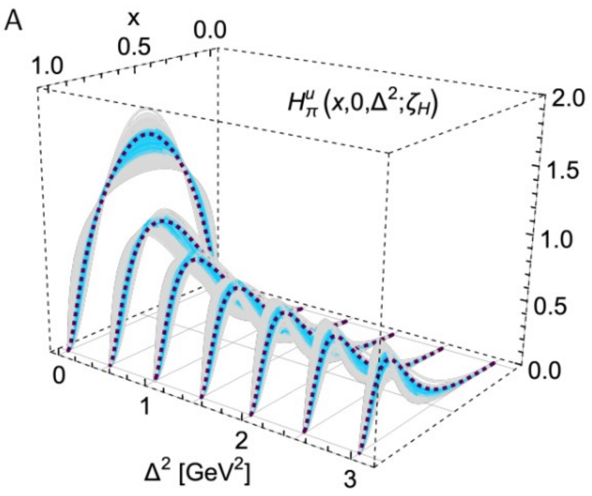
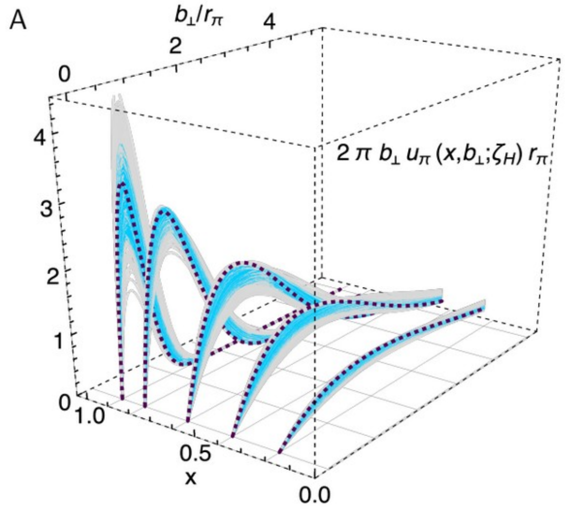
From 1D distributions to GPDs

Khépani Raya Montaño





Universidad de Huelva

POETIC 2023

May 2-6, 2023. Sao Paulo (Brasil)

QCD: Basic Facts

- QCD is characterized by two **emergent** phenomena: confinement and dynamical generation of mass (DGM).

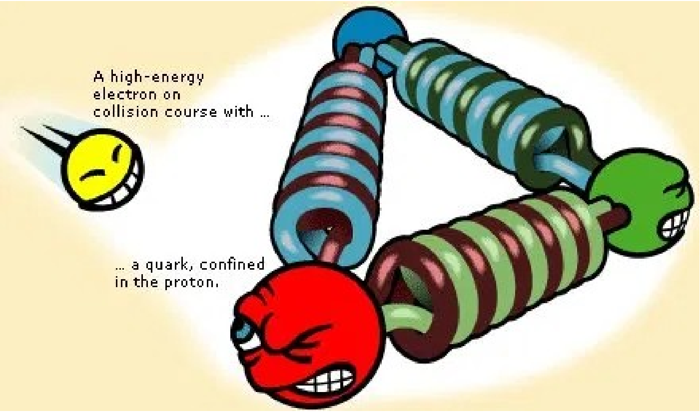
$$\mathcal{L}_{\rm QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j [\gamma_\mu D_\mu + m_j] q_j + \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu},$$

$$D_\mu = \partial_\mu + i g \tfrac{1}{2} \lambda^a A^a_\mu ,$$

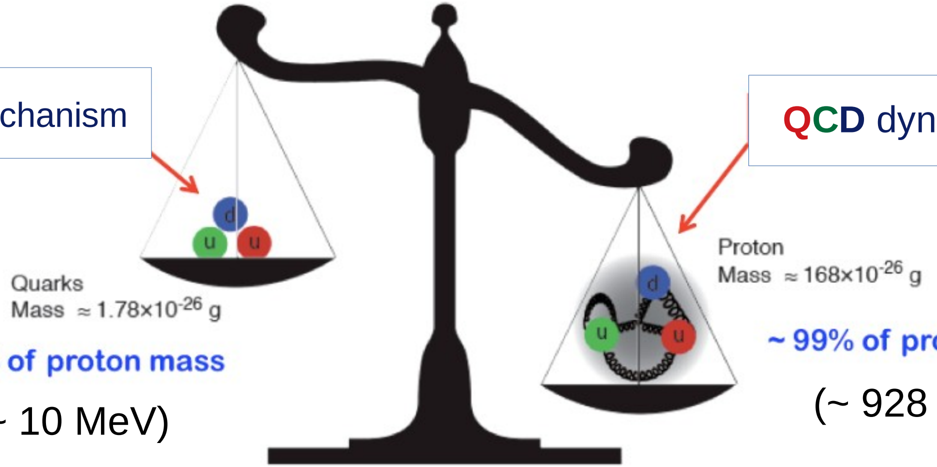
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - \underline{g f^{abc} A^b_\mu A^c_\nu},$$

- Quarks and gluons not *isolated* in nature.
  - Formation of colorless bound states: "**Hadrons**"
  - **1-fm scale** size of hadrons?

- Emergence of hadron masses (EHM) from QCD **dynamics**



Higgs mechanism



QCD dynamics

~ 1% of proton mass

(~ 10 MeV)

~ 99% of proton mass

(~ 928 MeV)

QCD: Basic Facts

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$$\mathcal{L}_{QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j[\gamma_\mu D_\mu + m_j]q_j + \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},$$
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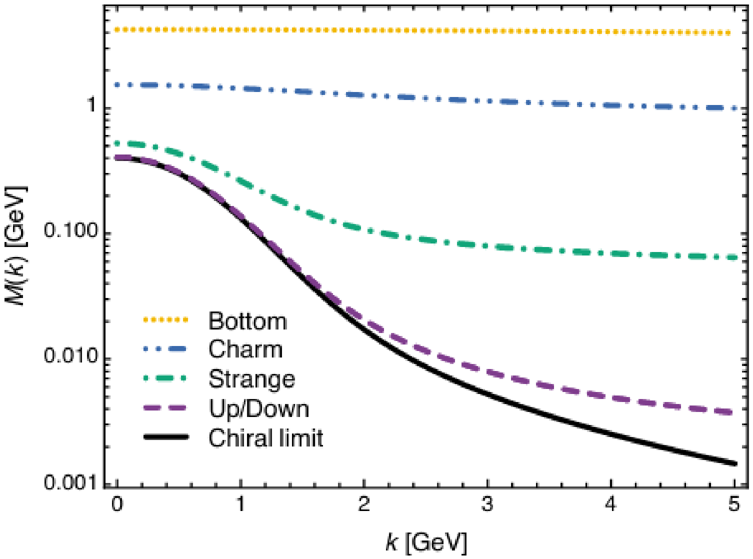
Can we trace them down to fundamental d.o.f?

Dynamical masses

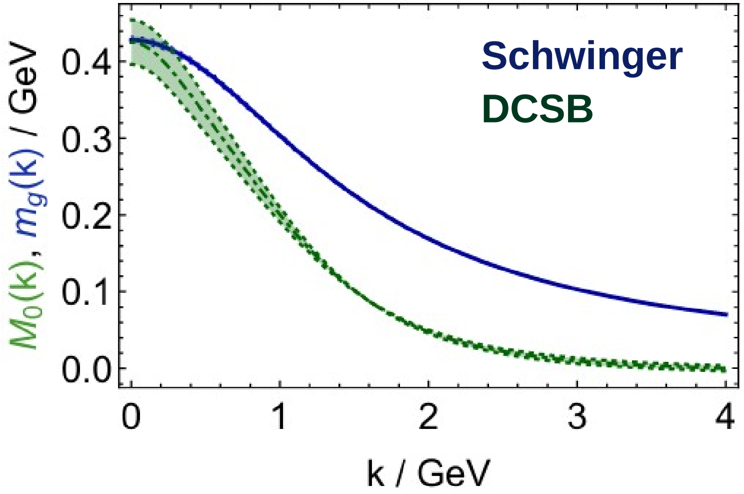
(**D**ynamical **C**hiral **S**ymmetry **B**reaking)



"Higgs" masses

$$S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma\cdot p + \mathbf{M_f(p^2)})$$

- Emergence of hadron masses (EHM) from QCD **dynamics**



Gluon and quark *running masses*

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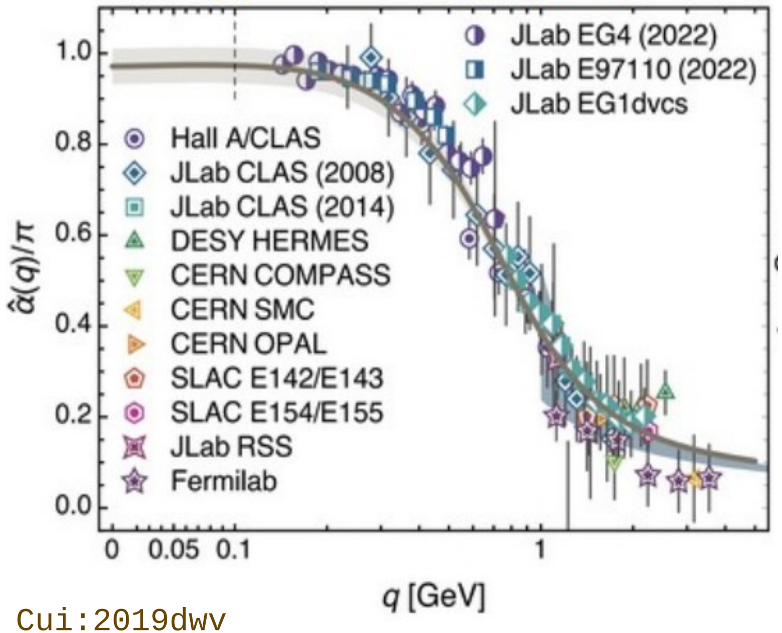
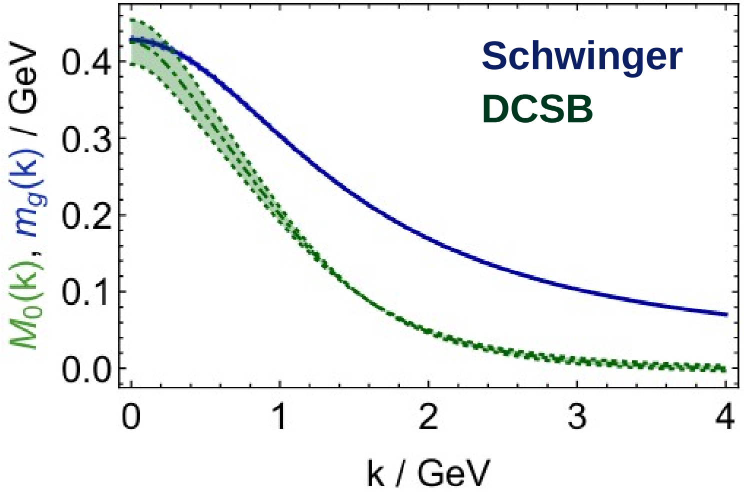
- Emergence of hadron masses (EHM) from QCD **dynamics**

(figure: D. Binosi's courtesy!)



Cui:2019dwv

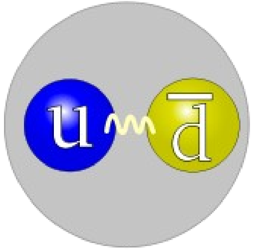


Gluon and quark *running masses*

Pion **Structure**

Pion

- **“Two”** quark **bound-state**

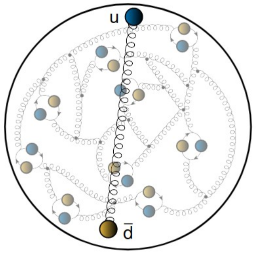


- **Archetype** of nuclear exchange **forces**
  - *Our favorite mediator*
- The **lightest** hadron in nature

Pion Structure

Pion

- **“Two”** quark **bound-state**

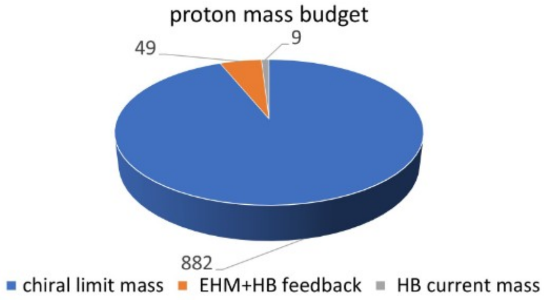


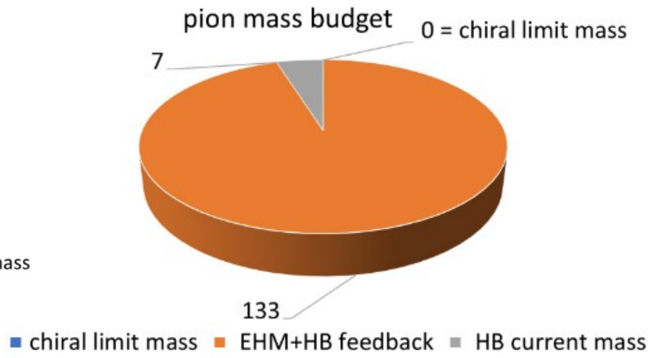
$$M_{u,d} \sim 310 \text{ MeV}$$

$$M_u + M_d \neq 140 \text{ MeV}$$

- **Archetype** of nuclear exchange **forces**
  - *Our favorite mediator*
- The **lightest** hadron in nature

➢ Unlike the proton, pion is **massless** in the absence of **Higgs** mass generation



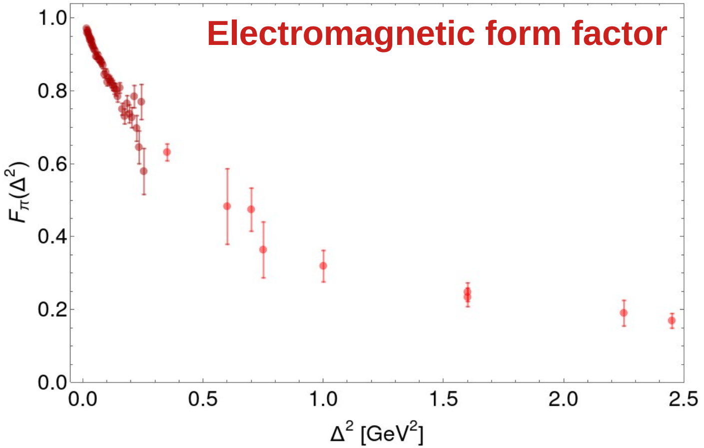


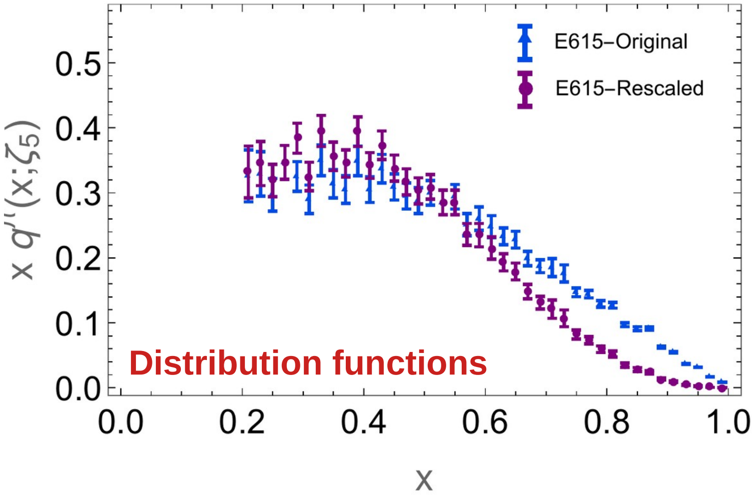
➔ Both a quark-antiquark **bound-state** and a **Golstone Boson**

➔ Its mere **existence** is connected with **mass** generation in the **SM**

Pion **Structure**

- The experimental access to the pion structure is via electromagnetic probes, yielding e.g.:





- Generalized parton distributions (**GPDs**) encode them **both** (and many more):

$$F_\pi(\Delta^2) = \int H_\pi^u(x,\xi,-\Delta^2;\zeta)\,, \quad u^\pi(x;\zeta) = H_\pi^u(x,0,0;\zeta)$$

→ But the experimental access and theoretical derivation is **far more complicated**.

Pion **GPD**

- **Question**:
  From 1-dimensional distributions (**EFF** and **PDF**), can we obtain the 3-dimensional **GPD**?

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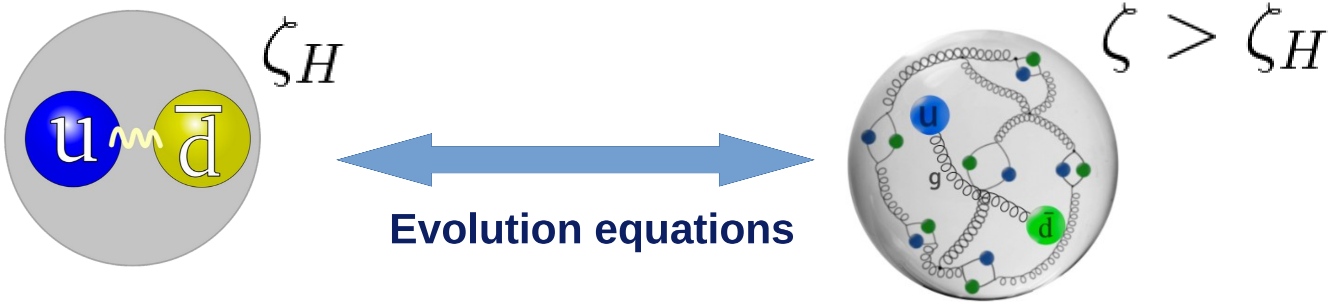
  From 1-dimensional distributions (**EFF** and **PDF**), can we obtain the 3-dimensional **GPD**?

- ~~Partial~~ **Answer**:
  Yes, we can. Under two premises:

  - *There exists at least one effective charge such that, when used to integrate the leading-order perturbative DGLAP equations, an evolution scheme for parton DFs is defined that is all-order exact.*



Thus, we can plainly connect the **parton** and **quasiparticle** picture in a well determined manner.

Pion GPD

- **Question**:
  From 1-dimensional distributions (**EFF** and **PDF**), can we obtain the 3-dimensional **GPD**?

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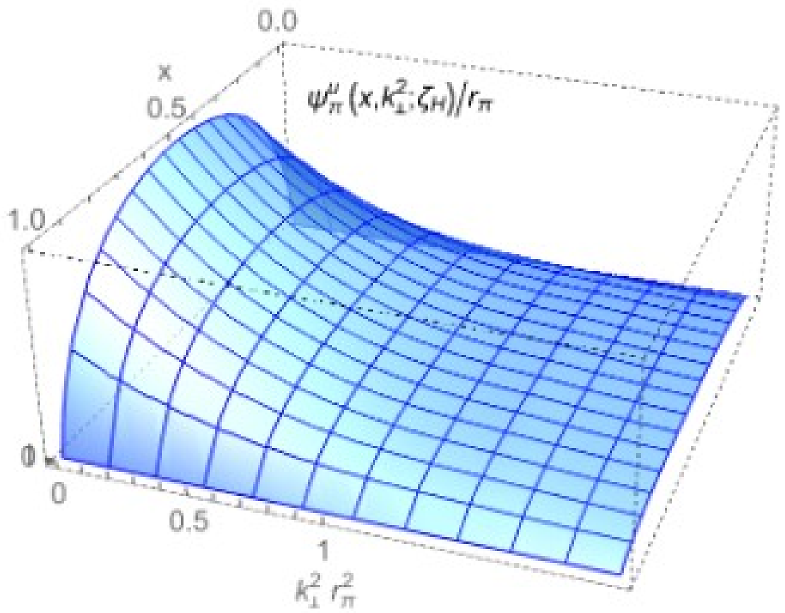
  - *A* ***factorised*** *representation of the pion light-front wave function (****LFWF****), from which the (DGLAP)* ***GPD*** *is derived, at the hadronic scale, is a sensible approximation.*

Overlap:
$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\psi_\pi^{u*}(x_-,k_{\perp-}^2;\zeta_H)\psi_\pi^u(x_+,k_{\perp+}^2;\zeta_H)$$

Factorization:
$$\psi_\pi^u(x,k_\perp^2;\zeta_H) = \tilde{\psi}_\pi^u(k_\perp^2)[u^\pi(x;\zeta_H)]^{1/2}$$

Raya:2021zrz
Raya:2022eqa

Light-front wave functions

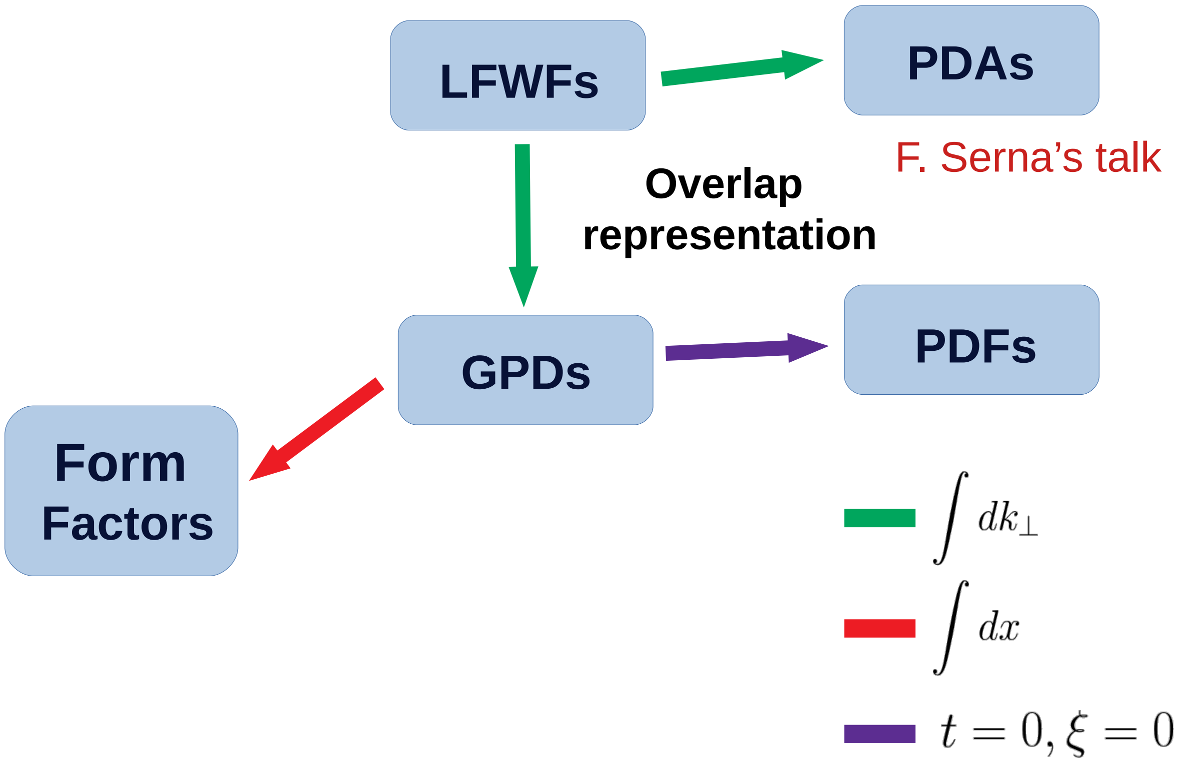


$$\psi_{\mathsf{P}}^{u}(x, k_\perp^2; \zeta)$$

"One ring to rule them all"

Light-front wave functions

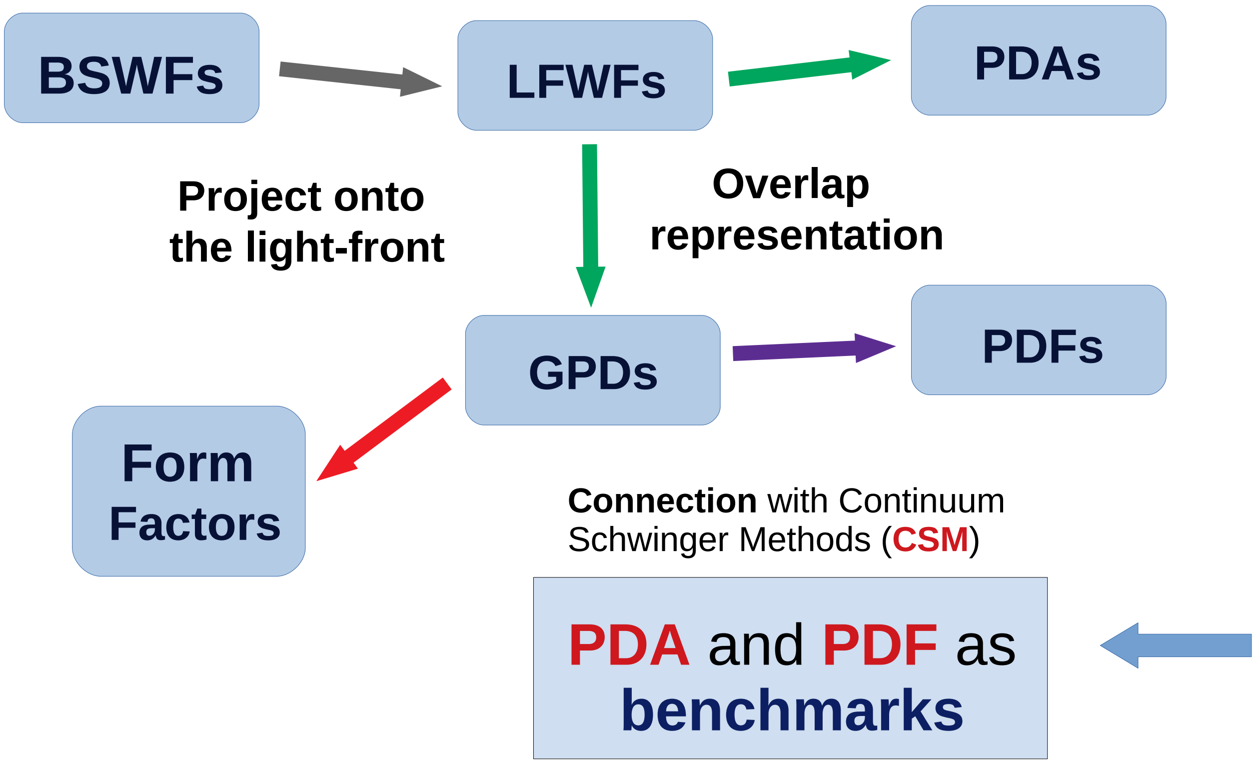
- **Goal:** get a **broad picture** of the pion and Kaon structure.



The idea:
Compute ***everything*** from the **LFWF.**

Light-front wave functions

- **Goal:** get a **broad picture** of the pion and Kaon structure.



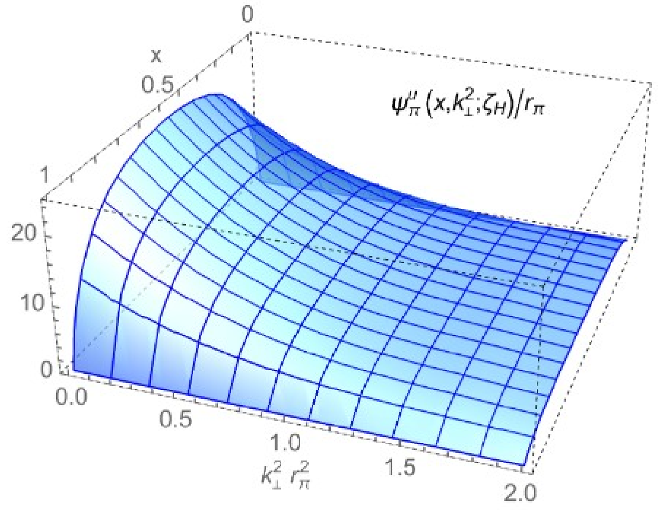
The idea:
Compute ***everything*** from the **LFWF.**

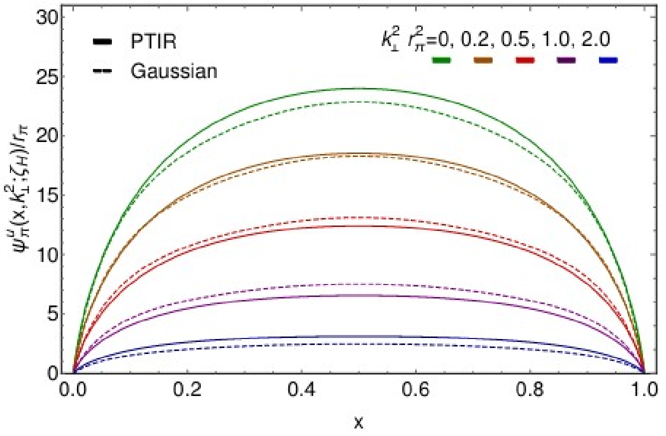
The inputs:
Solutions from quark **DSE** and meson **BSE**.

The alternative inputs:
Construct **BSWF** from realistic DSE ***predictions***.

A. Bashir's talk

About PTIRs
and LFWFs





LFWF: PTIR approach

- A perturbation theory integral representation for the **BSWF**:

$$n_K \chi_K(k_-^K, P_K) = \mathcal{M}(k,P) \int_{-1}^{1} dw\, \rho_K(w) \mathcal{D}(k,P)$$

(Kaon as example)

1 **2** **3**

1: Matrix structure (leading BSA):

$$\mathcal{M}(k;P_K) = -\gamma_5[\gamma \cdot P_K M_u + \gamma \cdot k(M_u - M_s) + \sigma_{\mu\nu} k_\mu P_{K\nu}]\,,$$

2: Spectral weight: Tightly connected with the meson properties.

3: Denominators: $\mathcal{D}(k;P_K) = \Delta(k^2, M_u^2)\Delta((k-P_K)^2, M_s^2)\hat{\Delta}(k_{\omega-1}^2, \Lambda_K^2)\,,$

where: $\Delta(s,t) = [s+t]^{-1}$, $\hat{\Delta}(s,t) = t\Delta(s,t)\,.$

Raya:2021zrz
Raya:2022eqa

LFWF: PTIR approach

- Recall the expression for the **LFWF**:

$$\psi_{\rm M}^{q}\left(x,k_\perp^2\right) = \mathrm{tr}\int_{dk_{||}} \delta_n^x(k_{\rm M})\gamma_5\gamma\cdot n\,\chi_{\rm M}(k_-,P) \qquad <x>_{\rm M}^q := \int_0^1 dx\, x^m \psi_{\rm M}^q(x,k_\perp^2)$$

- Algebraic manipulations yield:

+ Uniqueness of Mellin moments ➡

$$\Rightarrow \psi_{\rm M}^q(x,k_\perp) \sim \int dw\, \rho_{\rm M}(w)\cdots$$

✓ Compactness of this result is a merit of the AM.

- Thus, $\rho_{\rm M}(w)$ determines the profiles of, *e.g.* **PDA** and **PDF**: (it also works the **other way around**)

$$f_{\rm M}\phi_{\rm M}^q(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\psi_{\rm M}^q(x,k_\perp;\zeta_H)$$

$$q_{\rm M}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}|\psi_{\rm M}^q(x,k_\perp;\zeta_H)|^2$$

LFWF: PTIR approach

Raya:2021zrz
Raya:2022eqa

- More **explicitly**:

$$\psi_{\rm M}^{q}(x,k_\perp^2;\zeta_H) = 12\left[M_q(1-x)+M_{\bar h}x\right]X_{\rm P}(x;\sigma_\perp^2)$$

$$\sigma_\perp = k_\perp^2+\Omega_{\rm P}^2$$

$$X_{\rm M}(x;\sigma_\perp^2) = \left[\int_{-1}^{1-2x}dw\int_{1+\frac{2x}{w-1}}^{1}dv + \int_{1-2x}^{1}dw\int_{\frac{w-1+2x}{w+1}}^{1}dv\right]\frac{\rho_{\rm M}(w)}{n_{\rm M}}\frac{\Lambda_{\rm M}^2}{\sigma_\perp^2}$$

$$\Omega_{\rm M}^2 = vM_q^2+(1-v)\Lambda_{\rm P}^2 + (M_{\bar h}^2-M_q^2)\left(x-\tfrac{1}{2}[1-w][1-v]\right) + \left(x[x-1]+\tfrac{1}{4}[1-v][1-w^2]\right)m_{\rm M}^2$$

- Model **parameters**:

P	$m_{\rm P}$	M_u	M_h	$\Lambda_{\rm P}$	$b_0^{\rm P}$	$\omega_0^{\rm P}$	$v_{\rm P}$
π	0.14	0.31	M_u	M_u	0.275	1.23	0
K	0.49	0.31	$1.2M_u$	$3M_s$	0.1	0.625	0.41

$$\rho_{\rm P}(\omega) = \frac{1+\omega\, v_{\rm P}}{2a_{\rm P}b_0^{\rm P}}\left[\operatorname{sech}^2\left(\frac{\omega-\omega_0^{\rm P}}{2b_0^{\rm P}}\right)+\operatorname{sech}^2\left(\frac{\omega+\omega_0^{\rm P}}{2b_0^{\rm P}}\right)\right]$$

LFWF: Factorized case

- In the **chiral limit**, the **PTIR** reduces to:

$$\psi^q_{\mathrm{M}}(x,k_\perp^2;\zeta_H) \sim \tilde{f}(k_\perp)\phi^q_{\mathrm{M}}(x;\zeta_H) \sim f(k_\perp)[q_{\mathrm{M}}(x;\zeta_H)]^{1/2}$$

"Factorized model"

$$[\phi^q_{\mathrm{M}}(x;\zeta_H)]^2 \sim q_{\mathrm{M}}(x;\zeta_H)$$

- ✓ Sensible assumption as long as:

$m_{\mathrm{M}}^2 \approx 0$ (meson mass)

$M_{\bar{h}}^2 - M_q^2 \approx 0$ (*h*-antiquark, *q*-quark masses)

ζ_H

→ Produces **identical** results as PTIR model for pion

- Therefore:

$$\psi^q_{\mathrm{M}}(x,k_\perp^2;\zeta_H) = [q^{\mathrm{M}}(x;\zeta_H)]^{1/2}\left[4\sqrt{3}\pi\frac{M_q^3}{(k_\perp^2+M_q^2)^2}\right]$$

Single parameter!

$M_q \sim r_{\mathrm{M}}^{-1}$

(charge radius)

No need to determine the spectral weight !

LFWF: Factorized case

- In the **chiral limit**, the **PTIR** reduces to:

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"Factorized model"

$$[\phi_{\rm M}^{q}(x;\zeta_H)]^2 \sim q_{\rm M}(x;\zeta_H)$$

✓ Sensible assumption as long as:

$m_{\rm M}^2 \approx 0$ (meson mass) $\quad M_{\bar{h}}^2 - M_q^2 \approx 0$ (flavor asymmetry) $\quad \zeta_H$

→ Produces **identical** results as PTIR model for pion

- Therefore:

$$\psi_{\rm M}^{q}(x,k_\perp^2;\zeta_H) = [q^{\rm M}(x;\zeta_H)]^{1/2}\left[4\sqrt{3}\pi\frac{M_q^3}{(k_\perp^2+M_q^2)^2}\right]$$

Single parameter!

$$M_u = 0.31\ \text{GeV} \Leftrightarrow r_\pi = 0.66\ \text{fm}$$

Constituent mass value compatible with realistic estimations.

AtifSultan:2018end

LFWF: PTIR approach II

- A perturbation theory integral representation for the **BSWF**:

$$n_{\mathrm{M}}\chi_{\mathrm{M}}(k_-,P)=\underbrace{\mathcal{M}_{q,\bar{h}}(k,P)}_{1}\int_{-1}^{1}dw\,\underbrace{\tilde{\rho}_{\mathrm{M}}^{\nu}(w)}_{2}\underbrace{\mathcal{D}^{\nu}_{q,\bar{h}}(k,P)}_{3}$$

(Meson M)

1: Matrix structure (leading BSA):

$$\mathcal{M}_{q,\bar{h}}(k,P)\equiv-\gamma_5\left[M_q\gamma\cdot P+\gamma\cdot k(M_{\bar{h}}-M_q)+\sigma_{\mu\nu}k_\mu P_\nu-i\left(k\cdot p+M_qM_{\bar{h}}\right)\right]$$

2: Profile function: $\tilde{\rho}_{\mathrm{M}}^{\nu}(w)\equiv\rho_{\mathrm{M}}(w)\Lambda_w^{2\nu}\,.$

3: Denominators: $\mathcal{D}^{\nu}_{q,\bar{h}}(k,P)\equiv\Delta\left(k^2,M_q^2\right)\Delta\left(k^2_{w-1},\Lambda_w^2\right)^{\nu}\Delta\left(p^2,M_{\bar{h}}^2\right)\,.$

The crucial difference:

$$\Lambda^2(w):=\Lambda_w^2=M_q^2-\frac{1}{4}\left(1-w^2\right)m_{\mathrm{M}}^2+\frac{1}{2}\left(1-w\right)\left(M_{\bar{h}}^2-M_q^2\right)\,.$$

LFWF: PTIR approach II

A. Bashir's talk

- Then a series of algebraic results follows.

1. For the **BSWF**:

$$n_M \chi_M(k_-, P) = \mathcal{M}_{q,\bar{h}}(k, P) \int_0^1 d\alpha \mathcal{F}_M(\alpha, \sigma^{\nu+2}) \,, \quad \sigma = (k - \alpha P)^2 + \Lambda_{1-2\alpha}^2 ,$$

$$\mathcal{F}_M(\alpha, \sigma^{\nu+2}) = 2^\nu (\nu + 1) \Big[\int_{-1}^{1-2\alpha} dw \left(\frac{\alpha}{1-w} \right)^\nu + \int_{1-2\alpha}^{1} dw \left(\frac{1-\alpha}{1+w} \right)^\nu \Big] \frac{\tilde{\rho}_M^\nu(w)}{\sigma^{\nu+2}}$$

2. **LFWF** in terms of **PDA/PDF:**

$$f_M \phi_M^q(x; \zeta_H) = \int \frac{d^2 k_\perp}{16\pi^3} \psi_M^q(x, k_\perp; \zeta_H) \Rightarrow$$

$$\psi_M^q(x, k_\perp^2) = 16\pi^2 f_M \frac{\nu \Lambda_{1-2x}^{2\nu}}{(k_\perp^2 + \Lambda_{1-2x}^2)^{\nu+1}} \phi_M^q(x)$$

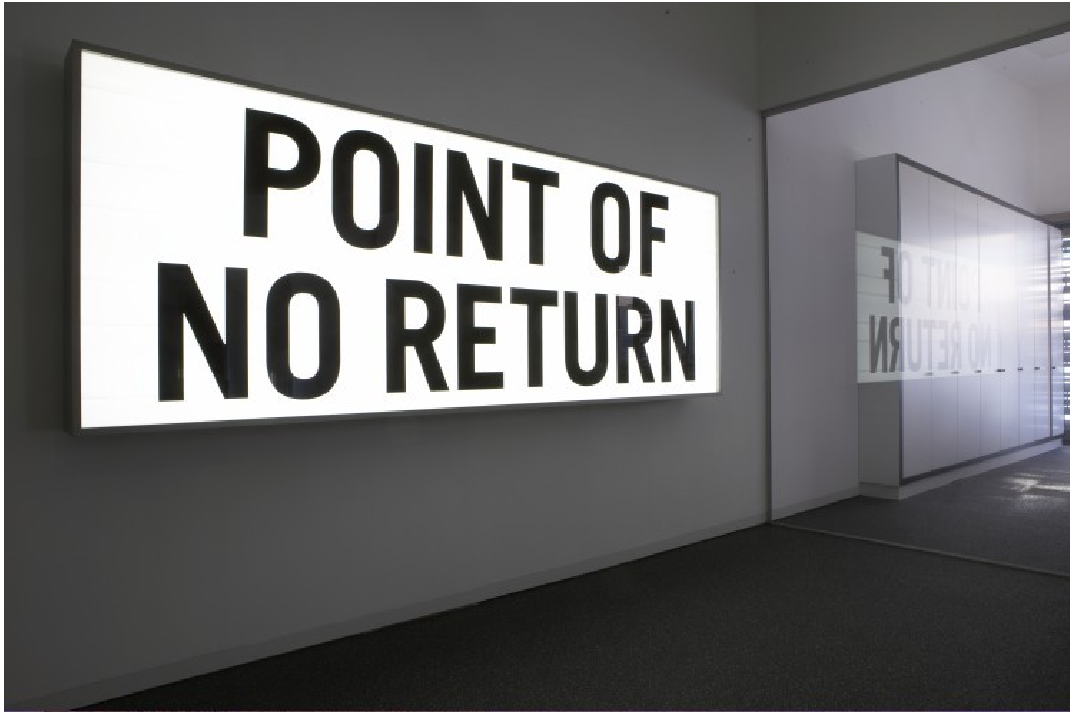
$$\Lambda_{1-2x}^2 = M_q^2 + x(M_{\bar{h}}^2 - M_q^2) - m_H^2 x(1-x)$$

Flavor asymmetry — Meson mass

Encodes the breaking of **factorization.**

- Completely factorized in the chiral limit.

Coming back
to the point...



Light-front wave functions

- Many **distributions** are related via the leading-twist light-front wave function (**LFWF**), e.g.:

Distribution amplitudes

$$f_{\mathrm{P}}\varphi^{u}_{\mathrm{P}}(x,\zeta_{\mathcal{H}}) = \int \frac{dk_{\perp}^{2}}{16\pi^{3}}\psi^{u}_{\mathrm{P}}\left(x,k_{\perp}^{2};\zeta_{\mathcal{H}}\right)$$

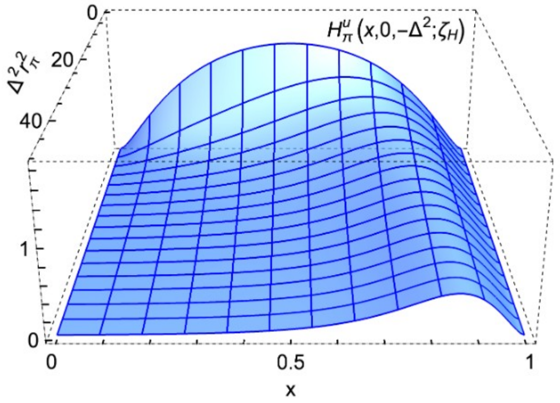
Distribution functions

$$u^{\mathrm{P}}(x;\zeta_{\mathcal{H}}) = \int \frac{d^{2}k_{\perp}}{16\pi^{3}}\left|\psi^{u}_{\mathrm{P}}\left(x,k_{\perp}^{2};\zeta_{\mathcal{H}}\right)\right|^{2}$$

- In the **DGLAP** kinematic domain, this is also the case of the valence-quark **GPD**:

$$H^{u}_{\mathrm{P}}(x,\xi,t;\zeta_{\mathcal{H}}) = \int \frac{d^{2}k_{\perp}}{16\pi^{3}}\psi^{u*}_{\mathrm{P}}\left(x_{-},k_{\perp -}^{2};\zeta_{\mathcal{H}}\right)\psi^{u}_{\mathrm{P}}\left(x_{+},k_{\perp +}^{2};\zeta_{\mathcal{H}}\right)$$

$$x_{\mp} = (x\mp\xi)/(1\mp\xi)\,,\; t=-\Delta^{2}$$
$$k_{\perp\mp} = k_{\perp} \pm (\Delta_{\perp}/2)(1-x)/(1\mp\xi)$$



- ✔ The overlap approach guarantees the **positivity** of the **GPD**.
- It is, in principle, limited to the **DGLAP** kinematic region. $|x| \geq |\xi|$
- ✔ Nonetheless, it can be exteded to the **ERBL** domain.

Chouika:2017rzs
Chavez:2021koz

Light-front wave functions

Albino:2022gzs
Raya:2021zrz
Raya:2022eqa

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Distribution functions

$$u^{\mathsf{P}}(x;\zeta_{\mathcal{H}}) = \int \frac{d^2k_\perp}{16\pi^3}\left|\psi^u_{\mathsf{P}}\left(x,k_\perp^2;\zeta_{\mathcal{H}}\right)\right|^2$$

- This connection already **suggests** that:

$$u^{\mathbf{P}}(x;\zeta_H) \sim \left[\varphi^u_{\mathbf{P}}(x;\zeta_H)\right]^2$$

is a fair approximation, implying:

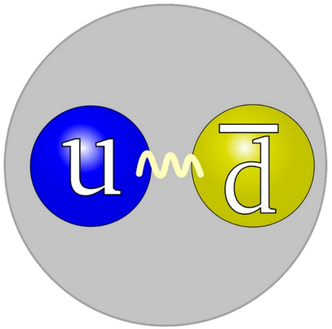
$$\psi^{\uparrow\downarrow}_{\mathbf{P}_u}(x,k_\perp^2;\zeta_H) = \tilde{\psi}^{\mathbf{u}}_{\mathbf{P}_u}(k_\perp^2)\left[u^{\mathbf{P}}(x;\zeta_H)\right]^{1/2}$$

- In fact, <u>we have learned</u> that *x*-*k* crossed terms are weighted by: $M_{\mathbf{P}}^2,\;\; M_h^2 - M_q^2$

➔ So a **factorized Ansatz** should be sensible for the **pion,** implying:

$$H^u_{\mathsf{P}}(x,\xi,t;\zeta_{\mathcal{H}}) = \Theta(x_-)\sqrt{u^{\mathbf{P}}(x_-;\zeta_H)u^{\mathbf{P}}(x_+;\zeta_H)}\,\Phi_{\mathbf{P}}(z;\zeta_H) \qquad z = \frac{(1-x)^2}{(1-\xi^2)^2}\Delta_\perp^2$$

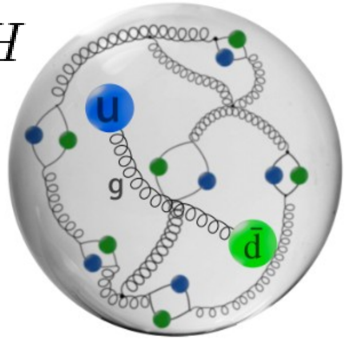
PARTON DISTRIBUTIONS



ζ_H

Resolution Scale

$\zeta > \zeta_H$



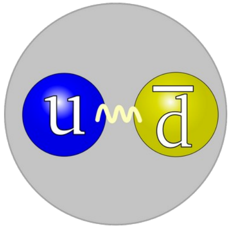
- **Fully-dressed valence quarks**

  (quasiparticles)

- **Unveiling of glue and sea d.o.f**

  (partons)

Pion PDF: hadronic scale



$(M_u = M_d)$

- **Fully-dressed valence quarks** (quasiparticles)

ζ_H : hadronic scale

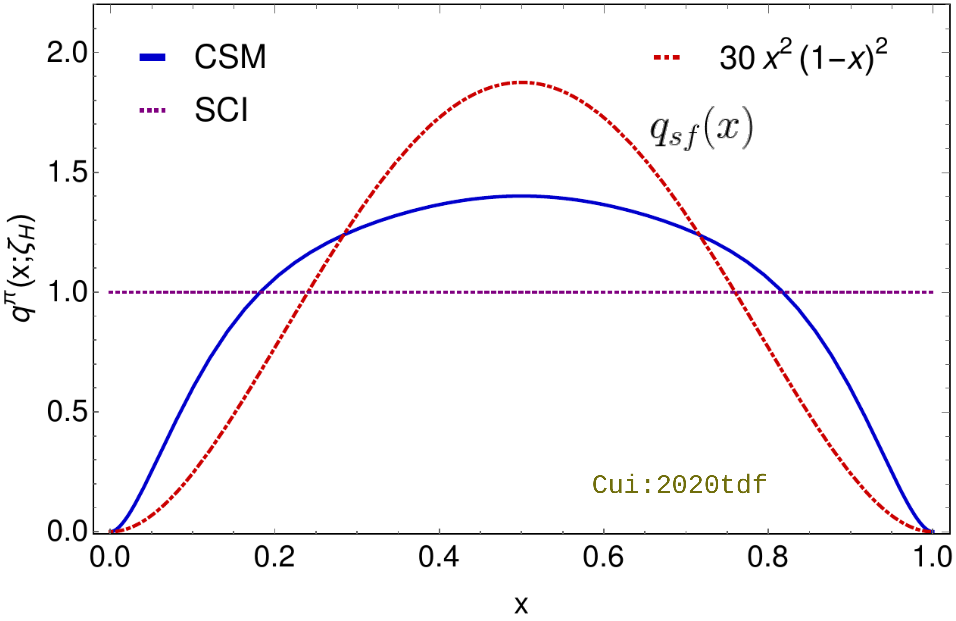
- At this scale, **all properties** of the hadron are contained within their valence quarks.

  - Equally massive quarks means a **50-50** share of the total momentum:

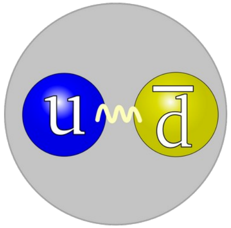
$$<x(\zeta_H)>_q = 0.5$$

  - This implies symmetric distributions:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$



Pion PDF: hadronic scale



$(M_u = M_d)$

- **Fully-dressed valence quarks** (quasiparticles)

ζ_H : hadronic scale

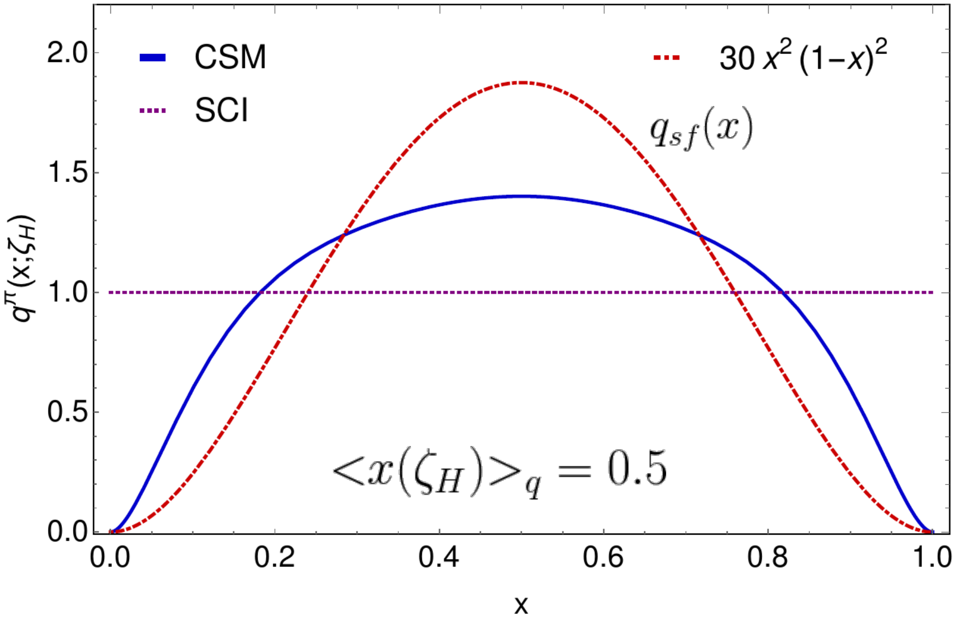
➢ At this scale, **all properties** of the hadron are contained within their valence quarks.

"**Physical**" boundaries:

$$\frac{1}{2^n} \overset{(i)}{\leq} \langle x^n \rangle_{u_\pi}^{\zeta_H} \overset{(ii)}{\leq} \frac{1}{1+n}$$

Produced by
$q(x;\zeta_H) = \delta(x - 1/2)$
(infinitely heavy valence quarks)

Produced by
$q(x;\zeta_H) = 1$
(massless SCI case)

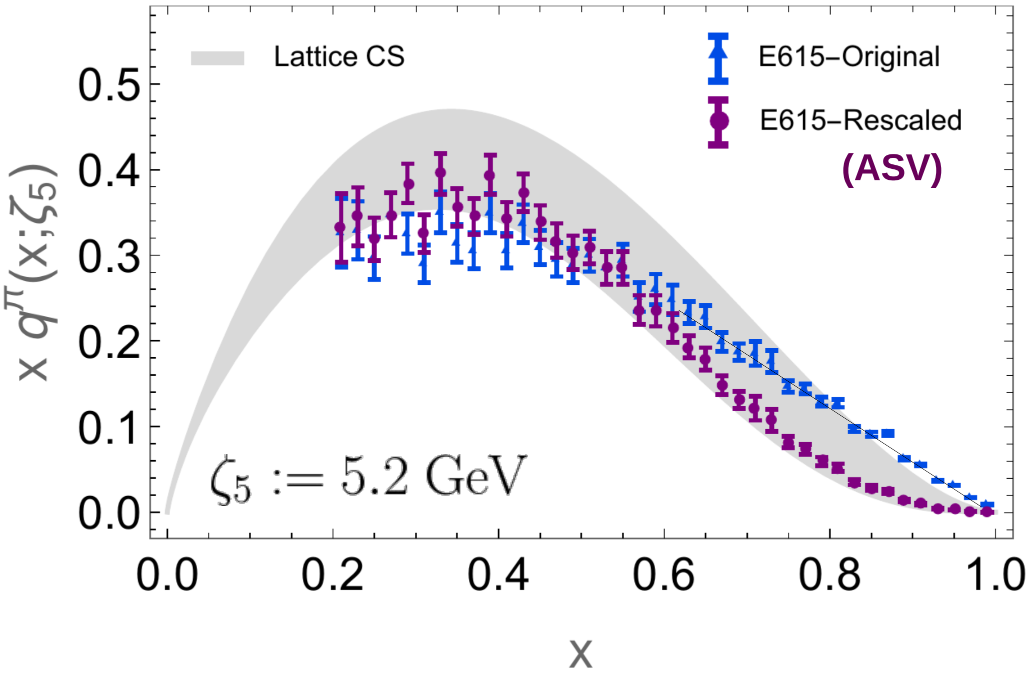


➔ Equally massive quarks means a **50-50** share of the total momentum.

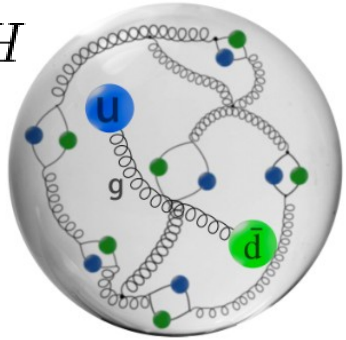
➔ This implies symmetric distributions.

Pion PDF: experimental scale

Conway:1989fs Aicher:2010cb Sufian:2020vzb



$\zeta > \zeta_H$

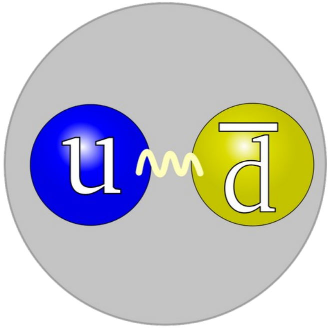


- **Unveiling of glue and sea d.o.f** (partons)

- **Experimental** data is given **here**.
- **Lattice QCD** results are also quoted beyond the **hadronic scale**.
  - The interpretation of parton distributions from cross sections demands **special care**.

Barry:2021osv

Pion PDF: energy scales

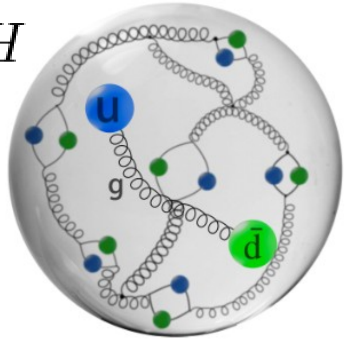


ζ_H

Resolution Scale

Evolution equations

$\zeta > \zeta_H$



- **Fully-dressed valence quarks**

  (quasiparticles)

  - Theoretical calculations are perfomed at *some* low energy scale.

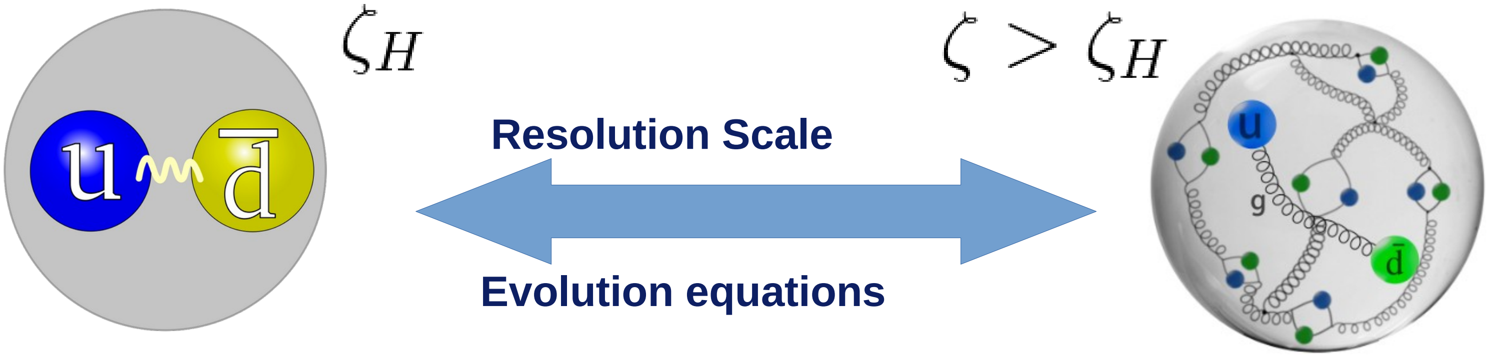
- **Unveiling of glue and sea d.o.f**

  (partons)

  - Then evolved via **DGLAP** equations to compare with experiment and lattice.

Pion PDF: energy scales

Rodriguez-Quintero:2019fyc



- **Fully-dressed valence quarks**

  (quasiparticles)

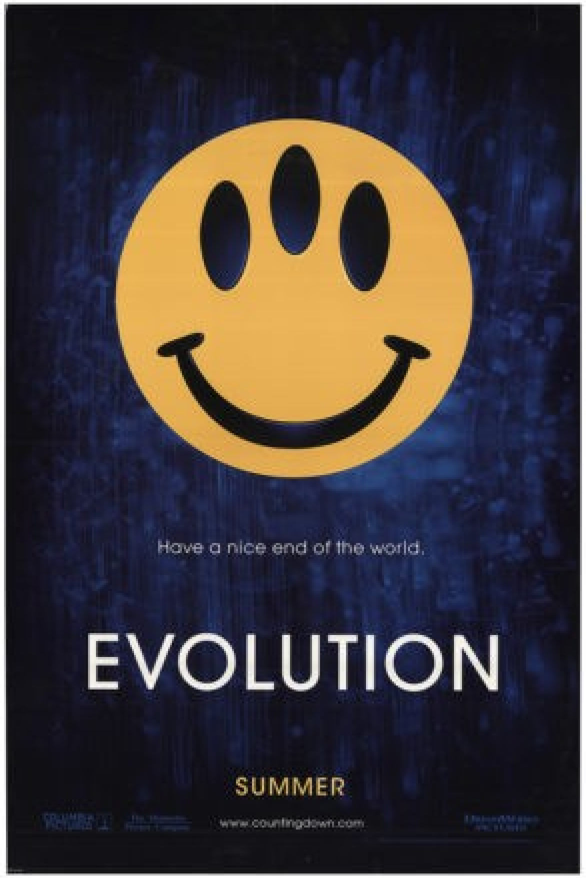
  - Theoretical calculations are perfomed at *some* low energy scale.

- **Unveiling of glue and sea d.o.f**

  (partons)

  - Then evolved via **DGLAP** equations to compare with experiment and lattice.

- Following our **all orders** evolution, we can go **either way**.
- Besides, the **hadronic scale** becomes unambigously **determined**.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DGLAP: All orders evolution

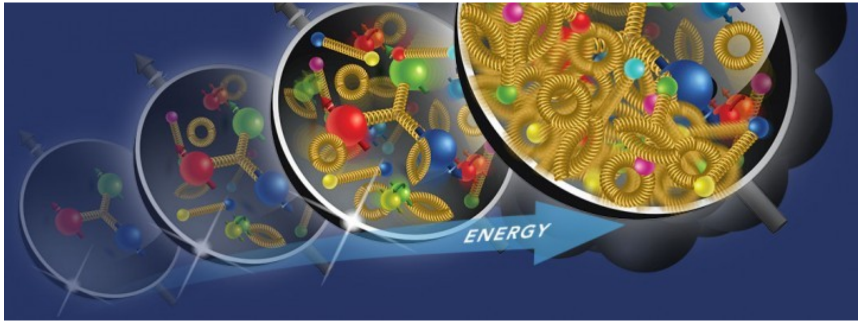
Idea. Define an **effective** coupling such that:

"All orders evolution"

Starting from fully-dressed **quasiparticles**, at ζ_H

Sea and **Gluon** content unveils, as prescribed by **QCD**

$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \int_0^1 dy \delta(y-x) - \boxed{\frac{\alpha(\zeta^2)}{4\pi}} \int_x^1 \frac{dy}{y} \begin{pmatrix} P_{qq}^{\rm NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\rm S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right) \end{pmatrix} \right\} \begin{pmatrix} H_\pi^{\rm NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\rm S}(y,t;\zeta) \end{pmatrix} = 0$$



- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- And connecting with the **hadron scale.**

Raya:2021zrz Cui:2020tdf

DGLAP: All orders evolution

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi}S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q$$

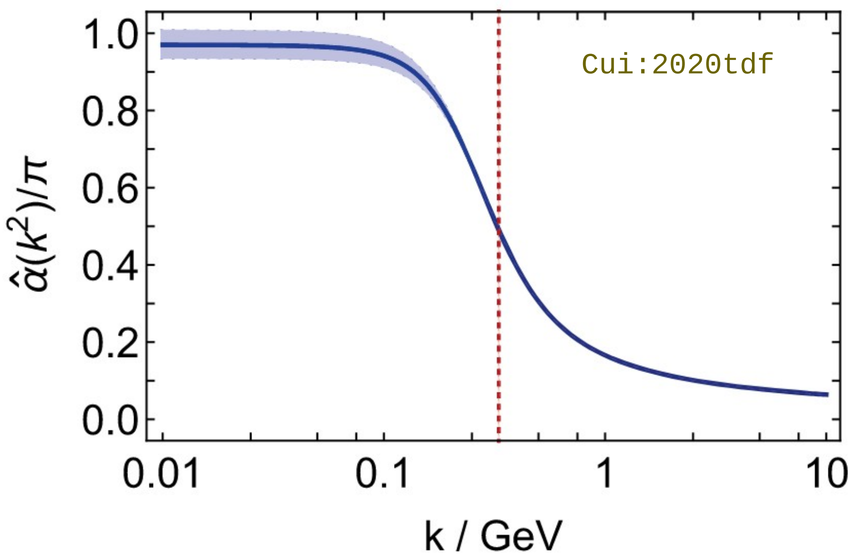
$$S(\zeta_0,\zeta_f) = \int_{2\ln(\zeta_0/\Lambda_{\rm QCD})}^{2\ln(\zeta_f/\Lambda_{\rm QCD})} dt\,\alpha(t)$$

Explicitly depending on the **effective charge**

$$\langle x^n(t;\zeta)\rangle_F = \int_0^1 dx\, x^n\, F(x,t;\zeta)$$

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$

- The **QCD PI effective charge** is our best candidate to accommodate our **all orders scheme**.

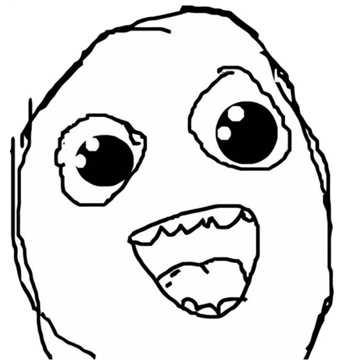


$$\hat{\alpha}(k^2) = \frac{\gamma_m \pi}{\ln\left[\frac{\mathcal{M}^2(k^2)}{\Lambda_{\rm QCD}^2}\right]} \Rightarrow \zeta_H = 0.331\ \text{GeV}$$

DGLAP: All orders evolution

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi}S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$



$$S(\zeta_0,\zeta_f) = \int_{2\ln(\zeta_0/\Lambda_{\rm QCD})}^{2\ln(\zeta_f/\Lambda_{\rm QCD})} dt\,\alpha(t)$$

This contains, ***implicitly***, the information of the **effective charge**

- No actual ***need*** to know it. Assuming its existence is **sufficient**.
- ***Unambiguous*** definition of the **hadron scale**:

$$\langle x(\zeta_H)\rangle_q = 0.5 \;\Rightarrow\; \langle x^n(\zeta_f)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\langle 2x(\zeta_f)\rangle_q\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

(flavor symmetric case)

DGLAP: All orders evolution

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi}S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q\left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

Information on the charge is here

- Details of the effective charge are **encoded** in the ratio of first moments.
- Natural connection with the **hadron scale**.

Implication 2:

$$\begin{aligned}
\langle 2x(\zeta_f)\rangle_q &= \exp\left(-\frac{8}{9\pi}S(\zeta_H,\zeta_f)\right), \qquad q = u, \bar{d};\\
\langle x(\zeta_f)\rangle_{\text{sea}} &= \langle x(\zeta_f)\rangle_{\sum_q q+\bar{q}} - (\langle x(\zeta_f)\rangle_u + \langle x(\zeta_f)\rangle_{\bar{d}}),\\
&= \frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u\\
\langle x(\zeta_f)\rangle_g &= \frac{4}{7}\left(1 - \langle 2x(\zeta_f)\rangle_u^{7/4}\right);
\end{aligned}$$

- **Sea** and **gluon** determined from valence-quark moments

DGLAP: All orders evolution

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi}S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q\left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

Information on the charge is here

- Can **jump** from one scale to another (both ways)
- Natural connection with the **hadron scale**.

Implication 2:

$$\begin{aligned}
\langle 2x(\zeta_f)\rangle_q &= \exp\left(-\frac{8}{9\pi}S(\zeta_H,\zeta_f)\right), \qquad q=u,\bar{d};\\
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&= \frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u\\
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\end{aligned}$$

- **Sea** and **gluon** determined from valence-quark moments
- **Asymptotic** (massless) limits are evident.

DGLAP: All orders evolution

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi}S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

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&= \frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u \\
\langle x(\zeta_f)\rangle_g &= \frac{4}{7}\left(1 - \langle 2x(\zeta_f)\rangle_u^{7/4}\right);
\end{aligned}$$

- **Sea** and **gluon** determined from valence-quark moments
- **Asymptotic** (massless) limits are evident.
- And, of course, the momentum **sum rule**:

$$\langle 2x(\zeta_f)\rangle_q + \langle x(\zeta_f)\rangle_{\text{sea}} + \langle x(\zeta_f)\rangle_g = 1$$

DGLAP: All orders evolution

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi}S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

Information on the charge is here

- Can **jump** from one scale to the another (even downwards)
- Natural connection with the **hadron scale**.

Implication 3: Recurrence relation

$$\langle x^{2n+1}\rangle_{u_\pi}^{\zeta} = \frac{(\langle 2x\rangle_{u_\pi}^{\zeta})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle_{u_\pi}^{\zeta} (\langle 2x\rangle_{u_\pi}^{\zeta})^{-\gamma_0^j/\gamma_0^1} \,.$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left.

DGLAP: All orders evolution

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi}S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q\left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

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Implication 3: Recurrence relation

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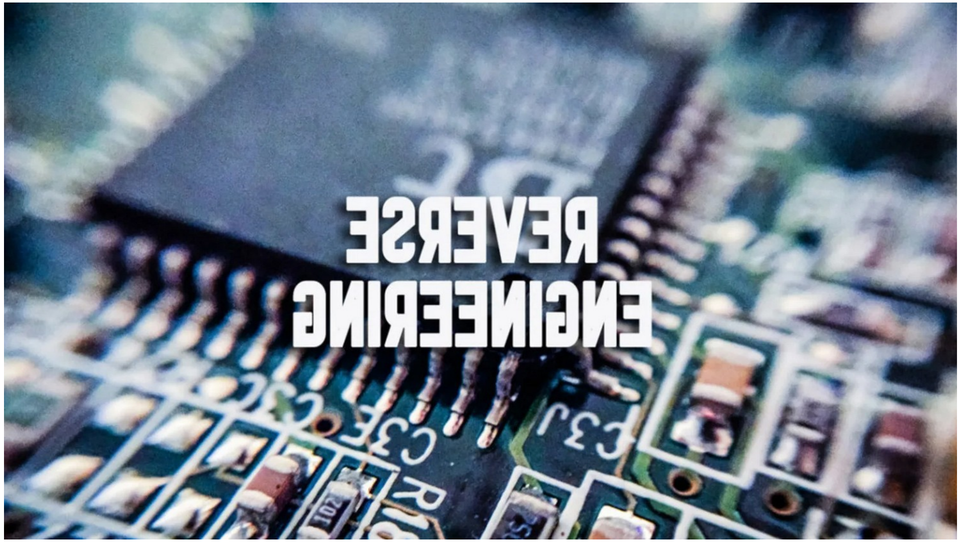
	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$	
n	Lattice input	Recurrence relation
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7		0.0078

Reverse engineering the PDF data



Pion **PDF**

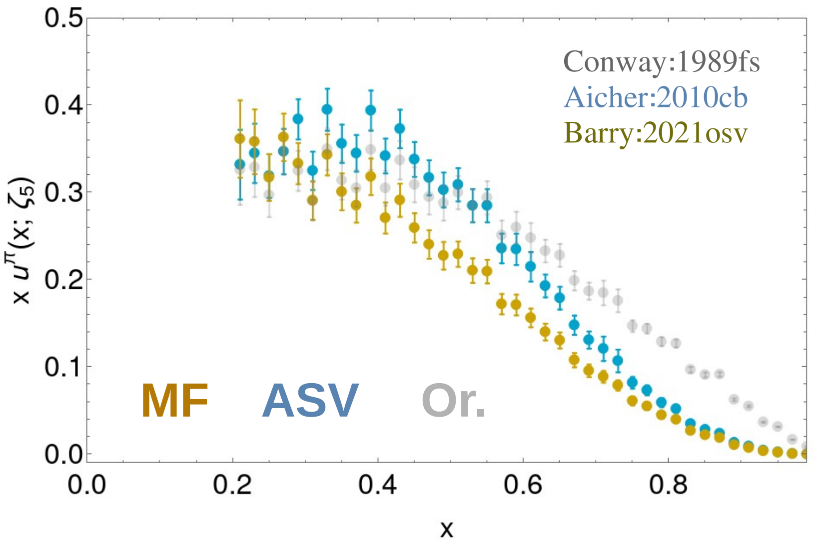
- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^\pi(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Normalization

$$\{\alpha_i^\zeta | i = 1,2,3\}$$

Free parameters



- Then, we proceed as follows:

1) Determine the **best values α$_i$** via least-squares fit to the data.

2) Generate new **values α$_i$,** distributed randomly around the best fit.

3) Using the latter set, evaluate:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

Data point with error

4) Accept a replica with probability:

$$\mathcal{P} = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}\,,\quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

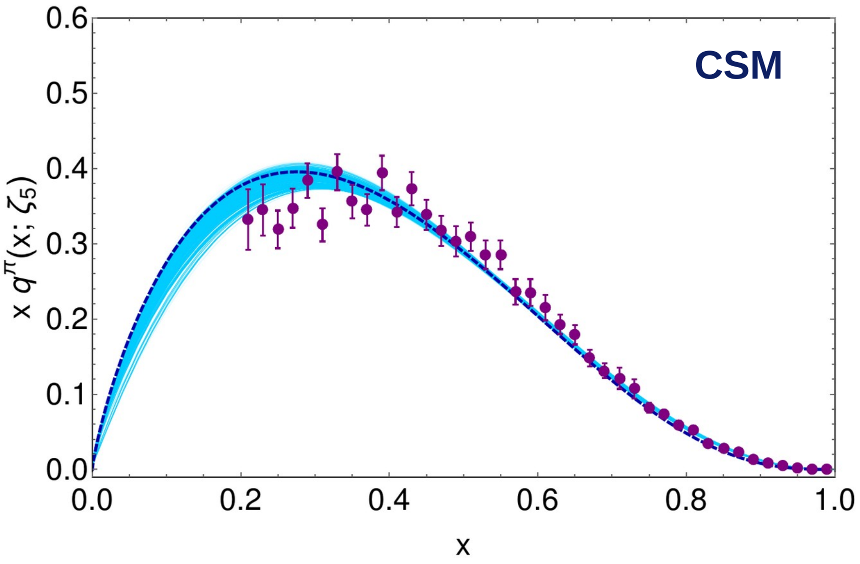
5) Evolve back to ζ_H

Repeat (2-5).

Pion PDF: ASV Data

➢ Applying this algorithm to the **ASV data** yields:

(average)



Mean values (of moments) and errors

{{0.5, 2.75144 × 10^-17}, {0.299833, 0.00647045}, {0.199907, 0.00735448}, {0.142895, 0.0068623}, {0.107274, 0.00608759}, {0.0835168, 0.00532834}, {0.0668711, 0.0046596}, {0.0547511, 0.00409028}, {0.0456496, 0.00361041}, {0.0386394, 0.00320609}}

Moments from SCI, ζ_H

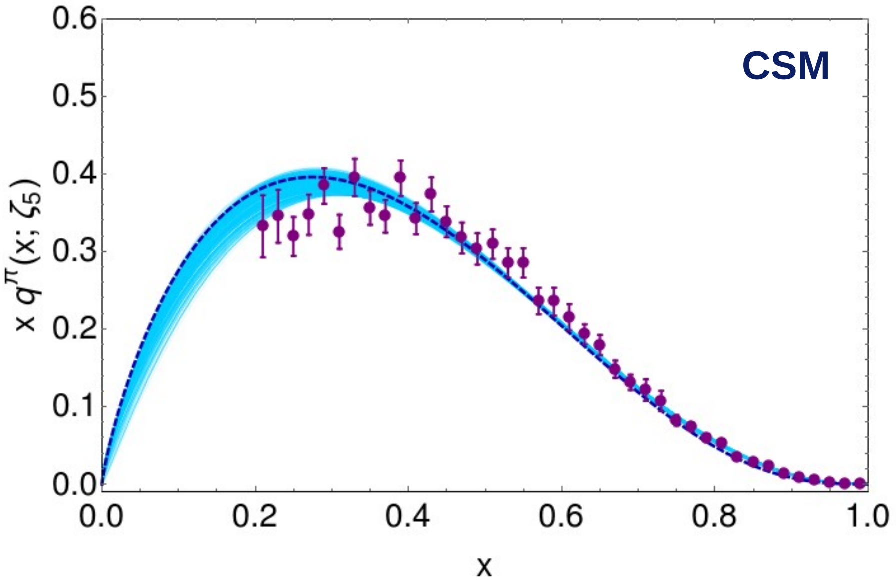
{0.5, 0.332885, 0.249327, 0.199231, 0.165865, 0.142056, 0.124215, 0.11035, 0.0992657, 0.090203, 0.0826552, 0.0762721, 0.0708035, 0.0660661, 0.0619225}

✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✔ Not at all similar to those from SCI

Pion PDF: ASV Data

- Applying this algorithm to the **ASV** **data** yields:



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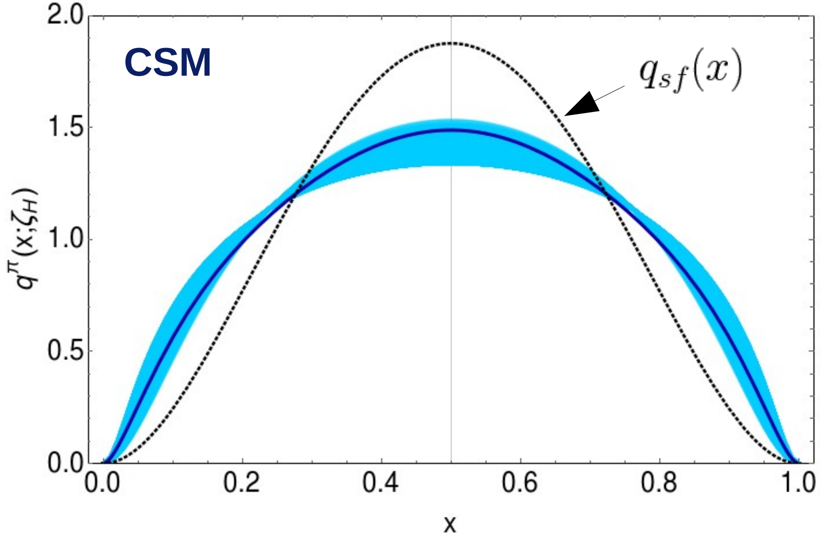
✔ Then, we can **reconstruct** the moments produced by each replica, using the single-parameter **Ansatz**:

$$u^{\pi}(x;\zeta_{\mathcal{H}}) = n_0 \ln(1 + x^2(1-x)^2/\rho^2)$$



✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

Pion PDF: Lattice Data

- We can follow an analogous procedure to infer, based upon **lattice data**, how the **hadronic scale PDF** should look like.

- Let us consider the list of **lattice QCD** moments:

n	[61]	[62]	[63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4	Joo:2019bzr		0.023(05)(06)
5	Sufian:2019bol		0.014(04)(05)
6	Alexandrou:2021mmi		0.009(03)(03)

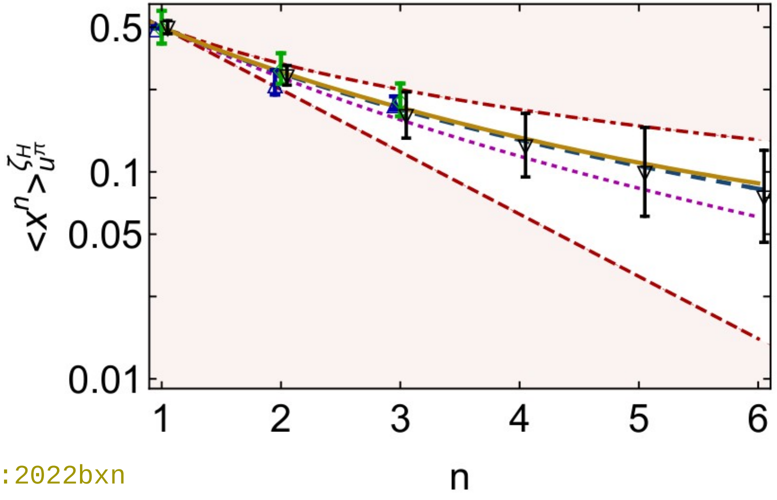
- Those verify the recurrence relation, thus being compatible with a **symmetric PDF** at ζ_H

- While also falling within the **physical bounds**.

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} (\langle 2x \rangle^{\zeta}_{u_\pi})^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

Produced by $q(x;\zeta_H) = \delta(x - 1/2)$ (infinitely heavy valence quarks)

Produced by $q(x;\zeta_H) = 1$ (massless SCI case)



Cui:2022bxn

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- While also falling within the **physical bounds**.

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

Produced by

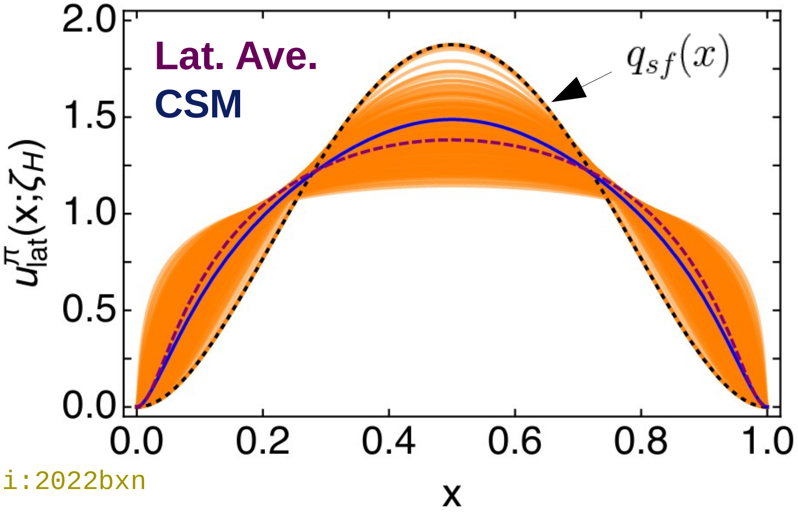
$$q(x;\zeta_H) = \delta(x - 1/2)$$

(infinitely heavy valence quarks)

Produced by

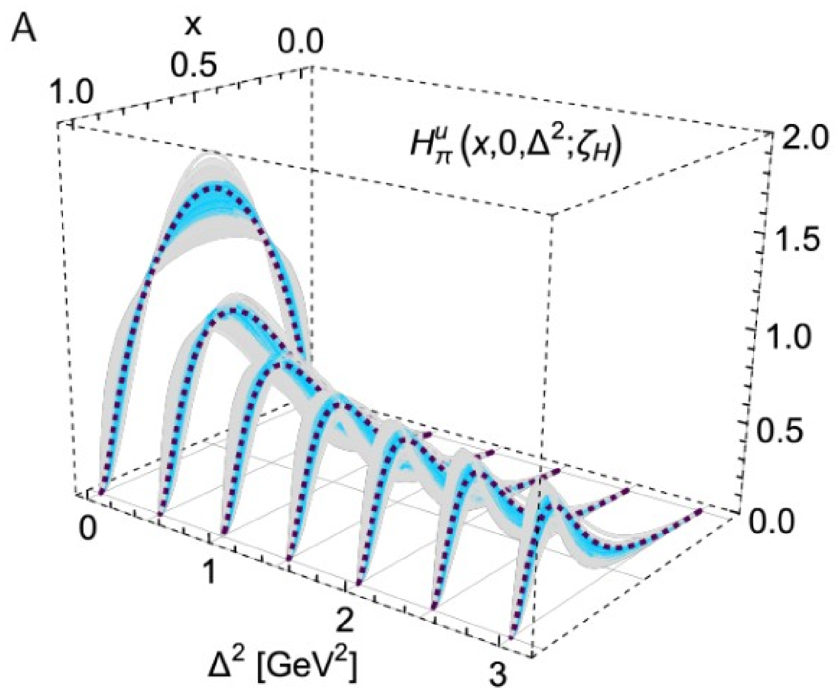
$$q(x;\zeta_H) = 1$$

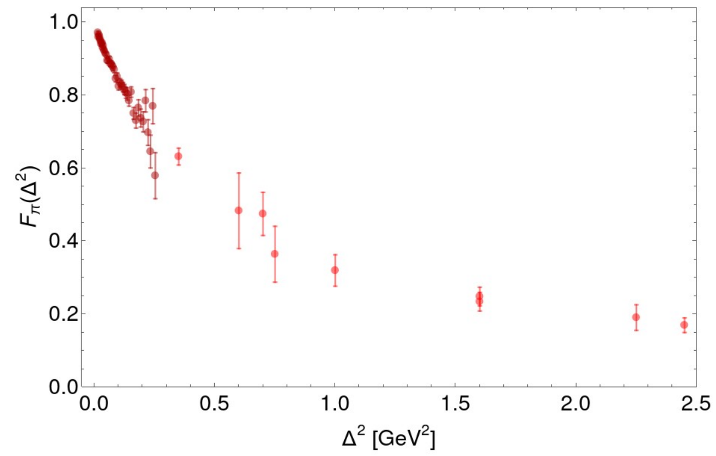
(massless SCI case)

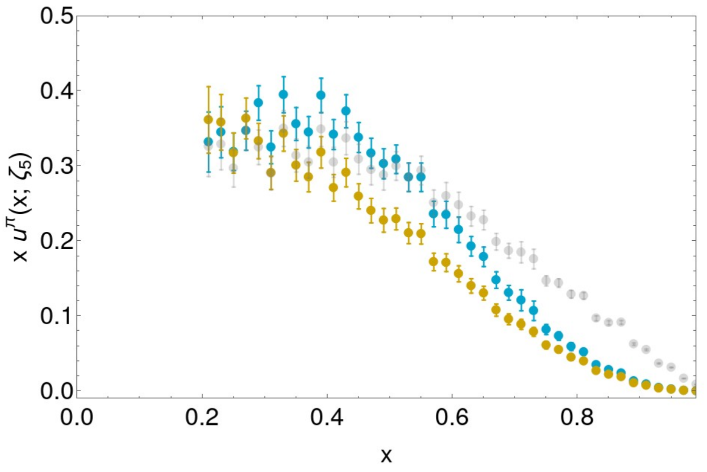


Cui:2022bxn

GPD from PDF and EFF







Setting the Stage

Raya:2021zrz

- Starting with a **factorized LFWF**, $\psi^{\uparrow\downarrow}_{\mathbf{P}u}(x, k_\perp^2; \zeta_H) = \tilde{\psi}^{\mathbf{u}}_{\mathbf{P}u}(k_\perp^2)\,[u^{\mathbf{P}}(x;\zeta_H)]^{1/2}$

This one shall be obtained as described previously

- The overlap representation for the **GPD** entails:

$$H^u_{\mathsf{P}}(x,\xi,t;\zeta_{\mathcal{H}}) = \int \frac{d^2k_\perp}{16\pi^3}\psi^{u*}_{\mathsf{P}}\left(x_-, k^2_{\perp -};\zeta_{\mathcal{H}}\right)\psi^u_{\mathsf{P}}\left(x_+, k^2_{\perp +};\zeta_{\mathcal{H}}\right)$$

$$= \Theta(x_-)\sqrt{u^{\mathbf{P}}(x_-;\zeta_H)u^{\mathbf{P}}(x_+;\zeta_H)}\Phi_{\mathbf{P}}(z;\zeta_H)$$

Heaviside Theta

This dictates the off-forward behavior of the GPD

- Where $z = s_\perp^2 = -t(1-x)^2/(1-\xi^2)^2$ and:

$$\Phi^u_{\mathbf{P}}(z;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3}\tilde{\psi}^{u*}_{\mathbf{P}}\left(\mathbf{k}_\perp^2;\zeta_H\right)\tilde{\psi}^{u}_{\mathbf{P}}\left((\mathbf{k}_\perp - \mathbf{s}_\perp)^2;\zeta_H\right)$$

… will be driven by the electromagnetic form factor

Setting the Stage

➢ Recall a **GPD** arising from a factorised **LFWF** adopts the form:

$$H^u_\pi(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

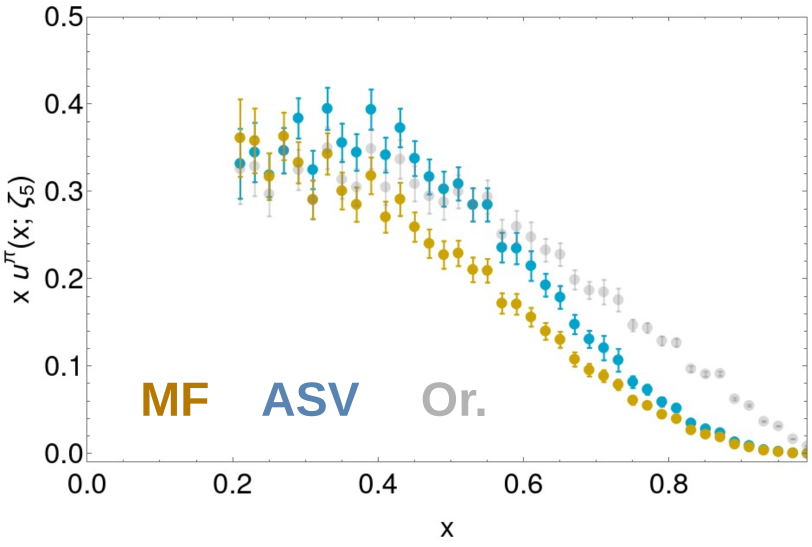
$$u^\pi(x;\zeta_H) = n_0 \ln(1+x^2(1-x)^2/\rho^2)$$

➢ The empirical data on **PDF** to contrast with:

- **ASV** analysis.
- **MF** resummation.
- **Lattice** QCD moments.

For references, see:

Cui:2022bxn
Cui:2021mom



Setting **the Stage**

$$F_\pi(t) = \int_0^1 dx\, u^\pi(x;\zeta_H)\Phi_\pi(z;\zeta_H)$$

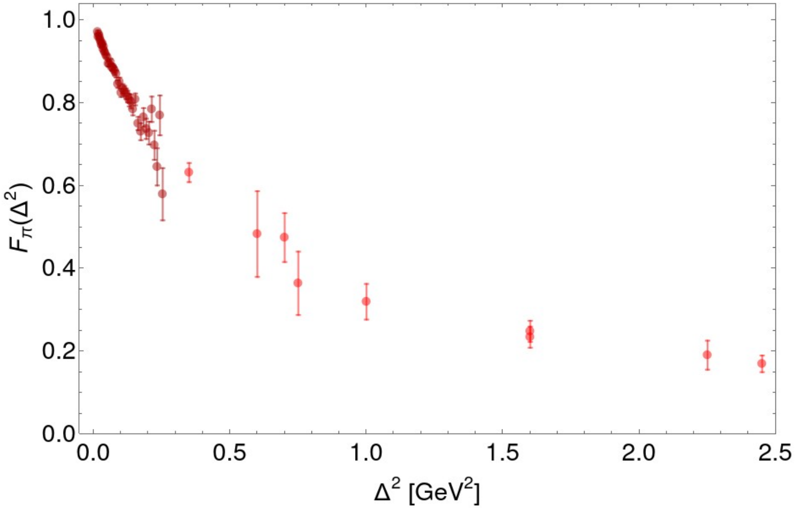
➢ We thus employ a **3-parameter** model for the **GPD:**:

$$\{\rho, \beta, \gamma\}$$

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

$$\lambda = \beta - \frac{r_\pi^2}{6\langle x^2\rangle_{u_\pi}^{\zeta_H}}$$

$$\Phi^\pi(y;\zeta_H) = \frac{1+\lambda y}{1+\beta y+\gamma^2 y^2}$$



➢ The empirical data on **EFF**:

- **JLab** data.
- **Charge** radius: $r_\pi = 0.64(2)\,\mathrm{fm}$

SPM extraction

Cui:2021aee

Conservative "Gaussian" error

- Given r_π, low-Q^2 data is redundant.

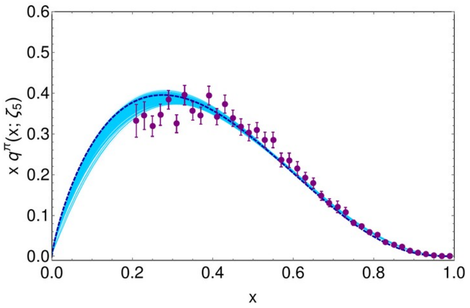
The **Algorithm**

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

1. For the chosen **PDF data** set, generate a **replica**. The replica would be accepted following the aforementioned *chi-2* criteria.

2. After acceptance, **evolve** it to the **hadronic scale** using several Mellin moments. The *de-evolved* PDF shall be reconstructed using the functional form:

$$u^\pi(x;\zeta_H) = n_0 \ln(1 + x^2(1-x)^2/\rho^2)$$



3. Store both the value $\boldsymbol{\rho_i}$ and the probability of acceptance $\mathbf{P(\rho_i)}$.

The Algorithm

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

4. Keeping the selected **PDF**, we now constrast **Φ** with the **EFF** data, via:

$$F_\pi(t) = \int_0^1 dx\, u^\pi(x;\zeta_H)\Phi_\pi(z;\zeta_H)$$

$$\Phi^\pi(y;\zeta_H) = \frac{1+\lambda y}{1+\beta y+\gamma^2 y^2}$$

$$\lambda = \beta - \frac{r_\pi^2}{6\langle x^2\rangle_{u_\pi}^{\zeta_{\mathcal{H}}}}$$

5. Employing a *chi-2* criteria, we compute the probability of acceptance $\mathbf{P(\Phi_i|\rho_i)}$.

6. The **GPD** is accepted with probability $\mathbf{P(\Phi_i|\rho_i)P(\rho_i)}$.

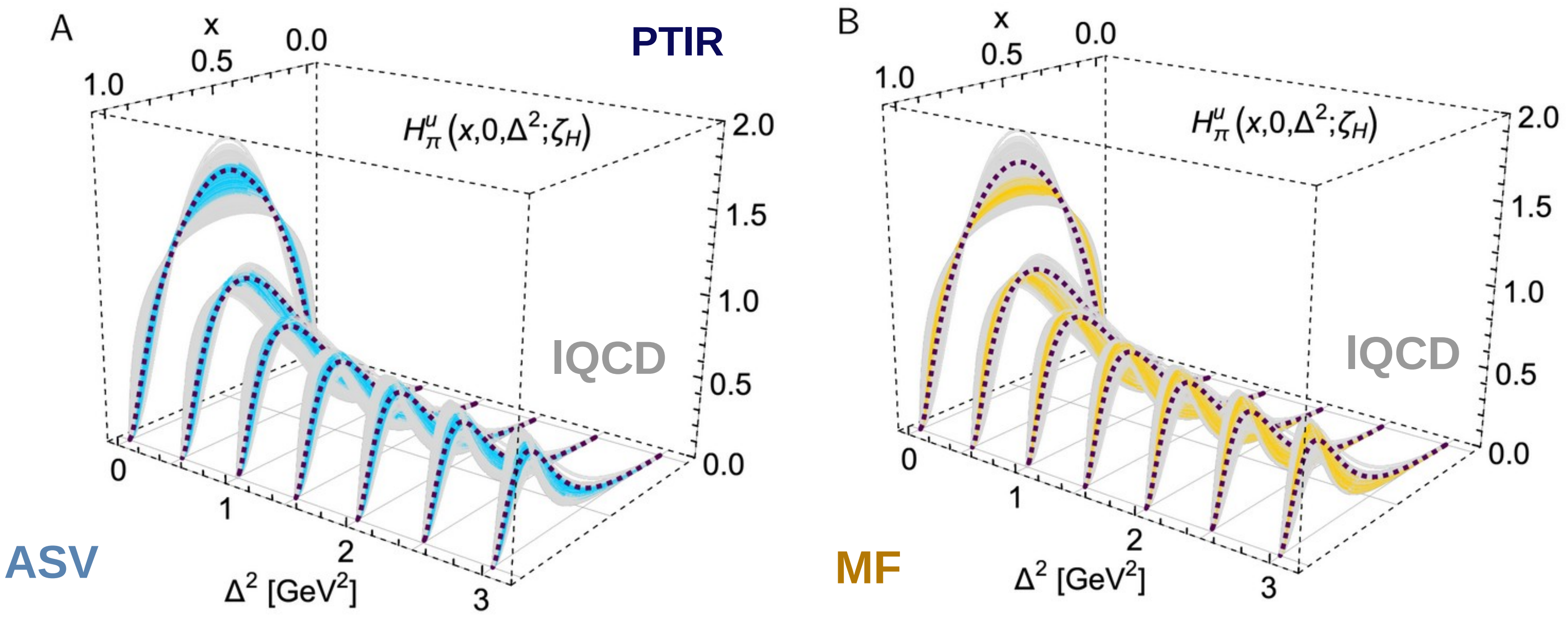
REPEAT

Numerical Results

Pion GPD

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

- Applying this procedure, from the pion **PDF** and **EFF** empirical data, one gets the **GPDs**:



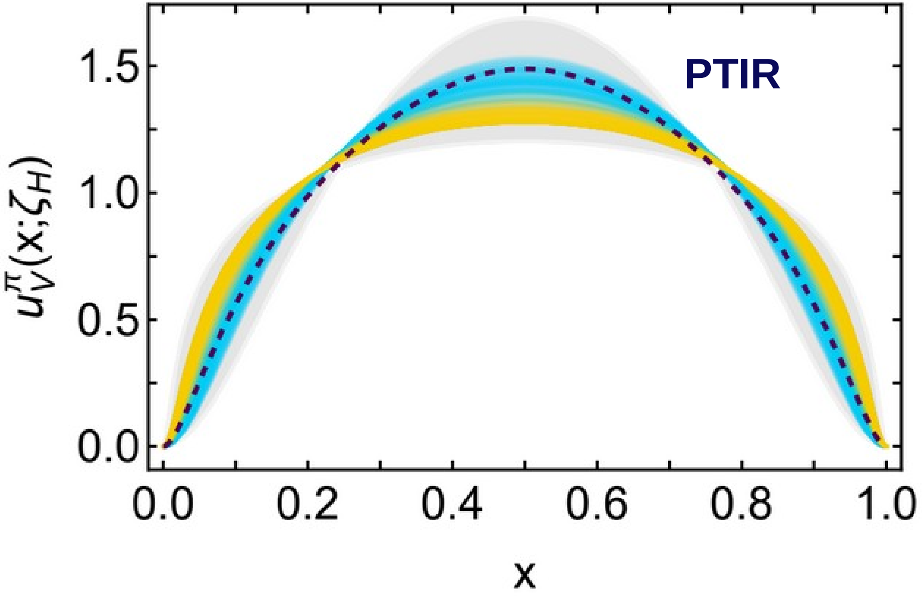
CSM:

Raya:2021zrz
Raya:2022eqa

Pion PDF

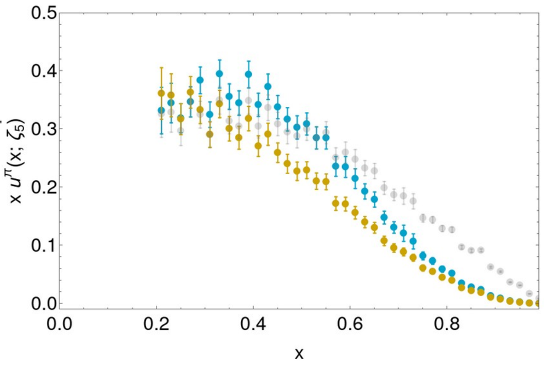
$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

- The **PDFs** agree within errors, but...



- **Lattice** QCD **cannot distinguish** between **ASV**, **MF** or the *parton-like* profiles.

n	[61]	[62]	[63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

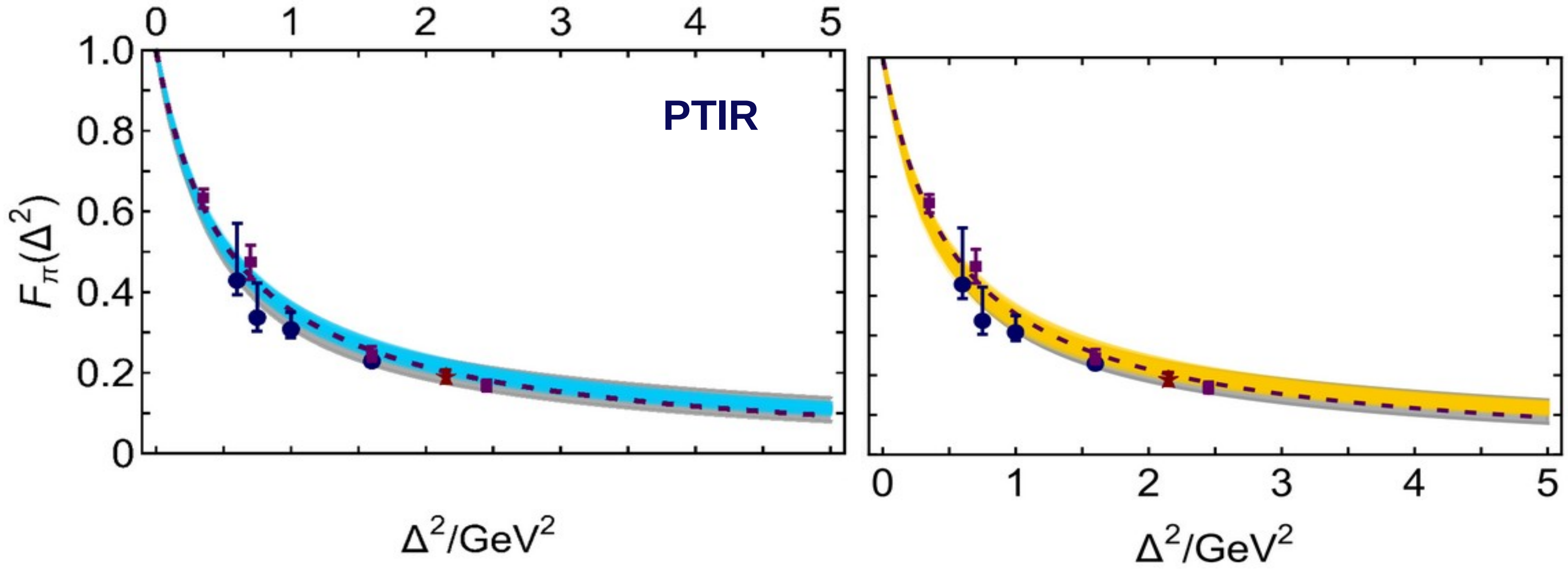


Pion EFF

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

- For the **EFF**, we essentially arrive at the same output.

→ **PTIR** model faithfully reproduces the **Data-Driven** result



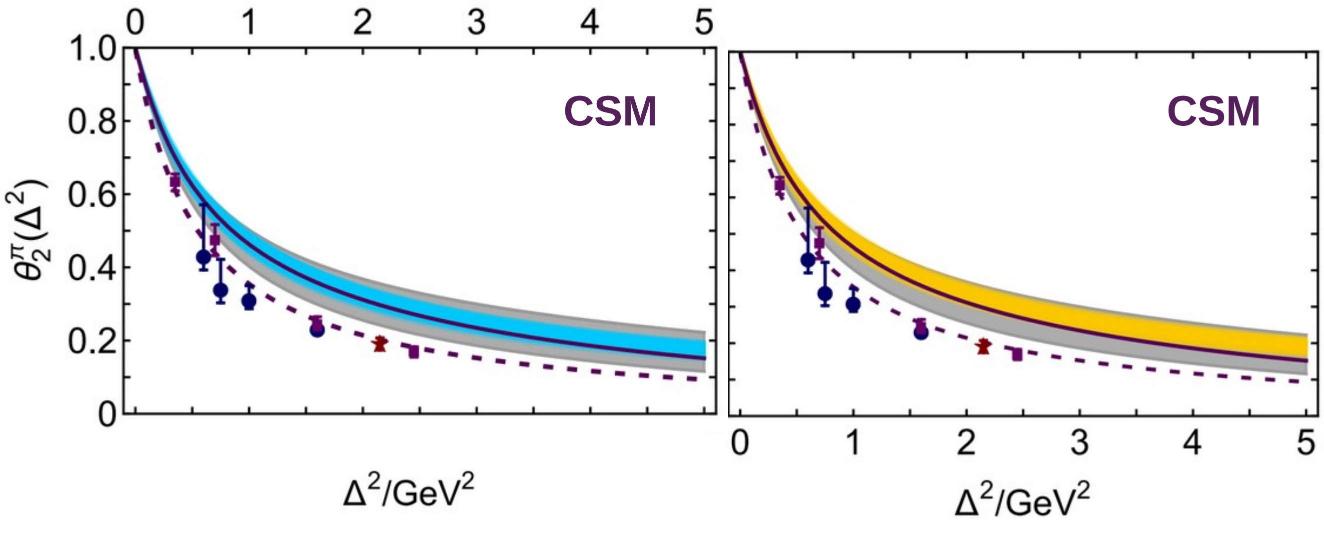
	r_π/fm
ASV	0.640(20)
MF	0.638(18)
lQCD	0.639(19)

Mass Distribution

- The first Mellin moment of the GPD yields the **gravitational form factors:**

$$\int_{-1}^{1} dx\, 2H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta_2^\pi(\Delta^2) - \xi^2\theta_1^\pi(\Delta^2)$$

- θ_1 currently escapes our approach, but θ_2 is within reach: $\theta_2^\pi(\Delta^2) = \int_0^1 dx\, 2xH_\pi(x,\xi=0,-\Delta^2)$



θ_2 is associated with the **mass distribution.**

- We found the **mass** radii:

	ASV	MF	lQCD
$r_\pi^{\theta_2}$	0.518(16)	0.498(14)	0.512(21)

Producing: $r_\pi^{\theta_2}/r_\pi = 0.79(3)$

mass/charge ratio

About **Radii**

$$H^u_{\mathrm{P}}(x,\xi,t;\zeta_H) = \theta(x_-)\left[u^{\mathrm{P}}(x_-;\zeta_H)u^{\mathrm{P}}(x_+;\zeta_H)\right]^{1/2}\Phi_{\mathrm{P}}(z;\zeta_H)$$

- In the **factorized** models:

$$\frac{\partial^n}{\partial^n z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = \frac{1}{\langle x^{2n}\rangle^{\zeta_{\mathcal{H}}}_{\bar{h}}}\frac{d^n F^u_{\mathrm{P}}(\Delta^2)}{d(\Delta^2)^n}\Big|_{\Delta^2=0}$$

PDF moments

Derivatives of **EFF**

$$\frac{\partial}{\partial z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = -\frac{r^2_{\mathrm{P}}}{4\chi^2_{\mathrm{P}}(\zeta_{\mathcal{H}})},$$

$$\frac{\partial}{\partial z}\Phi^{\bar{h}}_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = (1-d_{\mathrm{P}})\frac{\partial}{\partial z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0}$$

Asymmetry term = **0** for **pion**

- Therefore, the **mass *radius***:

$$r^{\theta_2 2}_{\mathrm{P}_u} = \frac{3r^2_{\mathrm{P}}}{2\chi^2_{\mathrm{P}}}\langle x^2(1-x)\rangle_{\mathrm{P}_{\bar{h}}},$$

$$r^{\theta_2 2}_{\mathrm{P}_{\bar{h}}} = \frac{3r^2_{\mathrm{P}}}{2\chi^2_{\mathrm{P}}}(1-d_{\mathrm{P}})\langle x^2(1-x)\rangle_{\mathrm{P}_u}$$

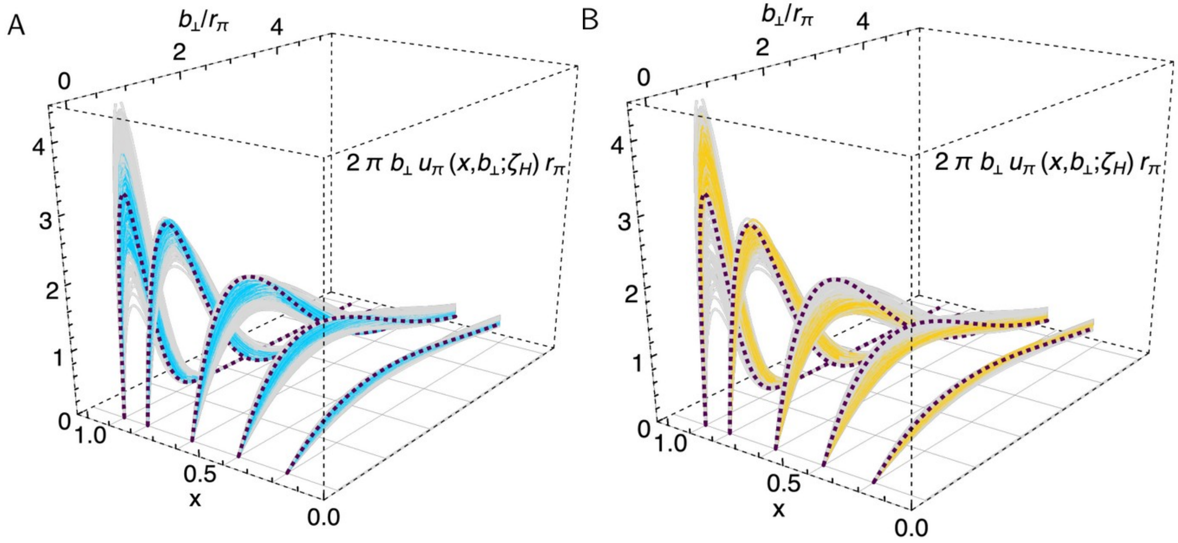
$$\left(\frac{r^{\theta_2}_{\pi}}{r^E_{\pi}}\right)^2 = \frac{<x^2(1-x)>^q_{\zeta_H}}{<x^2>^q_{\zeta_H}} \approx \left(\frac{4}{5}\right)^2$$

Determined from **PDF** moments!

IPS GPDs

- **Impact parameter** space **GPDs** are defined as:

$$u^\pi(x, b_\perp^2; \zeta_{\mathcal{H}}) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(|b_\perp|\Delta)\, H_\pi^u(x, 0, -\Delta^2; \zeta_{\mathcal{H}})$$



- Such that, in **factorized** models:

$$u^\pi(x, b_\perp^2; \zeta_H) = \frac{u^\pi(x;\zeta_H)}{(1-x)^2} \Psi^\pi\left(\frac{|b_\perp|}{1-x}; \zeta_H\right)$$

- The location and values of the **maxima**:

	x	$b_\perp/r_\pi$	i_π
CSM [57]	0.88	0.13	3.29
ASV	0.89(2)	0.10(2)	3.21(30)
MF	0.95(1)	0.05(1)	4.58(50)
lQCD	0.91(6)	0.08(5)	4.04(1.67)

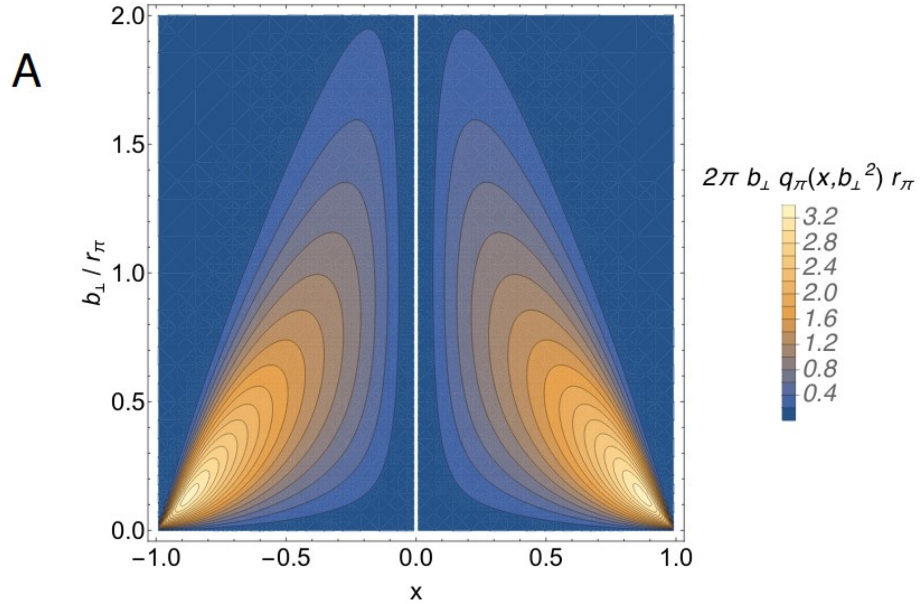
- Furthermore:

$$\langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_u^\pi = \frac{2}{3} r_\pi^2 = \langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_d^\pi$$

Algebraic result !

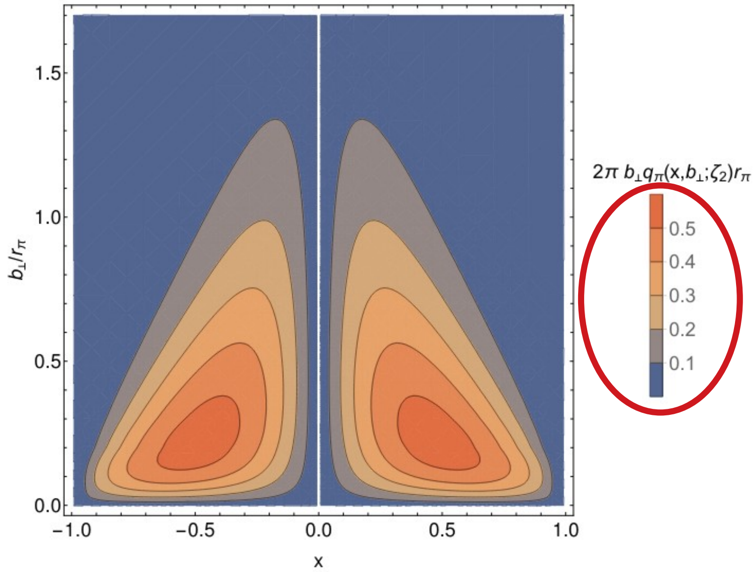
Evolved IPS-GPD: Pion Case

$$u^{\mathrm{P}}(x, b_\perp^2; \zeta_{\mathcal{H}}) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta)\, H_{\mathrm{P}}^u(x, 0, -\Delta^2; \zeta_{\mathcal{H}})$$



- **Likelihood** of finding a parton with LF momentum **x** at transverse position **b**

$$\zeta_H \to \zeta = 2\ \text{GeV}$$



- Peaks **broaden** and **maximum drifts:**

$$\max: \ 3.29 \to 0.55$$

$$(|x|, b) = (0.88, 0.13) \to (0.47, 0.23)$$

Evolved IPS-GPD: Pion Case

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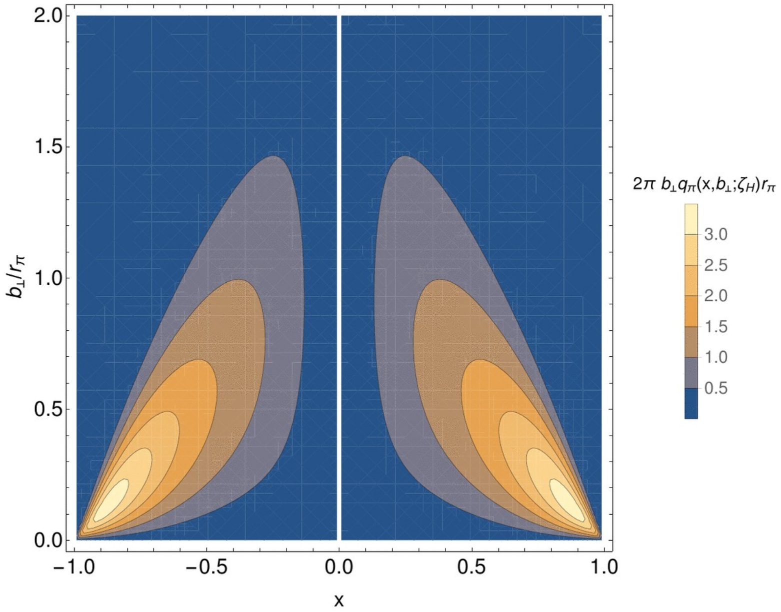

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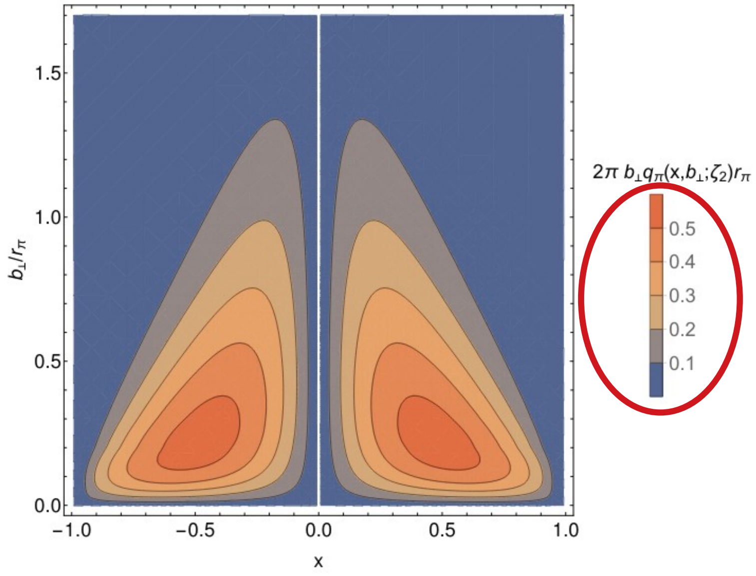

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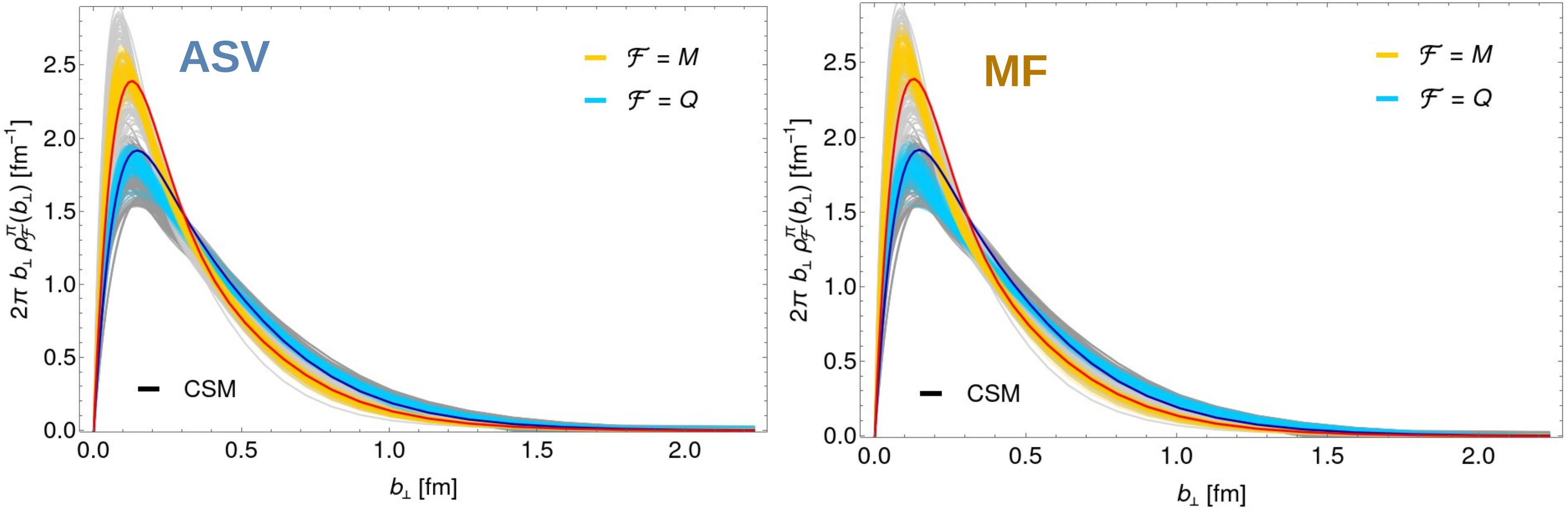
Distributions: Mass & Charge

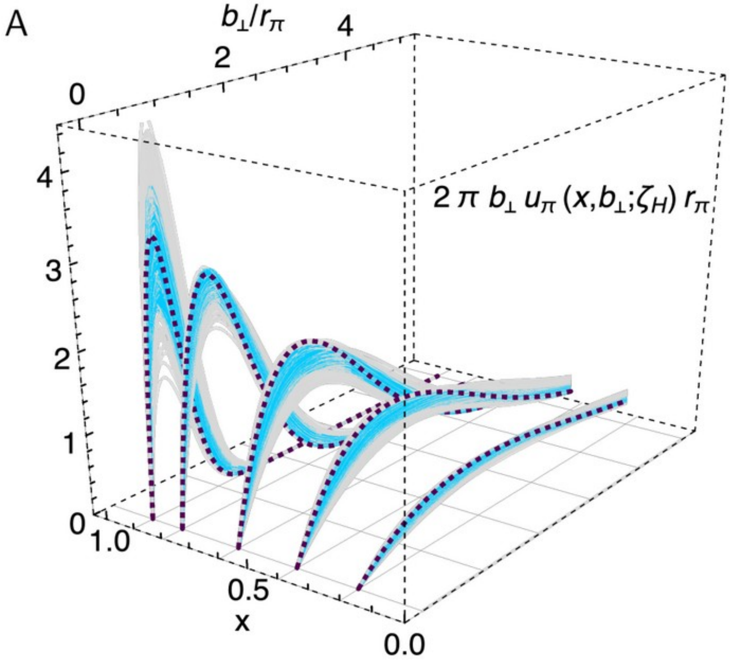
- **Density** distributions are obtained by integrating the **IPS-GPD.**

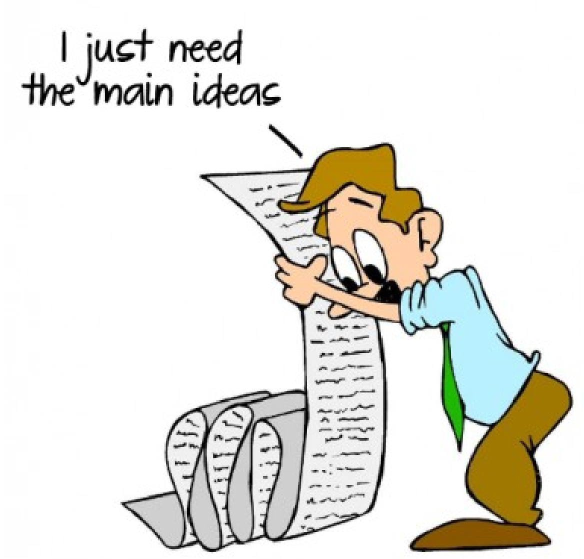
$$\rho^{\pi}_{\{F,\theta_2\}}(|b_\perp|) = \int_{-1}^{1} dx\,\{1,2x\}u^{\pi}(x,b_\perp^2;\zeta_{\mathcal{H}}) = \int_0^{\infty}\frac{d\Delta}{2\pi}\Delta J_0(|b_\perp|\Delta)\{F_\pi(\Delta^2),\theta_2(\Delta^2)\}$$



- The narrower curves correspond to the mass distribution, demonstrating that: **Charge** effects span over a **larger** domain than **mass** effects.

Conclusions and Scope

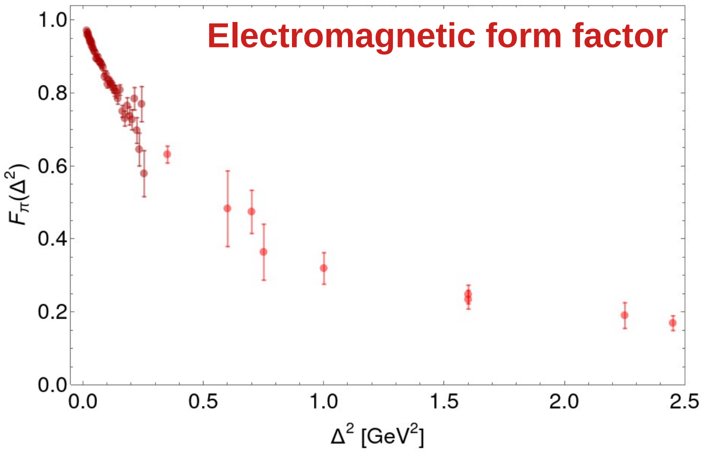


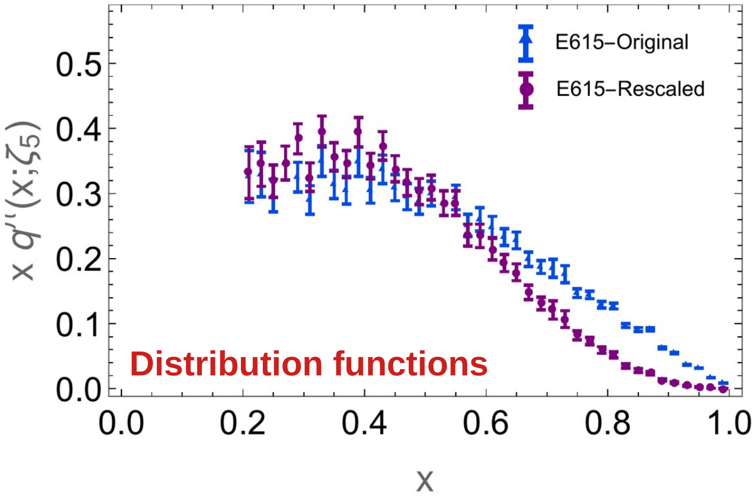


Conclusions and Scope

- **Question**:
  From the empirical knowledge of 1-dimensional distributions (**EFF** and **PDF**), can we obtain the 3-dimensional **GPD**?

$$u^\pi(x;\zeta_{e/l})\ ,\ F_\pi(\Delta^2) \longrightarrow H_\pi(x,\xi,-\Delta^2;\zeta)$$ **???**





Conclusions and Scope

- **Question:**
  From the empirical knowledge of 1-dimensional distributions (**EFF** and **PDF**), can we obtain the 3-dimensional **GPD**?

$$u^\pi(x;\zeta_{e/l})\,,\ F_\pi(\Delta^2) \longrightarrow H_\pi(x,\xi,-\Delta^2;\zeta)$$ **???**

- ~~Partial~~ **Answer:**

DGLAP GPD

All orders evolution: $u^\pi(x;\zeta_{e/l})$

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

Factorized LFWF

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) \sim \int_{k_\perp} \psi^*\psi$$

Sum rule

$$F_\pi(\Delta^2) = \int_0^1 dx\, H_\pi^u(x,0,-\Delta^2)$$

Conclusions and Scope

- **Question**:
  From the empirical knowledge of 1-dimensional distributions (**EFF** and **PDF**), can we obtain the 3-dimensional **GPD**?

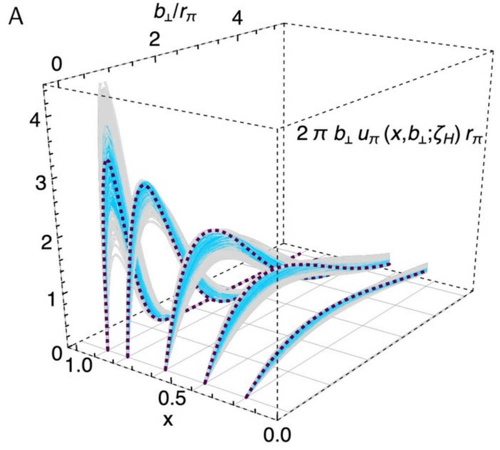
- **Answer**:
  Yes, but so far we are limited to the **DGLAP** region.

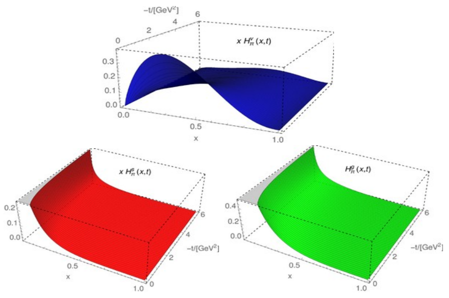
  - **Nevertheles...**

    - **Charge**, **Mass** and **Spatial** distributions are already within the reach of **DGLAP** domain.

    - In this domain, we can also evolve the **GPDs** to disentangle **valence**, **glue** and sea **content**.

    - Sophisticated covariant extensions to the **ERBL** domain are known.

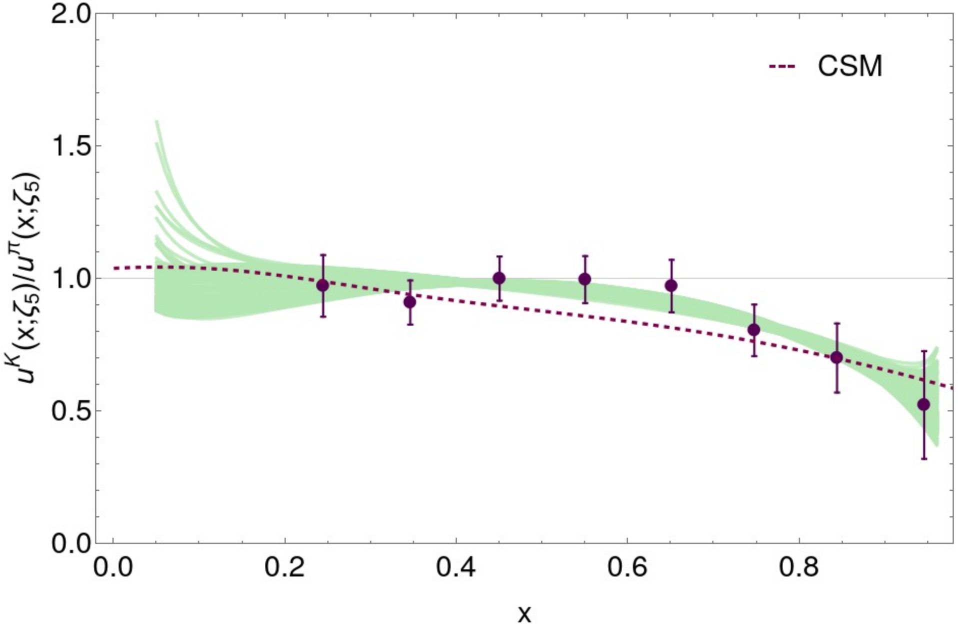


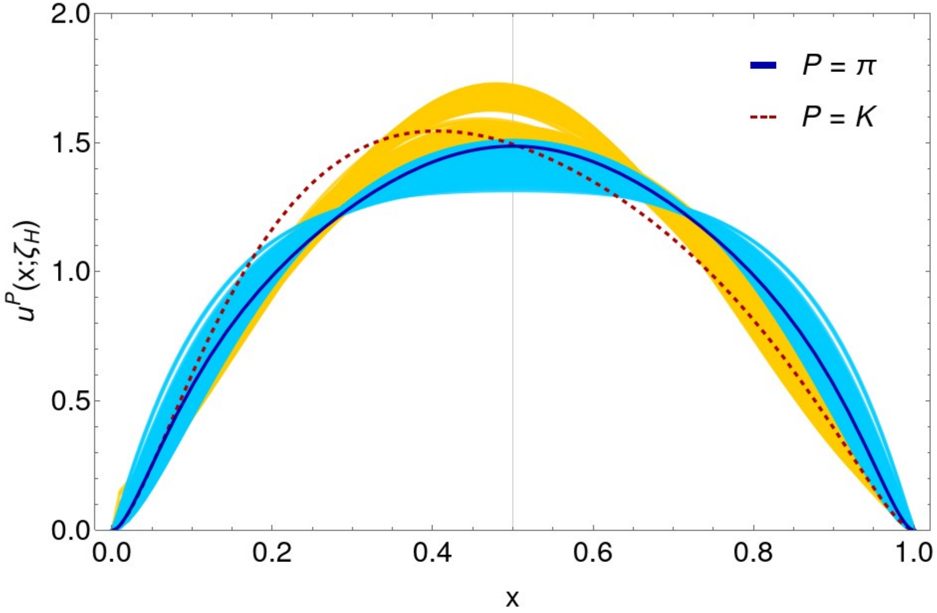


(notably, the preliminary CSM computation of the GFFs, shows agreement with the Data-Driven result)

Conclusions and Scope

- Even though analogous empirical information on the kaon is scarce, we can perform an **analogous exploration** of the **kaon**.





Conclusions and Scope

- With the **EFF** determined from experimental data, and further validated by a **completely independent** observable (the **PDF**), we can safely rely on the produced ensamble to derive other quantities.

- Such is the case of the **pion-box** contribution to the **muon's** anomalous magnetic moment:

$$\alpha_\mu^{\mathbf{P}-box} = \frac{\alpha_{em}^3}{432\pi^2}\int_\Omega \sum_i^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i^{\mathbf{P}-box}(Q_1,Q_2,\tau),$$

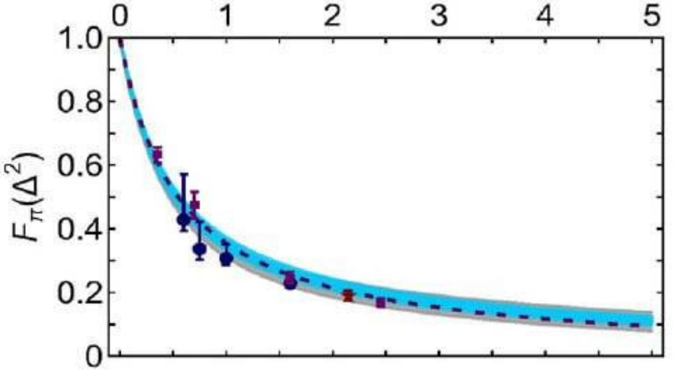
$$\bar{\Pi}_i^{\mathbf{P}-box}(Q_1^2,Q_2^2,Q_3^2) = \underline{F_{\mathbf{P}}(Q_1^2)F_{\mathbf{P}}(Q_2^2)F_{\mathbf{P}}(Q_3^2)} \times \mathcal{I}_i$$

- An **exploratory** calculation yields: **(with P. Roig)**

$$a_\mu^{\pi-\text{box}} = -(15.1)^{+0.5}_{-0.3} \times 10^{-11}$$

In fair agreement with modern estimates.

Eichmann:2019bqf
Miramontes:2021exi

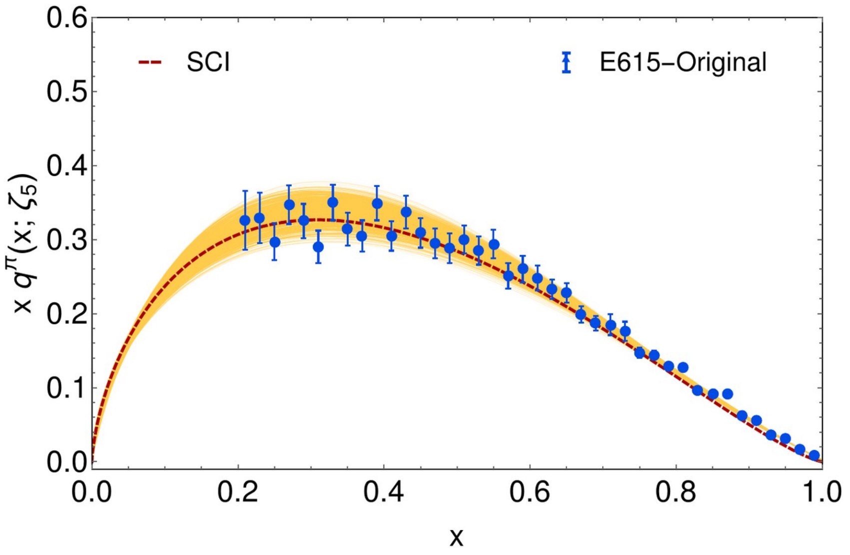


$$F_\pi(\Delta^2) = \int_0^1 dx\, H_\pi^u(x,0,-\Delta^2)$$

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \theta(x_-)\sqrt{u^\pi(x_-;\zeta_H)u^\pi(x_+;\zeta_H)}\,\Phi^\pi(z^2;\zeta_H)$$

Pion PDF: Original Data

- Applying this algorithm to the original data yields:



(average)

Mean values (of moments) and errors, ζ_H

{{0.5, 2.52187 × 10^-17}, {0.331527, 0.00803273}, {0.247615, 0.0110893}, {0.19784, 0.0121977}, {0.165066, 0.0124911}, {0.141928, 0.0124198}, {0.124755, 0.0121811}, {0.111521, 0.0118683}, {0.101021, 0.0115275}, {0.0924926, 0.0111824}, {0.085431, 0.010845}, {0.0794897, 0.0105214}, {0.0744232, 0.0102142}, {0.0700521, 0.00992435}, {0.0662432, 0.00965182}}

(SCI)

Moments from SCI, ζ_H

{0.5, 0.332885, 0.249327, 0.199231, 0.165865, 0.142056, 0.124215, 0.11035, 0.0992657, 0.090203, 0.0826552, 0.0762721, 0.0708035, 0.0660661, 0.0619225}

✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✗ But also exhibit agreement with the **SCI results.**

$$q_{\text{SCI}}(x;\zeta_H) \approx 1$$

Thus, given the **expectation**

$$u^{\pi}(x;\zeta) \overset{x\simeq 1}{\sim} (1-x)^{\beta = 2+\gamma(\zeta)}$$

We shall **discard** this for the upcoming construction of the valence quark GPD