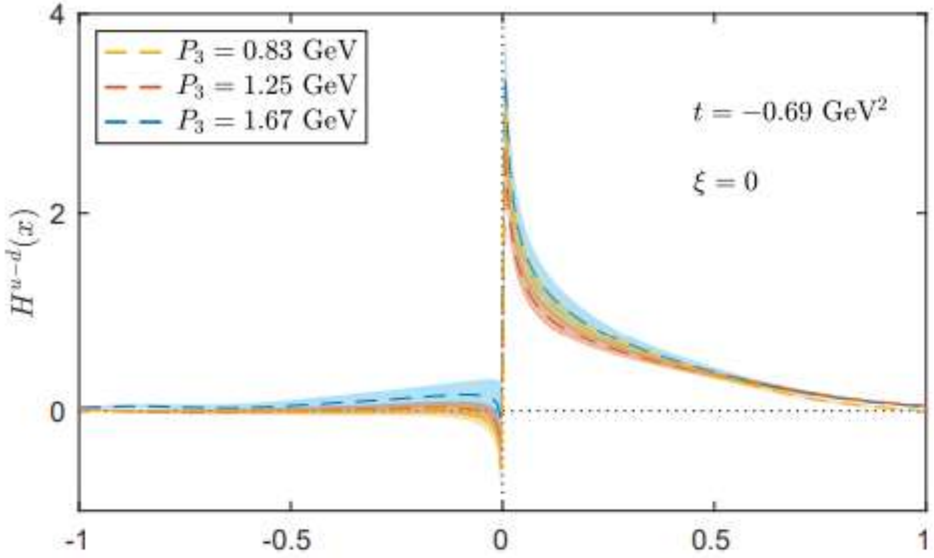
Name	β	N_f	$L^3 \times L_T$	a [fm]	M_π	$m_\pi L$
cA211.32	1.726	u, d, s, c	$32^3 \times 64$	0.093	260 MeV	4

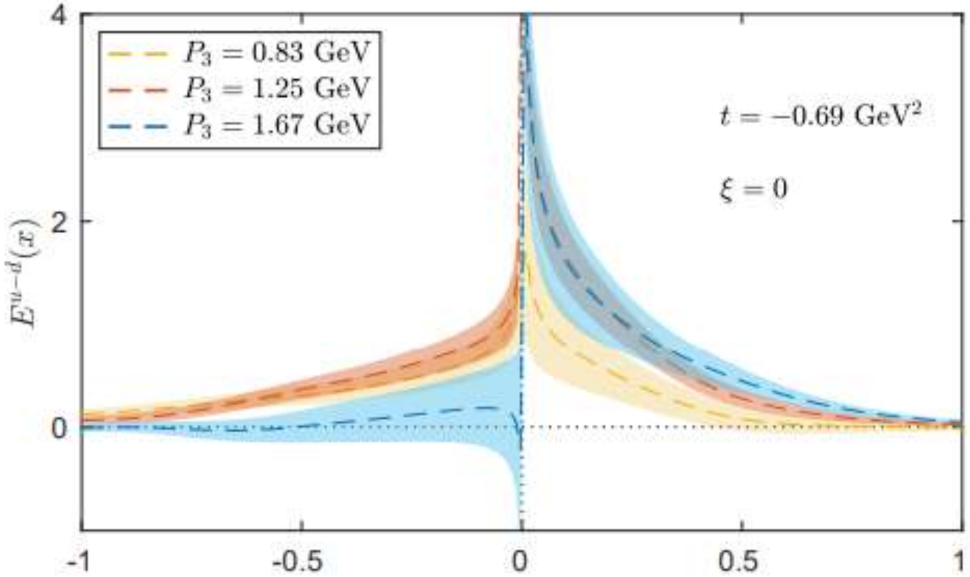
Latt

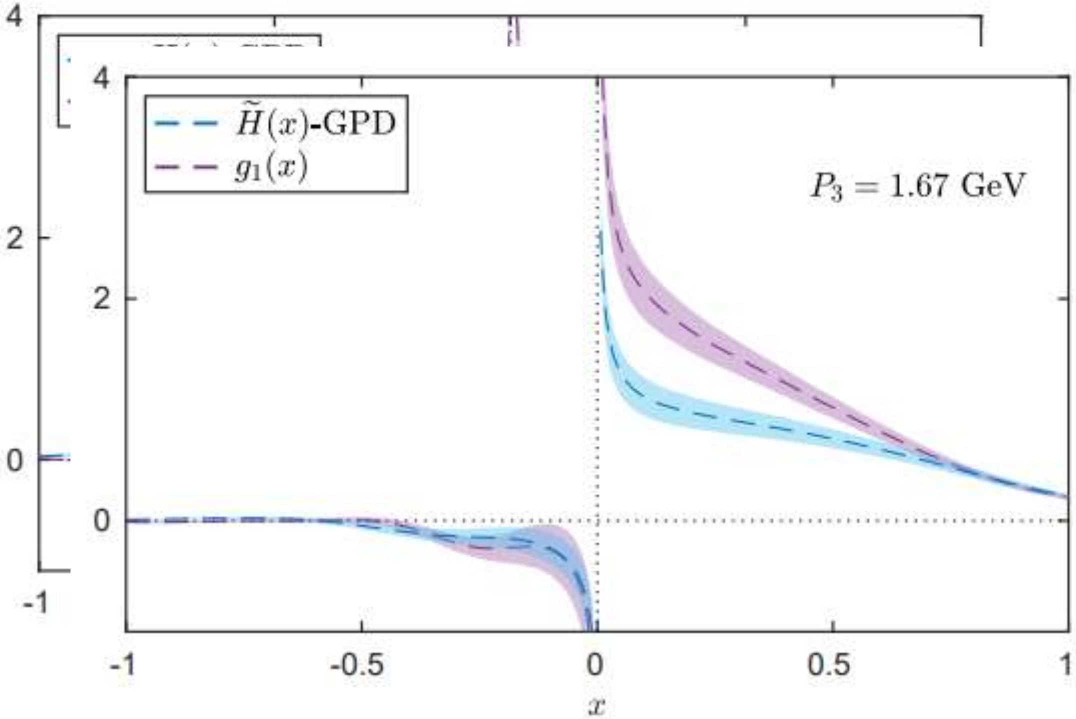
- f
- g
- g
- l
- $32^3 \times 64$ lattice, $L \approx 3$ fm
-

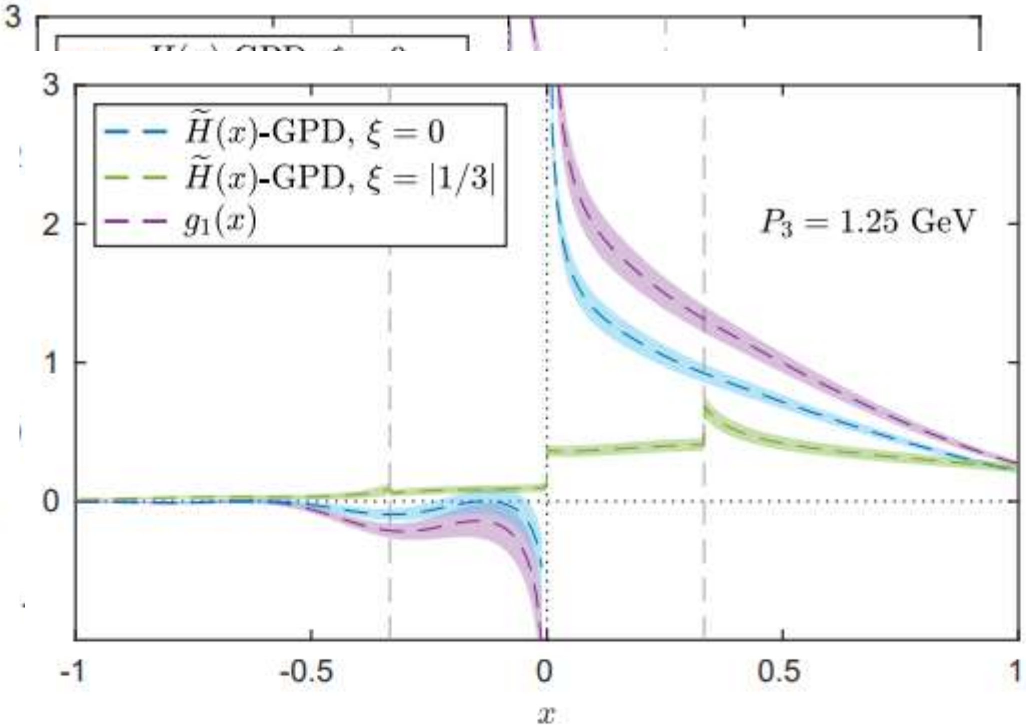
GeV

V²

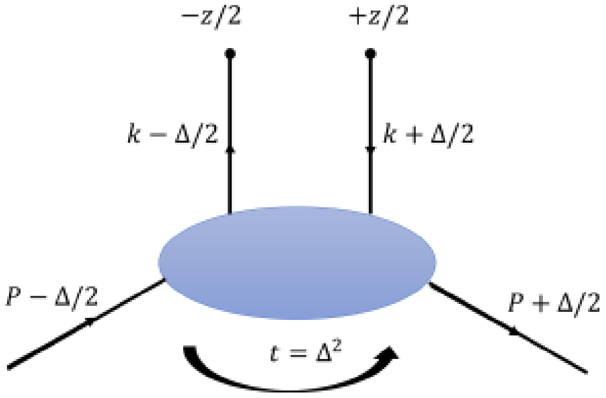


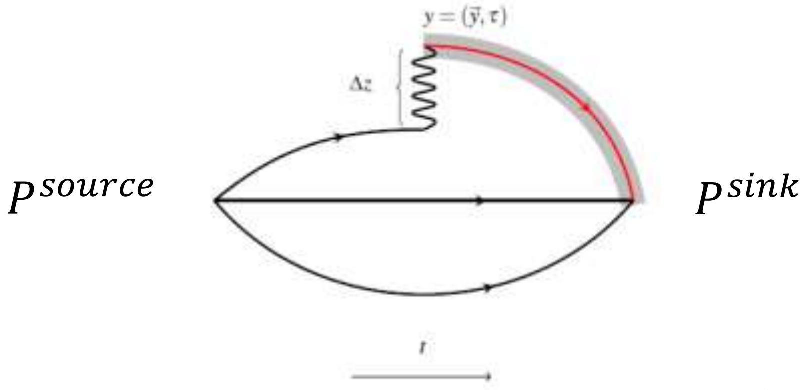






Problem with the current approach: not efficient



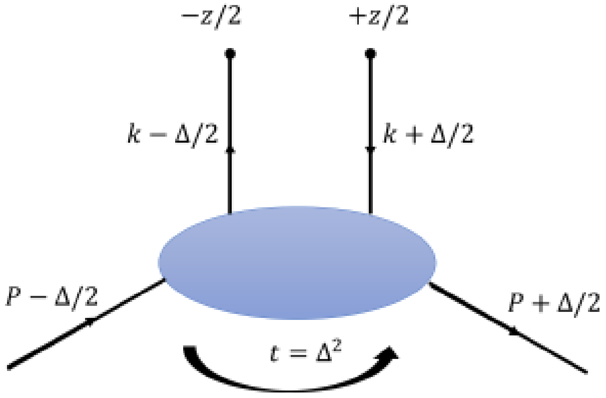


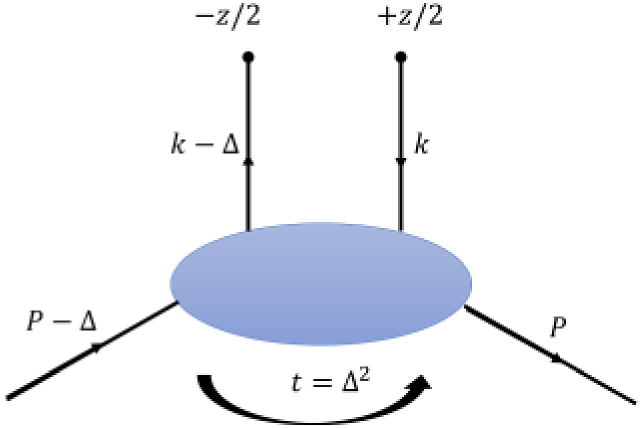
$$P^{source} = \left(-\frac{\Delta_x}{2}, -\frac{\Delta_y}{2}, P_3\right) \qquad P^{sink} = \left(-\frac{\Delta_x}{2}, -\frac{\Delta_y}{2}, P_3\right)$$

- Separate calculation for each momentum transfer: $P^{sink} = \left(-\frac{\Delta_x}{2}, -\frac{\Delta_y}{2}, P_3\right)$
- Much more efficient if $P^{sink} = (0,0,P_3)$

Symmetric and asymmetric frames

S. Bhattacharya et al., arXiv: 2209.05373





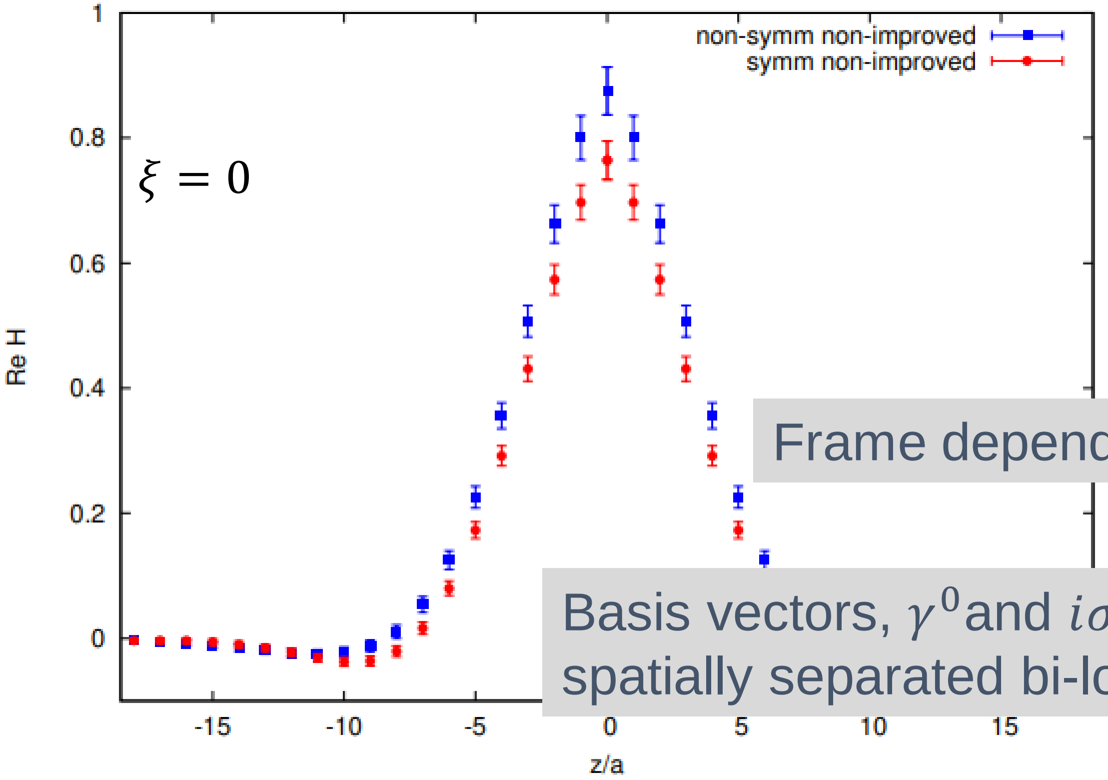
$$\begin{pmatrix} E_{i,s} \\ p^1_{i,s} \\ p^2_{i,s} \\ p^3_{i,s} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{i,a} \\ -\Delta^1_a \\ 0 \\ P^3 \end{pmatrix}$$

Transverse boost

→ $\langle\bar{\psi}\gamma^0\psi\rangle^s = \gamma\langle\bar{\psi}\gamma^0\psi\rangle^a - \gamma\beta\langle\bar{\psi}\gamma^1\psi\rangle^a$

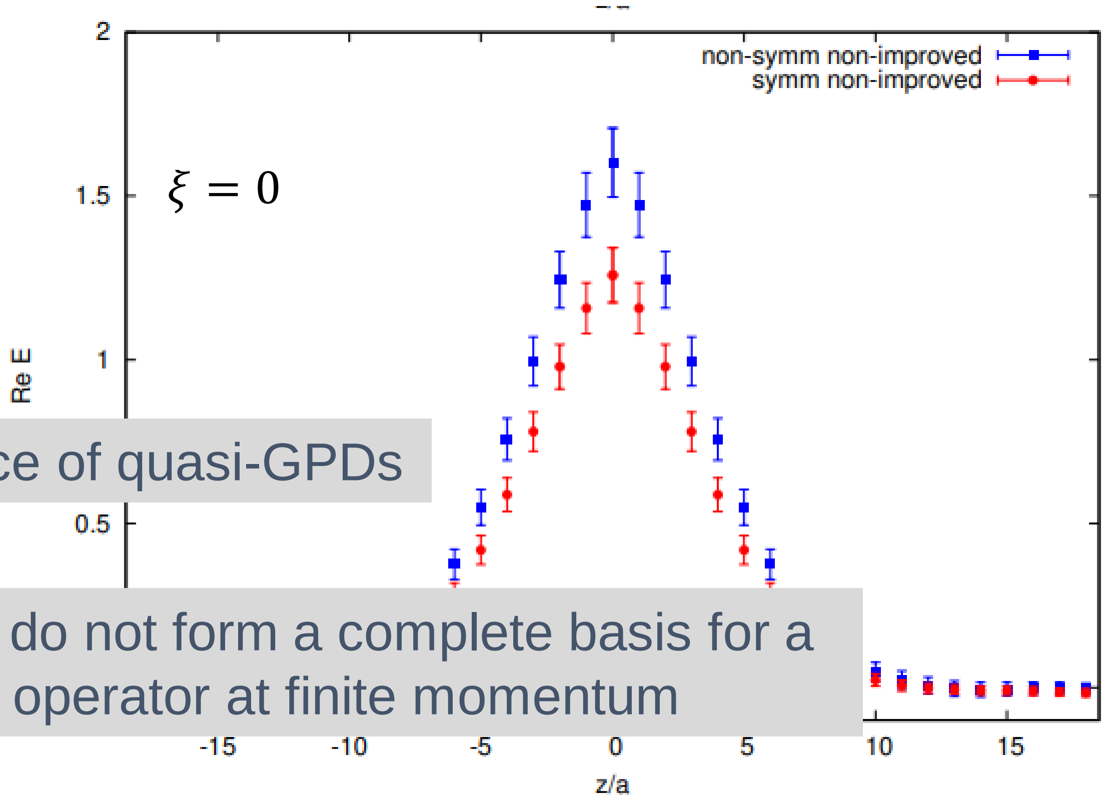
Historical definitions of quasi-GPD

$$F^0(z,P,\Delta) = \left\langle p';\lambda' \middle| \bar{\psi}\left(-\frac{z^3}{2}\right)\gamma^0 \mathcal{W}\left(-\frac{z^3}{2},\frac{z^3}{2}\right)\psi\left(\frac{z^3}{2}\right) \middle| p;\lambda \right\rangle$$



$\frac{\Delta_{\nu,s}}{\imath}$

$\frac{{}^{\nu}\Delta_\nu}{m}$



Frame dependence of quasi-GPDs

Basis vectors, $\gamma^0$and $i\sigma^{0\mu}$, do not form a complete basis for a spatially separated bi-local operator at finite momentum

New parametrization of position-space matrix elements

$$F^{\mu}(z,P,\Delta) = \bar{u}\left(p_f,\lambda'\right)\left[\frac{P^{\mu}}{m}A_1 + mz^{\mu}A_2 + \frac{\Delta^{\mu}}{m}A_3 + im\sigma^{\mu\nu}z_{\nu}A_4 + \frac{i\sigma^{\mu\nu}\Delta_{\nu}}{m}A_5 + \frac{P^{\mu}i\sigma^{\mu\nu}z_{\mu}\Delta_{\nu}}{m}A_6 + mz^{\mu}i\sigma^{\mu\nu}z_{\mu}\Delta_{\nu}A_7 + \frac{\Delta^{\mu}i\sigma^{\mu\nu}z_{\mu}\Delta_{\nu}}{m}A_8\right]u(p_i,\lambda)$$

- General structure of matrix elements based on constraints from Parity
- 8 linearly independent Dirac structures
- 8 Lorentz invariant amplitudes (Form Factors): $A_i = A_i(z\cdot P, z\cdot\Delta, t, z^2)$

Light cone case

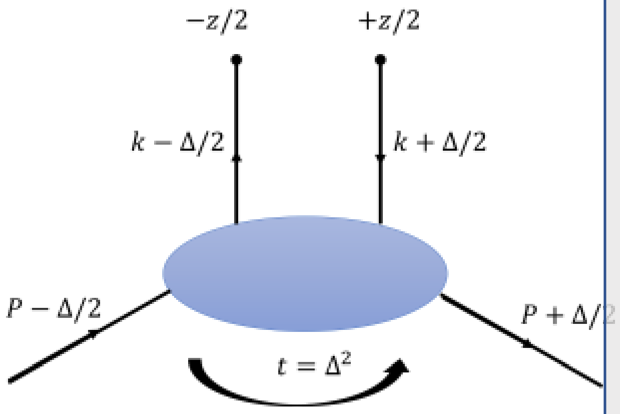
$$F^{+}(z,P,\Delta) = \bar{u}^{s/a}\left(p_f^{s/a},\lambda'\right)\left[\gamma^{+}H\left(z,P^{s/a},\Delta^{s/a}\right) + \frac{i\sigma^{+\nu}\Delta_{\nu}}{m}E\left(z,P^{s/a},\Delta^{s/a}\right)\right]u^{s/a}(p_i^{s/a},\lambda)$$

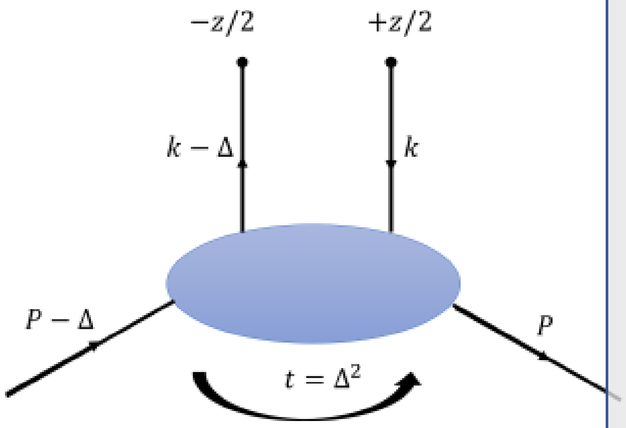
$$H\left(z,P^{s/a},\Delta^{s/a}\right) = A_1 + \frac{\Delta^{+,s/a}}{P^{+,s/a}}A_3$$

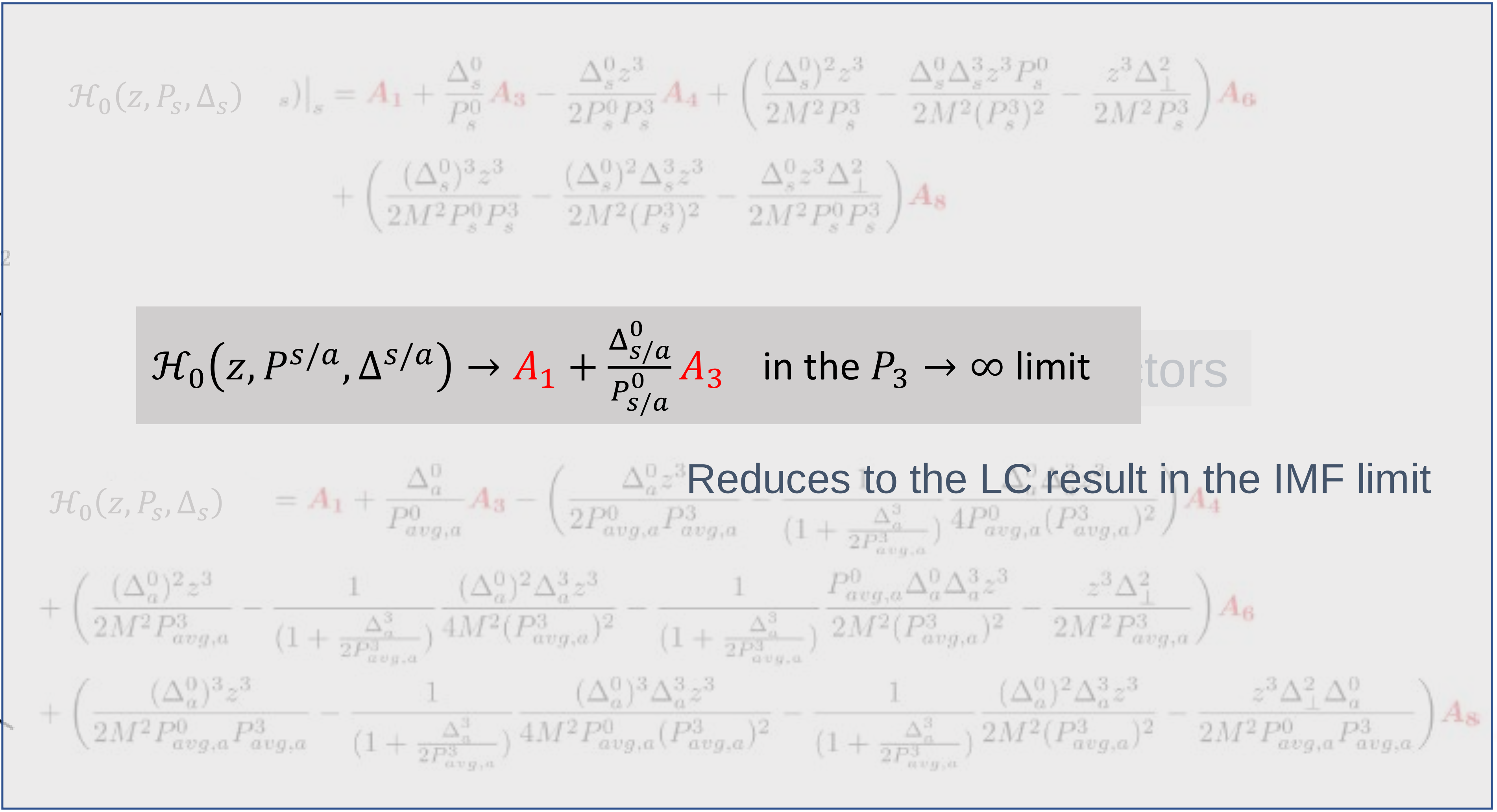
$$H\left(z\cdot P_{s/a}, z\cdot\Delta_{s/a}, t_{s/a}, z^2\right) = A_1 + \frac{z\cdot\Delta^{s/a}}{z\cdot P^{s/a}}A_3$$

Lorentz invariant

Quasi case:







$$\mathcal{H}_0(z,P_s,\Delta_s) \quad {}_s)\big|_s = A_1 + \frac{\Delta_s^0}{P_s^0}A_3 - \frac{\Delta_s^0 z^3}{2P_s^0P_s^3}A_4 + \left(\frac{(\Delta_s^0)^2z^3}{2M^2P_s^3} - \frac{\Delta_s^0\Delta_s^3z^3P_s^0}{2M^2(P_s^3)^2} - \frac{z^3\Delta_\perp^2}{2M^2P_s^3}\right)A_6$$
$$+\left(\frac{(\Delta_s^0)^3z^3}{2M^2P_s^0P_s^3} - \frac{(\Delta_s^0)^2\Delta_s^3z^3}{2M^2(P_s^3)^2} - \frac{\Delta_s^0z^3\Delta_\perp^2}{2M^2P_s^0P_s^3}\right)A_8$$

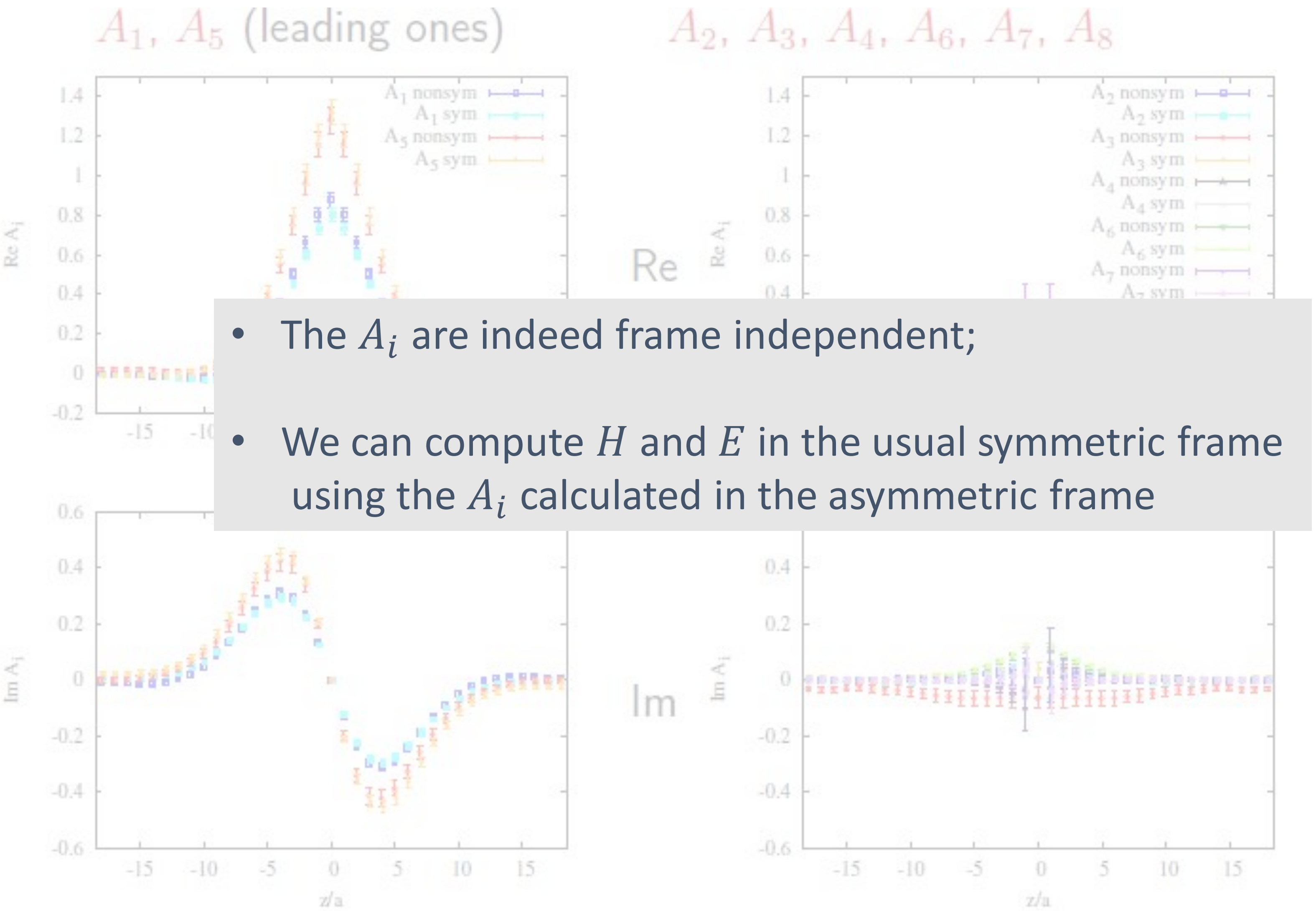
$$\mathcal{H}_0\left(z,P^{s/a},\Delta^{s/a}\right) \rightarrow A_1 + \frac{\Delta^0_{s/a}}{P^0_{s/a}}A_3 \quad \text{in the } P_3 \rightarrow \infty \text{ limit}$$

tors

Reduces to the LC result in the IMF limit

$$\mathcal{H}_0(z,P_s,\Delta_s) \quad = A_1 + \frac{\Delta_a^0}{P_{avg,a}^0}A_3 - \left(\frac{\Delta_a^0z^3}{2P_{avg,a}^0P_{avg,a}^3} - \frac{1}{(1+\frac{\Delta_a^3}{2P_{avg,a}^3})}\frac{\text{[illegible]}}{4P_{avg,a}^0(P_{avg,a}^3)^2}\right)A_4$$
$$+\left(\frac{(\Delta_a^0)^2z^3}{2M^2P_{avg,a}^3} - \frac{1}{(1+\frac{\Delta_a^3}{2P_{avg,a}^3})}\frac{(\Delta_a^0)^2\Delta_a^3z^3}{4M^2(P_{avg,a}^3)^2} - \frac{1}{(1+\frac{\Delta_a^3}{2P_{avg,a}^3})}\frac{P_{avg,a}^0\Delta_a^0\Delta_a^3z^3}{2M^2(P_{avg,a}^3)^2} - \frac{z^3\Delta_\perp^2}{2M^2P_{avg,a}^3}\right)A_6$$
$$+\left(\frac{(\Delta_a^0)^3z^3}{2M^2P_{avg,a}^0P_{avg,a}^3} - \frac{1}{(1+\frac{\Delta_a^3}{2P_{avg,a}^3})}\frac{(\Delta_a^0)^3\Delta_a^3z^3}{4M^2P_{avg,a}^0(P_{avg,a}^3)^2} - \frac{1}{(1+\frac{\Delta_a^3}{2P_{avg,a}^3})}\frac{(\Delta_a^0)^2\Delta_a^3z^3}{2M^2(P_{avg,a}^3)^2} - \frac{z^3\Delta_\perp^2\Delta_a^0}{2M^2P_{avg,a}^0P_{avg,a}^3}\right)A_8$$

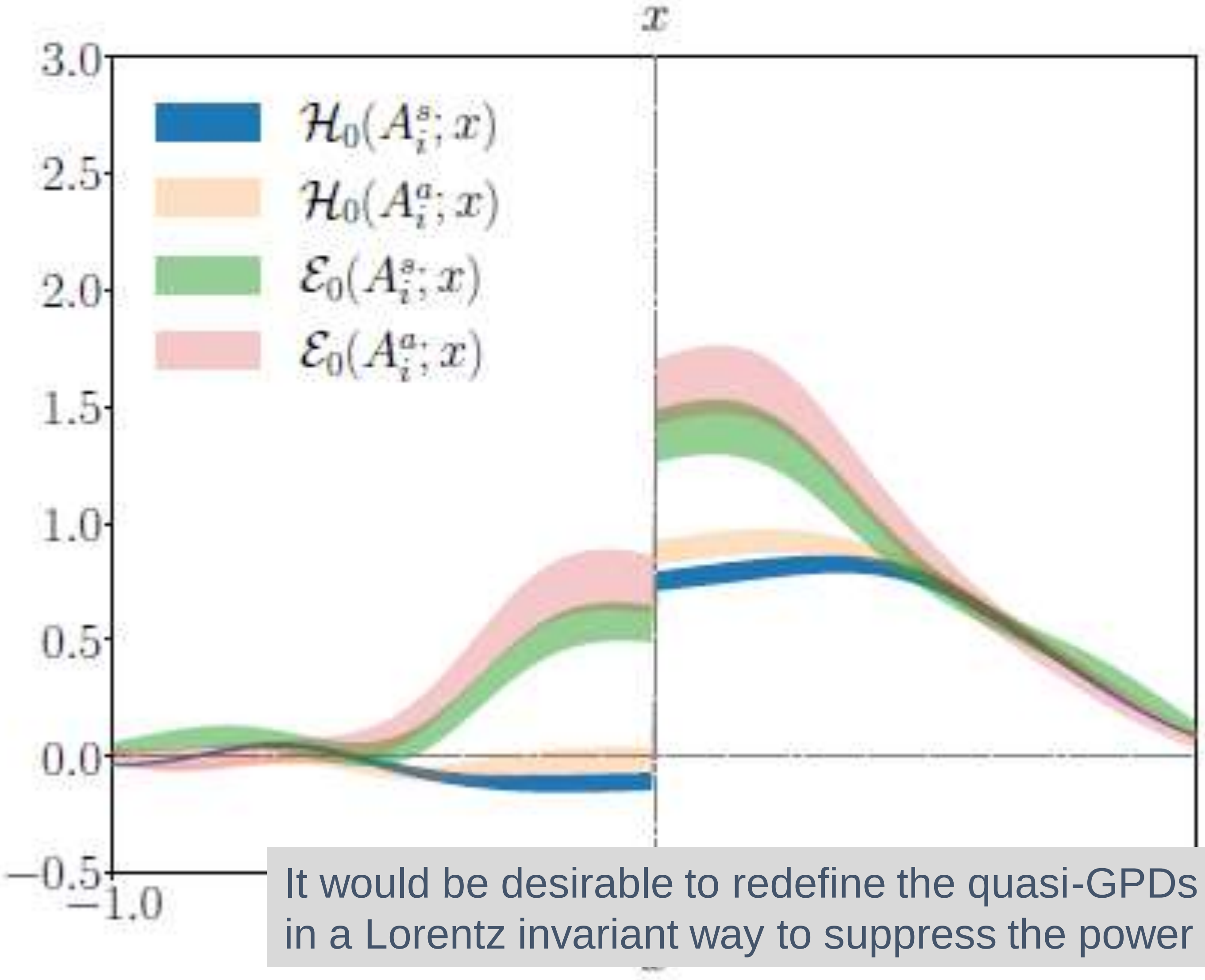
Extraction of the A_i in different frames



- The A_i are indeed frame independent;

- We can compute H and E in the usual symmetric frame using the A_i calculated in the asymmetric frame

Computing $\mathcal{H}_0$ and $\mathcal{E}_0$ in the two frames, with $\xi = 0$



It would be desirable to redefine the quasi-GPDs
in a Lorentz invariant way to suppress the power corrections

The Light-cone Lorentz Invariant definitions:

$$H\left(z \cdot P_{s/a}, z \cdot \Delta_{s/a}, t_{s/a}, z^2\right) = A_1 + \frac{z \cdot \Delta_{s/a}}{z \cdot P_{s/a}} A_3 \underset{\xi = 0}{\underline{\underline{\to A_1}}}$$

$$A_i \equiv A_i(z^2 = 0)$$

$$E\left(z \cdot P_{s/a}, z \cdot \Delta_{s/a}, t_{s/a}, z^2\right) = -A_1 - \frac{z \cdot \Delta_{s/a}}{z \cdot P_{s/a}} A_3 + 2A_5 + 2\mathrm{z} \cdot P_{s/a} A_6 + 2\mathrm{z} \cdot \Delta_{s/a} A_8 \underset{\xi = 0}{\underline{\underline{\to -A_1 + 2A_5 + 2\mathrm{z} \cdot P_{s/a} A_6}}}$$

Lorentz Invariant definitions for quasi $(z^2 \neq 0)$:

$$\mathcal{H}\left(z \cdot P_{s/a}, z \cdot \Delta_{s/a}, t_{s/a}, z^2\right) = A_1 + \frac{z \cdot \Delta_{s/a}}{z \cdot P_{s/a}} A_3 \to A_1$$

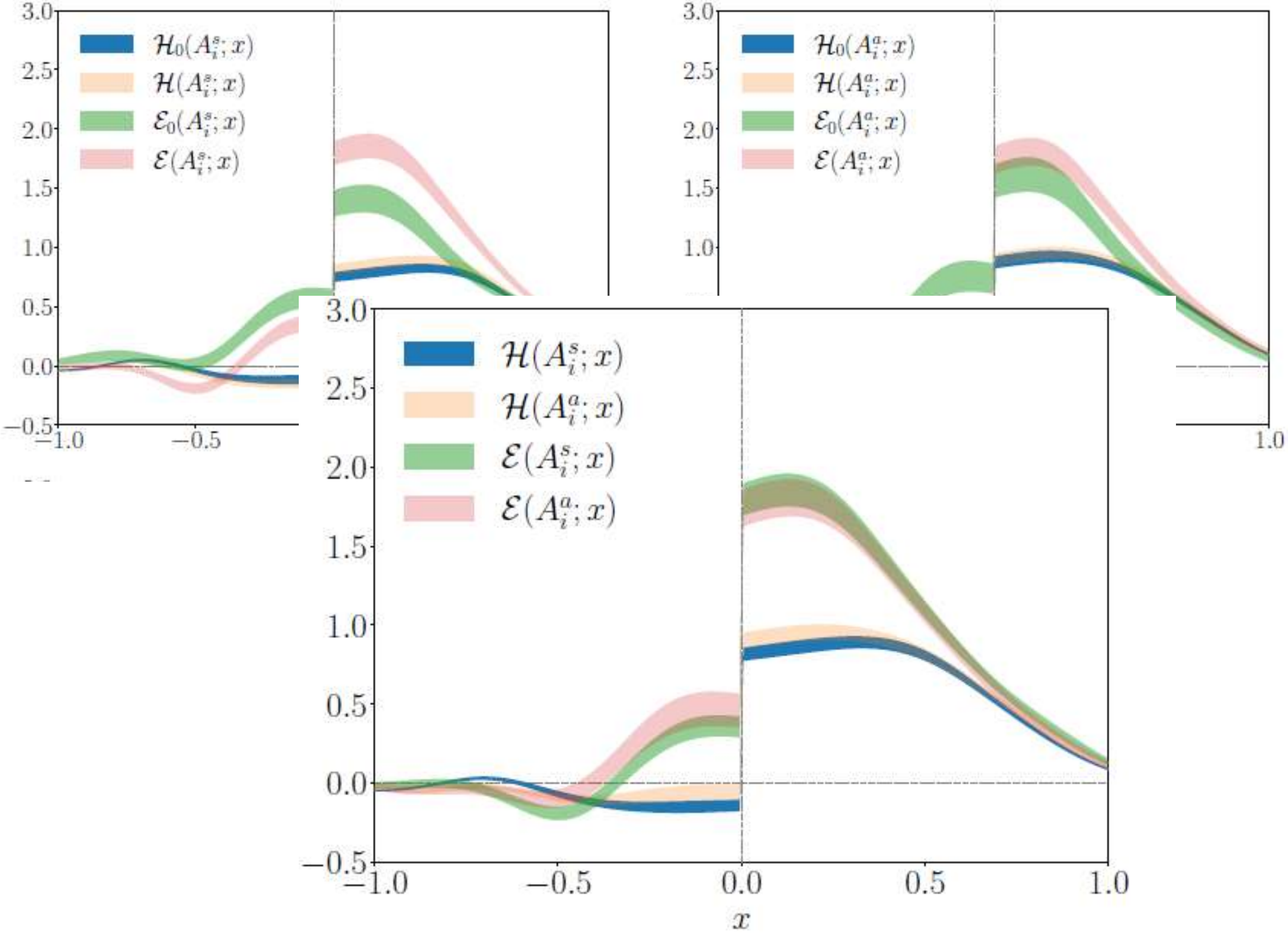
Equivalent to adding extra structures:

$$\mathcal{H}_0 \to c_0 \langle \bar{\psi} \gamma^0 \psi \rangle$$

$$\mathcal{H} \to c_0 \langle \bar{\psi} \gamma^0 \psi \rangle + c_1 \langle \bar{\psi} \gamma^1 \psi \rangle + c_2 \langle \bar{\psi} \gamma^2 \psi \rangle$$

$$\mathcal{E}\left(z \cdot P_{s/a}, z \cdot \Delta_{s/a}, t_{s/a}, z^2\right) = -A_1 - \frac{z \cdot \Delta_{s/a}}{z \cdot P_{s/a}} A_3 + 2A_5 + 2\mathrm{z} \cdot P_{s/a} A_6 + 2\mathrm{z} \cdot \Delta_{s/a} A_8 \to -A_1 + 2A_5 + 2\mathrm{z} \cdot P_{s/a} A_6$$

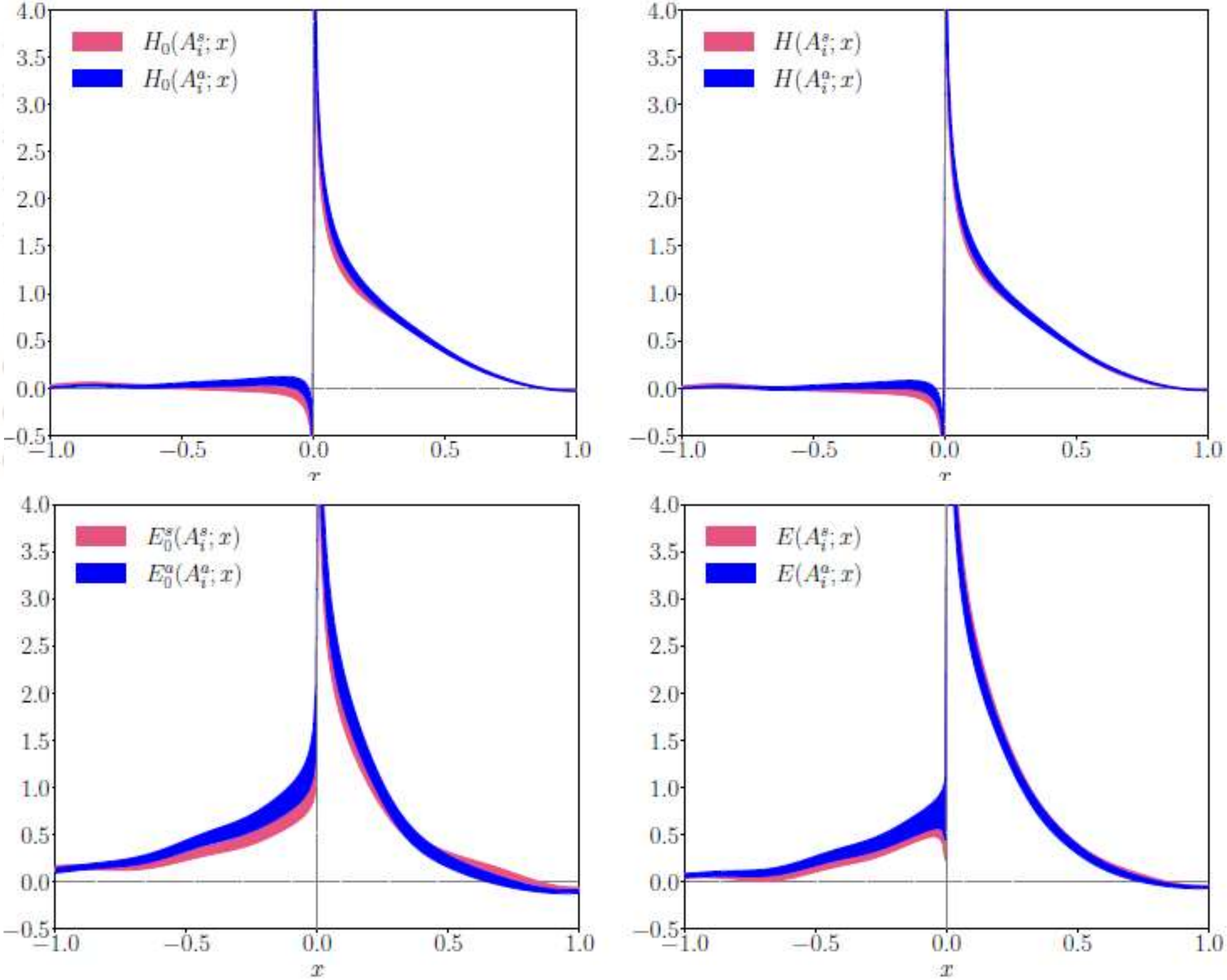
Using the LI definitions



Matching to the LC GPDs

We use the $RI \to \overline{MS}$ matching as computed in

Y.-S. Liu et al., Phys. Rev D 100, 034006 (2019), arXiv:1902.00307


$H_0(A_i^s;x)$
$H_0(A_i^a;x)$
$H(A_i^s;x)$
$H(A_i^a;x)$
$E_0^s(A_i^s;x)$
$E_0^a(A_i^a;x)$
$E(A_i^s;x)$
$E(A_i^a;x)$
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−0.5
0.0
0.5
1.0
x

TMD PDFs

ETMC + PKU

PRL 128, 062002 (2022) - TMD soft function
PoS Lattice2022 (2023) 123 - TMD beam function

Peking University

Yuan Li

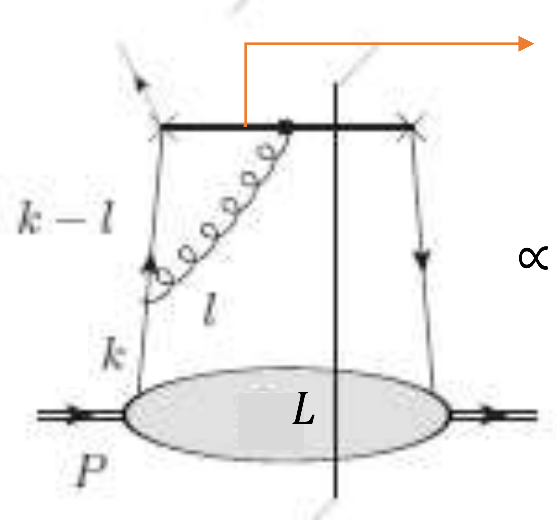
Shi-Cheng Xia

Xu Feng

Chuan Liu

TMDPDFs

How to define a TMDPDF? Is it enough to use the usual unintegrated PDFs ?



$$\propto \int dk^- d^2 l_T \int_0 \frac{dl^+}{l^+} \frac{L(k^+, k^-, \vec{k}_T)}{l_T^2 + m_g^2 + \cdots}$$

3 divergences

- UV
- IR
- $\boldsymbol{l^+ = 0}$ (soft gluons)

Origin of the extra divergence: light-like Wilson line

Can be rewritten in terms of rapidity: $y \equiv ln \frac{l^+}{l^-}$ → rapidity divergence

For the usual PDFs, these divergences cancel between the virtual and real corrections

For the transverse momentum PDFs, there is no cancellation.

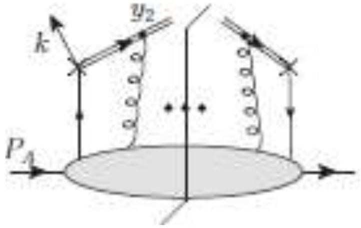
We can not use light-like Wilson lines.

And we have to subtract the soft part: Introduction of soft functions

TMDPDFs and soft functions from lattice

Using a simplified notation

$$f(x,\vec{b}_T,\zeta,\mu) \equiv \lim_{y_2\to-\infty} \frac{f^{unsub}(x,\vec{b}_T;y_{P_A}-y_2)}{\sqrt{S(b_T,y_n,y_2)}} Z_{UV}$$



$\zeta \equiv 2(xP_A^+e^{-y_n})^2$ Collins-Sopper scale

y_n is effectively the rapidity regulator

In principle, one can use the same idea of quasi-PDFs and compute purely spatial matrix elements of a hadron with momentum $\vec{P} = (0,0,P^z)$, to obtain quasi-TMDs:

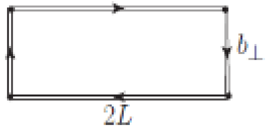
$$\tilde{f}^{unsub}(x,\vec{b}_T,\zeta_z,\mu) = \int \frac{d\omega^z}{2\pi} \lim_{L\to\infty} \frac{1}{\sqrt{Z_E(2L,b_T,\mu)}} e^{-ixP_A^z\omega^z} \left\langle P \left| \bar{\psi}\left(\frac{\omega}{2}\right) W_{n_2}\left(\frac{\omega}{2};L\right)^\dagger \frac{\gamma^z}{2} W_\perp W_{n_2}\left(-\frac{\omega}{2};L\right) \psi\left(-\frac{\omega}{2}\right) \right| P \right\rangle$$

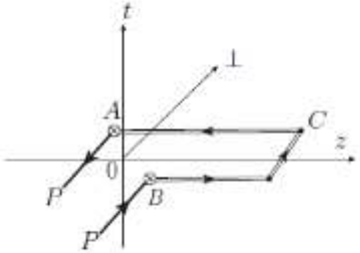
$\omega = (0,\vec{b}_T,\omega^z)$

$\zeta_z \equiv 2(xP^z)^2$,

P^z Plays the role of the rapidity

$Z_E(2L,b_T)$ =





For large rapidities (or P^z, in our case), we take advantage of the following Relation for the soft function: (Ji, Liu, Liu, arXiv:191011415)

$$S(b_T,y_1,y_2) = \frac{e^{(y_1+y_2)K(b_T)}}{S_r(b_T)}$$

K is the Collins-Sopper evolution kernel

S_r is rapidity independent and in principle can be compute using lattice

Matching Equation

$$f^{TMD}\,(x,b_\perp,\mu,\zeta) \;=\; H(\zeta_z\,,\mu)e^{-\ln\left(\frac{\zeta_z}{\zeta}\right)K(b_\perp,\mu)}S_r^{\frac{1}{2}}\,(b_\perp,\mu)f^{qTMD}(x,b_\perp,\mu,\zeta_z),$$

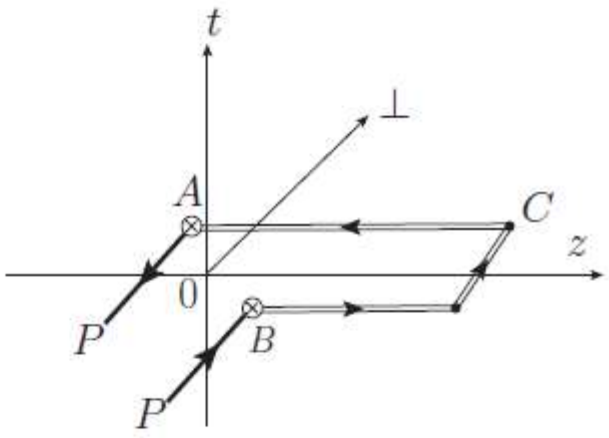
Desired quantity

Perturbative matching kernel

Reduced Soft function, also computed in LQCD

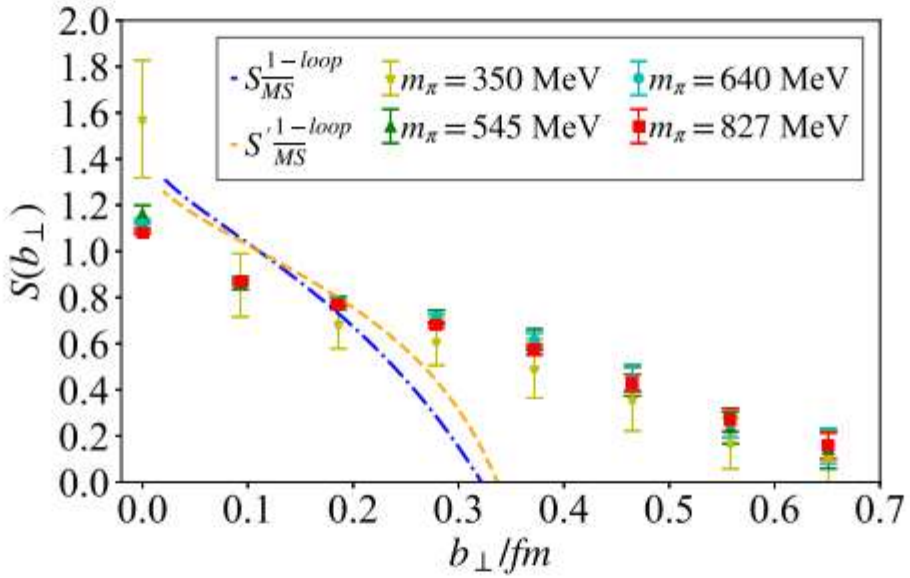
Computed in LQCD, with staple-shaped link

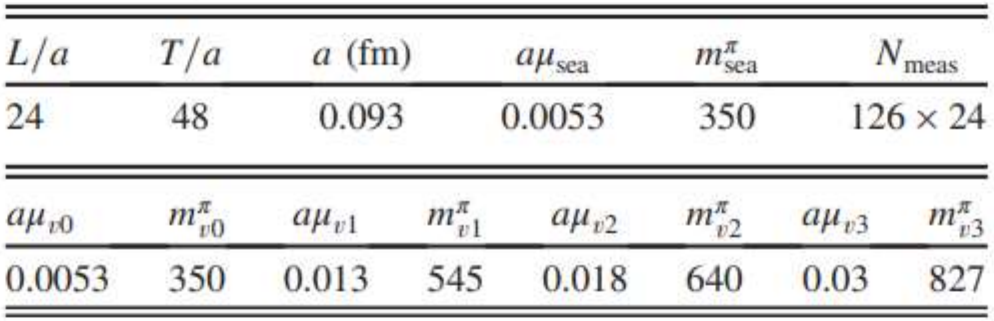
$\zeta \equiv 2(xP^+e^{-y_n})^2$	Collins-Sopper scale
	y_n is effectively the rapidity regulator
$\zeta_z \equiv 2(xP^z)^2\,,$	P^z Plays the role of the rapidity
$K(b_\perp,\mu)$	Collins-Sopper evolution kernel



From arXiv: 1911.03840

Intrinsic soft function as a function of the transverse separation $b_\perp$

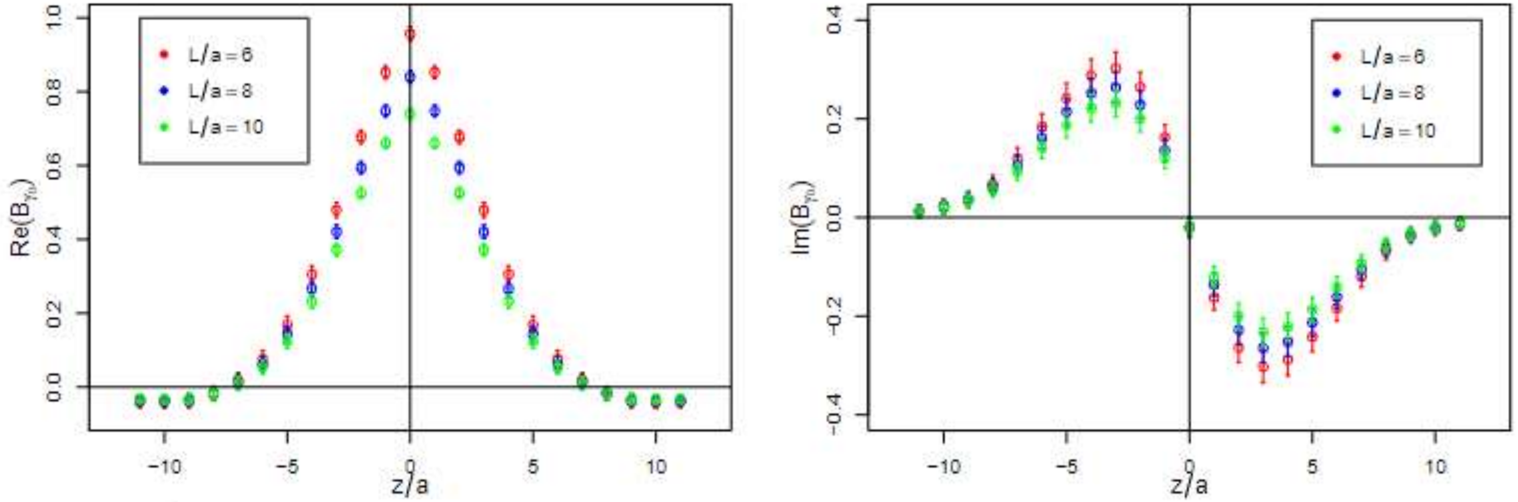


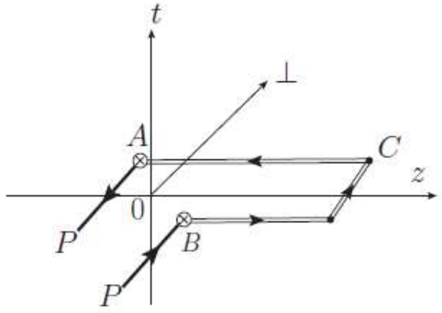


L/a	T/a	a (fm)	$a\mu_{\text{sea}}$	m^π_{sea}	N_{meas}
24	48	0.093	0.0053	350	126 × 24

$a\mu_{v0}$	m^π_{v0}	$a\mu_{v1}$	m^π_{v1}	$a\mu_{v2}$	m^π_{v2}	$a\mu_{v3}$	m^π_{v3}
0.0053	350	0.013	545	0.018	640	0.03	827

Beam function: $$B(z,P^z) = \left\langle P \left| \bar{\psi}\left(\frac{z}{2}\right) W_{n_2}\left(\frac{z}{2};L\right)^{\dagger} \frac{\gamma^0}{2} W_{\perp} W_{n_2}\left(-\frac{z}{2};L\right) \psi\left(-\frac{z}{2}\right) \right| P \right\rangle$$





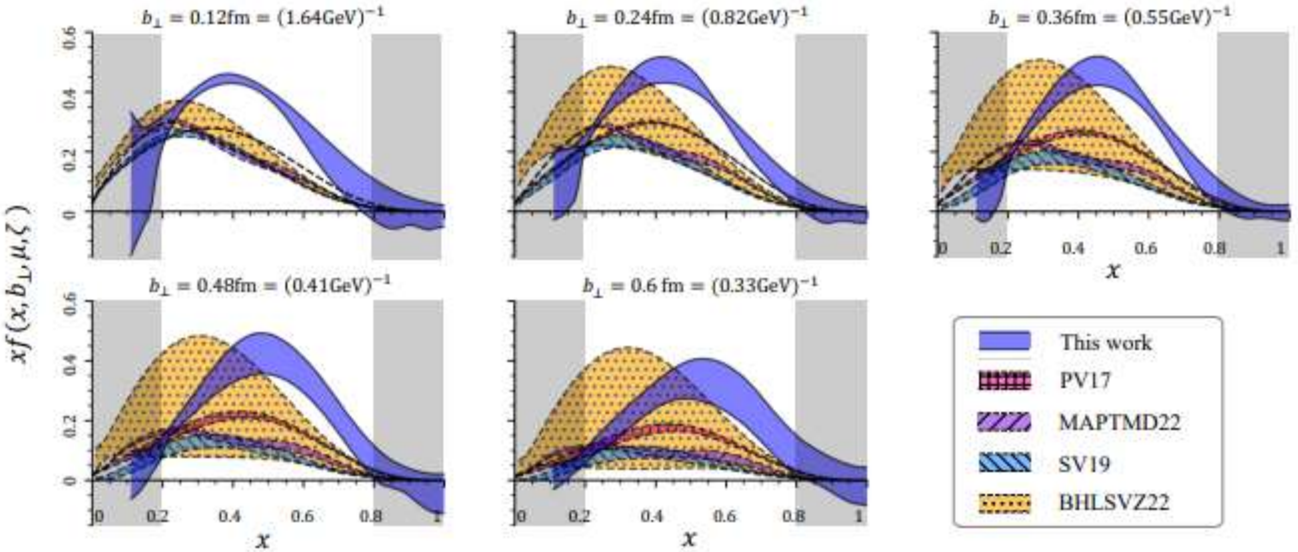
Linear divergence from the Wilson line connecting the quark fields

Log divergences from the end points of the staple

Log divergences from the cusps of the staple

As $L \to \infty$, pinch-pole singularities in positive powers of L, coming from the gluon exchange from the transverse Wilson lines

LPC is the only computation so far arXiv:2211.02340



Summary

- Huge developments on first principles for PDFs, GPDs, and TMDPDFs calculations
- First results for twist-3 PDFs
- For GDPs, formalism developed to compute them in symmetry or asymmetric frames developed
- Renormalization of TMDPDFs seems to be under control
- Non-perturbative calculation of the soft function performed
- First results on TMDPDFs are on the way

Many more works already done,

Pion and Kaon PDFs

Meson DA

Delta PDF

Gluon PDF

Transversity GPDs

Synergy between lattice and phenomenology

Many improvements can be made:

Higher boost

Discretization effects

Finite volume effects

Higher twist contamination

Truncation effects in the matching

The problem of x reconstruction

Road towards precision is open!