From 1D distributions to GPDs

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QCD: Basic Facts

- QCD is characterized by two emergent phenomena: confinement and dynamical generation of mass (DGM).
- Quarks and gluons not isolated in nature.
- Formation of colorless bound states: “Hadrons”
- 1-fm scale size of hadrons?

Emergence of hadron masses (EHM) from QCD dynamics

Higgs mechanism

QCD dynamics

Quarks
Mass = 1.78×10⁻²⁶ g

~ 1% of proton mass
(~ 10 MeV)

Proton
Mass = 168×10⁻²⁶ g

~ 99% of proton mass
(~ 928 MeV)
QCD: Basic Facts

- QCD is characterized by two emergent phenomena: confinement and dynamical generation of mass (DGM).

Can we trace them down to fundamental d.o.f?

- Emergence of hadron masses (EHM) from QCD dynamics

\[
L_{\text{QCD}} = \sum_{j=u,d,s,...} \bar{q}_j [\gamma_\mu D_\mu + m_j] q_j + \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu},
\]

\[
D_\mu = \partial_\mu + ig\frac{1}{2} \lambda^a A_\mu^a,
\]

\[
G_{\mu\nu}^a = \partial_\mu A_\nu^a + \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c,
\]

\[
S_f^{-1}(p) = Z_f^{-1}(p^2) (i \gamma \cdot p + M_f(p^2))
\]

Gluon and quark *running masses*
QCD: Basic Facts

- QCD is characterized by two emergent phenomena: confinement and dynamical generation of mass (DGM).

Can we trace them down to fundamental d.o.f?

- Emergence of hadron masses (EHM) from QCD dynamics

Cui:2019dwv
Pion Structure

Pion

- "Two" quark bound-state

- **Archetype** of nuclear exchange **forces**
  - *Our favorite mediator*

- The **lightest** hadron in nature
Pion Structure

Pion
- “Two” quark bound-state
  
  \[ M_{u,d} \sim 310 \text{ MeV} \]

  \[ M_u + M_d \neq 140 \text{ MeV} \]

- Archetype of nuclear exchange forces
  - Our favorite mediator

- The lightest hadron in nature

Unlike the proton, pion is **massless** in the absence of Higgs mass generation

> Both a quark-antiquark bound-state and a Golstone Boson

> Its mere existence is connected with mass generation in the SM
Pion Structure

➢ The experimental access to the pion structure is via electromagnetic probes, yielding e.g.:

Electromagnetic form factor

Distribution functions

➢ Generalized parton distributions (GPDs) encode them both (and many more):

\[ F_\pi(\Delta^2) = \int \mathcal{H}_u^\pi(x, \xi, -\Delta^2; \zeta), \quad u^\pi(x; \zeta) = \mathcal{H}_\pi^u(x, 0, 0; \zeta) \]

➢ But the experimental access and theoretical derivation is far more complicated.
Question:
From 1-dimensional distributions (EFF and PDF), can we obtain the 3-dimensional GPD?
**Pion GPD**

- **Question:**
  From 1-dimensional distributions (EFF and PDF), can we obtain the 3-dimensional GPD?

- **Partial Answer:**
  Yes, we can. Under two premises:
  
  - There exists at least one effective charge such that, when used to integrate the leading-order perturbative DGLAP equations, an evolution scheme for parton DFs is defined that is all-order exact.

Thus, we can plainly connect the parton and quasiparticle picture in a well determined manner.
**Question:**
From 1-dimensional distributions (EFF and PDF), can we obtain the 3-dimensional GPD?

**Partial Answer:**
Yes, we can. Under two premises:

1. There exists at least one effective charge such that, when used to integrate the leading-order perturbative DGLAP equations, an evolution scheme for parton DFs is defined that is all-order exact.
2. A factorised representation of the pion light-front wave function (LFWF), from which the (DGLAP) GPD is derived, at the hadronic scale, is a sensible approximation.

**Overlap:**
\[ H^u_\pi(x, \xi, -\Delta^2; \zeta_H) = \int \frac{d^2 k_\perp}{16\pi^3} \psi^u_\pi(x^-, k^2_\perp; \zeta_H) \psi^u_\pi(x^+, k^2_\perp; \zeta_H) \]

**Factorization:**
\[ \psi^u_\pi(x, k^2_\perp; \zeta_H) = \tilde{\psi}^u_\pi(k^2_\perp) [u^\pi(x; \zeta_H)]^{1/2} \]
Light-front wave functions

\[ \psi^u_P(x, k^2_{\perp}; \zeta) \]

“One ring to rule them all”
Goal: get a broad picture of the pion and Kaon structure.

The idea: Compute everything from the LFWF.

Form Factors

GPDs

PDFs

PDAs

LFWFs

Overlap representation

Goal: get a broad picture of the pion and Kaon structure.

The idea: Compute everything from the LFWF.
Goal: get a broad picture of the pion and Kaon structure.

Project onto the light-front

Overlap representation

The idea: Compute everything from the LFWF.

The inputs: Solutions from quark DSE and meson BSE.

The alternative inputs: Construct BSWF from realistic DSE predictions.

A. Bashir’s talk
About PTIRs and LFWFs
LFWF: PTIR approach

- A perturbation theory integral representation for the BSWF:

\[ n_K \chi_K(k^K, P_K) = \mathcal{M}(k, P) \int_{-1}^{1} dw \rho_K(w) \mathcal{D}(k, P) \]

(Kaon as example)

1: Matrix structure (leading BSA):

\[ \mathcal{M}(k; P_K) = -\gamma_5 [\gamma \cdot P_K M_u + \gamma \cdot k (M_u - M_s) + \sigma_{\mu\nu} k_\mu P_{K\nu}] \]

2: Spectral weight: Tightly connected with the meson properties.

3: Denominators: \( \mathcal{D}(k; P_K) = \Delta(k^2, M_u^2) \Delta((k - P_K)^2, M_s^2) \Delta(k_{\omega-1}^2, \Lambda_K^2) \),

where: \( \Delta(s, t) = [s + t]^{-1} \), \( \hat{\Delta}(s, t) = t \Delta(s, t) \).
Recall the expression for the LFWF:

\[
\psi^q_M(x, k^2_\perp) = \text{tr} \int_{dk_\parallel} \delta_n^x(k_M) \gamma_5 \gamma \cdot n \chi_M(k_-, P)
\]

Algebraic manipulations yield:

\[
\Rightarrow \psi^q_M(x, k_\perp) \sim \int dw \rho_M(w) \ldots
\]

Uniqueness of Mellin moments

Compactness of this result is a merit of the AM.

Thus, \(\rho_M(w)\) determines the profiles of, e.g. PDA and PDF: (it also works the other way around)

\[
f_M \phi^q_M(x; \zeta_H) = \int \frac{d^2 k_\perp}{16\pi^3} \psi^q_M(x, k_\perp; \zeta_H)
\]

\[
q_M(x; \zeta_H) = \int \frac{d^2 k_\perp}{16\pi^3} |\psi^q_M(x, k_\perp; \zeta_H)|^2
\]
LFWF: PTIR approach

More explicitly:

\[
\psi^q_M(x, k_\perp^2; \zeta_H) = 12 \left[ M_q(1 - x) + M_h x \right] X_P(x; \sigma_\perp^2) \quad \sigma_\perp = k_\perp^2 + \Omega_P^2
\]

\[
X_M(x; \sigma_\perp^2) = \left[ \int_{-1}^{1-2x} dw \int_{1+\frac{2x}{w-1}}^{1} dv + \int_{1-2x}^{1} dw \int_{\frac{w-1+2x}{w+1}}^{1} dv \right] \frac{\rho_M(w) \Lambda_M^2}{n_M \sigma_\perp^2}
\]

\[
\Omega_M^2 = \nu M_q^2 + (1 - \nu) \Lambda_P^2
\]

\[
+ (M_h^2 - M_q^2) \left( x - \frac{1}{2} [1 - w][1 - \nu] \right)
\]

\[
+ (x[x - 1] + \frac{1}{4} [1 - \nu][1 - w^2]) m_M^2
\]

Model parameters:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( m_P )</th>
<th>( M_u )</th>
<th>( M_h )</th>
<th>( \Lambda_P )</th>
<th>( b_0^P )</th>
<th>( \omega_0^P )</th>
<th>( v_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi )</td>
<td>0.14</td>
<td>0.31</td>
<td>( M_u )</td>
<td>( M_u )</td>
<td>0.275</td>
<td>1.23</td>
<td>0</td>
</tr>
<tr>
<td>( K )</td>
<td>0.49</td>
<td>0.31</td>
<td>1.2M_u</td>
<td>3M_s</td>
<td>0.1</td>
<td>0.625</td>
<td>0.41</td>
</tr>
</tbody>
</table>

\[
\rho_P(\omega) = \frac{1 + \omega v_P}{2a_P b_0^P} \left[ \text{sech}^2 \left( \frac{\omega - \omega_0^P}{2b_0^P} \right) + \text{sech}^2 \left( \frac{\omega + \omega_0^P}{2b_0^P} \right) \right]
\]
LFWF: Factorized case

- In the **chiral limit**, the PTIR reduces to:

\[
\psi^q_M(x, k^2_\perp; \zeta_H) \sim \tilde{f}(k_\perp) \phi^q_M(x; \zeta_H) \sim f(k_\perp)[q_M(x; \zeta_H)]^{1/2}
\]

**“Factorized model”**

\[
[\phi^q_M(x; \zeta_H)]^2 \sim q_M(x; \zeta_H)
\]

- Sensible assumption as long as:

\[
m^2_M \approx 0 \quad \quad M^2_h - M^2_q \approx 0 \quad \quad \zeta_H
\]

- (meson mass) \quad (h-antiquark, q-quark masses)

- Produces **identical** results as PTIR model for pion

- Therefore:

\[
\psi^q_M(x, k^2_\perp; \zeta_H) = [q^M(x; \zeta_H)]^{1/2} \left[4\sqrt{3}\pi \frac{M^3_q}{(k^2_\perp + M^2_q)^2}\right]
\]

**Single parameter!**

\[
M_q \sim r^{-1}_M
\]

(charge radius)

No need to determine the spectral weight!
LFWF: Factorized case

- In the **chiral limit**, the PTIR reduces to:

\[
\psi^q_M(x, k_\perp^2; \zeta_H) \sim \tilde{f}(k_\perp)\phi^q_M(x; \zeta_H) \sim f(k_\perp)[q_M(x; \zeta_H)]^{1/2}
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**“Factorized model”**

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[\phi^q_M(x; \zeta_H)]^2 \sim q_M(x; \zeta_H)
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- Sensible assumption as long as:

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m_M^2 \approx 0 \quad M_h^2 - M_q^2 \approx 0 \quad \zeta_H
\]

  (meson mass) (flavor asymmetry)

- Produces **identical** results as PTIR model for pion

Therefore:

\[
\psi^q_M(x, k_\perp^2; \zeta_H) = [q^M(x; \zeta_H)]^{1/2} \left[4\sqrt{3\pi} \frac{M_q^3}{(k_\perp^2 + M_q^2)^2}\right]
\]

Constituent mass value compatible with **realistic** estimations.

\[M_u = 0.31 \text{ GeV} \quad \Leftrightarrow \quad r_\pi = 0.66 \text{ fm}\]

AtifSultan:2018end
LFWF: PTIR approach II

A perturbation theory integral representation for the BSWF:

\[ n_{M,M}(k^-, P) = \mathcal{M}_{q,\bar{h}}(k, P) \int_{-1}^{1} dw \tilde{\rho}_M^\nu(w) D_{q,\bar{h}}^\nu(k, P) \]

1: Matrix structure (leading BSA):

\[ \mathcal{M}_{q,\bar{h}}(k, P) \equiv -\gamma_5 [M_q \gamma \cdot P + \gamma \cdot k (M_{\bar{h}} - M_q) + \sigma_{\mu\nu} k_\mu P_\nu - i (k \cdot p + M_q M_{\bar{h}})] \]

2: Profile function:

\[ \tilde{\rho}_M^\nu(w) \equiv \rho_M(w) \Lambda_w^{2\nu} \]

3: Denominators:

\[ D_{q,\bar{h}}^\nu(k, P) \equiv \Delta (k^2, M_q^2) \Delta (k_{w-1}^2, \Lambda_w^{2\nu}) \Delta (p^2, M_{\bar{h}}^2) \]

The crucial difference:

\[ \Lambda^2(w) := \Lambda_w^2 = M_q^2 - \frac{1}{4} (1 - w^2) m_M^2 + \frac{1}{2} (1 - w) (M_{\bar{h}}^2 - M_q^2) \]
LFWF: PTIR approach II

- Then a series of algebraic results follows.

1. For the BSWF:

\[ n_{M\chi_M}(k_-, P) = \mathcal{M}_{q,\bar{q}}(k, P) \int_0^1 d\alpha \mathcal{F}_M(\alpha, \sigma^{\nu+2}), \quad \sigma = (k-\alpha P)^2 + \Lambda_{1-2\alpha}^2, \]

\[ \mathcal{F}_M(\alpha, \sigma^{\nu+2}) = 2^\nu (\nu + 1) \left[ \int_{-1}^{1-2\alpha} dw \left( \frac{\alpha}{1-w} \right)^\nu + \int_{1-2\alpha}^1 dw \left( \frac{1-\alpha}{1+w} \right)^\nu \right] \frac{\tilde{\rho}_M(w)}{\sigma^{\nu+2}} \]

2. LFWF in terms of PDA/PDF:

\[ f_M^q(x; \zeta_H) = \frac{d^2 k_\perp}{16\pi^3} \psi_M^q(x, k_\perp; \zeta_H) \]

\[ \psi_M^q(x, k_\perp^2) = 16\pi^2 f_M \frac{\nu \Lambda_{1-2x}^{2\nu}}{(k_\perp^2 + \Lambda_{1-2x}^2)^{\nu+1}} \phi_M^q(x) \]

\[ \Lambda_{1-2x}^2 = M_{q,\bar{q}}^2 + x(M_{\bar{q}}^2 - M_q^2) - m_H^2 x(1-x) \]

- Flavor asymmetry
- Meson mass

Encodes the breaking of factorization.

\[ \rightarrow \text{Completely factorized in the chiral limit.} \]
Coming back to the point...
Many **distributions** are related via the leading-twist light-front wave function (LFWF), e.g.:

\[ f_P \varphi_P^u (x, \zeta_H) = \int \frac{dk_{\perp}^2}{16\pi^3} \psi_P^u (x, k_{\perp}^2; \zeta_H) \]

Distribution amplitudes

\[ u_P^u (x; \zeta_H) = \int \frac{d^2k_{\perp}}{16\pi^3} |\psi_P^u (x, k_{\perp}^2; \zeta_H)|^2 \]

Distribution functions

In the **DGLAP** kinematic domain, this is also the case of the valence-quark **GPD**:

\[ H_P^{u} (x, \xi, t; \zeta_H) = \int \frac{d^2k_{\perp}}{16\pi^3} \psi_P^{u*} (x_-, k_{\perp-}^2; \zeta_H) \psi_P^u (x_+, k_{\perp+}^2; \zeta_H) \]

- The overlap approach guarantees the **positivity** of the GPD.
  - It is, in principle, limited to the DGLAP kinematic region. \( |x| \geq |\xi| \)
- Nonetheless, it can be extended to the **ERBL** domain.

\[ x_{\mp} = (x \mp \xi)/(1 \mp \xi), \quad t = -\Delta^2 \]

\[ k_{\perp \mp} = k_{\perp} \pm (\Delta_{\perp}/2)(1 - x)/(1 \mp \xi) \]
Many distributions are related via the leading-twist light-front wave function (LFWF), e.g.:

\[ f_P \varphi^u_P(x, \zeta_H) = \int \frac{dk^2}{16\pi^3} \psi^u_P(x, k^2_\perp; \zeta_H) \]

\[ u^P(x; \zeta_H) = \int \frac{d^2k^2_\perp}{16\pi^3} |\psi^u_P(x, k^2_\perp; \zeta_H)|^2 \]

This connection already suggests that:

\[ u^P(x; \zeta_H) \sim [\varphi^u_P(x; \zeta_H)]^2 \]

is a fair approximation, implying:

\[ \psi^{\uparrow \downarrow}_{P_u}(x, k^2_\perp; \zeta_H) = \tilde{\psi}^u_{P_u}(k^2_\perp) [u^P(x; \zeta_H)]^{1/2} \]

In fact, we have learned that x-k crossed terms are weighted by: \( M_P^2, M_h^2 - M_q^2 \)

So a factorized Ansatz should be sensible for the pion, implying:

\[ H^u_P(x, \xi, t; \zeta_H) = \Theta(x^-) \sqrt{u^P(x^-; \zeta_H)u^P(x^+; \zeta_H)} \Phi_P(z; \zeta_H) \]

\[ z = \frac{(1 - x)^2}{(1 - \xi^2)^2} \Delta^2 \]
PARTON DISTRIBUTIONS

- **Fully-dressed valence quarks** (quasiparticles)
- **Unveiling of glue and sea d.o.f** (partons)

Resolution Scale

\[ \zeta_H \]

\[ \zeta > \zeta_H \]
• **Fully-dressed valence quarks** (quasiparticles)

\[ (M_u = M_d) \]

\[ \zeta_H : \text{hadronic scale} \]

➢ At this scale, **all properties** of the hadron are contained within their valence quarks.

➢ Equally massive quarks means a **50-50** share of the total momentum:

\[ \langle x(\zeta_H) \rangle_q = 0.5 \]

➢ This implies symmetric distributions:

\[ q(x; \zeta_H) = q(1 - x; \zeta_H) \]
Pion PDF: hadronic scale

- **Fully-dressed valence quarks** (quasiparticles)

\[ (M_u = M_d) \quad \zeta_H : \text{hadronic scale} \]

- At this scale, **all properties** of the hadron are contained within their valence quarks.

"Physical" boundaries:

\[
\frac{1}{2^n} \overset{(i)}{\leq} \langle x^n \rangle_{u,\pi} \overset{(ii)}{\leq} \frac{1}{1 + n}
\]

Produced by

- \( q(x; \zeta_H) = \delta(x - 1/2) \) (infinitely heavy valence quarks)
- \( q(x; \zeta_H) = 1 \) (massless SCI case)

Produced by

- Equally massive quarks means a **50-50** share of the total momentum.
- This implies symmetric distributions.
Pion PDF: experimental scale

- Unveiling of glue and sea d.o.f (partons)
  - Experimental data is given here.
  - Lattice QCD results are also quoted beyond the hadronic scale.
  - The interpretation of parton distributions from cross sections demands special care.
Pion PDF: energy scales

- Fully-dressed valence quarks (quasiparticles)
  - Theoretical calculations are performed at some low energy scale.

- Unveiling of glue and sea d.o.f (partons)
  - Then evolved via DGLAP equations to compare with experiment and lattice.
Pion PDF: energy scales

- Fully-dressed valence quarks (quasiparticles)
  - Theoretical calculations are performed at some low energy scale.

- Unveiling of glue and sea d.o.f (partons)
  - Then evolved via DGLAP equations to compare with experiment and lattice.

- Following our all orders evolution, we can go either way.
- Besides, the hadronic scale becomes unambiguously determined.

Resolution Scale: $\zeta > \zeta_H$

Evolution equations:

- $\zeta_H$

Rodriguez-Quintero:2019fyc
EVOLUTION

Have a nice end of the world.
**DGLAP: All orders evolution**

**Idea.** Define an **effective** coupling such that:

“**All orders evolution**”

\[
\left\{ \zeta^2 \frac{d}{d\zeta^2} \int_0^1 dy \delta(y - x) - \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P^{NS}_{qq} \left( \frac{x}{y} \right) \right. \\
\left. \begin{pmatrix}
0 & 0 \\
0 \quad P^S \left( \frac{x}{y} \right)
\end{pmatrix} \right\} \left( \begin{array}{c}
H^{NS, +}_\pi(y, t; \zeta) \\
H^S_\pi(y, t; \zeta)
\end{array} \right) = 0
\]

→ **Not** the LO QCD coupling but an **effective** one.

→ Making this equation **exact**.

→ And connecting with the **hadron scale**.

Sea and Gluon content unveils, as prescribed by QCD.
DGLAP: All orders evolution

Implication 1:

\[ \langle x^n(\zeta_f) \rangle_q = \exp \left( -\frac{\gamma^{(n)}_{qq}}{4\pi} S(\zeta_H, \zeta_f) \right) \langle x^n(\zeta_H) \rangle_q \]

\[ S(\zeta_0, \zeta_f) = \int_{2 \ln(\zeta_0/\Lambda_{QCD})}^{2 \ln(\zeta_f/\Lambda_{QCD})} dt \alpha(t) \]

**Explicitly** depending on the effective charge

\[ \langle x^n(t; \zeta) \rangle_F = \int_0^1 dx \, x^n \, F(x, t; \zeta) \]

\[ \gamma^{(n)}_{AB} = -\int_0^1 dx \, x^n P^C_{AB}(x) \]

- The QCD PI effective charge is our best candidate to accommodate our all orders scheme.

\[ \hat{\alpha}(k^2) = \frac{\gamma_m \pi}{\ln \left[ \frac{M^2(k^2)}{\Lambda_{QCD}^2} \right]} \]

\[ \zeta_H = 0.331 \text{ GeV} \]
DGLAP: All orders evolution

Implication 1:

\[ \langle x^n(\zeta_f) \rangle_q = \exp \left( -\frac{\gamma^{(n)}_{qq}}{4\pi} S(\zeta_H, \zeta_f) \right) \langle x^n(\zeta_H) \rangle_q = \langle x^n(\zeta_H) \rangle_q \left( \frac{\langle x(\zeta_f) \rangle_q}{\langle x(\zeta_H) \rangle_q} \right)^{\gamma^{(n)}_{qq}} / \gamma^{(1)}_{qq} \]

\[ S(\zeta_0, \zeta_f) = \int_{2\ln(\zeta_0/\Lambda_{QCD})}^{2\ln(\zeta_f/\Lambda_{QCD})} dt \alpha(t) \]

This contains, *implicitly*, the information of the effective charge.

→ No actual *need* to know it. Assuming its existence is *sufficient*.

→ *Unambiguous* definition of the hadron scale:

\[ \langle x(\zeta_H) \rangle_q = 0.5 \Rightarrow \langle x^n(\zeta_f) \rangle_q = \langle x^n(\zeta_H) \rangle_q \left( \langle 2x(\zeta_f) \rangle_q \right)^{\gamma^{(n)}_{qq}} / \gamma^{(1)}_{qq} \]

(flavor symmetric case)
**DGLAP: All orders evolution**

**Implication 1:**

\[
\langle x^n(\zeta_f) \rangle_q = \exp \left( -\frac{\gamma_{qq}^{(n)}}{4\pi} S(\zeta_H, \zeta_f) \right) \langle x^n(\zeta_H) \rangle_q = \langle x^n(\zeta_H) \rangle_q \left( \frac{\langle x(\zeta_f) \rangle_q}{\langle x(\zeta_H) \rangle_q} \right)^{\gamma_{qq}^{(n)}} / \gamma_{qq}^{(1)}
\]

Information on the charge is here

- Details of the effective charge are **encoded** in the ratio of first moments.
- Natural connection with the **hadron scale**.

**Implication 2:**

\[
\begin{align*}
\langle 2x(\zeta_f) \rangle_q &= \exp \left( -\frac{8}{9\pi} S(\zeta_H, \zeta_f) \right), \quad q = u, d; \\
\langle x(\zeta_f) \rangle_{\text{sea}} &= \langle x(\zeta_f) \rangle_{\sum q + \bar{q}} - (\langle x(\zeta_f) \rangle_u + \langle x(\zeta_f) \rangle_d), \\
&= \frac{3}{7} + \frac{4}{7} \langle 2x(\zeta_f) \rangle_u^{7/4} - \langle 2x(\zeta_f) \rangle_u \\
\langle x(\zeta_f) \rangle_g &= \frac{4}{7} \left( 1 - \langle 2x(\zeta_f) \rangle_u^{7/4} \right); \\
\end{align*}
\]

- Sea and **gluon** determined from valence-quark moments
DGLAP: All orders evolution

Implication 1:

\[ \langle x^n(\zeta_f) \rangle_q = \exp \left( -\frac{\gamma_{qq}^{(n)}}{4\pi} S(\zeta_H, \zeta_f) \right) \langle x^n(\zeta_H) \rangle_q = \langle x^n(\zeta_H) \rangle_q \left( \frac{\langle x(\zeta_f) \rangle_q}{\langle x(\zeta_H) \rangle_q} \right)^{\gamma_{qq}^{(n)}} \gamma_{qq}^{(1)} \]

Information on the charge is here

- Can \textbf{jump} from one scale to another (both ways)
- Natural connection with the \textit{hadron scale}.

Implication 2:

- Sea and \textbf{gluon} determined from valence-quark moments
- \textbf{Asymptotic} (massless) limits are evident.
DGLAP: All orders evolution

Implication 1:

\[ \langle x^n(\zeta_f) \rangle_q = \exp \left( -\frac{\gamma^{(n)}_{qq}}{4\pi} S(\zeta_H, \zeta_f) \right) \langle x^n(\zeta_H) \rangle_q = \langle x^n(\zeta_H) \rangle_q \left( \frac{\langle x(\zeta_f) \rangle_q}{\langle x(\zeta_H) \rangle_q} \right)^{\gamma^{(n)}_{qq}/\gamma^{(1)}_{qq}} \]

Information on the charge is here

- Can jump from one scale to another (both ways)
- Natural connection with the hadron scale.

Implication 2:

\[ \langle 2x(\zeta_f) \rangle_q = \exp \left( -\frac{8}{9\pi} S(\zeta_H, \zeta_f) \right), \quad q = u, \bar{d}; \]

\[ \langle x(\zeta_f) \rangle_{\text{sea}} = \langle x(\zeta_f) \rangle_{q+\bar{q}} - (\langle x(\zeta_f) \rangle_u + \langle x(\zeta_f) \rangle_{\bar{d}}), \]

\[ = \frac{3}{7} + \frac{4}{7} \langle 2x(\zeta_f) \rangle_{u}^{7/4} - \langle 2x(\zeta_f) \rangle_{u} \]

\[ \langle x(\zeta_f) \rangle_{g} = \frac{4}{7} \left( 1 - \langle 2x(\zeta_f) \rangle_{u}^{7/4} \right); \]

- Sea and gluon determined from valence-quark moments
- Asymptotic (massless) limits are evident.
- And, of course, the momentum sum rule:

\[ \langle 2x(\zeta_f) \rangle_q + \langle x(\zeta_f) \rangle_{\text{sea}} + \langle x(\zeta_f) \rangle_{g} = 1 \]
DGLAP: All orders evolution

Implication 1:

\[
\langle x^n(\zeta_f) \rangle_q = \exp \left( -\frac{\gamma_{qq}^{(n)}}{4\pi} S(\zeta_H, \zeta_f) \right) \langle x^n(\zeta_H) \rangle_q = \langle x^n(\zeta_H) \rangle_q \left( \frac{\langle x(\zeta_f) \rangle_q}{\langle x(\zeta_H) \rangle_q} \right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}
\]

Information on the charge is here

- Can jump from one scale to the another (even downwards)
- Natural connection with the hadron scale.

Implication 3: Recurrence relation

\[
\langle x^{2n+1} \rangle_{u\pi}^{\zeta_u} = \frac{(2x \langle x \rangle_{u\pi}^{\zeta_u}) \gamma_0^{2n+1}/\gamma_0^1}{2(n + 1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n + 1)}{j} \langle x^j \rangle_{u\pi}^{\zeta_u} (2x \langle x \rangle_{u\pi}^{\zeta_u}) - \gamma_j^0/\gamma_0^1.
\]

- Since isospin symmetry limit implies:
  \[
  q(x; \zeta_H) = q(1 - x; \zeta_H)
  \]

- Odd moments can be expressed in terms of previous even moments.

- Thus arriving at the recurrence relation on the left.
DGLAP: All orders evolution

Implication 1:

\[ \langle x^n(\zeta_f) \rangle_q = \exp \left( -\frac{\gamma_{qq}^{(n)}}{4\pi} S(\zeta_H, \zeta_f) \right) \langle x^n(\zeta_H) \rangle_q = \langle x^n(\zeta_H) \rangle_q \left( \frac{\langle x(\zeta_f) \rangle_q}{\langle x(\zeta_H) \rangle_q} \right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}} \]

Information on the charge is here

- Can jump from one scale to the another (even downwards)
- Natural connection with the hadron scale.

Implication 3: Recurrence relation

\[ \langle x^{2n+1} \rangle_{u_{\pi}} = \frac{\langle 2x \rangle_{u_{\pi}} \gamma_0^{2n+1}/\gamma_0^1}{2(n+1)} \]

\[ \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j \rangle_{u_{\pi}} (\langle 2x \rangle_{u_{\pi}} - \gamma_0^j/\gamma_0^1) \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>Lattice input</th>
<th>Recurrence relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.230(3)(7)</td>
<td>0.230</td>
</tr>
<tr>
<td>2</td>
<td>0.087(5)(8)</td>
<td>0.087</td>
</tr>
<tr>
<td>3</td>
<td>0.041(5)(9)</td>
<td>0.041</td>
</tr>
<tr>
<td>4</td>
<td>0.023(5)(6)</td>
<td>0.023</td>
</tr>
<tr>
<td>5</td>
<td>0.014(4)(5)</td>
<td>0.015</td>
</tr>
<tr>
<td>6</td>
<td>0.009(3)(3)</td>
<td>0.009</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>0.0078</td>
</tr>
</tbody>
</table>
Reverse engineering the PDF data
Let us assume the data can be parameterized with a certain functional form, i.e.:

\[ u^\pi(x; [\alpha_i]; \zeta) = n_\zeta x^{\alpha_i^\zeta} (1 - x)^{\alpha_2^\zeta} (1 + \alpha_3^\zeta x^2) \]

\[ \{\alpha_i^\zeta | i = 1, 2, 3\} \]

Then, we proceed as follows:

1) **Determine** the best values \( \alpha_i \) via least-squares fit to the data.

2) **Generate** new values \( \alpha_i \), distributed randomly around the best fit.

3) **Using** the latter set, evaluate:

\[ \chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l; [\alpha_i]; \zeta) - u_j)^2}{\delta_l^2} \]

4) **Accept** a replica with probability:

\[ \mathcal{P} = \frac{P(\chi^2; d)}{P(\chi_0^2; d)}, \quad P(y; d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2} \]

5) **Evolve** back to \( \hat{\zeta}^H \) **Repeat** (2-5).
Applying this algorithm to the ASV data yields:

- The produced moments are compatible with a symmetric PDF at the hadronic scale.
- Not at all similar to those from SCI CSM

### CSM

<table>
<thead>
<tr>
<th>(x)</th>
<th>(q^T(x, \xi_5))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

### SCI (average)

- Not at all similar to those from SCI

Mean values (of moments) and errors

\[
\{0.5, 2.75144 \times 10^{-1}\}, \{0.299838, 0.00647945\}, \{0.199967, 0.00735448\}, \{0.142895, 0.0088623\}, \{0.107274, 0.006608759\}, \{0.0835168, 0.00532834\}, \{0.06668711, 0.0046596\}, \{0.0547511, 0.0039028\}, \{0.0456496, 0.00361041\}, \{0.0386394, 0.00328609\}\]
Applying this algorithm to the ASV data yields:

✔ The produced moments are compatible with a symmetric PDF at the hadronic scale.

Then, we can reconstruct the moments produced by each replica, using the single-parameter Ansatz:

\[ u^\pi(x; \zeta_H) = n_0 \ln(1 + x^2(1 - x)^2/\rho^2) \]
We can follow an analogous procedure to infer, based upon lattice data, how the hadronic scale PDF should look like.

Let us consider the list of lattice QCD moments:

<table>
<thead>
<tr>
<th>n</th>
<th>[61]</th>
<th>[62]</th>
<th>[63]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.254(03)</td>
<td>0.18(3)</td>
<td>0.23(3)(7)</td>
</tr>
<tr>
<td>2</td>
<td>0.094(12)</td>
<td>0.064(10)</td>
<td>0.087(05)(08)</td>
</tr>
<tr>
<td>3</td>
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<td>0.030(05)</td>
<td>0.041(05)(09)</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.023(05)(06)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Joo:2019bzr</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Sufian:2019bol</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Alexandrou:2021mmi</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Those verify the recurrence relation, thus being compatible with a symmetric PDF at $\zeta_H$.

While also falling within the physical bounds:

$$\frac{1}{2^n} \leq \langle x^n \rangle_{u_\pi} (\langle 2x \rangle_{u_\pi}^{\zeta})^{-\gamma_0^n / \gamma_0^1} \leq \frac{1}{1 + n}$$

Produced by:

- $q(x; \zeta_H) = \delta(x - 1/2)$
  - (infinitely heavy valence quarks)

- $q(x; \zeta_H) = 1$
  - (massless SCI case)
Pion PDF: Lattice Data

- We can follow an analogous procedure to infer, based upon lattice data, how the hadronic scale PDF should look like.

- Let us consider the list of lattice QCD moments:

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<tr>
<td>6</td>
<td>0.009(03)(03)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Those verify the recurrence relation, thus being compatible with a symmetric PDF at $\zeta_H$.

- While also falling within the physical bounds:

$$\frac{1}{2^n} \leq \langle x^n \rangle_{\zeta_H} (\langle 2x \rangle_{\zeta_H}^{1/\gamma_0} - \gamma_0^n / \gamma_0^1) \leq \frac{1}{1 + n}$$

Produced by

$$q(x; \zeta_H) = \delta(x - 1/2)$$

(infinitely heavy valence quarks)

Produced by

$$q(x; \zeta_H) = 1$$

(massless SCI case)

Cui:2022bxn
GPD from PDF and EFF
Setting the Stage

- Starting with a factorized LFWF, \( \psi_{P_u}^{\uparrow\downarrow}(x, k_\perp^2; \zeta_H) = \tilde{\psi}_{P_u}^{\uparrow\downarrow}(k_\perp^2) [u^P(x; \zeta_H)]^{1/2} \)

- The overlap representation for the GPD entails:

\[
H_P^u(x, \xi, t; \zeta_H) = \int \frac{d^2k_\perp}{16\pi^3} \psi_P^u(x_-, k_\perp^2; \zeta_H) \psi_P^u(x_+, k_\perp^2+; \zeta_H)
\]

\[
= \Theta(x_-) \sqrt{u^P(x_-; \zeta_H)u^P(x_+; \zeta_H)} \Phi_P(z; \zeta_H)
\]

- Where \( z = s_\perp^2 = -t(1-x)^2/(1-\xi^2)^2 \) and:

\[
\Phi_P^u(z; \zeta_H) = \int \frac{d^2k_\perp}{16\pi^3} \tilde{\psi}_P^u(k_\perp^2; \zeta_H) \tilde{\psi}_P^u((k_\perp - s_\perp)^2; \zeta_H)
\]

This one shall be obtained as described previously.

This dictates the off-forward behavior of the GPD.

... will be driven by the electromagnetic form factor.
Setting the Stage

➢ Recall a GPD arising from a factorised LFWF adopts the form:

\[ H^u_{\pi}(x, \xi, -\Delta^2; \zeta_H) = \theta(x_-) \sqrt{u^{\pi}(x_-; \zeta_H)u^{\pi}(x_+; \zeta_H)} \Phi^{\pi}(z^2; \zeta_H) \]

\[ u^{\pi}(x; \zeta_H) = n_0 \ln(1 + x^2(1 - x)^2/\rho^2) \]

➢ The empirical data on PDF to contrast with:
  - ASV analysis.
  - MF resummation.
  - Lattice QCD moments.

For references, see:

Cui:2022bxn
Cui:2021mom
Setting the Stage

We thus employ a 3-parameter model for the GPD:

$$F_\pi(t) = \int_0^1 dx \ u_\pi(x; \zeta_H) \Phi_\pi(z; \zeta_H)$$

$$H^u_\pi(x, \xi, -\Delta^2; \zeta_H) = \theta(x_-) \sqrt{u_\pi(x_-; \zeta_H) u_\pi(x_+; \zeta_H)} \Phi_\pi(z^2; \zeta_H)$$

$$\lambda = \beta - \frac{r_\pi^2}{6 \langle x^2 \rangle_{u_\pi}}$$

$$\Phi_\pi(y; \zeta_H) = \frac{1 + \lambda y}{1 + \beta y + \gamma^2 y^2}$$

The empirical data on EFF:

- JLab data.
- Charge radius: \( r_\pi = 0.64(2) \text{ fm} \)
  - SPM extraction
  - Conservative “Gaussian” error
- Given \( r_\pi \), low-\( Q^2 \) data is redundant.
The Algorithm

1. For the chosen PDF data set, generate a replica. The replica would be accepted following the aforementioned chi-2 criteria.

2. After acceptance, evolve it to the hadronic scale using several Mellin moments. The de-evolved PDF shall be reconstructed using the functional form:

$$H_u^\pi(x, \xi, -\Delta^2; \zeta_H) = \theta(x_-) \sqrt{u^\pi(x_-; \zeta_H)u^\pi(x_+; \zeta_H)} \Phi^\pi(z^2; \zeta_H)$$

3. Store both the value $\rho_i$ and the probability of acceptance $P(\rho_i)$. 

$$u^\pi(x; \zeta_H) = n_0 \ln(1 + x^2(1 - x)^2 / \rho^2)$$
The Algorithm

4. Keeping the selected PDF, we now contrast $\Phi$ with the EFF data, via:

$$H^u_\pi(x, \xi, -\Delta^2; \zeta_H) = \theta(x_-) \sqrt{u^\pi(x_-; \zeta_H)u^\pi(x_+; \zeta_H)} \Phi^\pi(z^2; \zeta_H)$$

5. Employing a chi-2 criteria, we compute the probability of acceptance $P(\Phi_i|\rho_i)$.

6. The GPD is accepted with probability $P(\Phi_i|\rho_i)P(\rho_i)$.

REPEAT
Numerical Results

SHOW ME RESULTS
Applying this procedure, from the pion PDF and EFF empirical data, one gets the GPDs:

\[ H^u_\pi(x, \xi, -\Delta^2; \zeta_H) = \theta(x) \sqrt{u^\pi(x^-; \zeta_H) u^\pi(x^+; \zeta_H)} \Phi^\pi(z^2; \zeta_H) \]
The PDFs agree within errors, but...

- **Lattice QCD cannot distinguish** between ASV, MF or the parton-like profiles.

\[
H^u_\pi(x, \xi, -\Delta^2; \zeta_H) = \theta(x_-) \sqrt{u^\pi(x_; \zeta_H) u^\pi(x_+; \zeta_H)} \Phi^\pi(z^2; \zeta_H)
\]
Pion EFF

For the EFF, we essentially arrive at the same output.

- PTIR model faithfully reproduces the Data-Driven result

\[
H^u_{\pi}(x, \xi, -\Delta^2; \zeta_H) = \theta(x_-) \sqrt{u^\pi(x_-; \zeta_H) u^\pi(x_+; \zeta_H)} \Phi^\pi(z^2; \zeta_H)
\]

<table>
<thead>
<tr>
<th>( r_\pi / \text{fm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASV</td>
</tr>
<tr>
<td>MF</td>
</tr>
<tr>
<td>lQCD</td>
</tr>
</tbody>
</table>
The first Mellin moment of the GPD yields the \textit{gravitational form factors}:

\[
\int_{-1}^{1} dx \, 2H_u^u(x, \xi, -\Delta^2; \zeta_H) = \theta_2^\pi(\Delta^2) - \xi^2 \theta_1^\pi(\Delta^2)
\]

\(\theta_1\) currently escapes our approach, but \(\theta_2\) is within reach:

\[
\theta_2^\pi(\Delta^2) = \int_{0}^{1} dx \, 2x H_\pi(x, \xi = 0, -\Delta^2)
\]

\(\theta_2\) is associated with the \textit{mass distribution}.

We found the \textbf{mass radii}: 

<table>
<thead>
<tr>
<th>(r_{\pi}^{\theta_2})</th>
<th>ASV</th>
<th>MF</th>
<th>lQCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.518(16))</td>
<td>0.498(14)</td>
<td>0.512(21)</td>
<td></td>
</tr>
</tbody>
</table>

Producing: 

\[
\frac{r_{\pi}^{\theta_2}}{r_{\pi}} = 0.79(3)
\]

\textit{mass/charge ratio}
About Radii

\[ H_P^u(x, \xi, t; \zeta_H) = \theta(x_-) \left[ u_P(x_-; \zeta_H)u_P(x_+; \zeta_H) \right]^{1/2} \Phi_P(z; \zeta_H) \]

- In the **factorized** models:
  \[ \frac{\partial^n}{\partial z^n} \Phi_P^u(z; \zeta_H) \bigg|_{z=0} = \frac{1}{\langle x^{2n} \rangle_{\zeta_H}} \frac{d^n F_P^u(\Delta^2)}{d(\Delta^2)^n} \bigg|_{\Delta^2=0} \]
  \[ \frac{\partial}{\partial z} \Phi_P^u(z; \zeta_H) \bigg|_{z=0} = -\frac{r_P^2}{4x_P^2(\zeta_H)} \]
  \[ \frac{\partial}{\partial z} \Phi_P^u(z; \zeta_H) \bigg|_{z=0} = (1 - d_P) \frac{\partial}{\partial z} \Phi_P^u(z; \zeta_H) \bigg|_{z=0} \]

- Therefore, the **mass radius**:
  \[ r_{P_u}^{\theta_2^2} = \frac{3r_P^2}{2\chi_p^2} \langle x^2(1-x) \rangle_{\zeta_H} \]
  \[ r_{P_h}^{\theta_2^2} = \frac{3r_P^2}{2\chi_p^2} (1 - d_P) \langle x^2(1-x) \rangle_{\zeta_H} \]

\[ \left( \frac{\gamma_{\pi}^E}{\gamma_{\pi}} \right)^2 = \frac{\langle x^2(1-x) \rangle_{\zeta_H}^q}{\langle x^2 \rangle_{\zeta_H}^q} \approx \left( \frac{4}{5} \right)^2 \]

Determined from PDF moments!

**Asymmetry term = 0 for pion**
IPS GPDs

- Impact parameter space GPDs are defined as:

\[ u^\pi(x, b_\perp^2; \zeta_H) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(|b_\perp|\Delta) H^u(x, 0, -\Delta^2; \zeta_H) \]

- Such that, in factorized models:

\[ u^\pi(x, b_\perp^2; \zeta_H) = \frac{u^\pi(x; \zeta_H)}{(1-x)^2} \Psi^\pi \left( \frac{|b_\perp|}{1-x}; \zeta_H \right) \]

- The location and values of the maxima:

<table>
<thead>
<tr>
<th></th>
<th>(x)</th>
<th>(b_\perp/r_\pi)</th>
<th>(i_\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSM [57]</td>
<td>0.88</td>
<td>0.13</td>
<td>3.29</td>
</tr>
<tr>
<td>ASV</td>
<td>0.89(2)</td>
<td>0.10(2)</td>
<td>3.21(30)</td>
</tr>
<tr>
<td>MF</td>
<td>0.95(1)</td>
<td>0.05(1)</td>
<td>4.58(50)</td>
</tr>
<tr>
<td>lQCD</td>
<td>0.91(6)</td>
<td>0.08(5)</td>
<td>4.04(1.67)</td>
</tr>
</tbody>
</table>

- Furthermore:

\[ \langle b_{\perp}^2(\zeta_H) \rangle^\pi_u = \frac{2}{3} r_{\pi}^2 = \langle b_{\perp}^2(\zeta_H) \rangle^\pi_d \]
Evolved IPS-GPD: Pion Case

\[ u^p(x, b_\perp^2; \zeta_H) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta) H^p_\perp(x, 0, -\Delta^2; \zeta_H) \]

- Peaks **broaden** and **maximum drifts**:
  \[ \max: 3.29 \to 0.55 \]
  \[ (|x|, b) = (0.88, 0.13) \to (0.47, 0.23) \]

- **Likelihood** of finding a parton with LF momentum \( x \) at transverse position \( b \)
Evolved IPS-GPD: Pion Case

\[ u^P(x, b^2; \zeta_H) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b\Delta) H^P_\perp(x, 0, -\Delta^2; \zeta_H) \]

- Peaks **broaden** and **maximum drifts**:

  \[
  \max : \ 3.29 \rightarrow 0.55 \quad (|x|, b) = (0.88, 0.13) \rightarrow (0.47, 0.23)
  \]

- **Likelihood** of finding a parton with LF momentum \( x \) at transverse position \( b \)
Distributions: Mass & Charge

- **Density** distributions are obtained by integrating the IPS-GPD.

\[
\rho_\{F,0\}^\pi(|b_\perp|) = \int_{-1}^{1} dx \{1, 2x\} u_\pi(x, b_\perp^2; \zeta_H) = \int_0^{\infty} \frac{d\Delta}{2\pi} \Delta J_0(|b_\perp|\Delta) \{F_\pi(\Delta^2), \theta_2(\Delta^2)\}
\]

- The narrower curves correspond to the mass distribution, demonstrating that: **Charge** effects span over a larger domain than mass effects.
Conclusions and Scope
Conclusions and Scope

➢ Question:
From the empirical knowledge of 1-dimensional distributions (EFF and PDF), can we obtain the 3-dimensional GPD?

\[ u_0^\pi(x; \zeta_{e/l}), \quad F_\pi(\Delta^2) \rightarrow H_\pi(x, \xi, -\Delta^2; \zeta) \]
Conclusions and Scope

➢ Question:
From the empirical knowledge of 1-dimensional distributions (EFF and PDF), can we obtain the 3-dimensional GPD?

\[ u^\pi(x; \zeta_{e/l}), \quad F_\pi(\Delta^2) \quad \rightarrow \quad H_\pi(x, \xi, -\Delta^2; \zeta) \]

➢ Partial Answer:

DGLAP GPD

\[ H^u_\pi(x, \xi, -\Delta^2; \zeta_H) = \theta(x_-) \sqrt{u^\pi(x_-; \zeta_H)} u^\pi(x_+; \zeta_H) \Phi^\pi(z^2; \zeta_H) \]

Factorized LFWF

\[ H^u_\pi(x, \xi, -\Delta^2; \zeta_H) \sim \int_{k_L} \psi^* \psi \]

All orders evolution

\[ u^\pi(x; \zeta_{e/l}) \]

Sum rule

\[ F_\pi(\Delta^2) = \int_0^1 dx \ H^u_\pi(x, 0, -\Delta^2) \]
Conclusions and Scope

➢ Question:
From the empirical knowledge of 1-dimensional distributions (EFF and PDF), can we obtain the 3-dimensional GPD?

➢ Answer:
Yes, but so far we are limited to the DGLAP region.

➔ Nevertheless...

➢ Charge, Mass and Spatial distributions are already within the reach of DGLAP domain.

➢ In this domain, we can also evolve the GPDs to disentangle valence, glue and sea content.

➢ Sophisticated covariant extensions to the ERBL domain are known.

(notably, the preliminary CSM computation of the GFFs, shows agreement with the Data-Driven result)
Even though analogous empirical information on the kaon is scarce, we can perform an analogous exploration of the kaon.
Conclusions and Scope

- With the EFF determined from experimental data, and further validated by a completely independent observable (the PDF), we can safely rely on the produced ensemble to derive other quantities.

- Such is the case of the pion-box contribution to the muon's anomalous magnetic moment:

\[
\alpha_{\mu}^{\text{p-box}} = \frac{\alpha_{\text{em}}^3}{432\pi^2} \sum_{i}^{12} T_i(Q_1, Q_2, \tau) \Pi_{i}^{\text{p-box}}(Q_1, Q_2, \tau),
\]

\[
\Pi_{i}^{\text{p-box}}(Q_1^2, Q_2^2, Q_3^2) = F_{\Pi}(Q_1^2) F_{\Pi}(Q_2^2) F_{\Pi}(Q_3^2) \times \mathcal{I}_i
\]

- An exploratory calculation yields: (with P. Roig)

\[
a_{\mu}^{\pi-\text{box}} = -(15.1^{+0.5}_{-0.3}) \times 10^{-11}
\]

In fair agreement with modern estimates.

Eichmann:2019bqf
Miramontes:2021exi

\[
H_{\pi}^{u}(x, \xi, -\Delta^2; \zeta_H) = \theta(x-) \sqrt{u_\pi(x_-; \zeta_H)} u_\pi(x_+; \zeta_H) \Phi^\pi(z^2; \zeta_H)
\]
Applying this algorithm to the original data yields:

✓ The produced moments are compatible with a symmetric PDF at the hadronic scale.

✗ But also exhibit agreement with the SCI results.

\[ q_{SCI}(x; \zeta_H) \approx 1 \]

Mean values (of moments) and errors, \( \zeta_H \)

\[
\{ \{0.5, 0.25187 \times 10^{-17}\}, \{0.331527, 0.00803273\}, \{0.247615, 0.0110893\}, \\
\{0.19784, 0.0121977\}, \{0.165066, 0.0124911\}, \{0.141928, 0.0124198\}, \\
\{0.124755, 0.0121811\}, \{0.11521, 0.0116068\}, \{0.101021, 0.0115275\}, \\
\{0.0924926, 0.0111824\}, \{0.085431, 0.0108454\}, \{0.0794897, 0.0105214\}, \\
\{0.0744232, 0.0102142\}, \{0.0700521, 0.00992435\}, \{0.0662432, 0.00965182\}\}
\]

Mean values (of moments) and errors, \( \zeta_H \)

\[
\{ \{0.5, 0.332885, 0.249327, 0.199231, 0.165865, 0.142056, 0.124215, 0.11035, \\
0.0992657, 0.090203, 0.0826552, 0.0762721, 0.0708335, 0.0660651, 0.0619225\}\}
\]

Thus, given the expectation

\[ u^\pi(x; \zeta) \approx \frac{1}{(1 - x)^{\beta}} = 2 + \gamma(\zeta) \]

We shall discard this for the upcoming construction of the valence quark GPD.