PDFs, GPDs and TMDPDFs from lattice QCD

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With thanks to my collaborators for the works presented here:

C. Alexandrou, S. Bhattacharya, K. Cichy, M. Constantinou, J. Dodson, X. Gao, K. Hadjiyiannakou, K. Jansen, A. Metz, S. Mukherjee, A. Scapellato, A. Sen, Y. Zhao
Cross sections are measured:

**Totally inclusive**

**Semi-inclusive**

**Uncircled PDF**

**Circled PDF**

**Transversity PDF**

**Unpolarized PDF**

**Helicity PDF**

**Light-cone coordinates**

\[ γ^+ = \frac{γ^0 + γ^3}{\sqrt{2}} \]

\[ x = x_B j = \frac{-q^2}{2P.q} \]

is the momentum fraction carried by a given parton

Have access to the chiral-even distributions \( f_1(x) \) (unpolarized) and \( g_1(x) \) (helicity)

Have access to the chiral-odd distribution \( h_1(x) \) (transversity). Naturally more difficult to obtain data on transversity

**Example:**

\[ f_1(x) = \frac{1}{4\pi} \int dz^- e^{-iP^+z^-} \langle P|\bar{\psi}(z^-)γ^+ψ(0)|P \rangle \]
Twist-3 PDFs

Twist expansion: $f_i(x) = f_i^{(0)} + \frac{f_i^{(1)}}{Q} + \frac{f_i^{(2)}}{Q^2} + \ldots$

Twist-2 + Twist-3 + Twist-4

Twist-3:
- $\hat{1}$ : $e(x)$  
- $\gamma^j \gamma_5$ : $g_T(x)$  
- $\sigma^{jk}$ : $h_L(x)$

No density interpretation;  
Contain information of quark-gluon-quark correlations;  
Possible zero mode contribution;  
Hard to determine experimentally.

Examples:

$g_T(x) \equiv g_1(x) + g_2(x)$  
Can be interpreted as a transverse force acting on the quark being scattered  
(Burkardt PRD88 (2013) 114502)

$g_T^{ww}(x) = \int_x^{+1} dy \: g_1(y)$  
Wandzura-Wilczek approximation  
Experimentally: Possible 15-40% violation  
(Accardi et al., JHEP 11, 093 (2009))

$\int_{-1}^{+1} dx \: g_T(x) = \int_{-1}^{+1} dx \: g_1(x)$  
Burkhardt-Cottingham sum rule

$\int_{-1}^{+1} dx (e^u(x) + e^d(x)) = \frac{\sigma_{\pi N}}{m}$  
The $e(x)$ PDF is related to the pion-nucleon sigma term
Generalised PDFs (GPDs)

Momentum transfer: \[ \Delta \equiv p'' - p', \quad t \equiv \Delta^2, \]

Fraction of the momentum transfer: \[ \xi \equiv -\frac{p''^+ - p'^+}{p''^+ + p'^+} = -\frac{2\Delta^+}{p^+}, \quad \xi \text{ is called skewness} \]

GPDs are multidimensional objects, depending on \( x, t, \xi \)
Transverse momentum dependent PDFs (TMDPDFs)

Why TMDPDFs?

If we measure only the invariant mass of the final lepton pair:

\[
\frac{d\sigma}{dQ^2} = \sum_{i,j} \int dx_1 dx_2 f_i(x_1) f_j(x_2) H(x_1, x_2) \left( 1 + O\left(\frac{m^2}{Q^2}\right) \right)
\]

If we measure the transverse momentum $q_T$ of the lepton pair, we have access to the transverse momentum of the quarks!

\[
\frac{d^2\sigma}{dQ^2 dq_T^2} = \sum_{i,j} \int dx_1 dx_2 \int d^2 b_T e^{ib_T q_T} f_i(x_1, b_T) f_j(x_2, b_T) H(x_1, x_2) \left( 1 + O\left(\frac{\Lambda_{QCD} q_T^2}{Q^2} \right) \right), \quad q_T \ll Q
\]

Transverse momentum dependent PDFs
Last 5 years witnessed enormous progress on first principles computations of both PDFs and GPDs

Theoretical papers
X. Ji, PRL 110, 262002 (2013) - Quasi
A.V. Radyushkin, PRD 96, 034025 (2017) - Pseudo
A. J. Chambers et al., PRL 118, 242001 (2017) - OPE without OPE
Yan-Qing Ma and Jian-Wei Qiu, PRL 120, 022003 (2018) - Good lattice cross sections

Exploratory studies
LP3, PRD 91, 054510 (2015)
ETMC, PRD 92, 014502 (2015)

Nucleon PDFs at physical pion mass using Quasi
ETMC, NPB 923, 394 (2017) - Nonperturbative renormalization
ETMC, PRL 121, 112001 (2018) - Unpolarized and helicity
LP3, PRL 121, 242003 (2018) - Helicity

Nucleon PDFs at physical pion mass using Pseudo
ETMC, PRD 103, 034510 (2021)
HadStruc., PRL 125, 232003 (2020) - Extrapolated to physical pion mass

Nucleon GPDs
ETMC, PRL 125, 262001 (2020) - Unpolarized and helicity
ETMC+Temple+BNL+ANL, PRD 106 125, 115412 (2022) Symmetric and asymmetric frames

Twist-3
ETMC/Temple, PRD 102, 111501 (2020)
ETMC/Temple, PRD 104 115410 (2021)

List restricted to physical pion mass results or exploratory studies. There are many more works on the subject and I apologize to authors of works not listed.
TMDPDs just starting

Theoretical papers
M. A. Ebert, I. W. Stewart, Y. Zhao, PRD 99, 034505 (2019)
M. A. Ebert et al., JHEP 37, 2019 (2019)
P. Shanahan, M. Wagman, Y. Zhao, PRD 102, 014511 (2020)
M. A. Ebert et al., arXiv:2201.08401

Exploratory studies – Soft function
LPC, PRL 125, 192001 (2020)
ETMC, PRL 128, 062002 (2022)

Exploratory studies – Collins-Soper kernel
ETMC, PRL 128, 062002 (2022)
LPC, arXiv: 2204.00200

Exploratory studies – Beam functions and TMDPDFs
ETMC+PKU, PoS Lattice2022 (2023) 123
ETMC+PKU, PoS Lattice2022 (2023) 733
LPC, arXiv: 2211.02340
Light-cone PDFs and quasi PDFs

\[ q(x) = \frac{1}{4\pi} \int dz^- e^{-iP^+z^-} \langle P | \bar{\psi}(z^-) \gamma^+ W(z^-, 0) \psi(0) | P \rangle \]

Dirac Structure  Wilson line

Quark distribution is given by a light-front correlation

\[ z^- = \frac{t - z}{\sqrt{2}}, \quad P^+ = \frac{E + P^z}{\sqrt{2}} \]
Our focus: isovector quark distributions, $q(x) \equiv u(x) - d(x)$

Perturbative correction to isovector quark distributions:

$$q(x, \Lambda) = 
\left\{ \begin{array}{c}
\delta(1 - x/y) \\
\Pi(\Lambda) \delta(1 - x/y) \\
\Gamma(x/y, \Lambda)
\end{array} \right\} 
+ \cdots + 
\left\{ \begin{array}{c}
\delta(1 - x/y) \\
\Pi(\Lambda) \delta(1 - x/y) \\
\Gamma(x/y, \Lambda)
\end{array} \right\} 
\otimes q_{bare}(y) + \mathcal{O}(\alpha_s^2)$$

Regulator of IR and UV divergences

$$q(x, \Lambda) = q_{bare}(x) \left\{ 1 + \frac{\alpha_s}{2\pi} \Pi(\Lambda) \right\} + \frac{\alpha_s}{2\pi} \int_x^1 \Gamma \left( \frac{x}{y}, \Lambda \right) q_{bare}(y) \frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$
Simplest diagram

\[ \frac{d^2 k_\perp \bar{u}(p) \gamma^+ k \cdot \gamma \gamma_\mu u(p)}{(2\pi)^2 k^2 (p - k)^2} \delta \left( y - \frac{k^+}{p^+} \right) \]

\[ p = (\xi P^+, 0, 0, 0); \quad \xi = \frac{p^+}{P^+} \]

For 0 < y < 1, one pole in the upper half and other in the lower half of the complex plane

For y > 1 or y < 0, the poles are either on the lower half or on the upper half of the complex plane

\[ k^2 + i\epsilon = 2yp^+ \left( k^- - \frac{k_\perp^2}{2yp^+} + i\epsilon \right) \]

\[ (p - k)^2 + i\epsilon = -2p^+(1 - y) \left( k^- + \frac{k_\perp^2}{2p^+(1 - y)} - i\epsilon \right) \]

\[ = 2\alpha_s C_F (1 - y) \int \frac{d^2 k_\perp \bar{u}(p) \gamma^+ u(p)}{(2\pi)^2 k_\perp^2} = \frac{\alpha_s}{2\pi} 4p^+(1 - y) \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} + \ln \left( \frac{\mu^2}{\mu_F^2} \right) \right) \]

With support only in the physical region, 0 < y < 1

DR used for IR and UV divergences
Infinite momentum frame (IMF)

\[ E = -i g^2 C_F \int \frac{dk^0 dk^3 d^2 k_\perp}{(2\pi)^4} \frac{\bar{u}(p) \gamma^\mu k \cdot \gamma^3 k \cdot \gamma_{\mu} u(p)}{(k^2)^2 (p - k)^2} \delta \left( y - \frac{k^3}{p^3} \right) \]

\[ k^2 + i \epsilon = \left( k^0 - \sqrt{k_\perp^2 + y^2 (p^3)^2} + i \epsilon \right) \left( k^0 + \sqrt{k_\perp^2 + y^2 (p^3)^2} - i \epsilon \right) \]

\[ (p - k)^2 + i \epsilon = \left( k^0 - p^3 - \sqrt{k_\perp^2 + (1 - y)^2 (p^3)^2} + i \epsilon \right) \left( k^0 - p^3 + \sqrt{k_\perp^2 + (1 - y)^2 (p^3)^2} - i \epsilon \right) \]

Integrating in \( k^0 \) and taking the \( p^3 \to \infty \) limit:

\[ = 2 \alpha_s C_F (1 - y) \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{\bar{u}(p) \gamma^3 u(p)}{k_\perp^2} = \frac{\alpha_s}{2\pi} 4 p_3 (1 - y) \left( \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} + \ln \left( \frac{\mu^2}{\mu_F^2} \right) \right) \]

with \( 0 < y < 1 \)

LC and IMF have the same IR and UV behaviour and are equivalent.

Unfortunately, they can not be computed within LQCD
What if $p_3$ is kept finite?

\[
\bar{q}(x, \Lambda) = q_{\text{bare}}(x) \left\{ 1 + \frac{\alpha_s}{2\pi} \bar{\Pi}(\Lambda) \right\} + \frac{\alpha_s}{2\pi} \int_x^1 \bar{\Gamma} \left( \frac{x}{y}, \Lambda \right) q_{\text{bare}}(y) \frac{dy}{y} + \mathcal{O}(\alpha_s^2)
\]

Regulator of IR and UV divergences
Keeping $p_3$ finite

\[ = -i g^2 C_F \left[ \frac{dk^0 dk^3 d^2 k_\perp}{(2\pi)^4} \bar{u}(p) \gamma^\mu k \cdot \gamma^3 k \cdot \gamma_\mu u(p) \right] \frac{\delta \left( y - \frac{k^3}{p^3} \right)}{(k^2)^2 (p - k)^2} \]

Integrating in $k^0$ and keeping $p_3$ finite, we have an integral over $k_T$ which is UV finite! But has an IR divergence. Using Dimensional Regularization:

\[
\frac{\alpha_s}{2\pi} \frac{4p_3}{p_3^2} \left[ (1 - y) \left( -\frac{1}{\epsilon_{IR}} + \ln \left( \frac{p_3^2}{\mu_F^2} \right) + \ln(4y(1 - y)) \right) + 1 \right], \quad 0 < y < 1
\]

\[
+ \frac{\alpha_s}{2\pi} \frac{4p_3}{p_3^2} \left[ (1 - y) \ln \left( \frac{x}{x - 1} \right) + 1 \right], \quad y > 1
\]

\[
+ \frac{\alpha_s}{2\pi} \frac{4p_3}{p_3^2} \left[ (1 - y) \ln \left( \frac{x - 1}{x} \right) - 1 \right], \quad y < 0
\]

Support outside the physical region!

Same IR pole as in the LC and IMF cases

UV divergence appears only when integrating over all parton momentum fraction $y$
We want to go from a purely spatial correlation to a light-front correlation.

\[ \frac{1}{2} \int \frac{dz^3}{2\pi} e^{ik^3 z^3} \left( P' \right| \bar{\psi} \left( -\frac{z^3}{2} \right) \Gamma W \left( -\frac{z^3}{2}, \frac{z^3}{2} \right) \psi \left( \frac{z^3}{2} \right) \right) P ; \lambda \]

Purely spatial correlation


\[ H_Q(x, \xi, P^3, \mu) = \int \frac{d\gamma}{|y|} C \left( \frac{x - \xi}{y}, \frac{\mu}{y P^3} \right) H(y, \xi, \mu) + \text{power corrections} \]

Computed in LQCD

Computed in pQCD
Results for Twist-2

ETMC

ETMC, NPB 923, 394 (2017) - Nonperturbative renormalization

ETMC, PRL 121, 112001 (2018) - Unpolarized and helicity quasi

ETMC, PRD 103, 034510 (2021) - Unpolarized and helicity pseudo
\( P_3 = 1.38 \text{ GeV} \)

Unpolarized

**Quarks**

\[ m_\pi \cong 130 \text{ MeV} \]

\[ 48^3 \times 96 \text{ lattice} \]

\[ a \cong 0.093 \text{ fm} \]
Pseudo-PDF approach

Uses same ensemble as the quasi approach

Different from quasi case, here a pheno inspired ansatz is used to reconstruct the x dependence

- Purple: Statistical error
- Blue: Quantified systematics
- Cyan: Estimated systematics
- Black: NNPDF3.1 parametrization

ETMC, PRD 103, 034510 (2021)
Results for Twist-3

ETMC + Temple

PRD 102 (2020) 11, 111501  - Lattice $g_T(x)$
PRD 102 (2020) 3, 034005  - Matching $g_T(x)$
PRD 102 (2020) 224025  - Matching $e(x)$ and $h_L(x)$
arXiv: 2107.02574  - Lattice $h_L(x)$

<table>
<thead>
<tr>
<th>Name</th>
<th>$\beta$</th>
<th>$N_f$</th>
<th>$L^3 \times L_T$</th>
<th>$a$ [fm]</th>
<th>$M_\pi$</th>
<th>$m_\pi L$</th>
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<tbody>
<tr>
<td>cA211.32</td>
<td>1.726</td>
<td>$u, d, s, c$</td>
<td>$32^3 \times 64$</td>
<td>0.093</td>
<td>260 MeV</td>
<td>4</td>
</tr>
</tbody>
</table>
The $g_T(x) = g_1(x) + g_2(x)$ distribution

$P_3 = 1.67 \, GeV$

The BC sum rule is verified:

$$\int_{-1}^{+1} dx \ g_T(x) - \int_{-1}^{+1} dx \ g_1(x) = 0.01(20)$$

The WW approximation

$$g_T^{\text{WW}}(x) = \int_x^{+1} dy \ g_1(y)$$

Up to $x < 0.5$, $g_T(x)$ agrees with $g_T^{\text{WW}}(x)$

Violations of 30-40% possible
The chiral-odd twist-3 distribution $h_L(x)$

The WW approximation relates $h_L(x)$ to its twist-2 counterpart $h_1(x)$

$$h_L^{ww}(x) = 2x \int_x^{+1} \frac{dy}{y^2} h_1(y)$$

Suggests that the twist-3 distribution can be determined from its twist-2 counterpart.
Twist-2 GPDs

ETMC

PRL 125, 262001 (2020) - Unpolarized ane helicity

PRD 105, 034501 (2022) - Transversity

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Lattice setup:

- **fermions:** \( N_f = 2 \) twisted mass fermions + clover term
- **gluons:** Iwasaki gauge action
- **gauge field configurations generated by ETMC**
- **lattice spacing** \( a \approx 0.093 \) fm
- **32 \times 64** lattice, \( L \approx 3 \) fm
- **\( m_\pi \approx 260 \) MeV**

Kinematics:

- **three nucleon boosts:** \( P_3 = 0.83, 1.25, 1.67 \) GeV
- **momentum transfers:** \(-t = 0, 0.69, 1.02 \) GeV
- **skewness:** \( \xi = 0, 1/3 \)
- **Twist-2 unpolarized + helicity GPDs:** Phys. Rev. Lett. 125 (2020) 2620
- **Twist-2 transversity GPDs:** Phys. Rev. D 105 (2022) 034501
Problem with the current approach: not efficient

\[ P_{\text{source}} = \left( -\frac{\Delta x}{2}, -\frac{\Delta y}{2}, P_3 \right) \]

\[ P_{\text{sink}} = \left( -\frac{\Delta x}{2}, -\frac{\Delta y}{2}, P_3 \right) \]

- Separate calculation for each momentum transfer: \[ P_{\text{sink}} = \left( -\frac{\Delta x}{2}, -\frac{\Delta y}{2}, P_3 \right) \]
- Much more efficient if \( P_{\text{sink}} = (0,0,P_3) \)
Symmetric and asymmetric frames

S. Bhattacharya et al., arXiv: 2209.05373

\[
\begin{align*}
\begin{pmatrix}
E_{i,s} \\
p_{1,s}^1 \\
p_{i,s}^2 \\
p_{i,s}^3
\end{pmatrix} &= \begin{pmatrix}
\gamma & -\gamma\beta & 0 & 0 \\
-\gamma\beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
E_{i,a} \\
-\Delta^1_a \\
0 \\
p_3
\end{pmatrix}
\end{align*}
\]

Transverse boost

\[
\langle \bar{\psi}\gamma^0\psi \rangle^s = \gamma \langle \bar{\psi}\gamma^0\psi \rangle^a - \gamma\beta \langle \bar{\psi}\gamma^1\psi \rangle^a
\]
Historical definitions of quasi-GPD

\[ F^0(z, P, \Delta) = \left| p'; \lambda' \left[ \bar{\psi} \left( -\frac{Z^3}{2} \right) \gamma^0 W \left( -\frac{Z^3}{2}, \frac{Z^3}{2} \right) \psi \left( \frac{Z^3}{2} \right) \right] p; \lambda \right| \]

Frame dependence of quasi-GPDs

Basis vectors, \( \gamma^0 \) and \( i \sigma^0 \mu \), do not form a complete basis for a spatially separated bi-local operator at finite momentum.
New parametrization of position-space matrix elements

\[
F^\mu(z, P, \Delta) = \bar{u}(p_f, \lambda') \left[ \frac{P^\mu}{m} A_1 + mz^\mu A_2 + \frac{\Delta^\mu}{m} A_3 + i\sigma^{\mu\nu}z_\nu A_4 + \frac{i\sigma^{\mu\nu}\Delta_\nu}{m} A_5 + \frac{P^\mu i\sigma^{\mu\nu}z_\nu \Delta_\nu}{m} A_6 + mz^\mu i\sigma^{\mu\nu}z_\nu \Delta_\nu A_7 + \frac{\Delta^\mu i\sigma^{\mu\nu}z_\nu \Delta_\nu}{m} A_8 \right] u(p_i, \lambda)
\]

- General structure of matrix elements based on constraints from Parity
- 8 linearly independent Dirac structures
- 8 Lorentz invariant amplitudes (Form Factors): \( A_i = A_i(z \cdot P, z \cdot \Delta, t, z^2) \)

Light cone case

\[
F^+(z, P, \Delta) = \bar{u}^{s/a}(p_f^{s/a}, \lambda') \left[ \gamma^+ H(z, P^{s/a}, \Delta^{s/a}) + \frac{i\sigma^{\nu\Delta_\nu}}{m} E(z, P^{s/a}, \Delta^{s/a}) \right] u^{s/a}(p_i^{s/a}, \lambda)
\]

\[
H(z, P^{s/a}, \Delta^{s/a}) = A_1 + \frac{\Delta^{+,s/a}}{P^{+,s/a}} A_3
\]

\[
H(z \cdot P_{s/a}, z \cdot \Delta_{s/a}, t_{s/a}, z^2) = A_1 + \frac{z \cdot \Delta^{s/a}}{z \cdot P^{s/a}} A_3
\]

Lorentz invariant
Quasi case:

\[ \mathcal{H}_0(z, P_s, \Delta_s) \big|_{s} = A_1 + \frac{\Delta_z^0}{P_s^0} A_3 + \frac{\Delta_z^0 \Delta_s^3}{2 P_s^0 P_s^0} A_4 + \left( \frac{(\Delta_z^0)^2 \Delta_s^3}{2 M^2 P_s^0} - \frac{\Delta_z^0 \Delta_s^3 \Delta_s^3 P_s^0}{2 M^2 (P_s^0)^2} - \frac{\Delta_s^3 \Delta_s^2}{2 M^2 P_s^0} \right) A_6 \]

\[ + \left( \frac{(\Delta_z^0)^2 \Delta_s^3}{2 M^2 P_s^0 P_s^0} - \frac{(\Delta_z^0)^2 \Delta_s^3 \Delta_s^3}{2 M^2 (P_s^0)^2} - \frac{\Delta_z^0 \Delta_s^3 \Delta_s^2}{2 M^2 P_s^0} \right) A_8 \]

Reduces to the LC result in the IMF limit

\[ \mathcal{H}_0(z, P_s^0, \Delta_s^0) \to A_1 + \frac{\Delta_z^0}{P_s^0} A_3 \quad \text{in the } P_s^0 \to \infty \text{ limit} \]
Extraction of the $A_i$ in different frames

- The $A_i$ are indeed frame independent;
- We can compute $H$ and $E$ in the usual symmetric frame using the $A_i$ calculated in the asymmetric frame.
Computing $\mathcal{H}_0$ and $\mathcal{E}_0$ in the two frames, with $\xi = 0$

It would be desirable to redefine the quasi-GPDs in a Lorentz invariant way to suppress the power corrections.
The Light-cone Lorentz Invariant definitions:

\[ H(z \cdot P_s/a, z \cdot \Delta s/a, t_s/a, z^2) = A_1 + \frac{z \cdot \Delta s/a}{z \cdot P_s/a} A_3 \rightarrow A_1 \]

\[ \xi = 0 \]

\[ A_i \equiv A_i(z^2 = 0) \]

\[ E(z \cdot P_s/a, z \cdot \Delta s/a, t_s/a, z^2) = -A_1 - \frac{z \cdot \Delta s/a}{z \cdot P_s/a} A_3 + 2A_5 + 2z \cdot P_s/a A_6 + 2z \cdot \Delta s/a A_8 \rightarrow -A_1 + 2A_5 + 2z \cdot P_s/a A_6 \]

\[ \xi = 0 \]

Lorentz Invariant definitions for quasi \((z^2 \neq 0)\):

\[ H(z \cdot P_s/a, z \cdot \Delta s/a, t_s/a, z^2) = A_1 + \frac{z \cdot \Delta s/a}{z \cdot P_s/a} A_3 \rightarrow A_1 \]

\[ \mathcal{H}_0 \rightarrow c_0 \langle \bar{\psi} \gamma^0 \psi \rangle \]

\[ \mathcal{H} \rightarrow c_0 \langle \bar{\psi} \gamma^1 \psi \rangle + c_1 \langle \bar{\psi} \gamma^2 \psi \rangle + c_2 \langle \bar{\psi} \gamma^3 \psi \rangle \]

\[ E(z \cdot P_s/a, z \cdot \Delta s/a, t_s/a, z^2) = -A_1 - \frac{z \cdot \Delta s/a}{z \cdot P_s/a} A_3 + 2A_5 + 2z \cdot P_s/a A_6 + 2z \cdot \Delta s/a A_8 \rightarrow -A_1 + 2A_5 + 2z \cdot P_s/a A_6 \]
Using the LI definitions
Matching to the LC GPDs

We use the $RI \to \overline{MS}$ matching as computed in

TMD PDFs

ETMC + PKU

PRL 128, 062002 (2022) - TMD soft function
PoS Lattice2022 (2023) 123 - TMD beam function

Peking University
Yuan Li
Shi-Cheng Xia
Xu Feng
Chuan Liu
TMDPDFs

How to define a TMDPDF? Is it enough to use the usual unintegrated PDFs?

\[
\mathcal{L} \propto \sum_{n, d, k} - \sum_{l, T, k} L + \sum_{n, l, T} \mathcal{L} + \sum_{m, g, 2} \cdots
\]

3 divergences
- UV
- IR
- \( l^+ = 0 \) (soft gluons)

Origin of the extra divergence: light-like Wilson line

Can be rewritten in terms of rapidity: \( y \equiv \ln \frac{l^+}{l^-} \rightarrow \text{rapidity divergence} \)

For the usual PDFs, these divergences cancel between the virtual and real corrections

For the transverse momentum PDFs, there is no cancellation.

We can not use light-like Wilson lines.

And we have to subtract the soft part: Introduction of soft functions
Using a simplified notation

\[
f(x, \vec{b}_T, \zeta, \mu) \equiv \lim_{y_2 \to -\infty} \frac{f^{\text{unsub}}(x, \vec{b}_T; y_{PA} - y_2)}{\sqrt{S(b_T, y_n, y_2)}} Z_{UV}
\]

\[
\zeta \equiv 2(xP_A^+ e^{-y_n})^2 \quad \text{Collins-Soper scale}
\]

\[y_n \text{ is effectively the rapidity regulator}\]

In principle, one can use the same idea of quasi-PDFs and compute purely spatial matrix elements of a hadron with momentum \( \vec{P} = (0,0,P^z) \), to obtain quasi-TMDs:

\[
f^{\text{unsub}}(x, \vec{b}_T, \zeta, \mu) = \int \frac{d\omega^z}{2\pi} \lim_{L \to \infty} \frac{1}{Z_E(2L, b_T, \mu)} e^{-ixP_A^z\omega^z} \left\langle P \left| \overline{\psi} \left( \frac{\omega}{2} \right) W_{n_2} \left( \frac{\omega}{2}; L \right) \frac{\gamma^z}{2} W_1 W_{n_2} \left( -\frac{\omega}{2}; L \right) \psi \left( -\frac{\omega}{2} \right) \right| P \right\rangle \quad \omega = (0, \vec{b}_T, \omega^z)
\]

\[
\zeta \equiv 2(xP^2)^z, \quad Z_E(2L, b_T) = \int \frac{d\omega^z}{2\pi} \lim_{L \to \infty} \frac{1}{Z_E(2L, b_T, \mu)} e^{-ixP_A^z\omega^z} \left\langle P \left| \overline{\psi} \left( \frac{\omega}{2} \right) W_{n_2} \left( \frac{\omega}{2}; L \right) \frac{\gamma^z}{2} W_1 W_{n_2} \left( -\frac{\omega}{2}; L \right) \psi \left( -\frac{\omega}{2} \right) \right| P \right\rangle
\]

\[P^z \text{ Plays the role of the rapidity}\]

For large rapidities (or \( P^z \), in our case), we take advantage of the following Relation for the soft function: \( \left( Ji, Liu, Liu, \text{arXiv:191011415} \right) \)

\[
S(b_T, y_1, y_2) = \frac{e^{(y_1+y_2)K(b_T)}}{S_r(b_T)} \quad K \text{ is the Collins-Soper evolution kernel}
\]

\[S_r \text{ is rapidity independent and in principle can be compute using lattice}\]
Matching Equation

\[ f^{TMD}(x, b_\perp, \mu, \zeta) = H(\zeta_z, \mu)e^{-\ln(\frac{\zeta_z}{\zeta})K(b_\perp, \mu)} \frac{1}{S} S_{r}^{\frac{1}{2}}(b_\perp, \mu)f^{qTMD}(x, b_\perp, \mu, \zeta_z), \]

- **Desired quantity**
- **Perturbative matching kernel**
- **Reduced Soft function, also computed in LQCD**

\[ \zeta \equiv 2(xP^+ e^{-y_n})^2 \]  
**Collins-Sopper scale** 
\[ y_n \text{ is effectively the rapidity regulator} \]

\[ \zeta_z \equiv 2(xP^z)^2, \]  
**P^z Plays the role of the rapidity**

\[ K(b_\perp, \mu) \]  
**Collins-Sopper evolution kernel**

Computed in LQCD, with staple-shaped link

From arXiv: 1911.03840
Intrinsic soft function as a function of the transverse separation $b_\perp$
Beam function: \[ B(z, P^z) = \left\langle P \left| \bar{\psi} \left( \frac{Z}{2} \right) W_{n_2} \left( \frac{Z}{2}; L \right) \frac{\gamma^0}{2} W_{\perp} W_{n_2} \left( -\frac{Z}{2}; L \right) \psi \left( -\frac{Z}{2} \right) \right| P \right\rangle \]

- Linear divergence from the Wilson line connecting the quark fields
- Log divergences from the end points of the staple
- Log divergences from the cusps of the staple
- As \( L \to \infty \), pinch-pole singularities in positive powers of \( L \), coming from the gluon exchange from the transverse Wilson lines
LPC is the only computation so far arXiv:2211.02340
Summary

- Huge developments on first principles for PDFs, GPDs, and TMDPDFs calculations
- First results for twist-3 PDFs
- For GDPs, formalism developed to compute them in symmetry or asymmetric frames developed
- Renormalization of TMDPDFs seems to be under control
- Non-perturbative calculation of the soft function performed
- First results on TMDPDFs are on the way
Many more works already done,

Pion and Kaon PDFs
Meson DA
Delta PDF
Gluon PDF
Transversity GPDs
Synergy between lattice and phenomenology

Many improvements can be made:

Higher boost
Discretization effects
Finite volume effects
Higher twist contamination
Truncation effects in the matching
The problem of $x$ reconstruction

Road towards precision is open!