SCHOOL ON NONLINEAR DYNAMICS, COMPLEX NETWORKS, INFORMATION THEORY, AND MACHINE LEARNING IN NEUROSCIENCE

INTRODUCTION TO SYNCHRONIZATION PHENOMENA AND THE KURAMOTO MODEL

GOTHAm lab

Universidad Zaragoza 1542

Institute for Biocomputation & Physics of Complex Systems (BIFI)

JESÚS GÓMEZ-GARDEÑES, UNIV. OF ZARAGOZA

GROUP OF THEORETICAL & APPLIED MODELING

JESÚS GÓMEZ-GARDEÑES, UNIV. OF ZARAGOZA

- **MS and Phd in Physics @ University of Zaragoza 2006**
- **Postdoc @ Scuola Superiore di Catania & URV 2007-2008**
- **Lecturer at U. Rey Juan Carlos in Madrid 2009-2010**
- **Ramón y Caja Fellow @ University of Zaragoza 2011-2015**
- **Full Professor @ University of Zaragoza since 2023**

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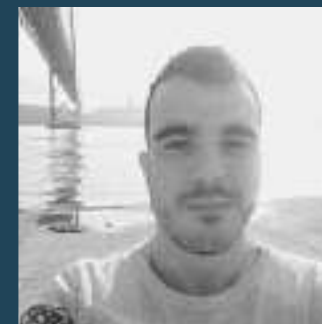
PERMANENT:


Jesús Gómez-Gardeñes


Luis Mario Floría

EXTERNAL:


David Soriano-Paños


Alessio Cardillo

PREDOC STUDENTS:


Francisco Bauza


Sergio Faci


Pablo Gallarta


Santiago Lamata


Hugo Pérez


Pablo Valgañón

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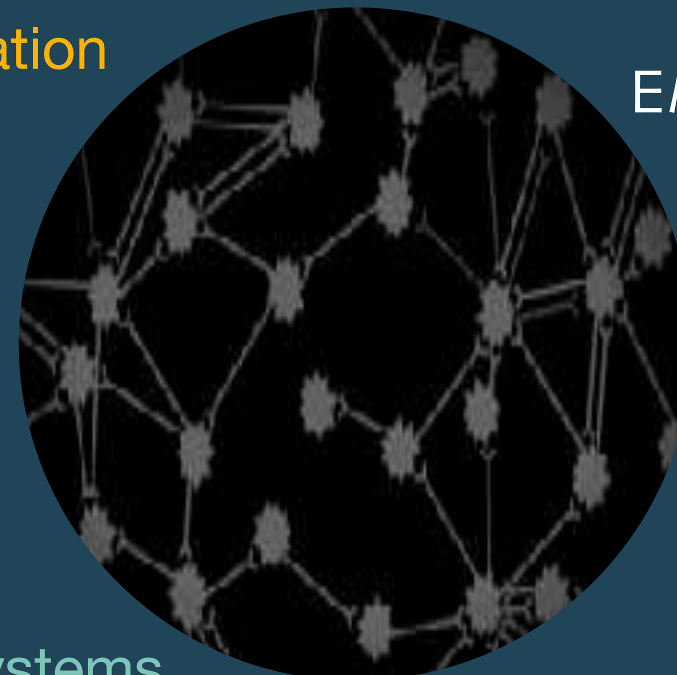
Culture transmission and spread of infrormation

Cooperation and evolutionary dynamics

Congestion in Communication systems

Epidemic dynamics in large populations

Synchronization of interacting dynamical systems



EMERGENCE OF COLLECTIVE PHENOMENA IN LARGE ENSEMBLES OF INTERACTING UNITS

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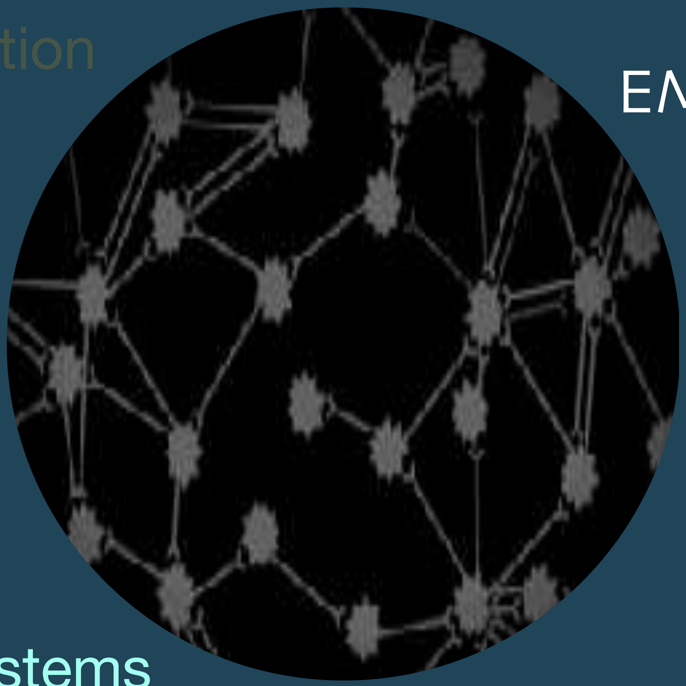

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LECTURE 1: INTRODUCTION TO SYNCHRONIZATION & THE KURAMOTO MODEL

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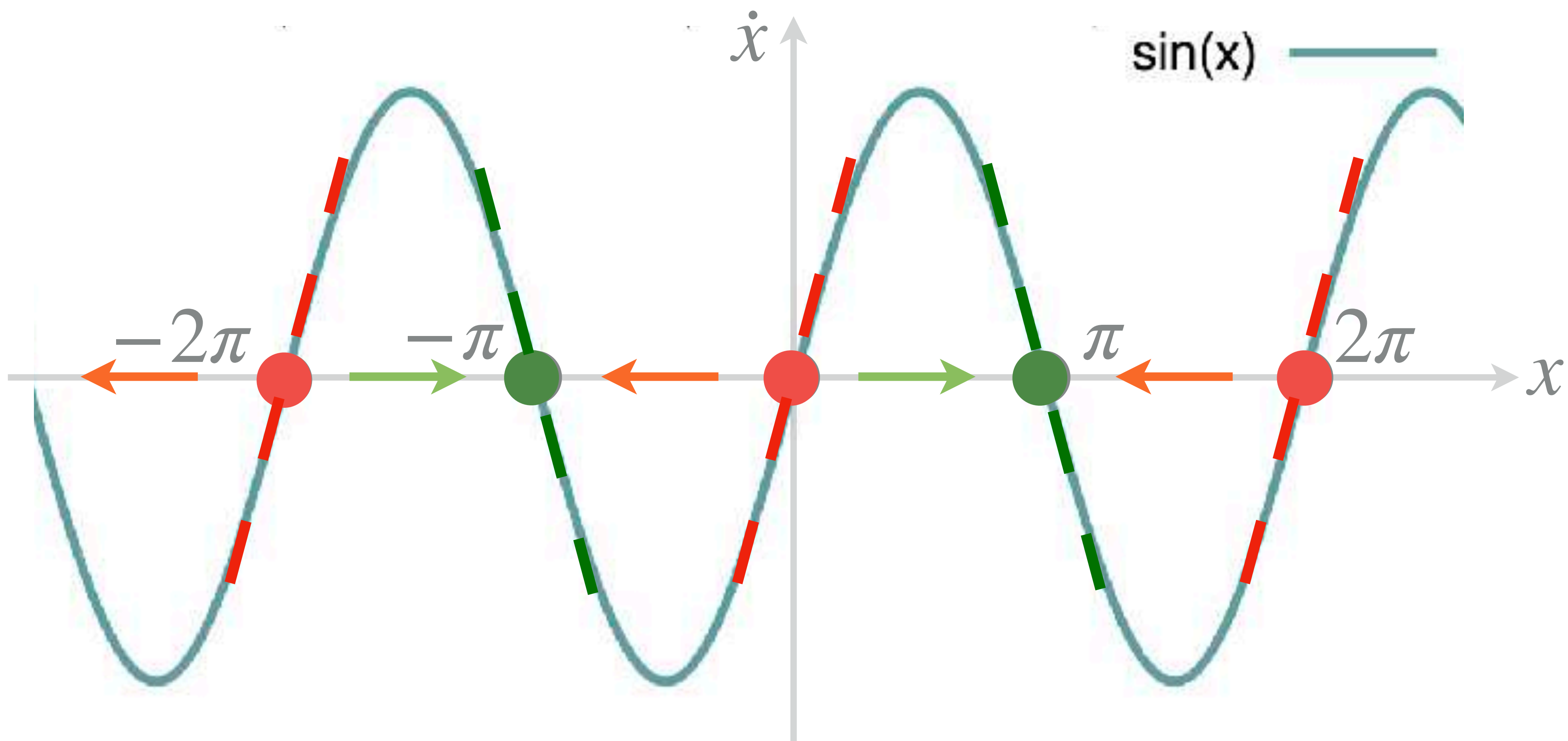
One dimensional Dynamical System:

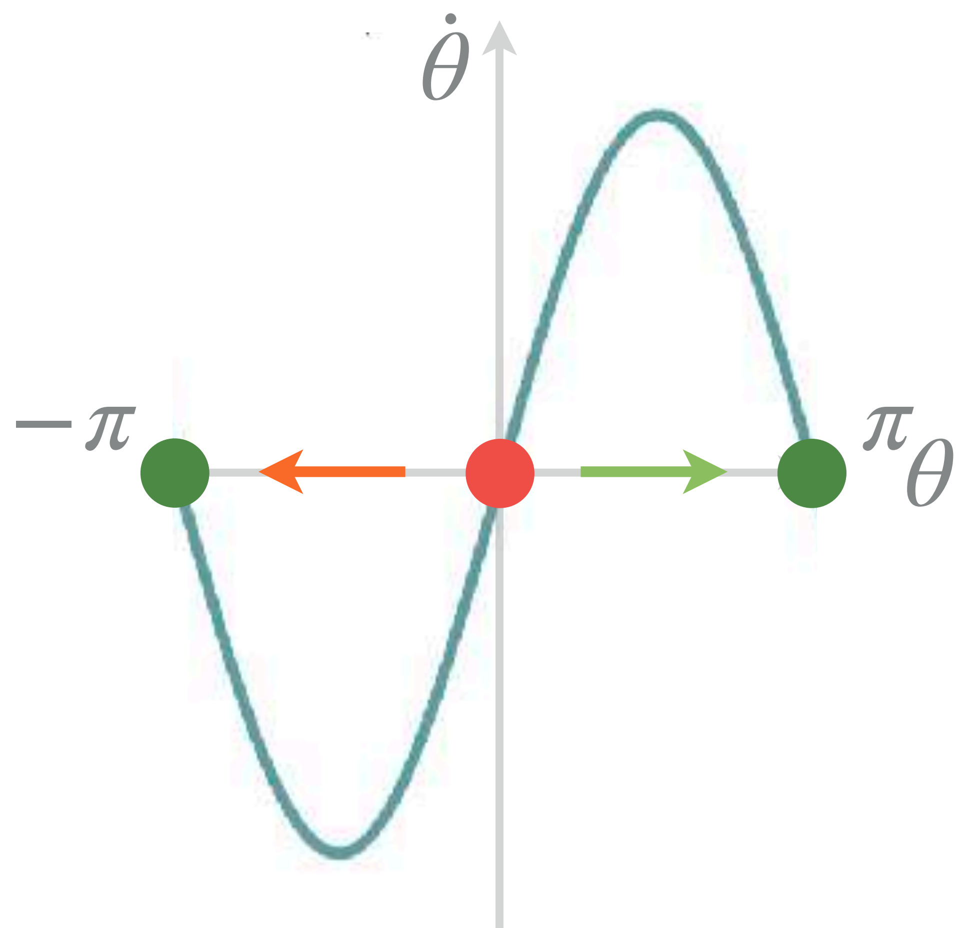
$$\frac{dx}{dt} = F(x) \longrightarrow \frac{dx}{dt} = \sin(x)$$



Fixed Points $(\dot{x} = 0)$ $\left\{ \begin{array}{l} \text{STABLE Fixed Points} \\ \text{UNSTABLE Fixed Points} \end{array} \right\} F'(x^*)$

One dimensional Dynamical System on the circle:

$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \sin(\theta)$$



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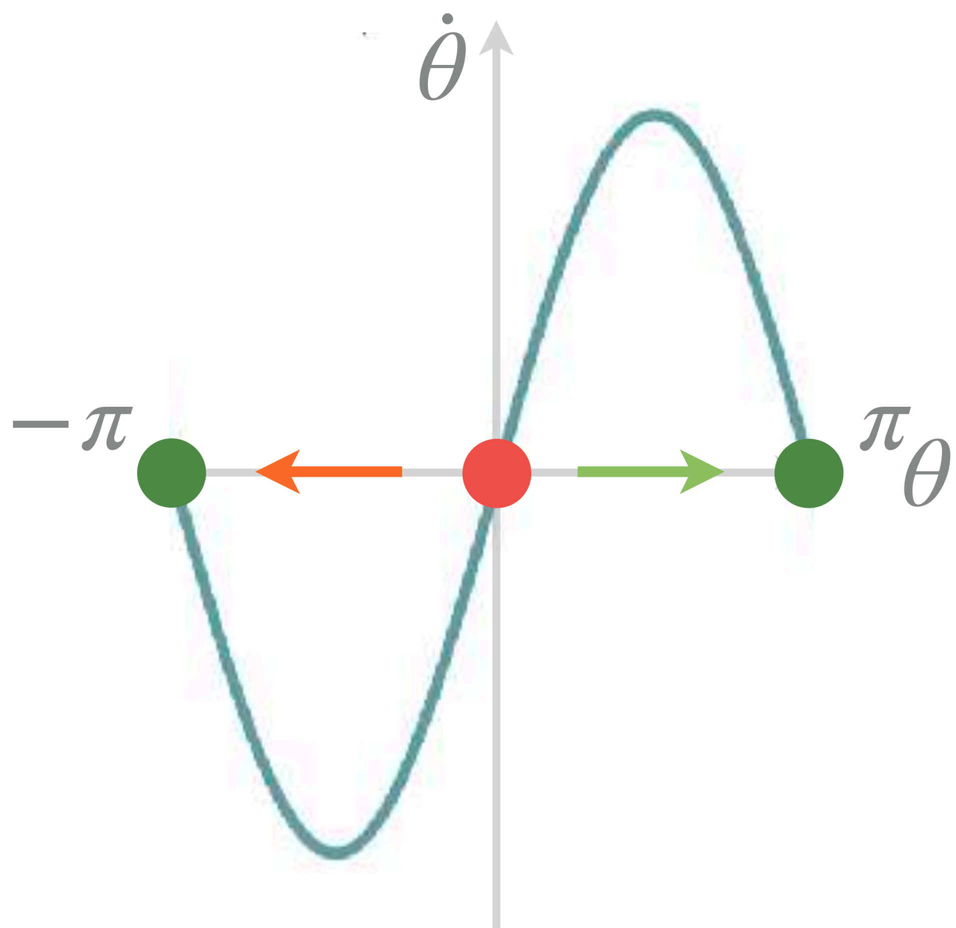

$F(\theta)$ must be a 2π-periodic function of θ:

$$F(\theta \pm n2\pi) = F(\theta)$$

Fixed Points $(\dot{\theta} = 0)$
- STABLE Fixed Points
- UNSTABLE Fixed Points

Index: From 2 to N coupled phase oscillators

- NONUNIFORM OSCILLATORS
- SYNCHRONIZATION WITH AN EXTERNAL STIMULUS
- SYNCHRONIZATION OF AN ENTIRE POPULATION:

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UNIFORM Oscillation:

$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \omega$$

Fixed Points $(\dot{\theta} = 0)$ $\begin{cases} \text{STABLE Fixed Points} \\ \text{UNSTABLE Fixed Points} \end{cases}$

UNIFORM Oscillation:

$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \omega$$

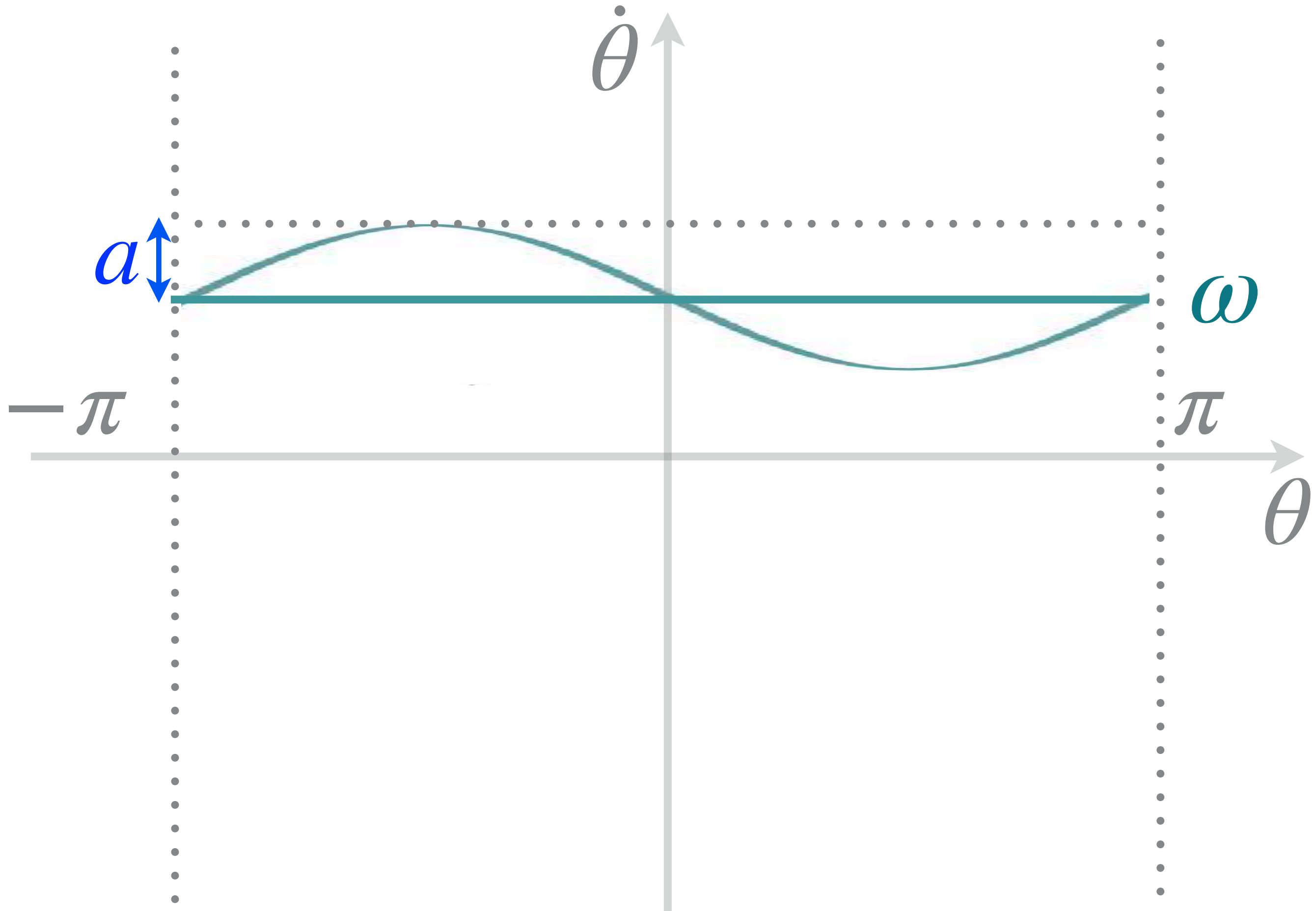
- NO Fixed Points!
- Periodic Oscillations with:

$$T = \frac{2\pi}{\omega}$$

Fixed Points $(\dot{\theta} = 0)$ $\begin{cases} \text{STABLE Fixed Points} \\ \text{UNSTABLE Fixed Points} \end{cases}$

NONUNIFORM Oscillation:

$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \omega - a\sin(\theta)$$



- NO Fixed Points!
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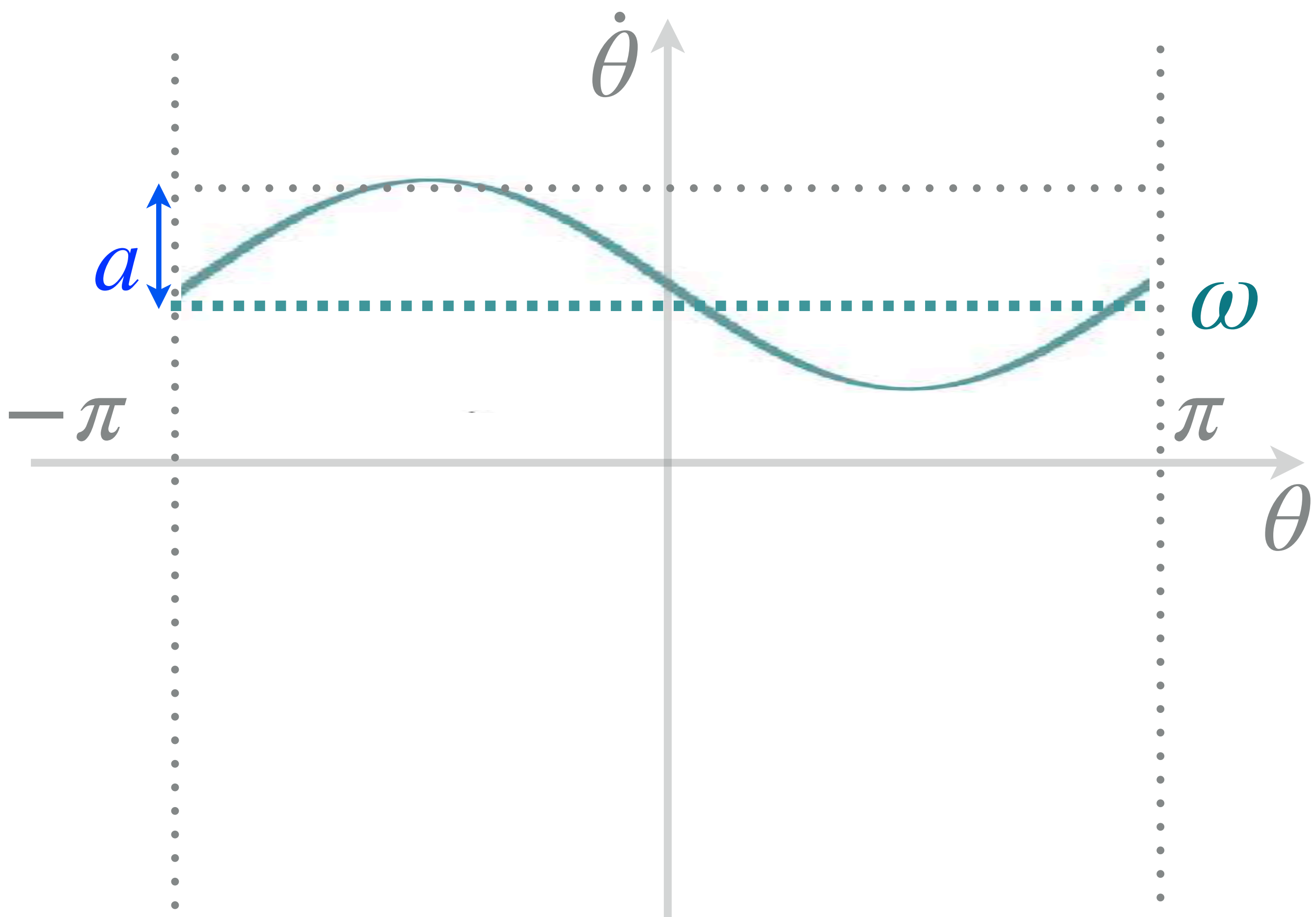

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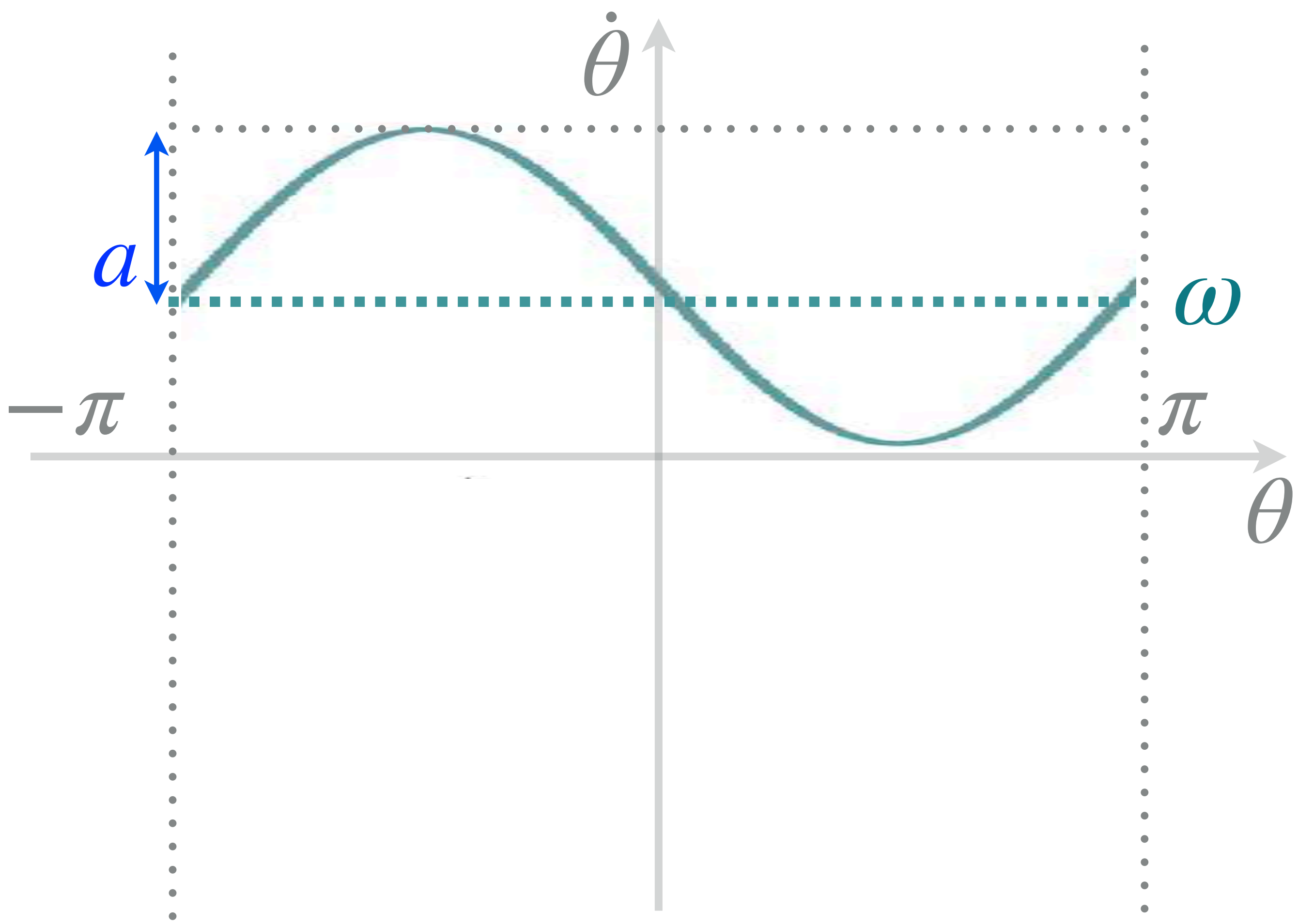

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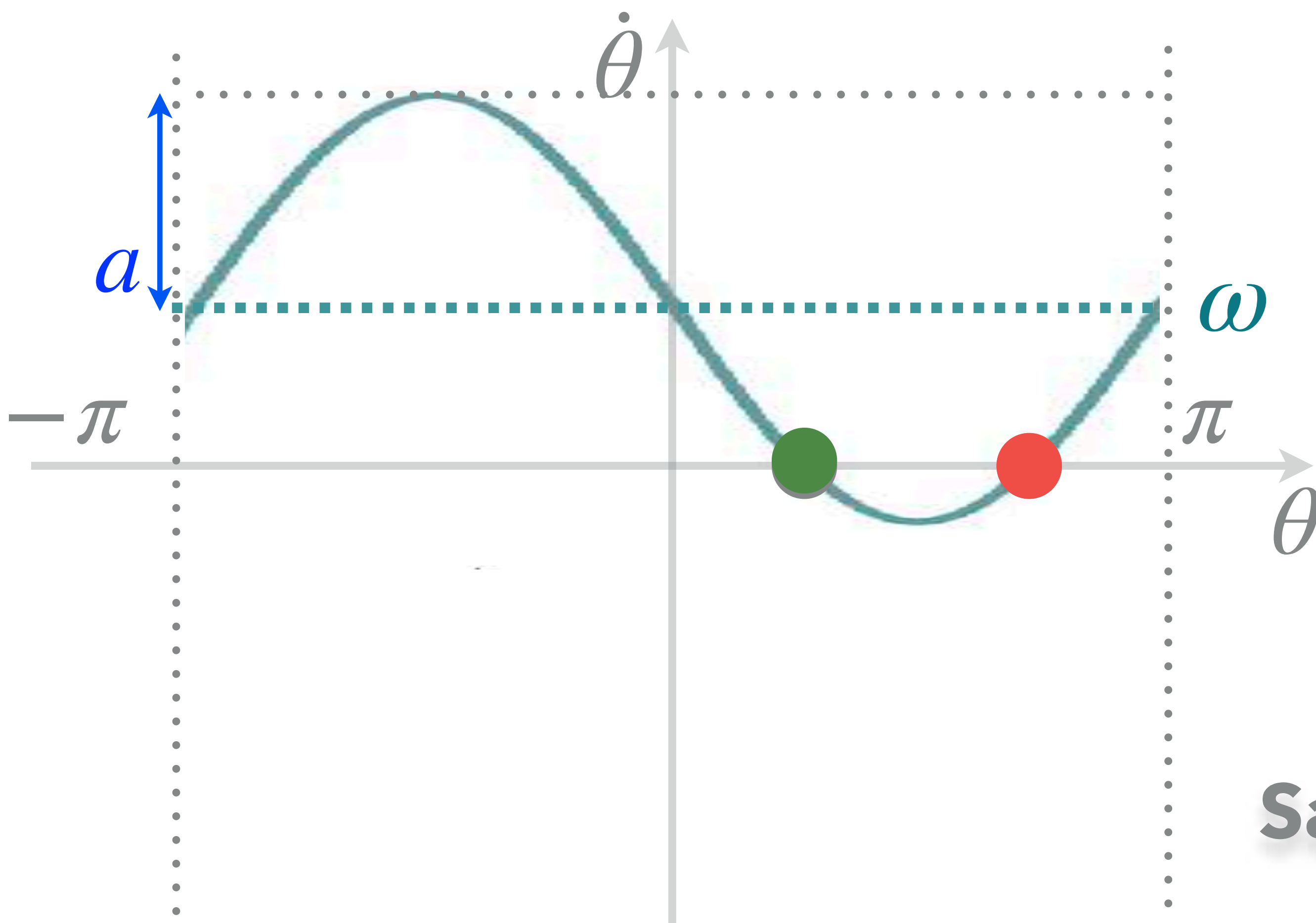

- ~~NO Fixed Points!~~
- Periodic Oscillations with: $\cancel{T = \frac{2\pi}{\omega}}$

Saddle node bifurcation with $a_c = \omega$

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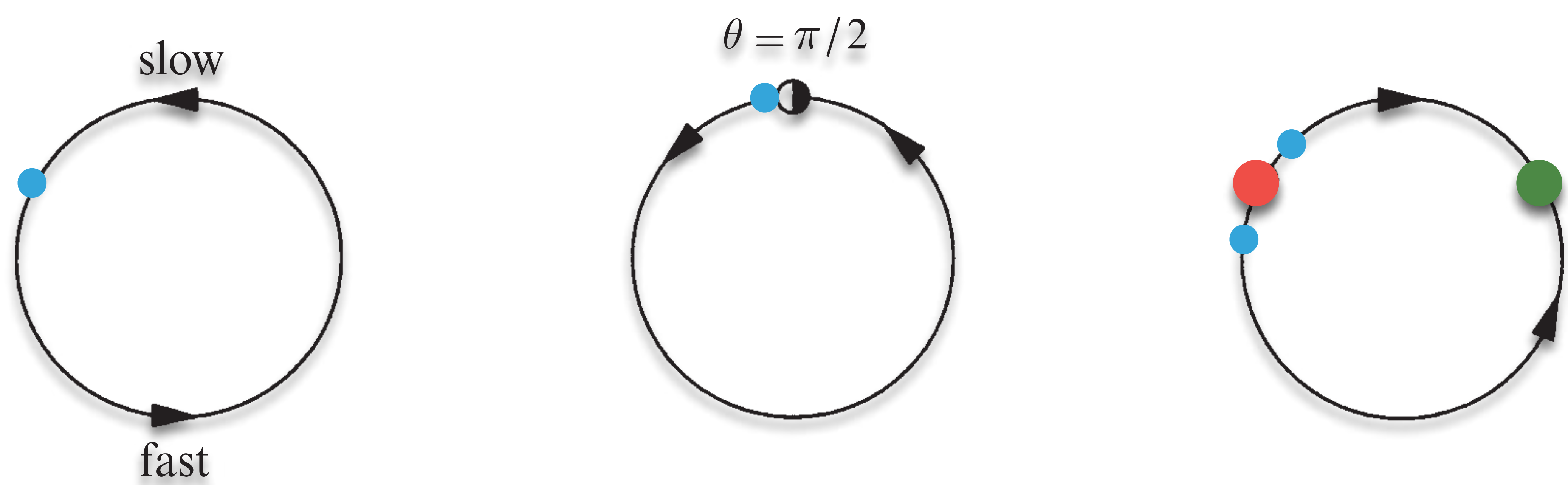

(a) $a < \omega$ (b) $a = \omega$ (c) $a > \omega$

NONUNIFORM Oscillation:

$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \omega - a\sin(\theta)$$

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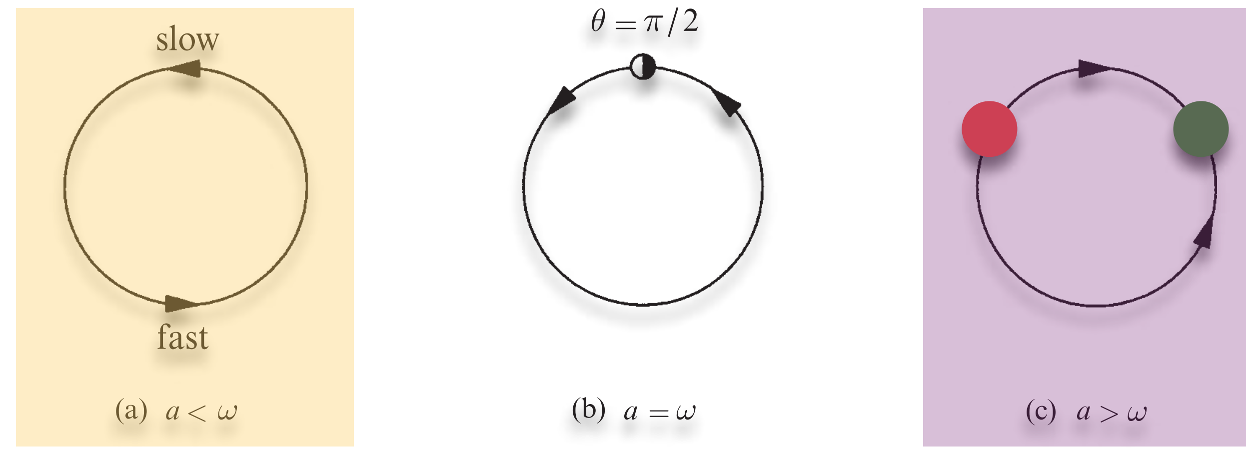


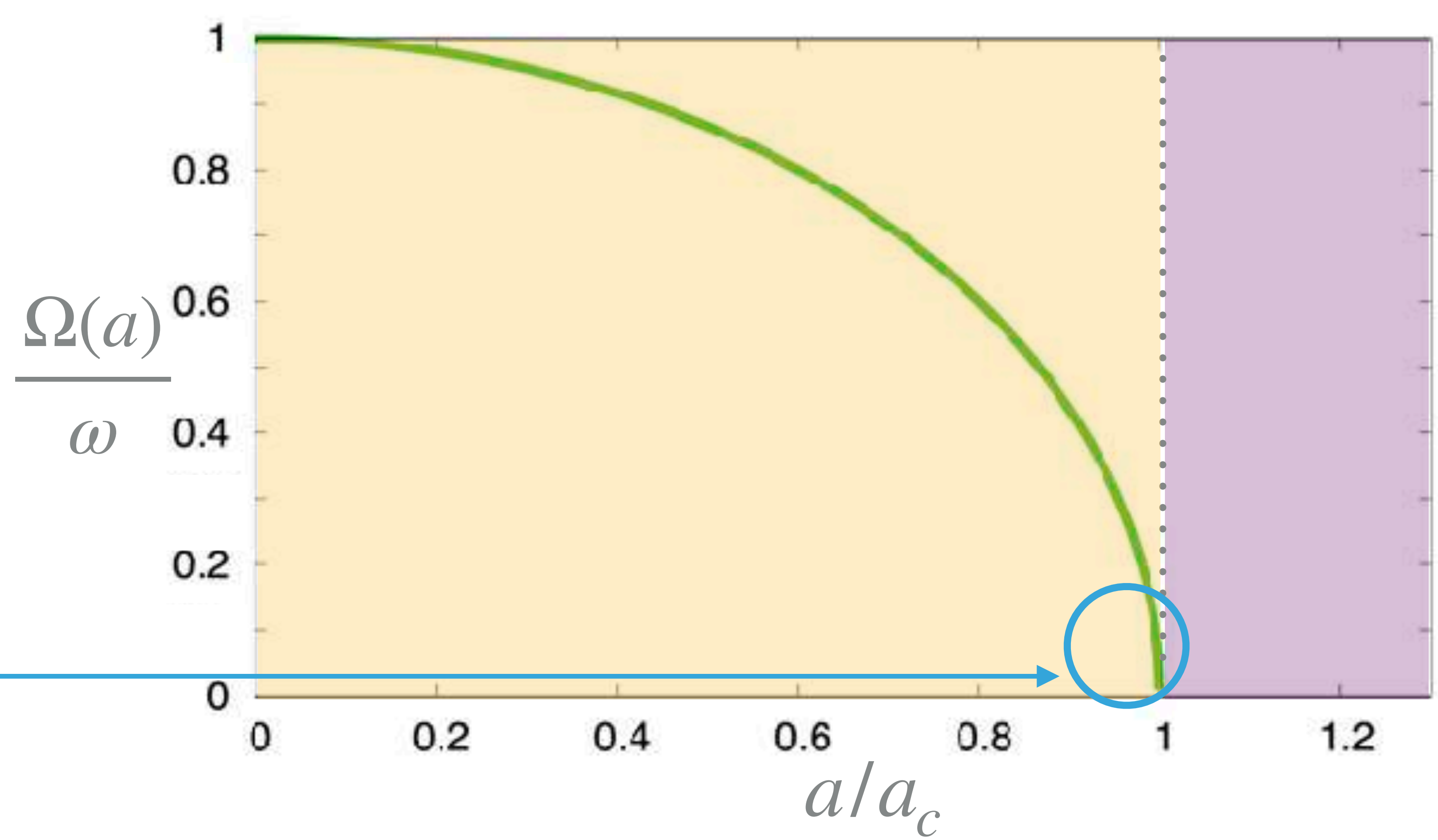
$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}$$

$$= \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

$$\Omega(a) = \sqrt{\omega^2 - a^2}$$

As a tends to $a_c = \omega$:

$$\Omega(a) \rightarrow \sqrt{2\omega}(a_c - a)^{1/2}$$



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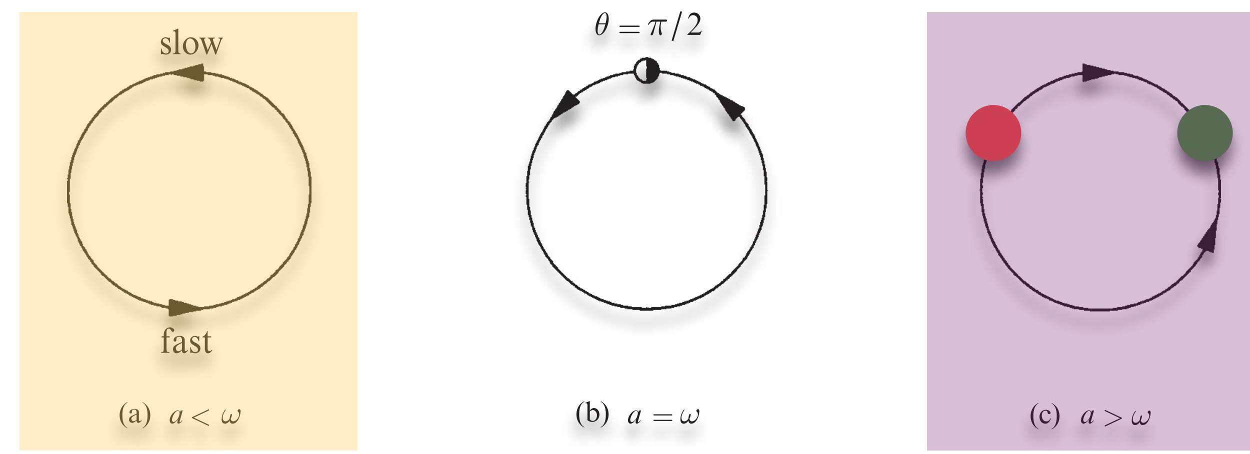

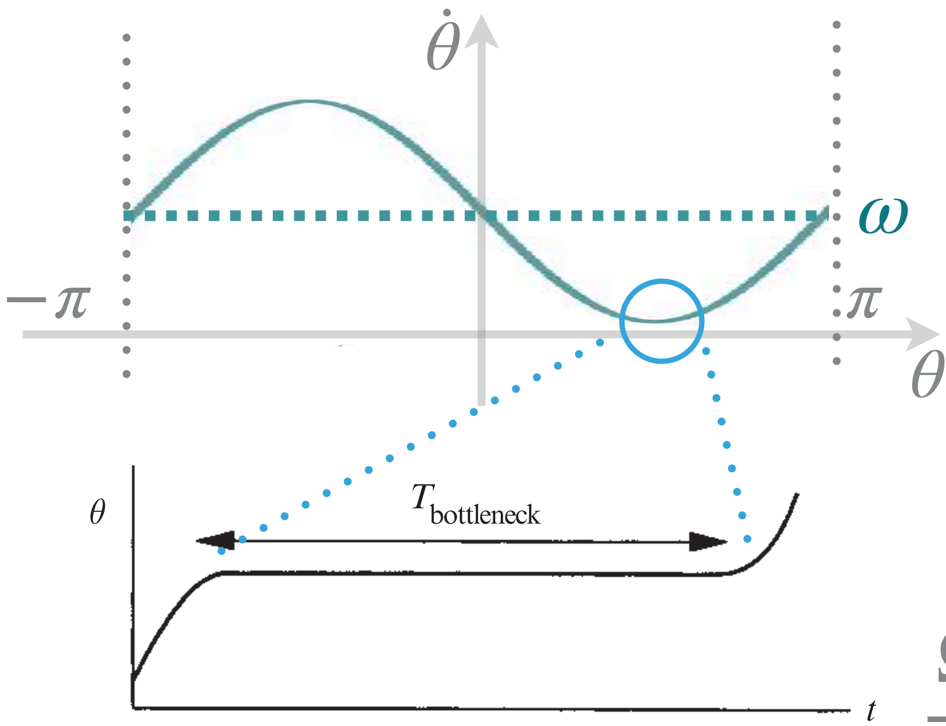

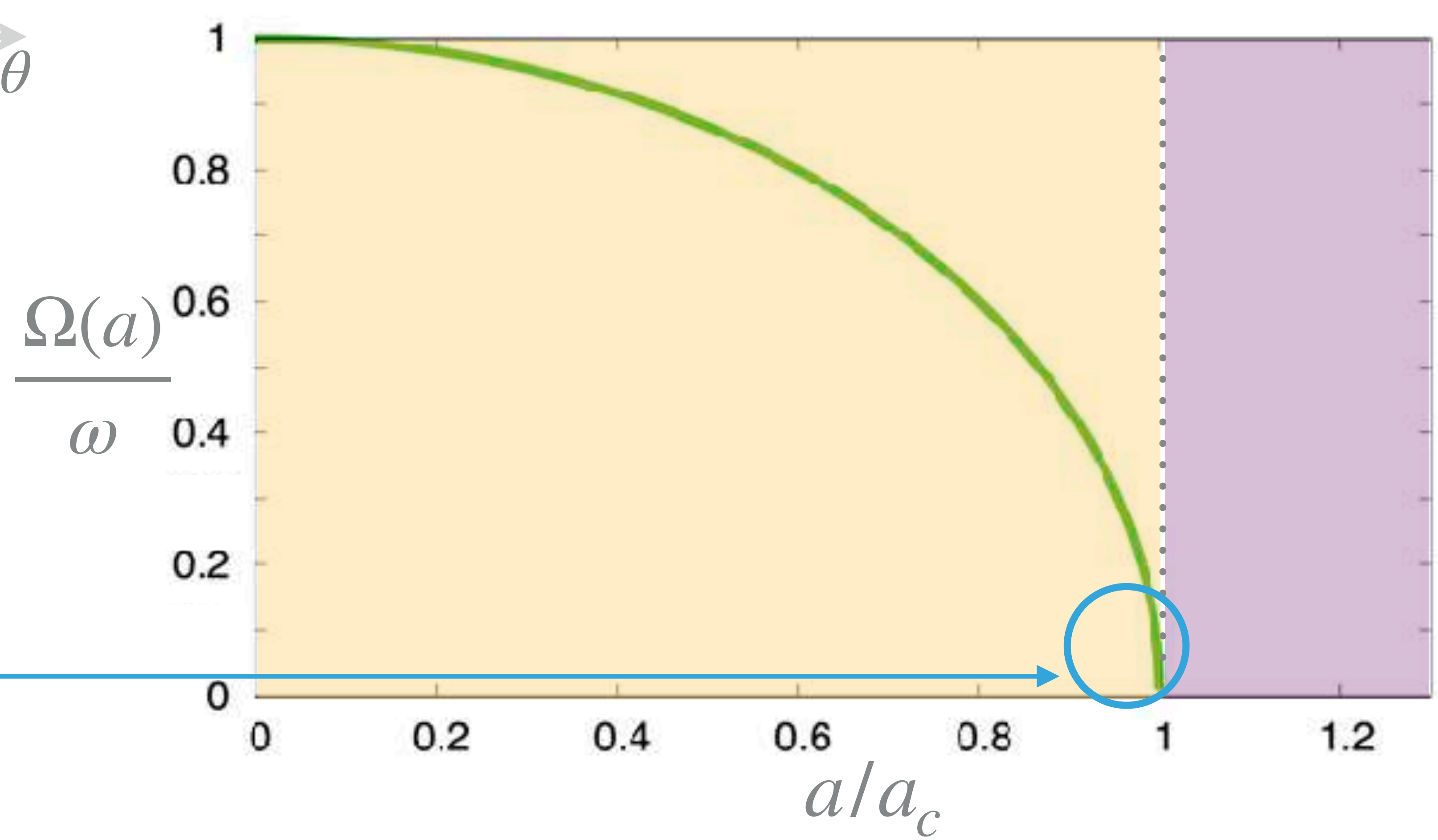

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SYNC WITH EXTERNAL STIMULUS

Ermentrout & Rinzel, *Am. J. Physiol. (1984)*:

$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \omega + a\sin(\Theta - \theta)$$

$\xleftrightarrow{\;a\;}$

The firefly attempts to synchronize with the overall rhythm

SYNC WITH EXTERNAL STIMULUS

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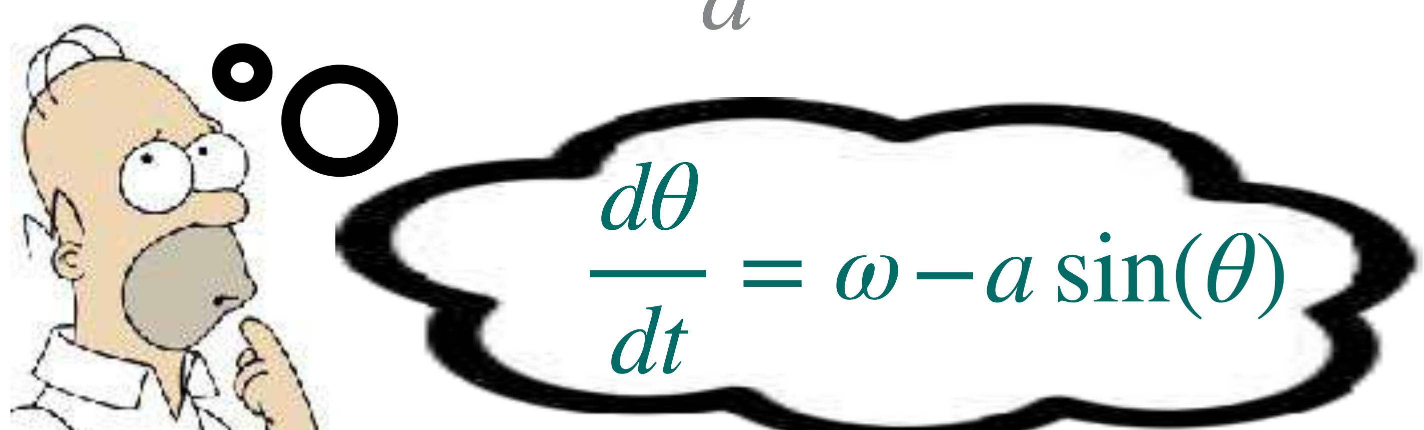
$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \omega + a\sin(\Theta - \theta)$$

- **Assumption:** The bulk remains unaltered by the effect of our firefly, so that it oscillates periodically as: $\dot{\Theta} = \Omega$
- Lets consider: $\phi = \Theta - \theta$

$$\frac{d\phi}{dt} = \frac{d\Theta}{dt} - \frac{d\theta}{dt} = \Omega - \omega - a\sin(\phi)$$

- Adimensionalization: $\tau = at$ & $\tilde{\omega} = \dfrac{\Omega - \omega}{a}$

$$\frac{d\phi}{d\tau} = \tilde{\omega} - \sin(\phi)$$



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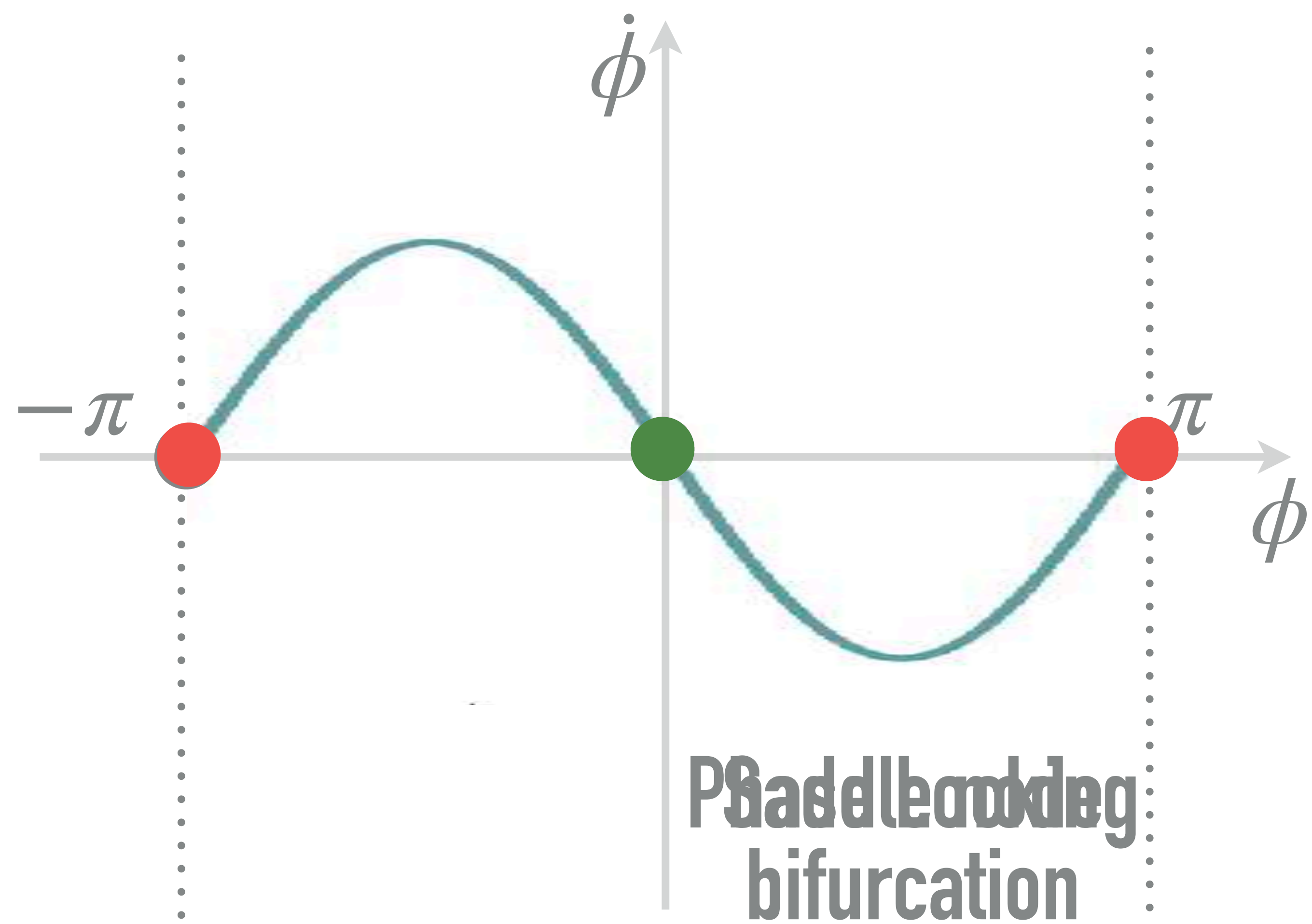
$$\frac{d\phi}{d\tau} = \tilde{\omega} - \sin(\phi) \quad \text{with} \quad \tilde{\omega} = \frac{\Omega - \omega}{a}$$

- Firefly synchronized with the population $\tilde{\omega} = 0$.
- What happens when $\tilde{\omega}$ increases?
- What does it mean that $\tilde{\omega}$ increases?



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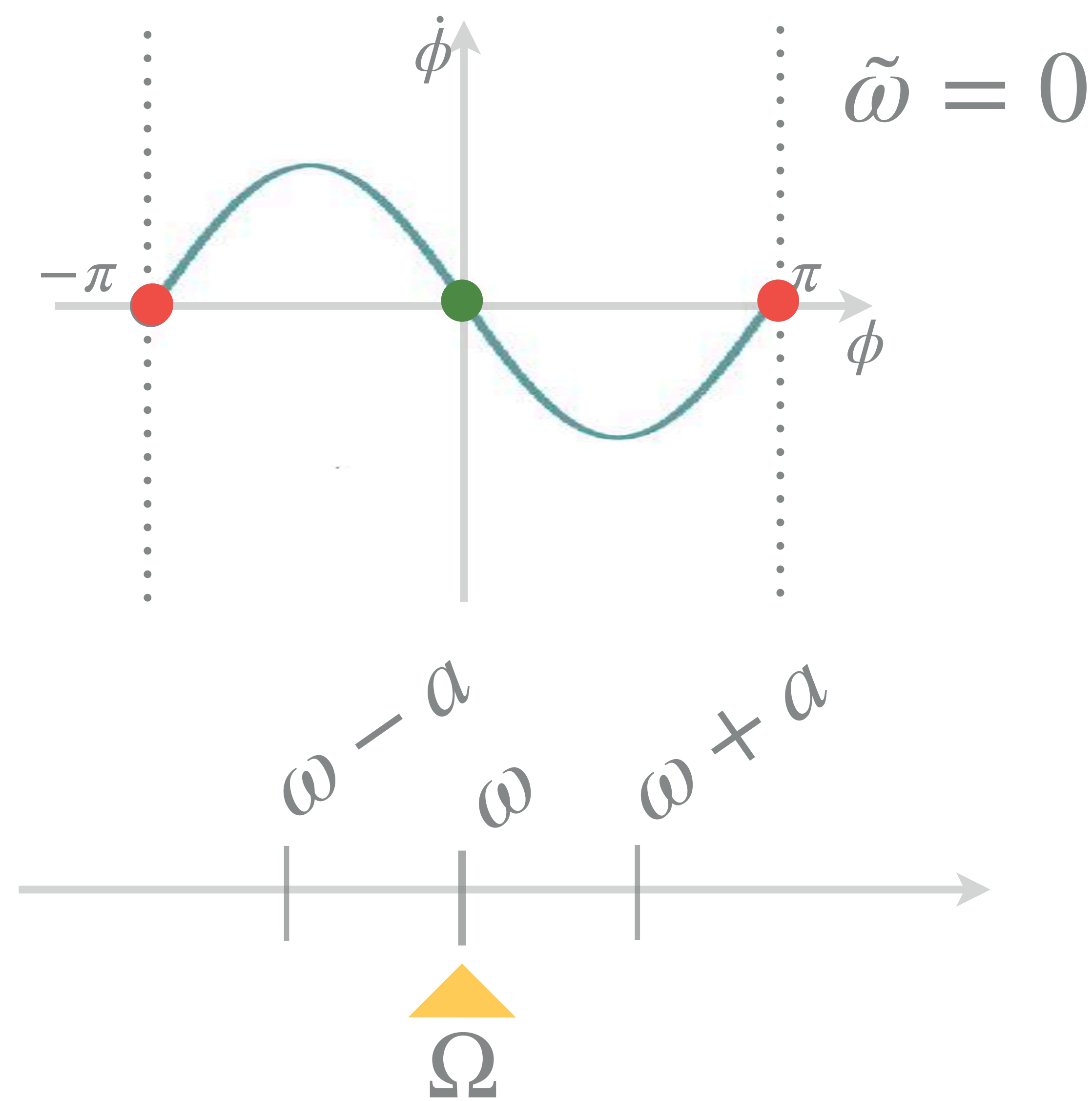

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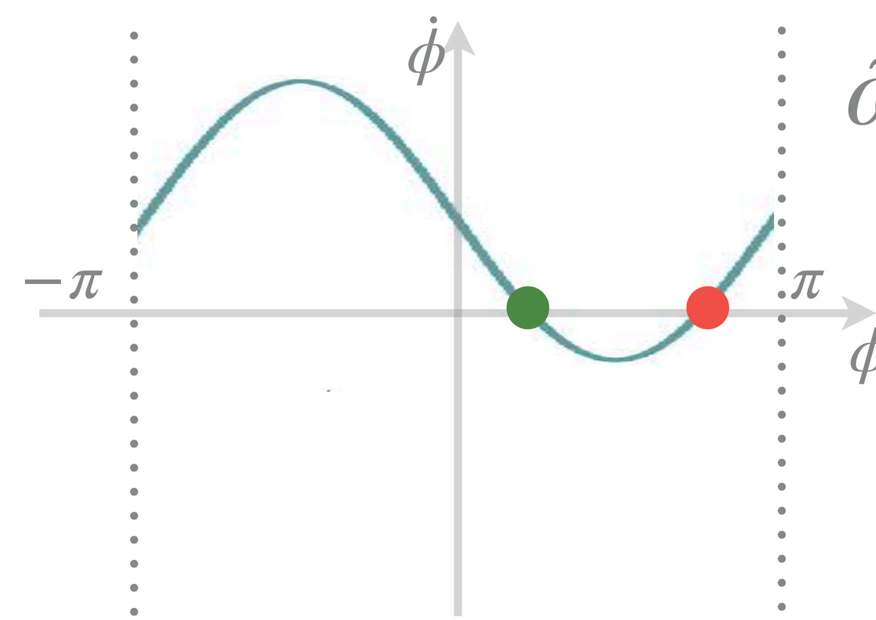
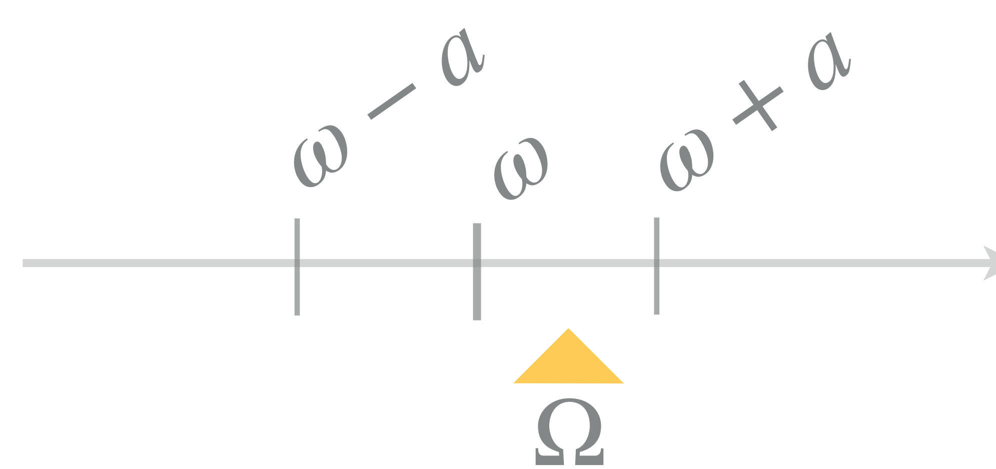


$\tilde{\omega} < 1$



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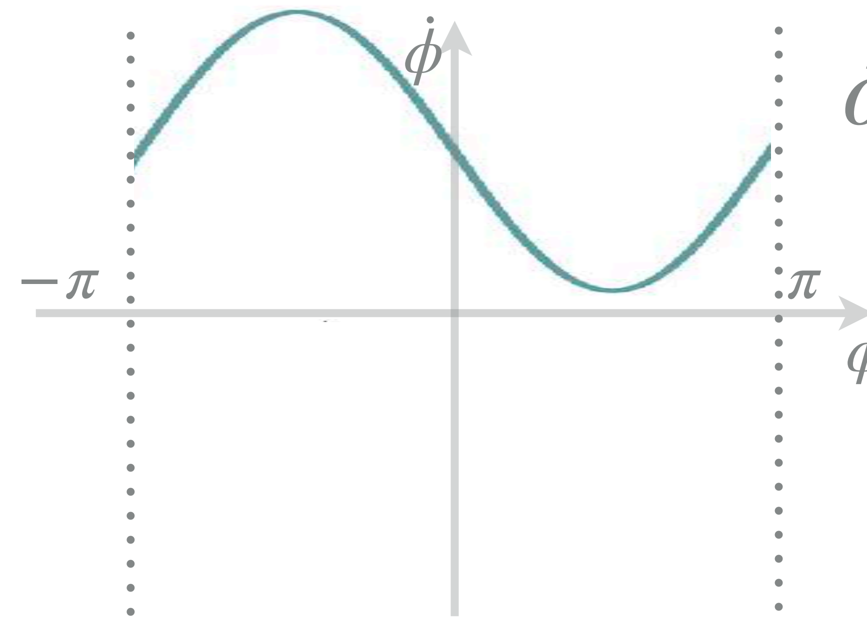
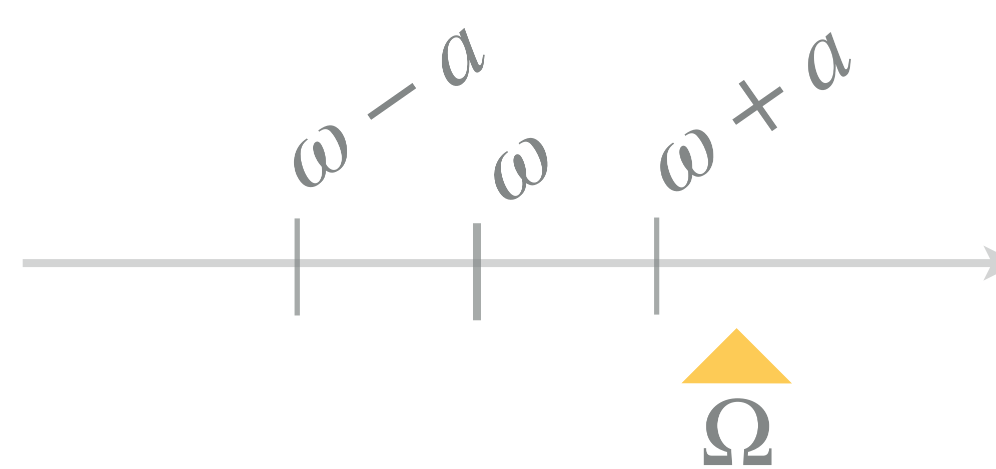
- What does it mean that $\tilde{\omega}$ increases?

$$\tilde{\omega} = \frac{\boxed{\Omega - \omega}}{a}$$ Fixed!



$\tilde{\omega} > 1$



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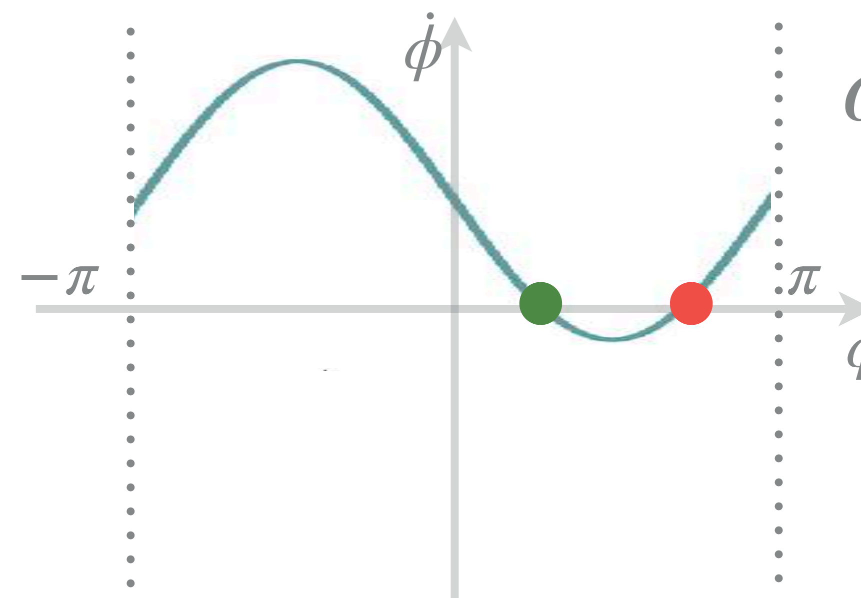
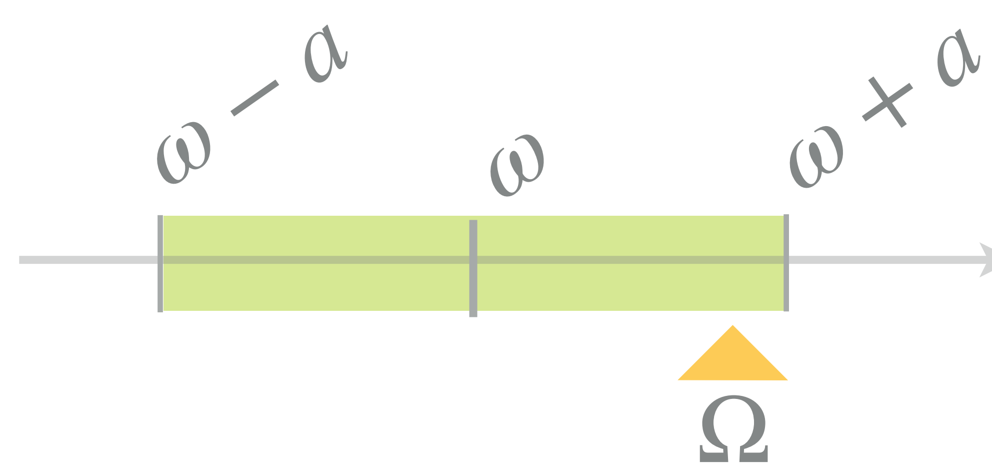


By varying the coupling of the firefly we can adjust the range of entrainment



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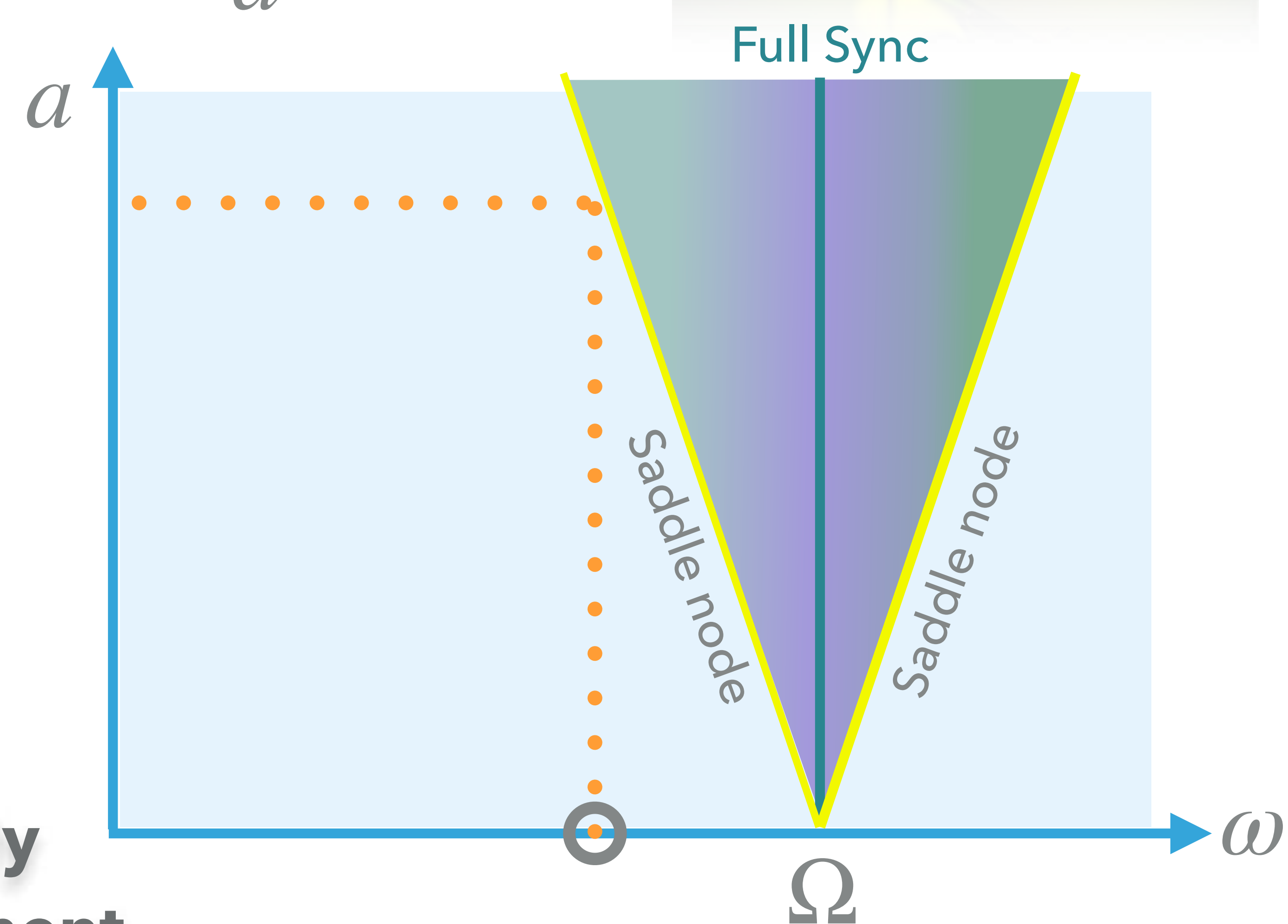


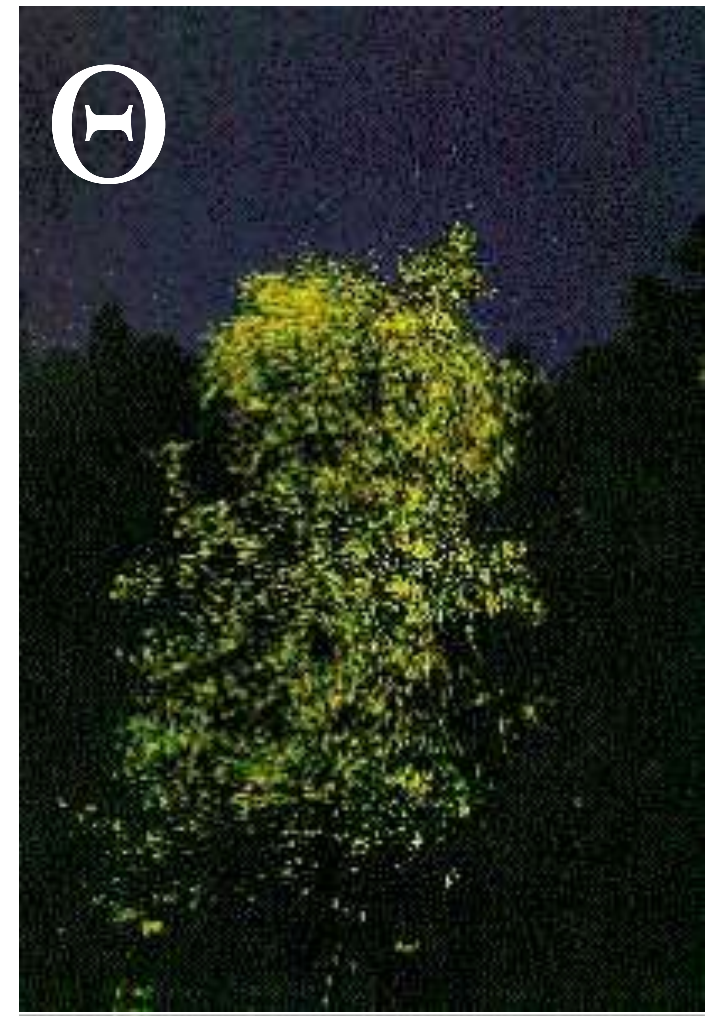
By varying the coupling of the firefly we can adjust the range of entrainment

SYNC WITH EXTERNAL STIMULUS

Ermentrout & Rinzel model (1984):

$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \omega - a\sin(\Theta - \theta)$$

a



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Ermentrout & Rinzel, *Am. J. Physiol. (1984)*:

$$\frac{d\theta}{dt} = F(\theta) \longrightarrow \frac{d\theta}{dt} = \omega - a\sin(\Theta - \theta)$$

Oversimplification: We have assumed that the bulk is already synchronized when our little firefly enters into play, but…

How do thousands of fireflies get synchronized?

OPEN PROBLEM:

SYNCHRONIZATION OF POPULATIONS OF COUPLED UNITS

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Synchronization

Ermentrout & Rinzel model (1984):

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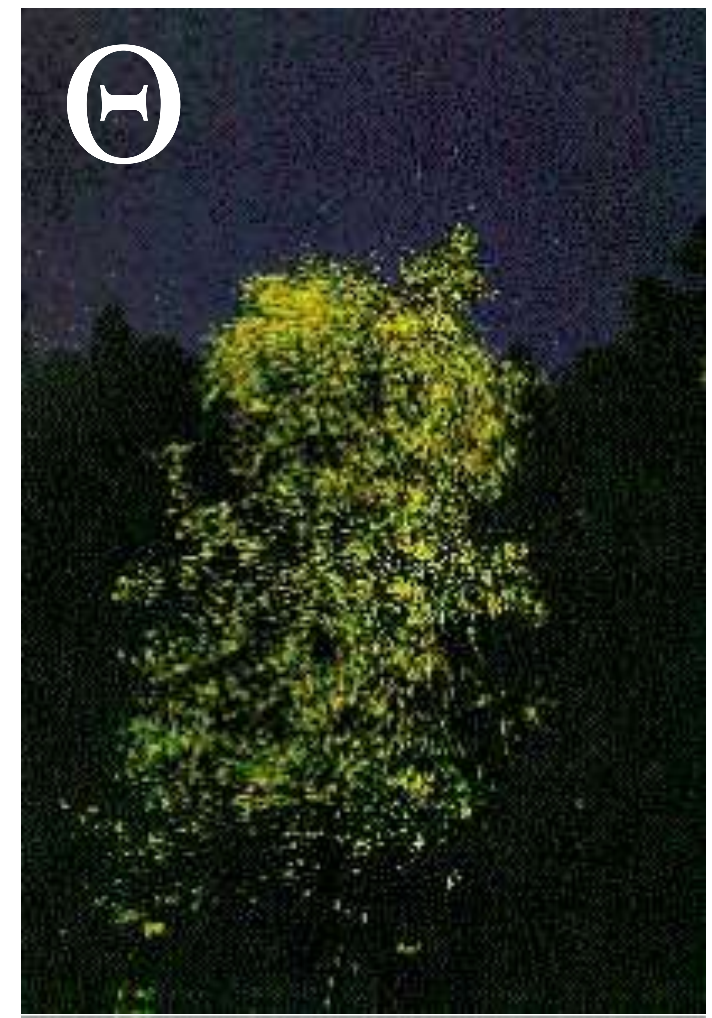
$\longleftarrow a \longrightarrow$

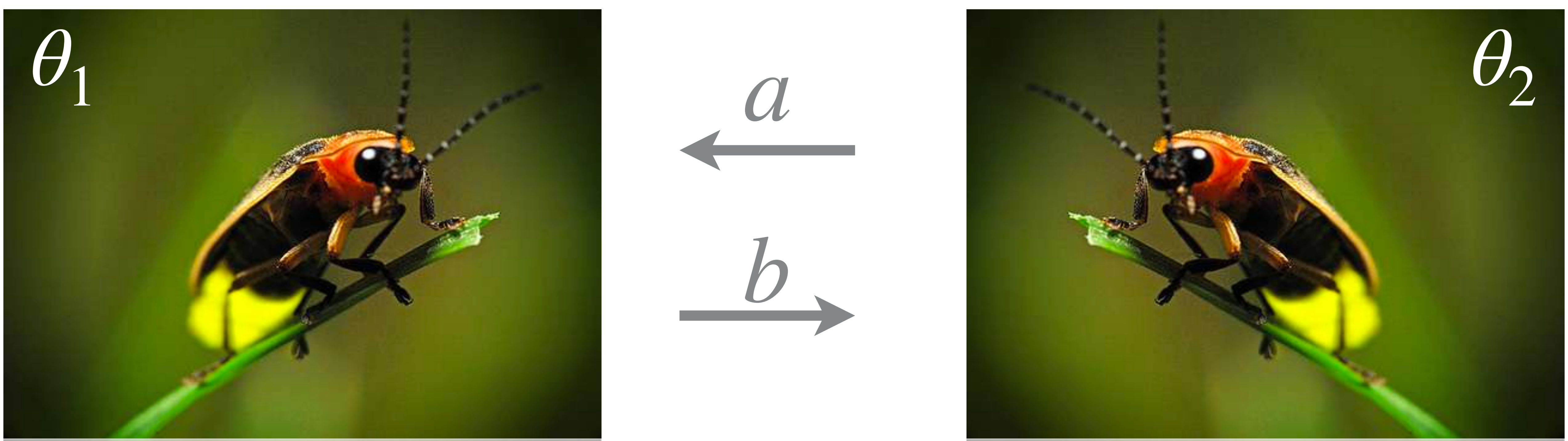


The firefly attempts to synchronize with the overall rhythm

Two coupled units:

$$\frac{d\theta_1}{dt} = \omega_1 + a\sin(\theta_2 - \theta_1)$$

$$\frac{d\theta_2}{dt} = \omega_2 + b\sin(\theta_1 - \theta_2)$$



Two different fireflies attempt to synchronize their rhythms

Two coupled units:

$$\frac{d\theta_1}{dt} = \omega_1 + a\sin(\theta_2 - \theta_1)$$

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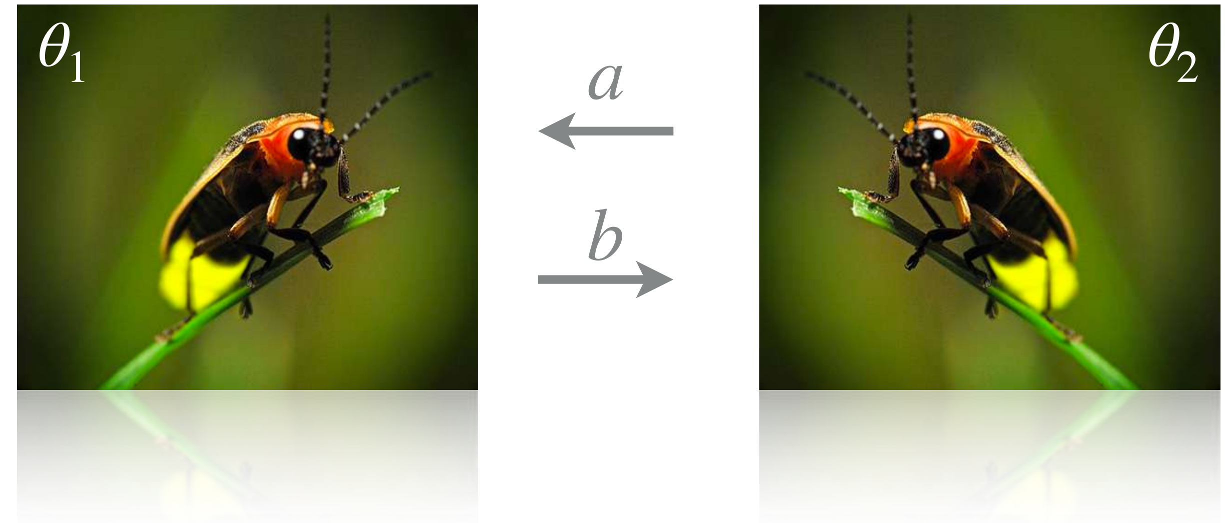
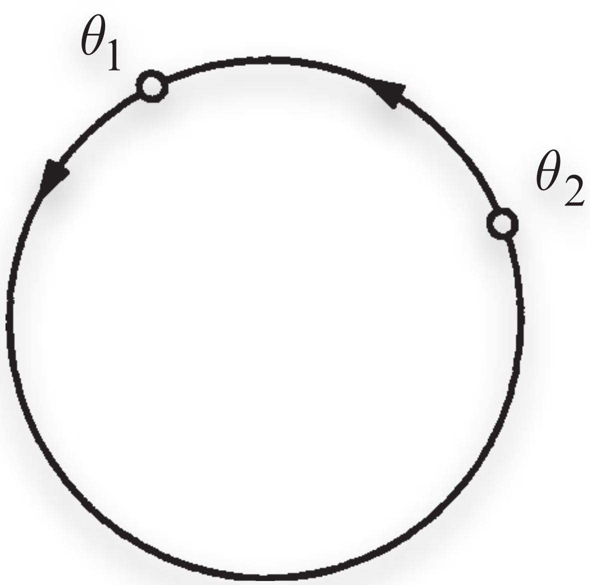
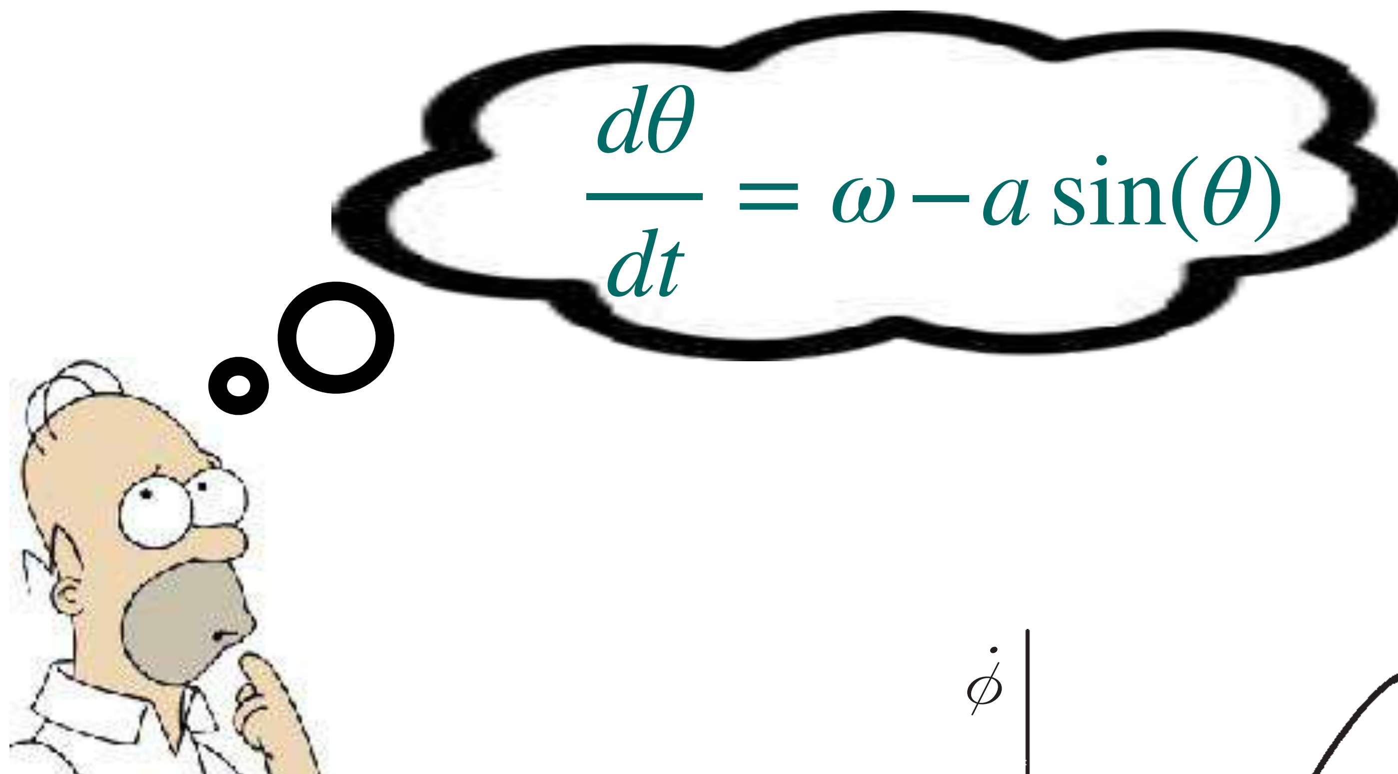
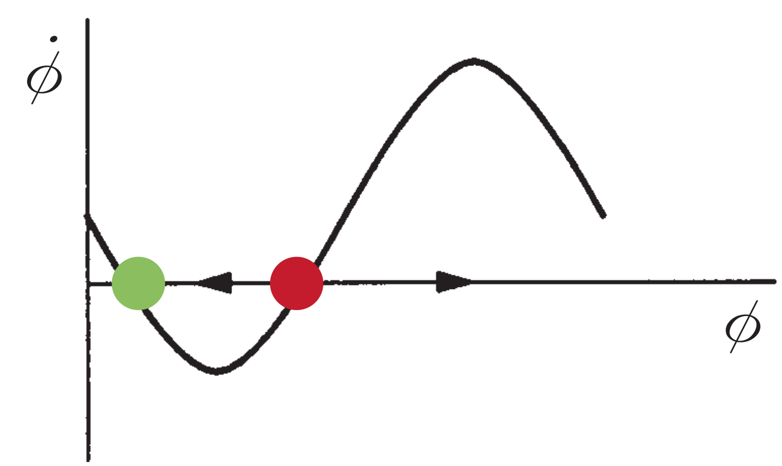






$$\phi = \theta_1 - \theta_2$$

$$\frac{d\phi}{dt} = \dot{\theta}_1 - \dot{\theta}_2 = (\omega_1 - \omega_2) - (a+b)\sin\phi$$



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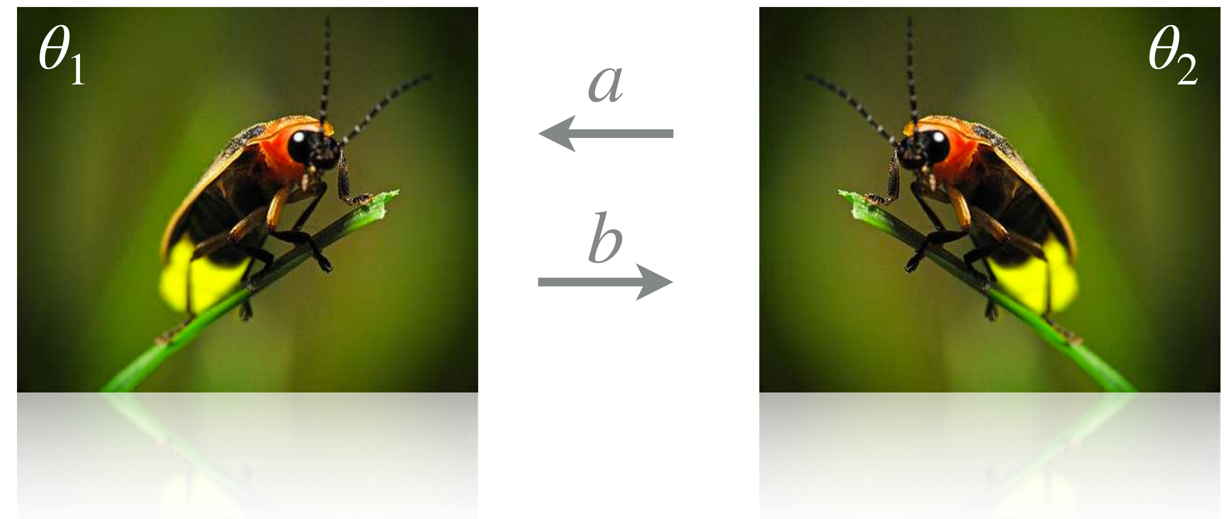

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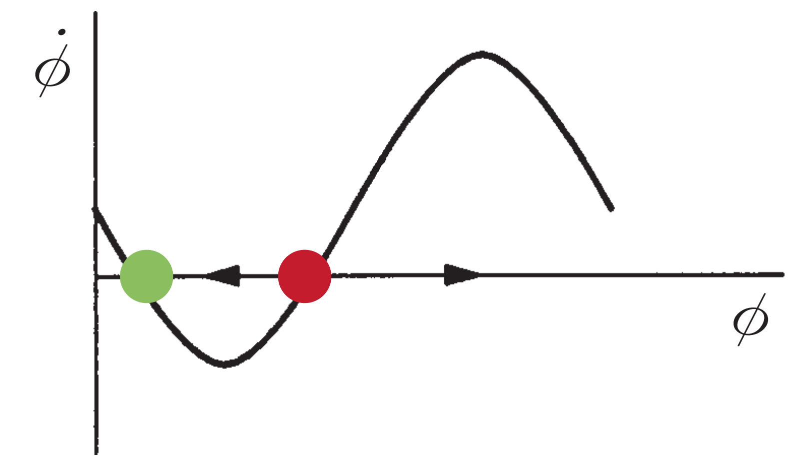
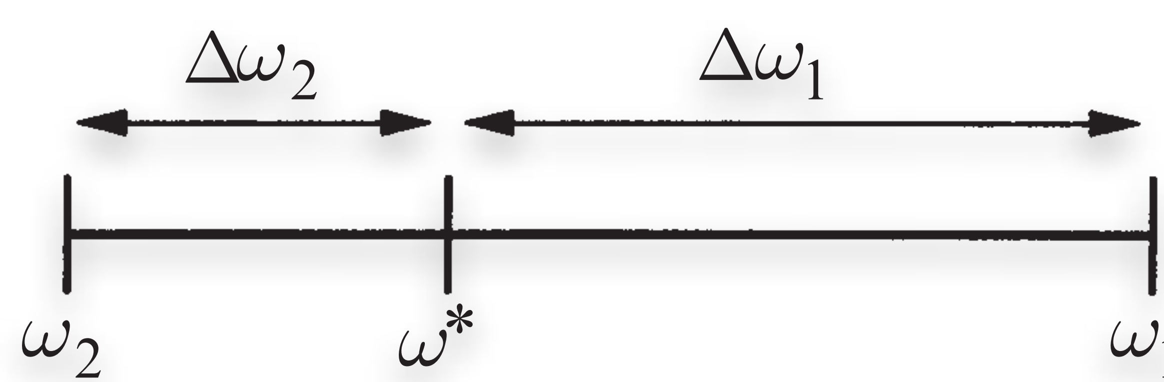


$$\dot{\phi} = 0: \ \sin\phi^* = \frac{\omega_1 - \omega_2}{a+b} \longrightarrow \dot{\theta}_1 = \dot{\theta}_2$$

Compromise Frequency:



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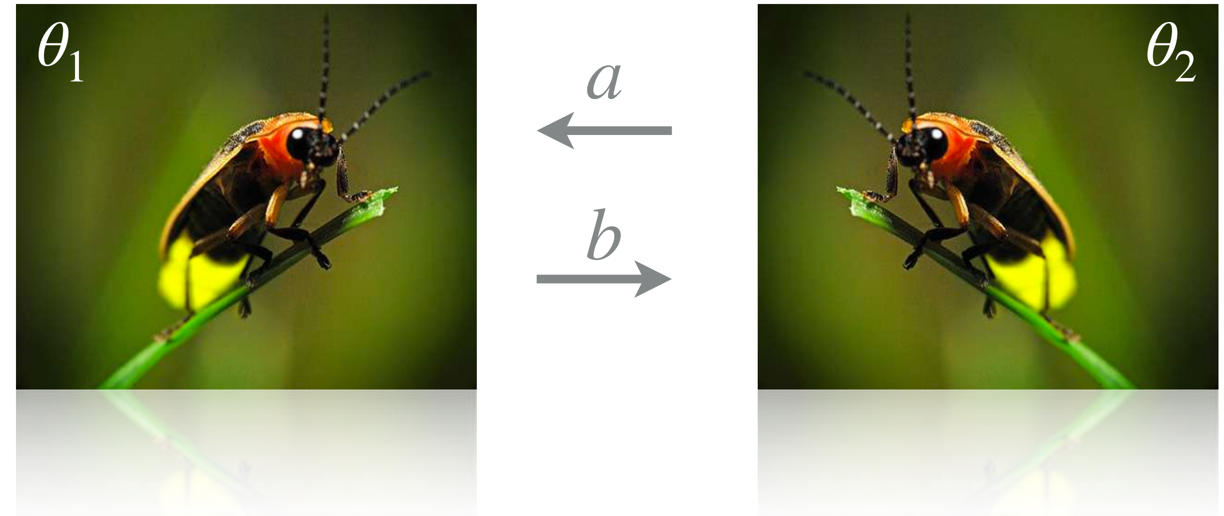
Two coupled units:

$$\frac{d\theta_1}{dt} = \omega_1 + a\sin(\theta_2 - \theta_1)$$

$$\frac{d\theta_2}{dt} = \omega_2 + b\sin(\theta_1 - \theta_2)$$



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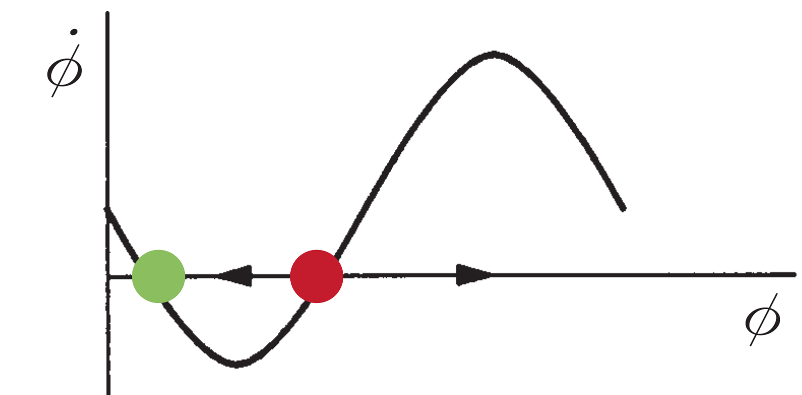
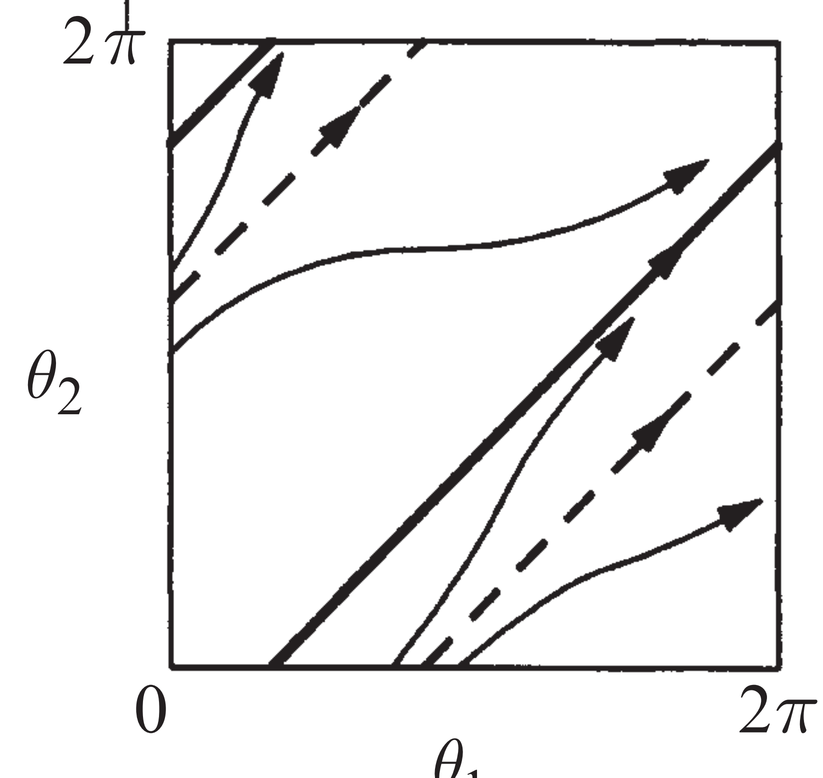


$$\sin\phi^* = \frac{\omega_1 - \omega_2}{a+b}$$

If $|\omega_1 - \omega_2|$ starts to increase until $|\omega_1 - \omega_2|$=(a+b)



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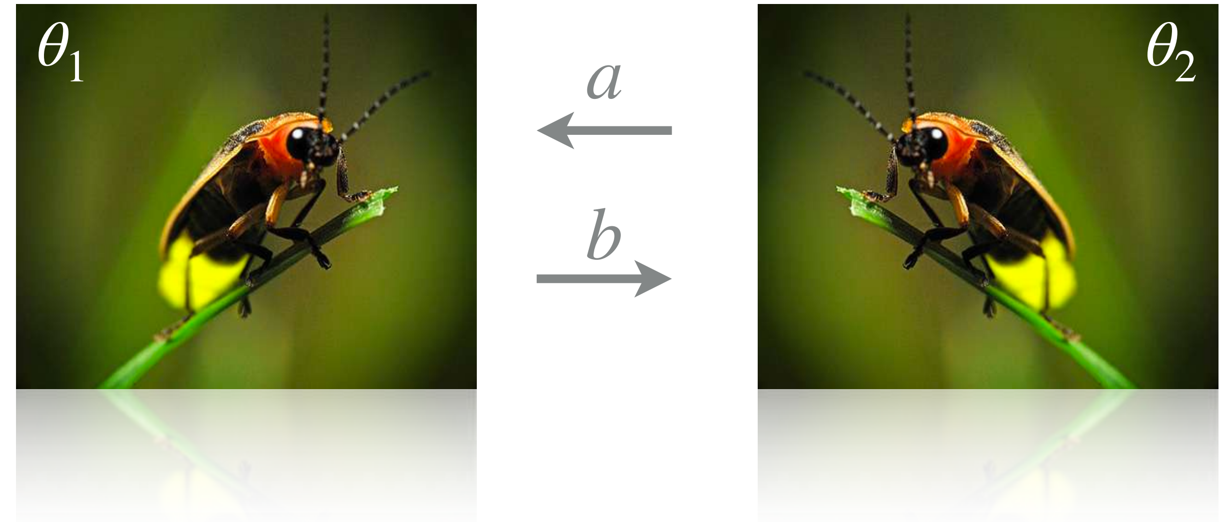

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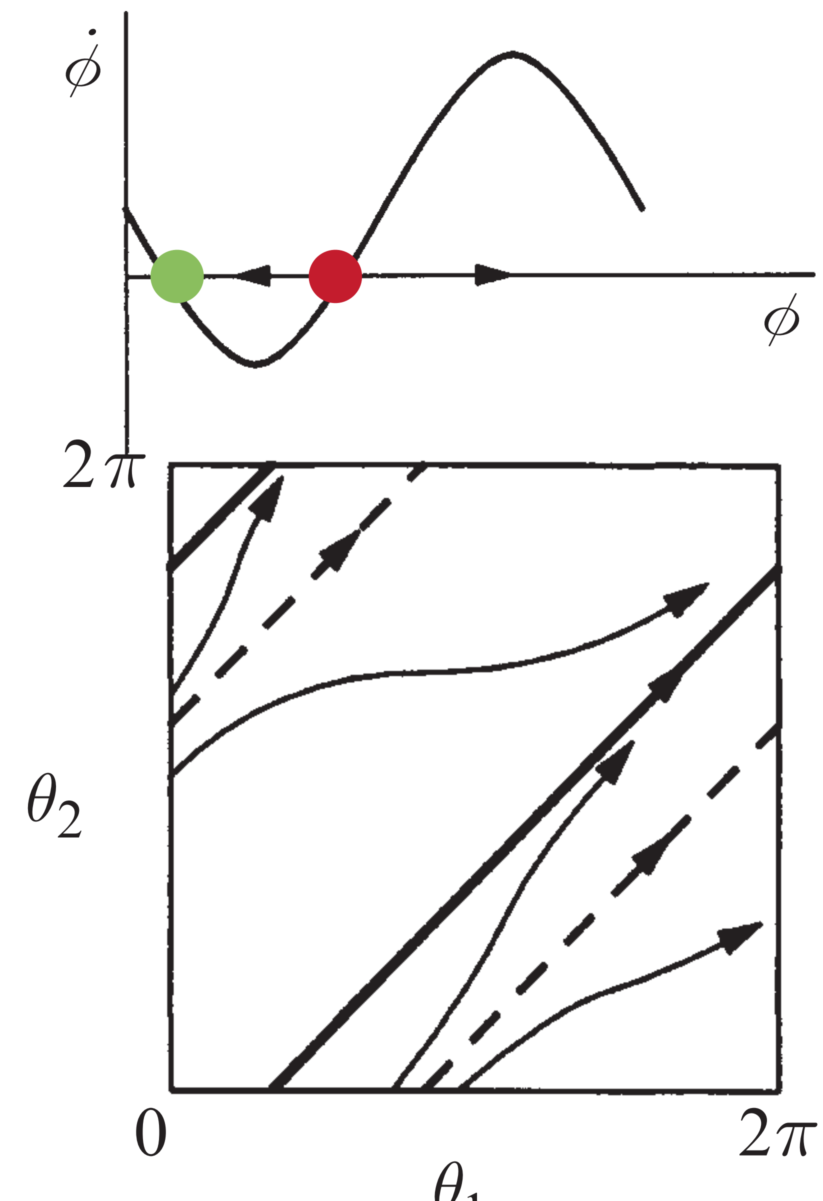
$$\sin\phi^* = \frac{\omega_1 - \omega_2}{a+b}$$

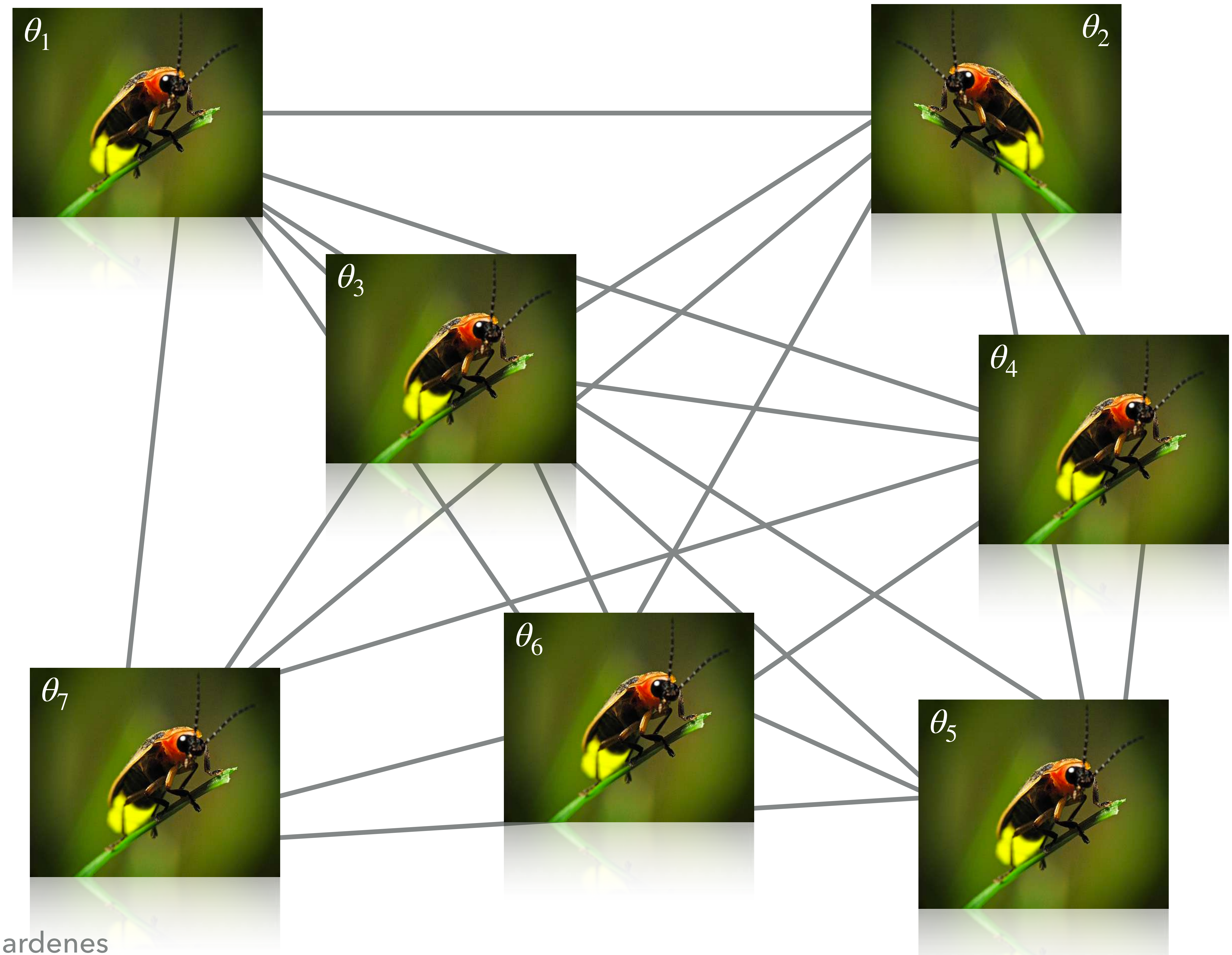


If $|\omega_1 - \omega_2|$ starts to increase until $|\omega_1 - \omega_2|$=(a+b)

SADDLE NODE BIFURCATION

SYNCHRONIZATION

SYNCHRONIZATION



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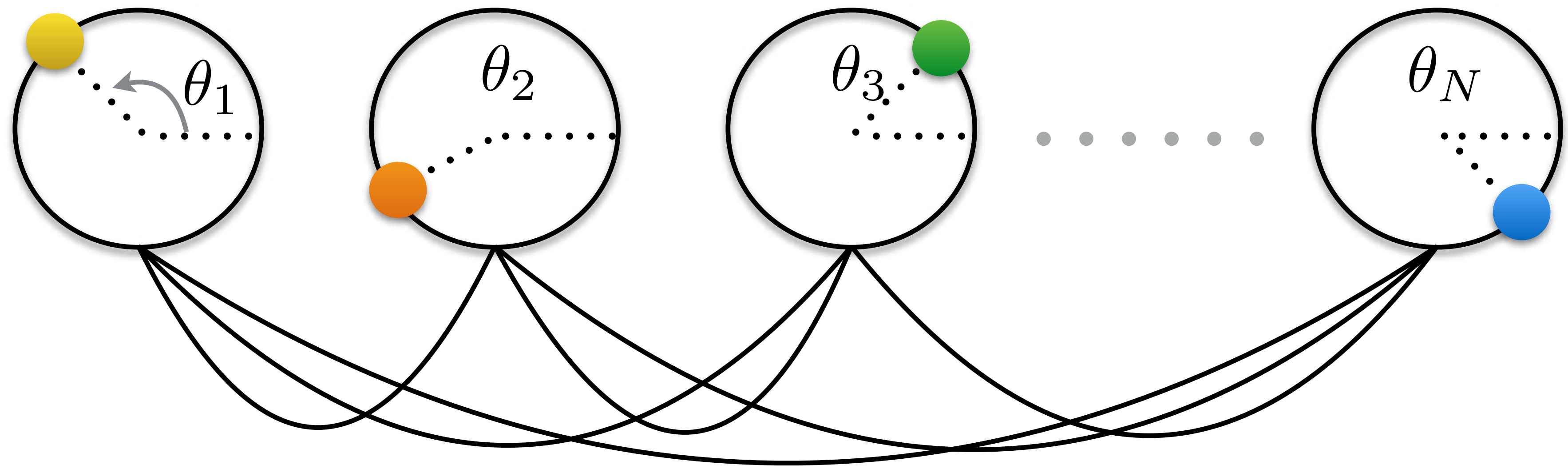
The Kuramoto model

Yoshiki Kuramoto (蔵本 由紀) @ Kyoto University

Wait… We were dealing with 1D flows and now we jump to N-dimensional dynamical systems?

The Kuramoto model

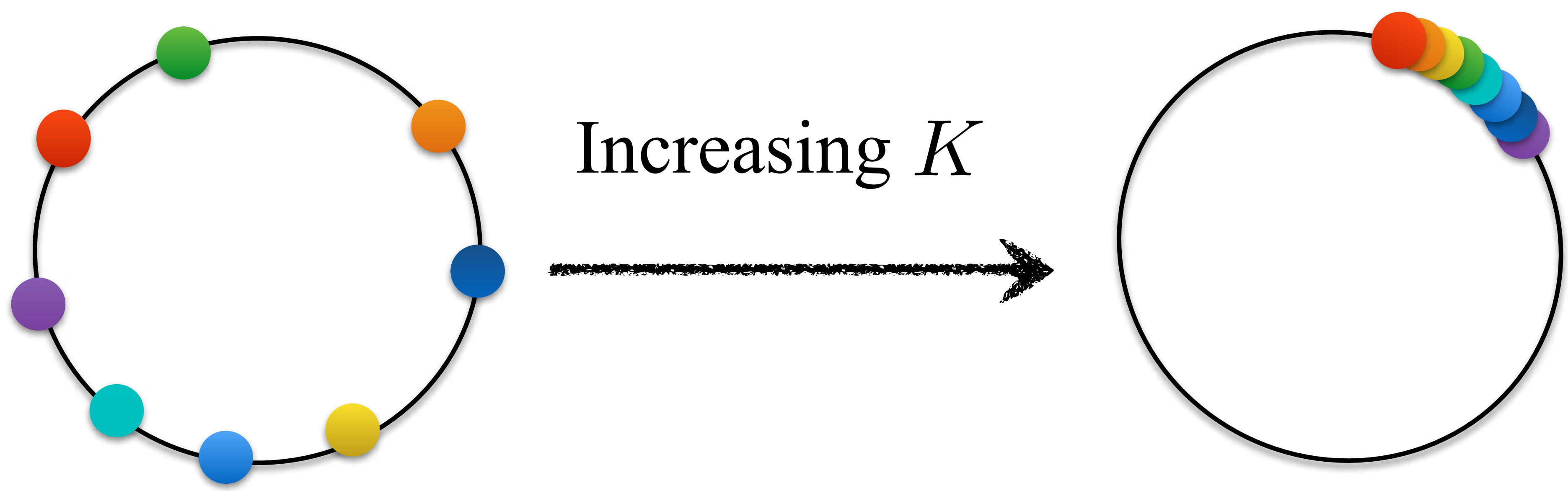
Yoshiki Kuramoto (蔵本 由紀)
@ Kyoto University



$$\dot{\theta}_i(t) = \omega_i + \boxed{\frac{K}{N}} \sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t))$$

$g(\omega)$: Unimodal and symmetric around $\langle \omega \rangle = 0$

Emergence of Synchronization



The Kuramoto model

Yoshiki Kuramoto (蔵本 由紀)
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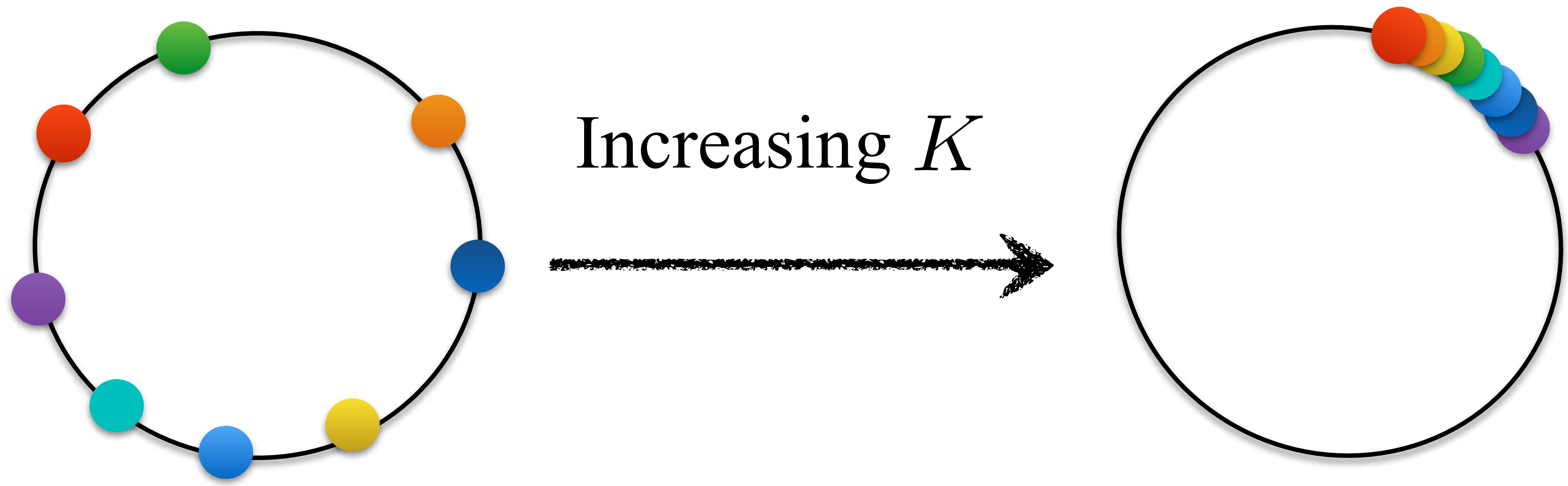
Emergence of Synchronization

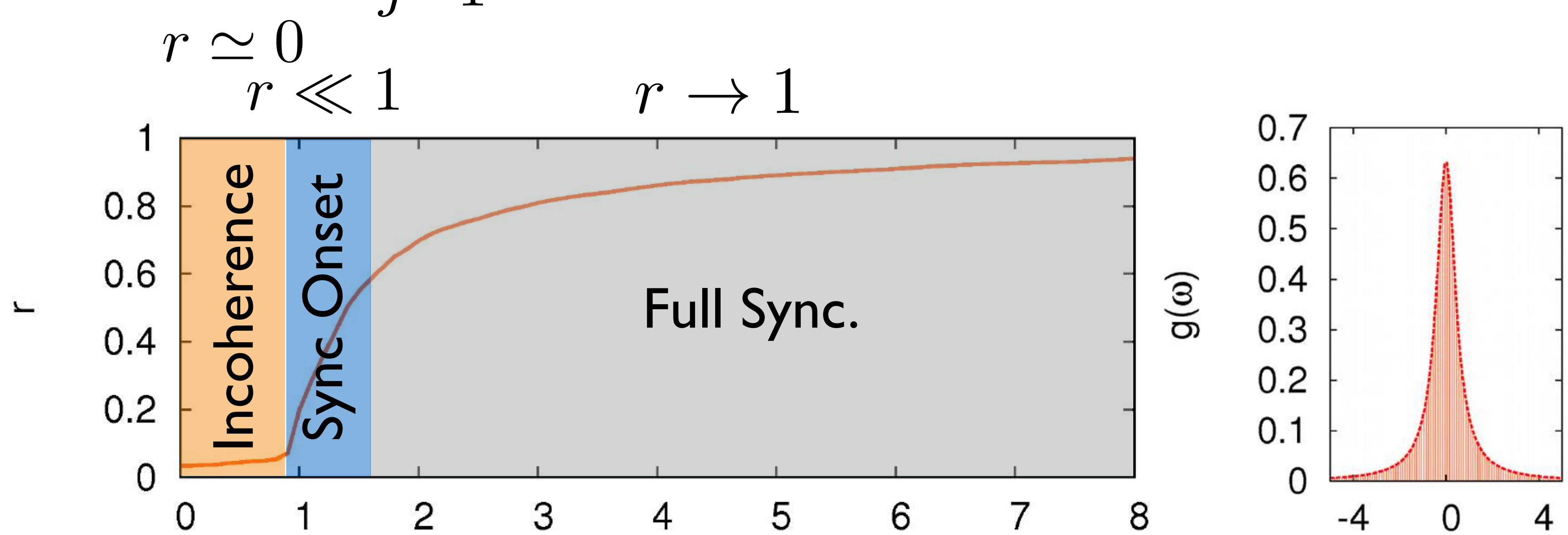


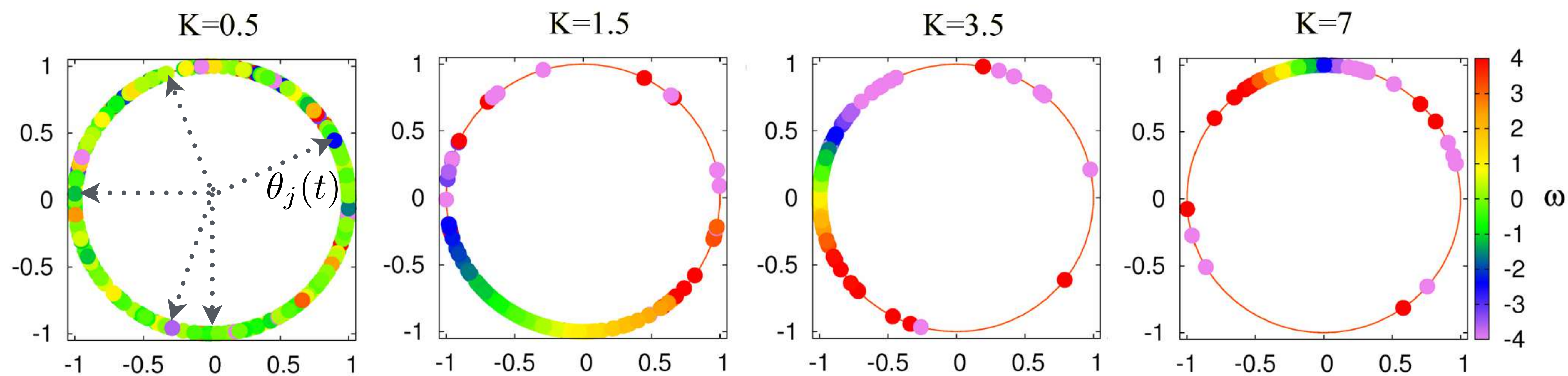
The Kuramoto model

Yoshiki Kuramoto (蔵本 由紀)
@ Kyoto University

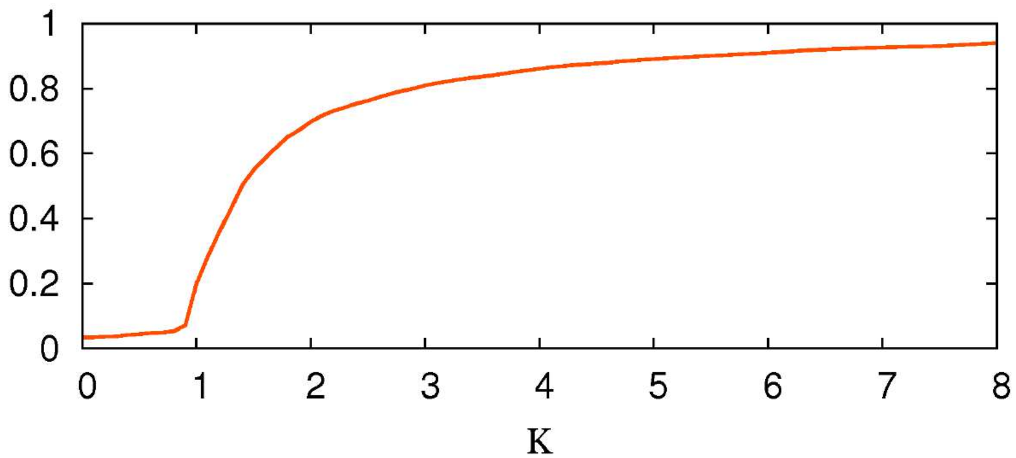
$$\frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} = r(t)\mathrm{e}^{\mathrm{i}\Psi(t)}$$

Kuramoto Order Parameter





The Kuramoto model

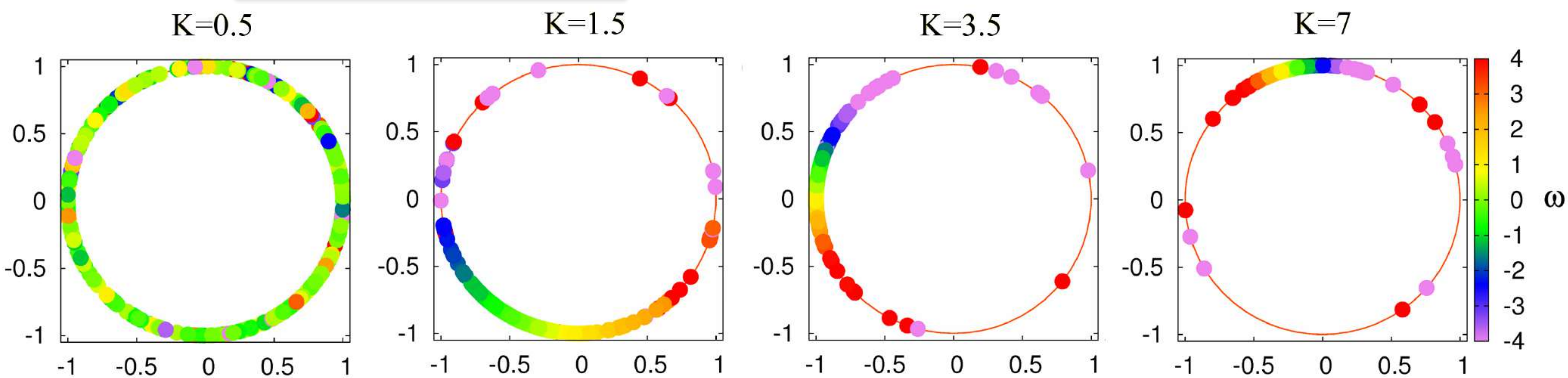


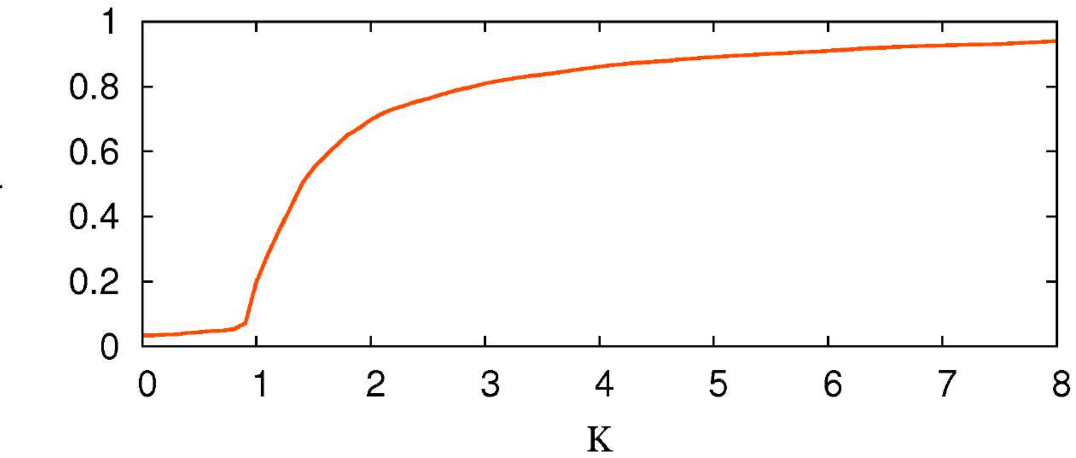
$$\frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} = r(t)\mathrm{e}^{\mathrm{i}\Psi(t)}$$

Kuramoto Order Parameter

$$r(t)\mathrm{e}^{\mathrm{i}(\Psi(t)-\theta_i(t))} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}(\theta_j(t)-\theta_i(t))} \longrightarrow r(t)\sin(\Psi(t)-\theta_i(t)) = \boxed{\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j(t)-\theta_i(t))}$$

$$\dot{\theta}_i(t) = \omega_i + \boxed{\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t)-\theta_i(t))} \longrightarrow \dot{\theta}_i(t) = \omega_i + Kr(t)\sin(\Psi(t)-\theta_i(t))$$





From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators

Steven H. Strogatz

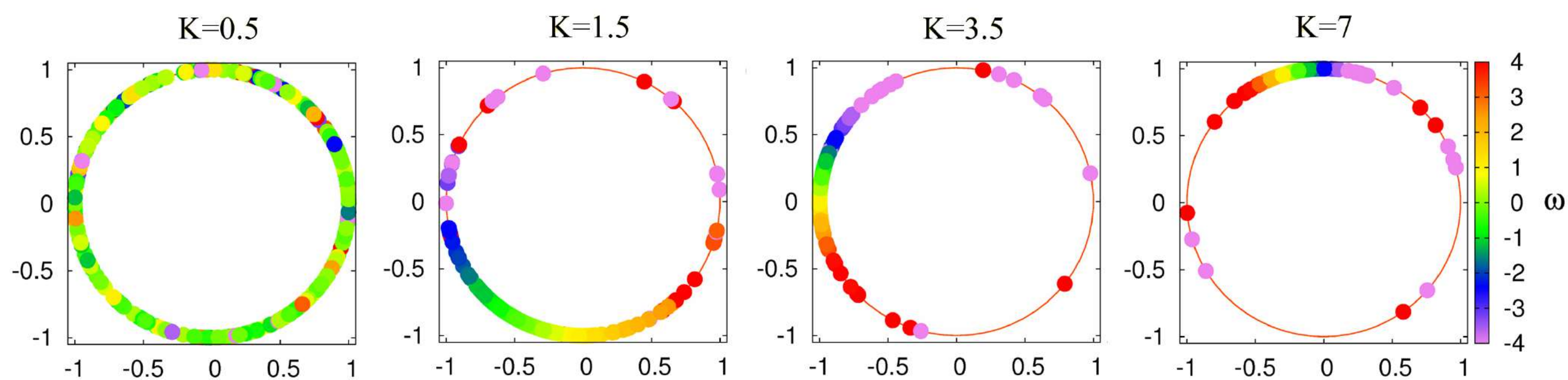
Center for Applied Mathematics and Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, NY 14853, USA

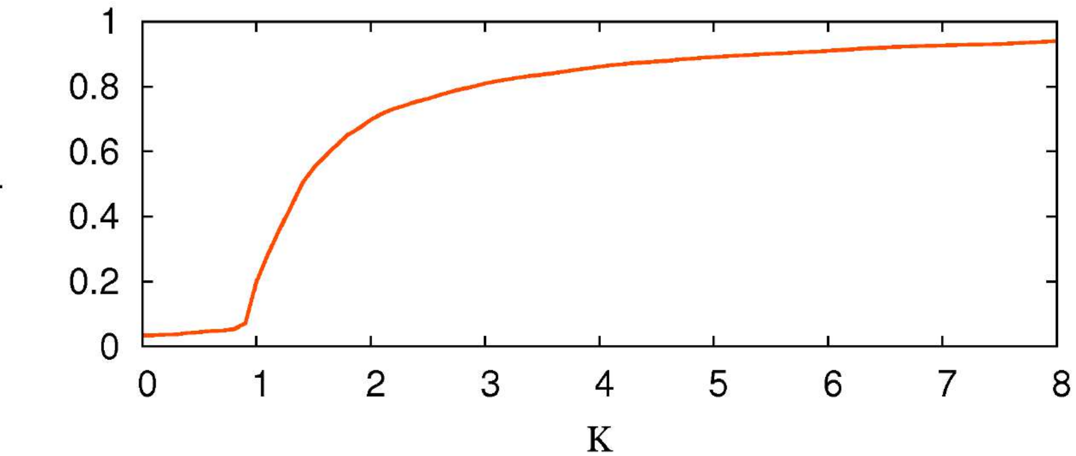
$$\dot{\theta}_i(t) = \omega_i + Kr(t)\sin(\Psi(t) - \theta_i(t))$$

In this form, the mean-field character of the model becomes obvious. Each oscillator appears to be uncoupled from all the others, although of course they are interacting, but only through the mean-field quantities r and ψ.

Steady State

$$\dot{\theta}_i = 0: \ \omega_i = Kr^* \sin(\theta_i^*) \longrightarrow \begin{cases} |\omega_i| \leq Kr^* & \text{Locked} \\ |\omega_i| > Kr^* & \text{Drifting} \end{cases}$$





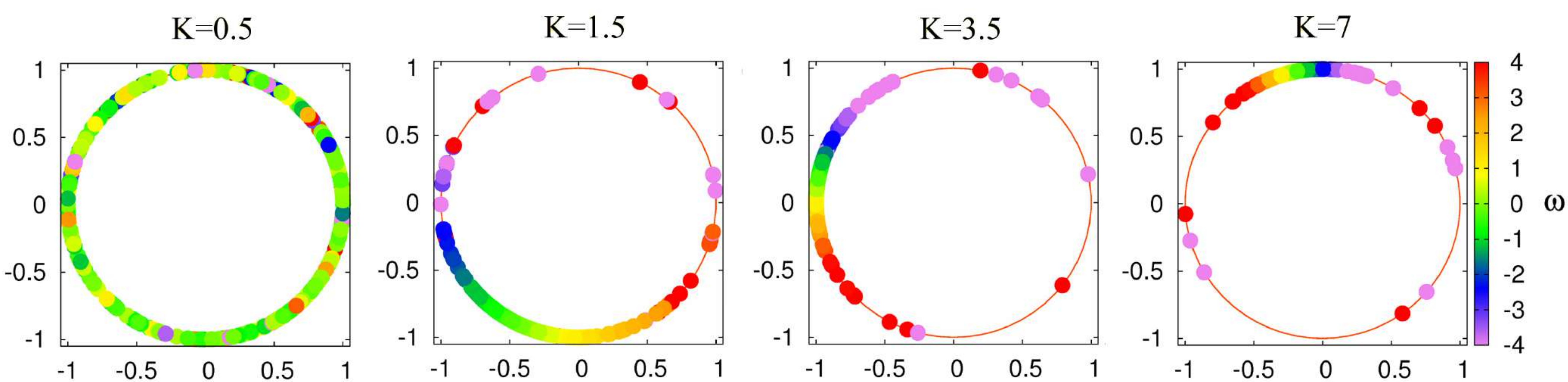
$$\frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} = r(t)\mathrm{e}^{\mathrm{i}\cancel{\Psi}(t)}$$

Kuramoto Order Parameter

$$r^* = \langle \mathrm{e}^{\mathrm{i}\theta_j^*}\rangle_{\mathrm{Lock}} + \cancel{\langle \mathrm{e}^{\mathrm{i}\theta_j(t)}\rangle_{\mathrm{Drift}}} = \int_{-Kr^*}^{+Kr^*} g(\omega)\mathrm{e}^{\mathrm{i}\theta^*(\omega)}\mathrm{d}\omega$$

Steady State

$$\omega_i = Kr^*\sin(\theta_i^*) \longrightarrow \begin{cases} |\omega_i| \leq Kr^* & \text{Locked} \\ |\omega_i| > Kr^* & \text{Drifting} \end{cases}$$



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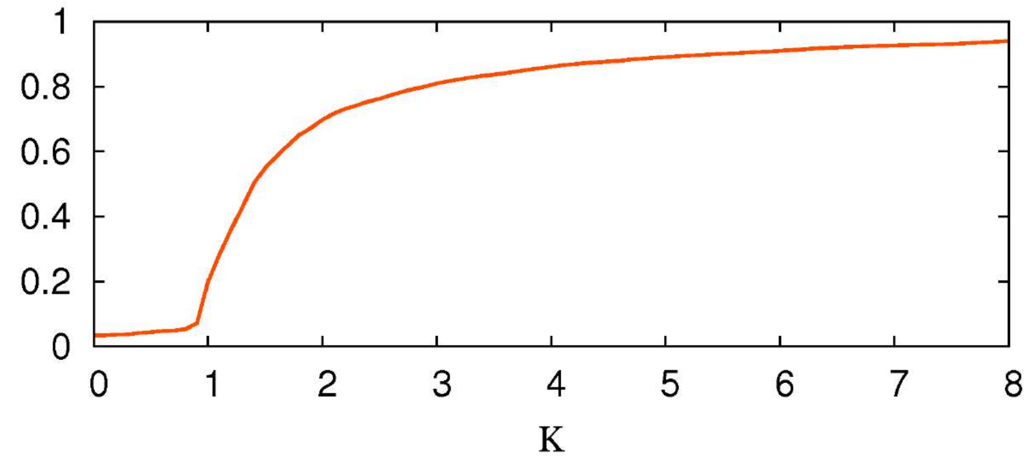
SYNCHRONIZATION



$$\frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} = r(t)\mathrm{e}^{\mathrm{i}\cancel{\Psi}(t)}$$

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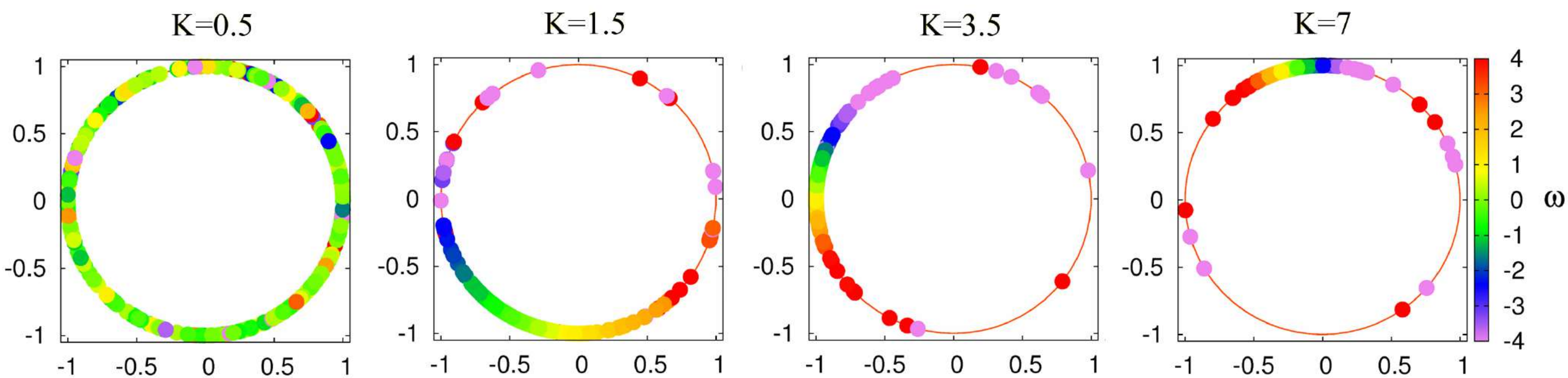
$$= \int_{-Kr^*}^{+Kr^*} g(\omega)\cos(\theta^*(\omega))\mathrm{d}\omega + \mathrm{i}\cancel{\int_{-Kr^*}^{+Kr^*} g(\omega) \frac{\omega}{Kr^*}}\mathrm{d}\omega$$



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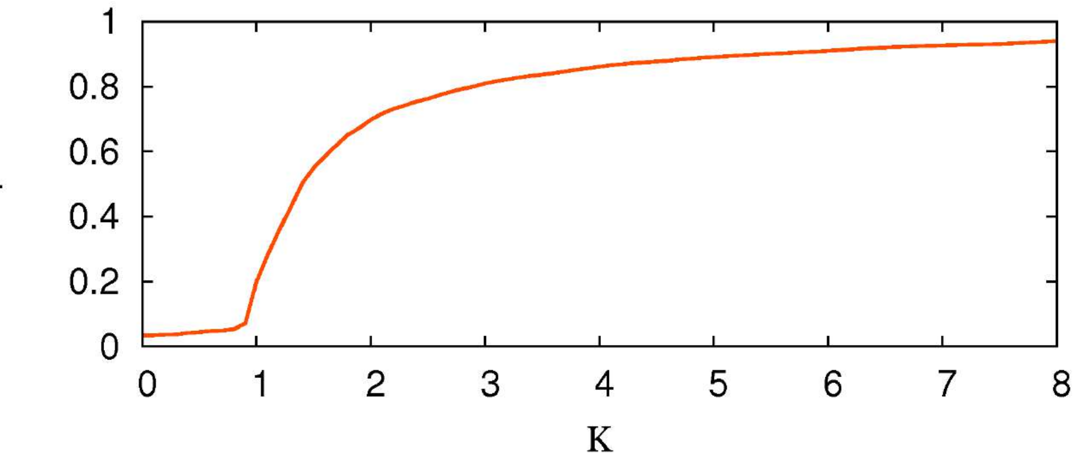
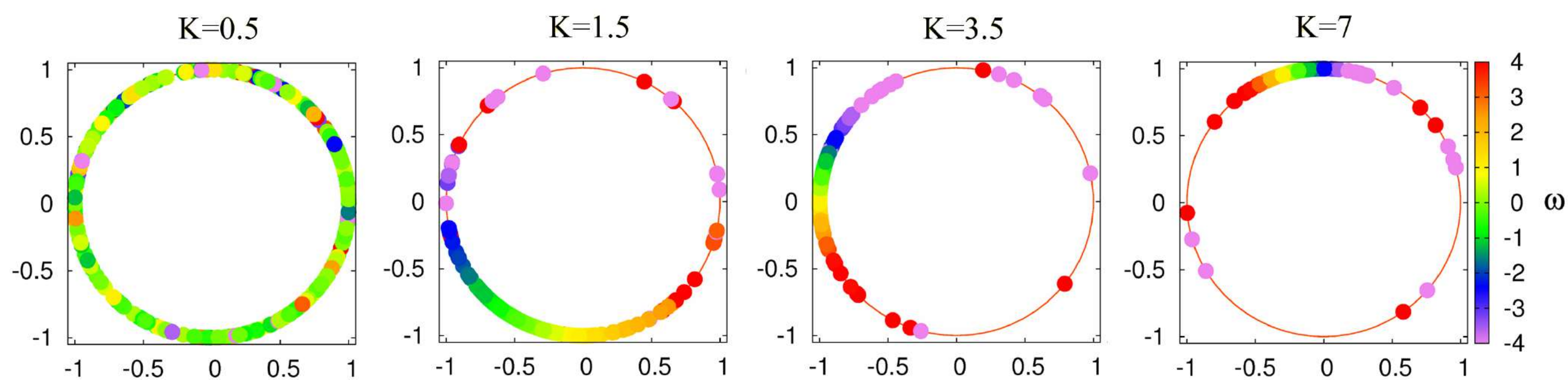


$$\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)} = r(t)e^{i\cancel{\Psi}(t)}$$

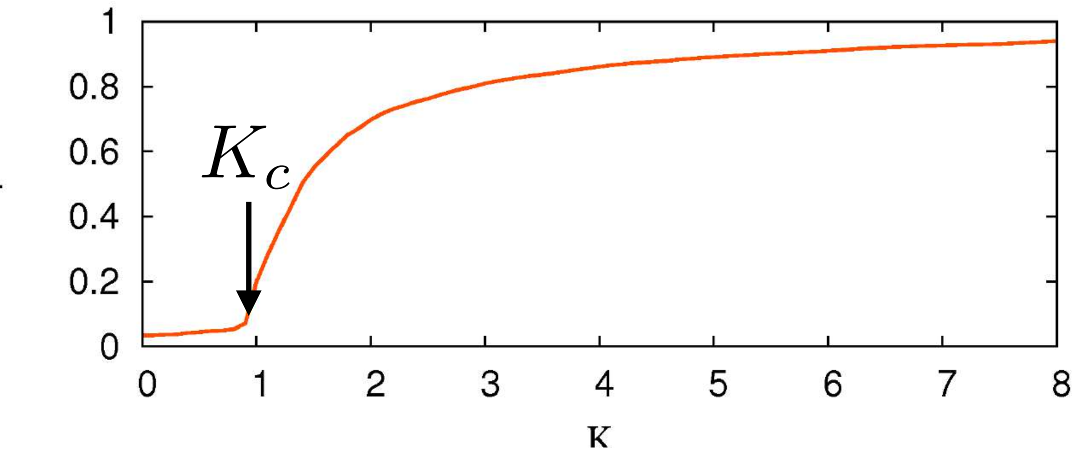
Kuramoto Order Parameter

$$r^* = \langle e^{i\theta_j^*}\rangle_{\mathrm{Lock}} + \cancel{\langle e^{i\theta_j(t)}\rangle_{\mathrm{Drift}}} = \int_{-Kr^*}^{+Kr^*} g(\omega)e^{i\theta^*(\omega)}\mathrm{d}\omega$$

$$= \int_{-Kr^*}^{+Kr^*} g(\omega)\cos(\theta^*(\omega))\mathrm{d}\omega = Kr^*\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} g(Kr^*\sin\theta^*)\cos^2(\theta^*)\mathrm{d}\theta^*$$



@gomezgardenes



$$\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)} = r(t)e^{i\cancel{\Psi}(t)}$$

Kuramoto Order Parameter

$$1\cancel{r^*} = K\cancel{r^*}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} g(Kr^*\sin\theta^*)\cos^2(\theta^*)d\theta^*$$

Non-trivial Solution?

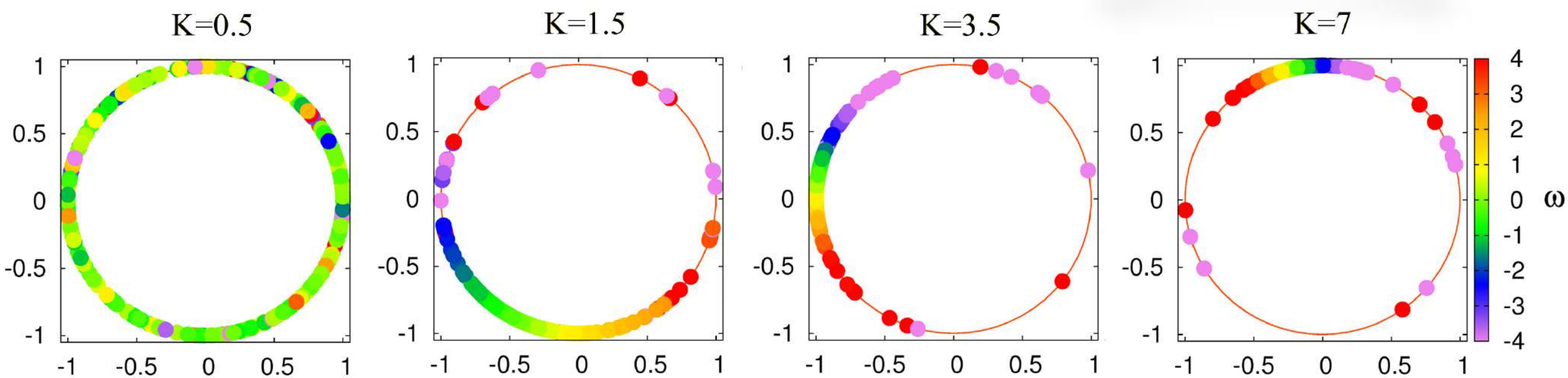
Trivial Solution

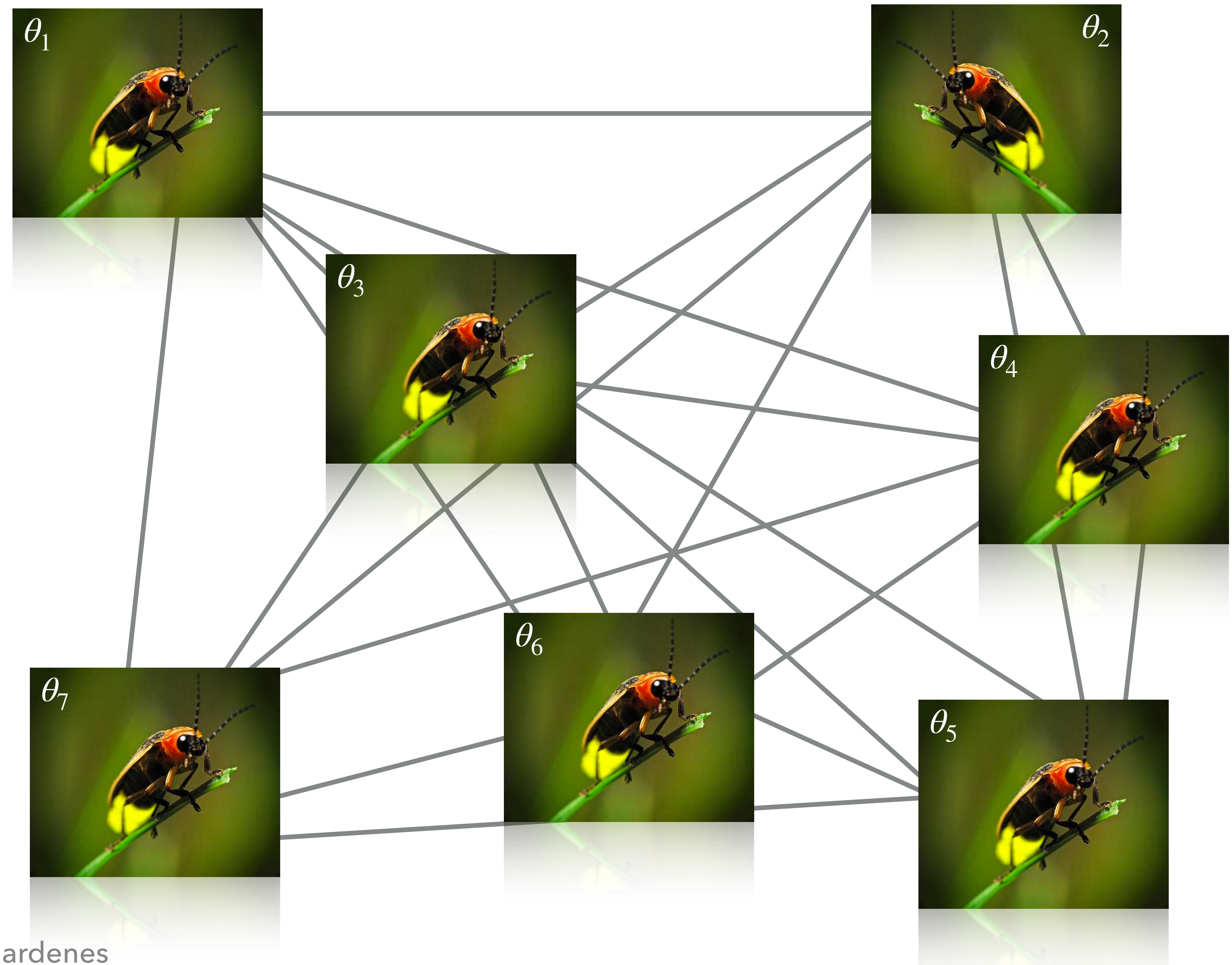
$$r^* = 0$$

At K_c $r \to 0^+$:

$$1 = K_c g(0)\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos^2(\theta^*)d\theta^* = K_c g(0)\frac{\pi}{2}$$

$$K_c = \frac{2}{\pi g(0)}$$




θ_1
θ_2
θ_3
θ_4
θ_6
θ_7
θ_5

Sync on networks