SCHOOL ON NONLINEAR DYNAMICS, COMPLEX NETWORKS, INFORMATION THEORY, AND MACHINE LEARNING IN NEUROSCIENCE

INTRODUCTION TO SYNCHRONIZATION PHENOMENA AND THE KURAMOTO MODEL

LECTURE 2: SYNCHRONIZATION IN COMPLEX NETWORKS

GOTHAm lab

Universidad Zaragoza 1542

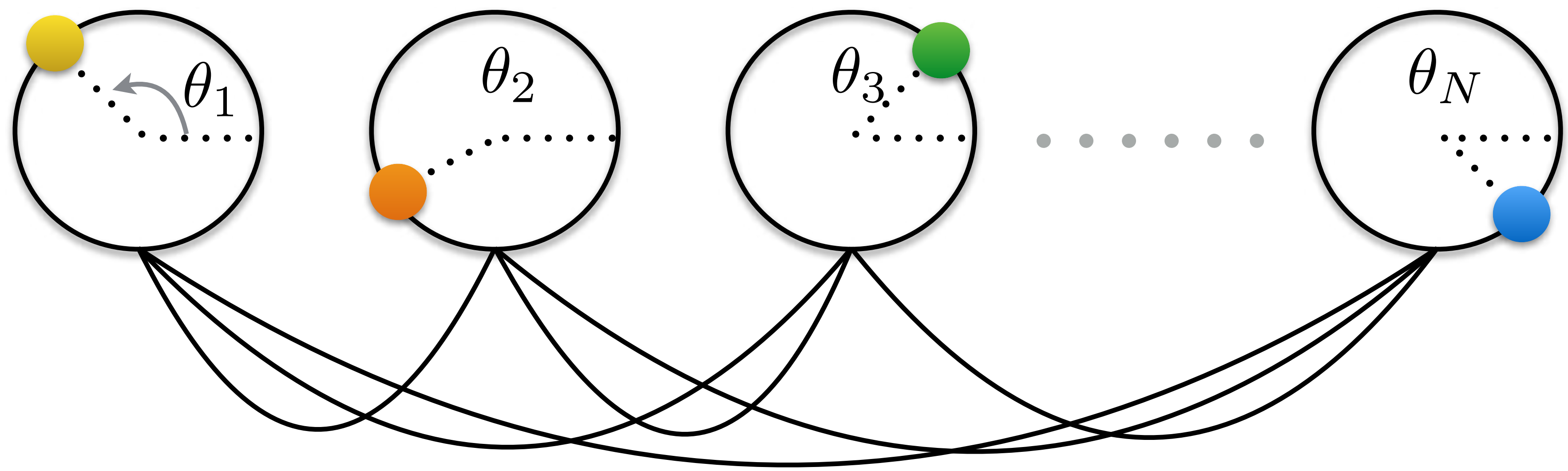
Institute for Biocomputation & Physics of Complex Systems (BIFI)

JESÚS GÓMEZ-GARDEÑES, UNIV. OF ZARAGOZA

GROUP OF THEORETICAL & APPLIED MODELING

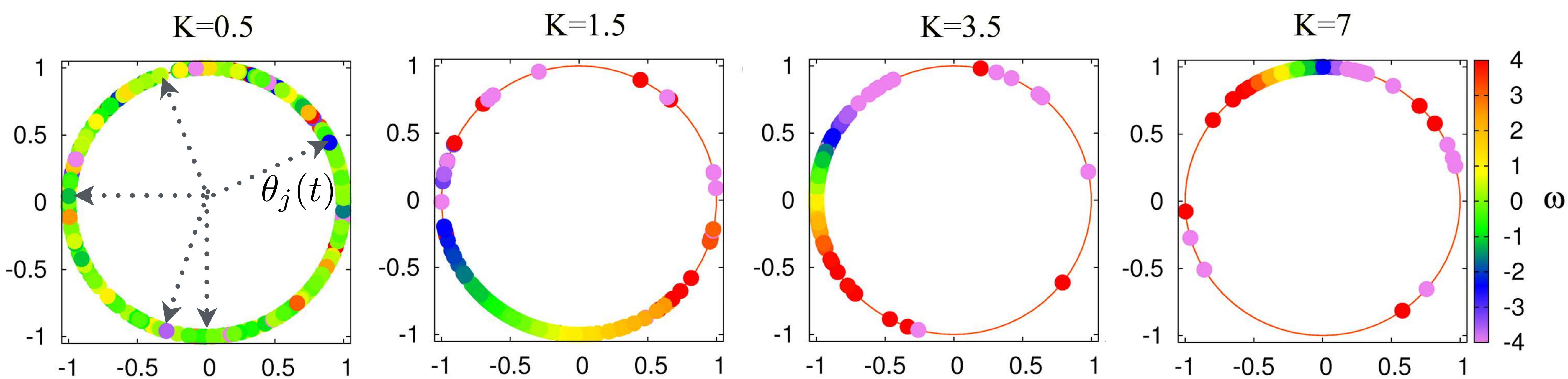
The Kuramoto model

Yoshiki Kuramoto (蔵本 由紀)
@ Kyoto University



$g(\omega)$

$$\dot{\theta}_i(t) = \omega_i + \boxed{\frac{K}{N}} \sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t))$$



The Kuramoto model

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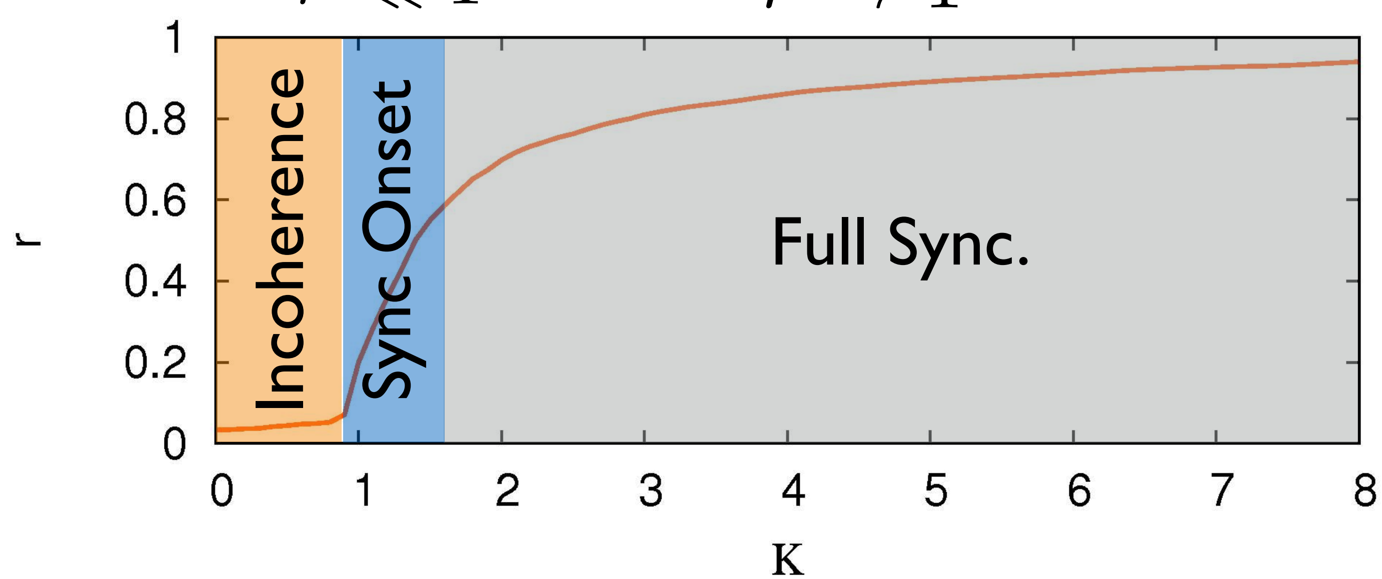
$$\frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} = r(t)\mathrm{e}^{\mathrm{i}\Psi(t)}$$

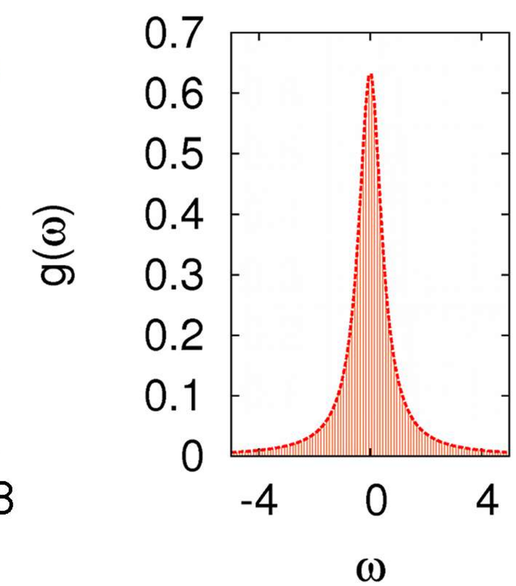
Kuramoto
Order Parameter

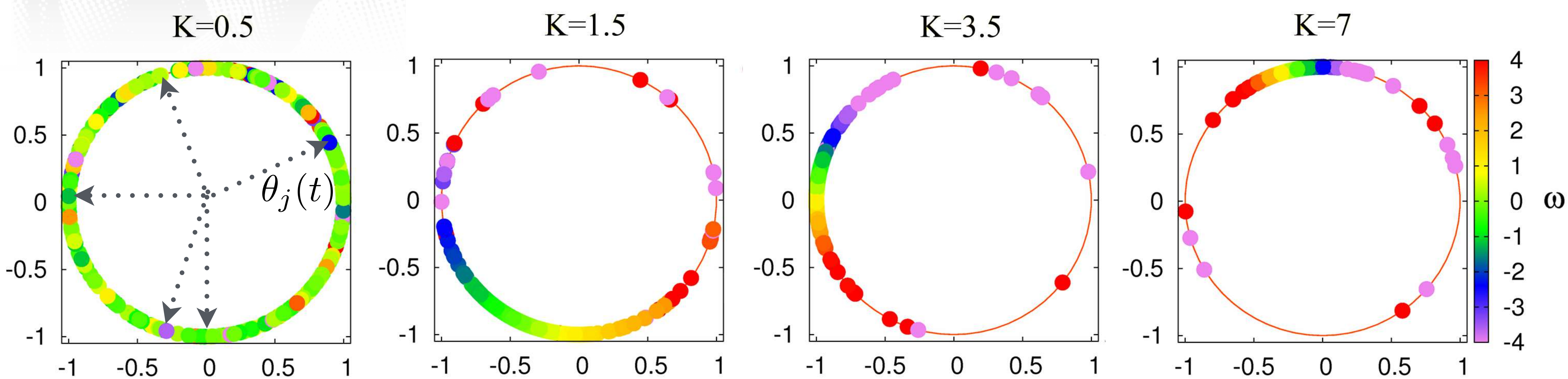
$r \simeq 0$

$r \ll 1$

$r \to 1$







The Kuramoto model

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From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators

Steven H. Strogatz

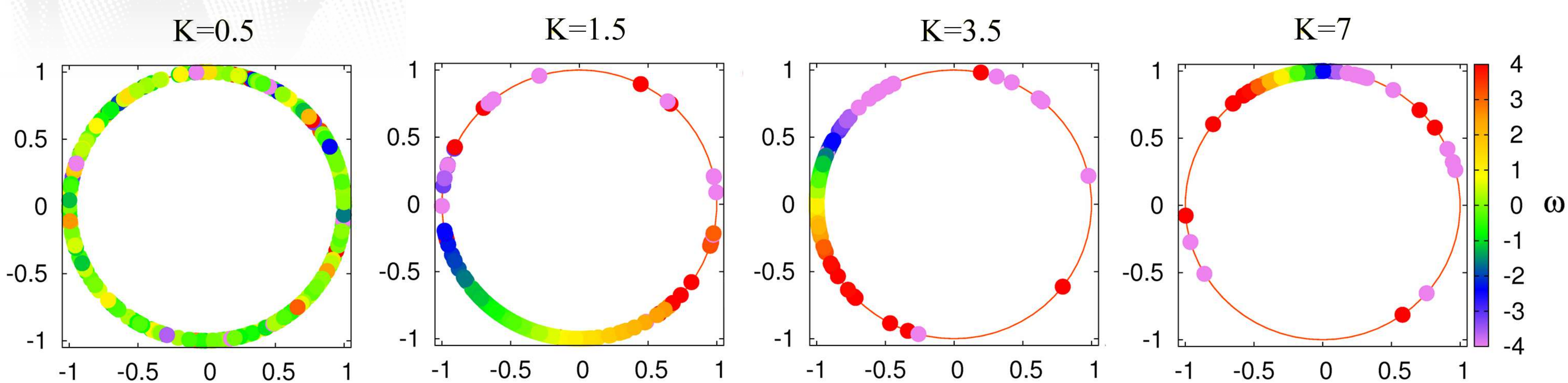
Center for Applied Mathematics and Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, NY 14853, USA

$$\dot{\theta}_i(t) = \omega_i + Kr(t)\sin(\Psi(t) - \theta_i(t))$$

In this form, the mean-field character of the model becomes obvious. Each oscillator appears to be uncoupled from all the others, although of course they are interacting, but only through the mean-field quantities r and ψ.

Steady State

$$\omega_i = Kr^*\sin(\theta_i^*) \longrightarrow \begin{cases} |\omega_i| \leq Kr^* & \text{Locked} \\ |\omega_i| > Kr^* & \text{Drifting} \end{cases}$$



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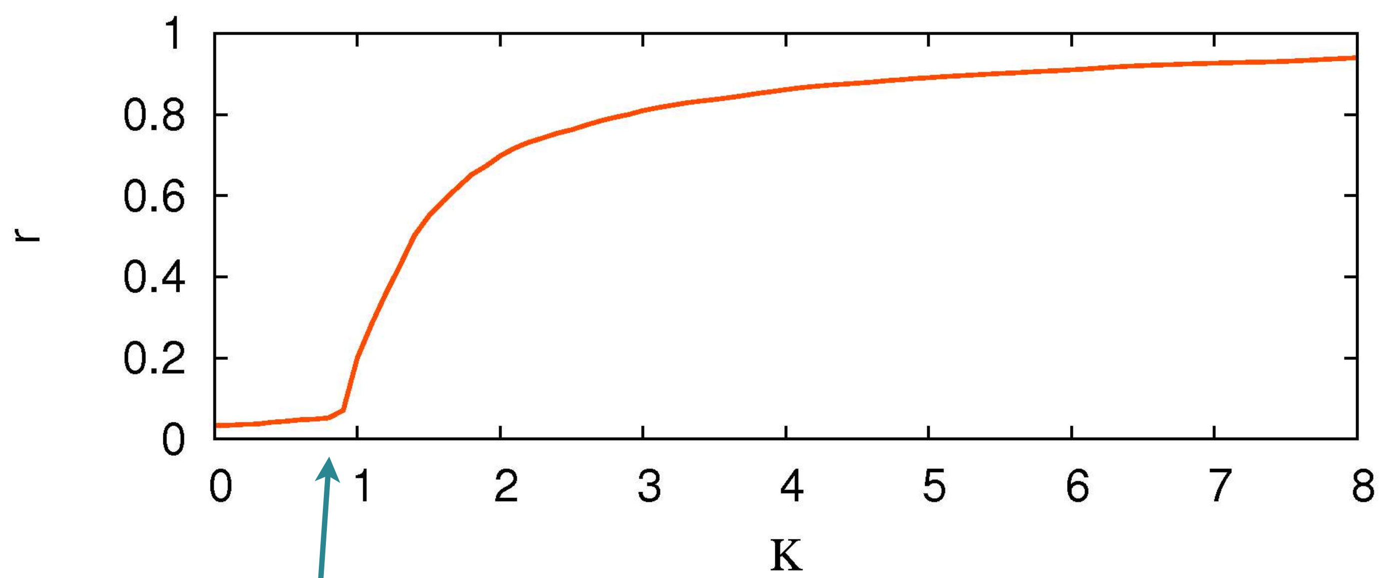
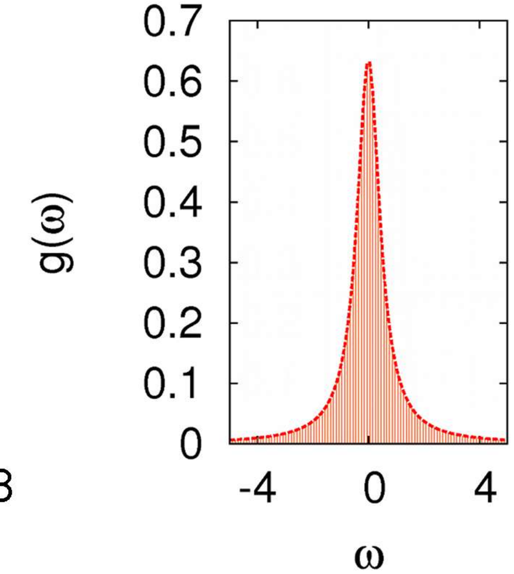
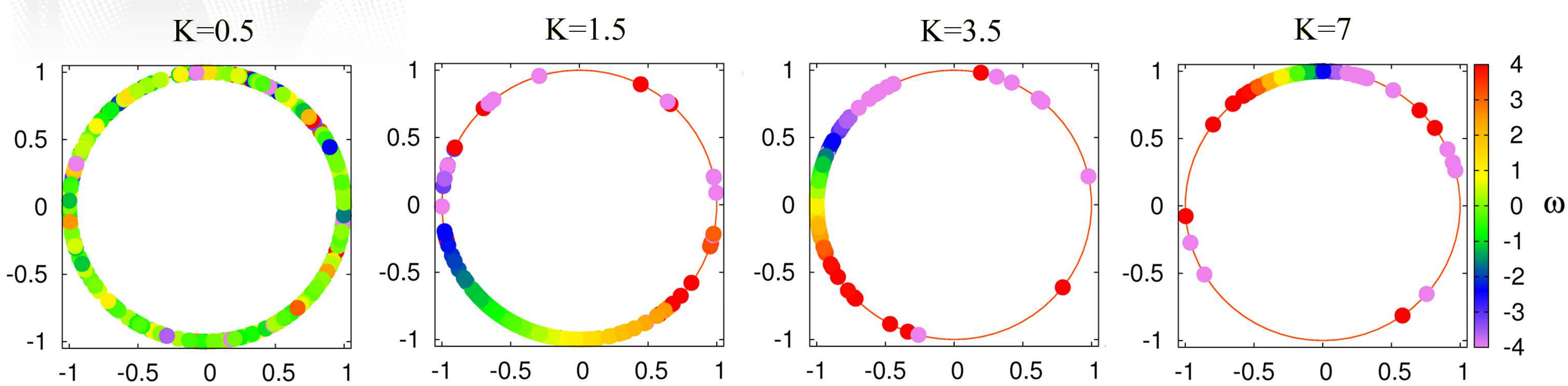
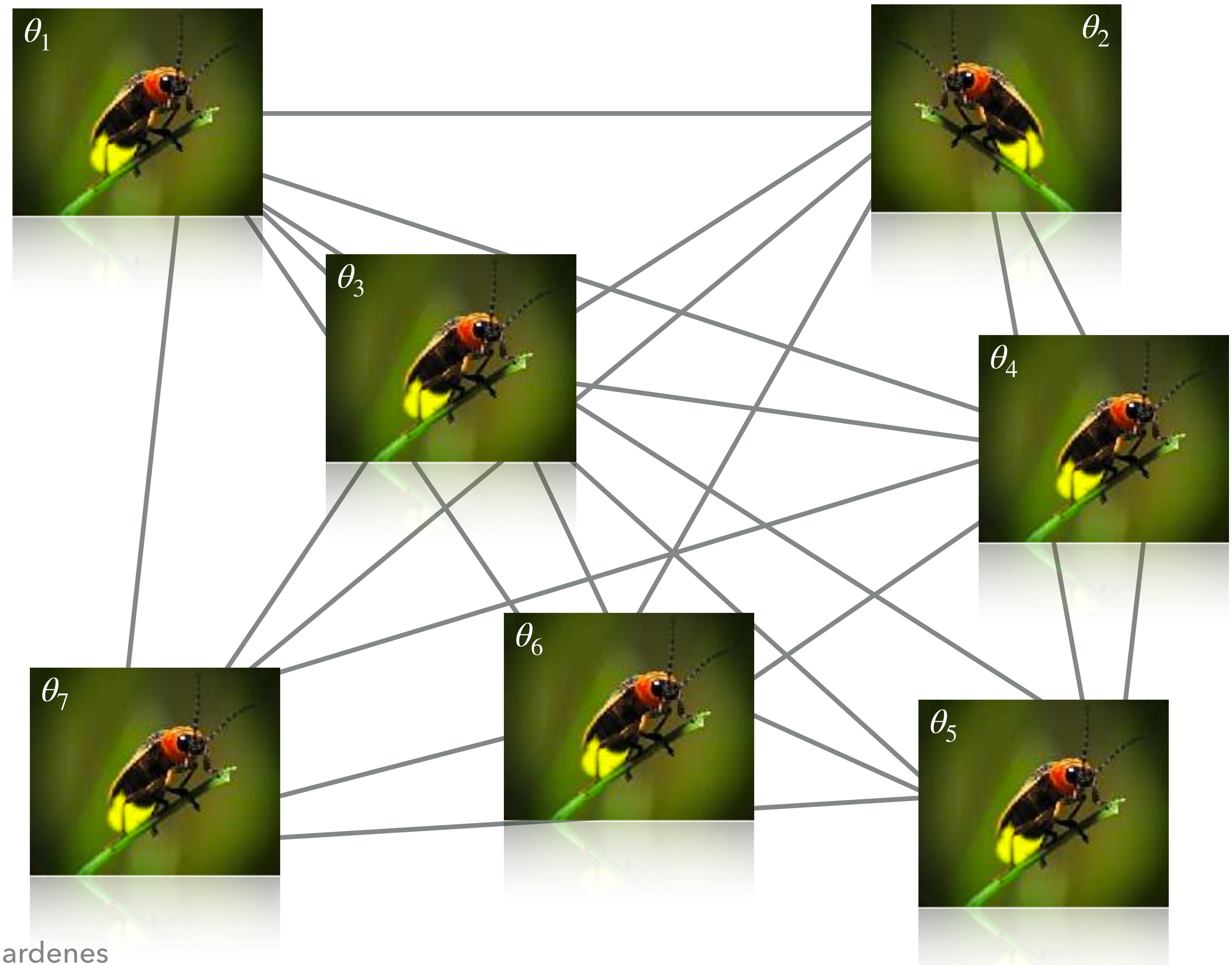




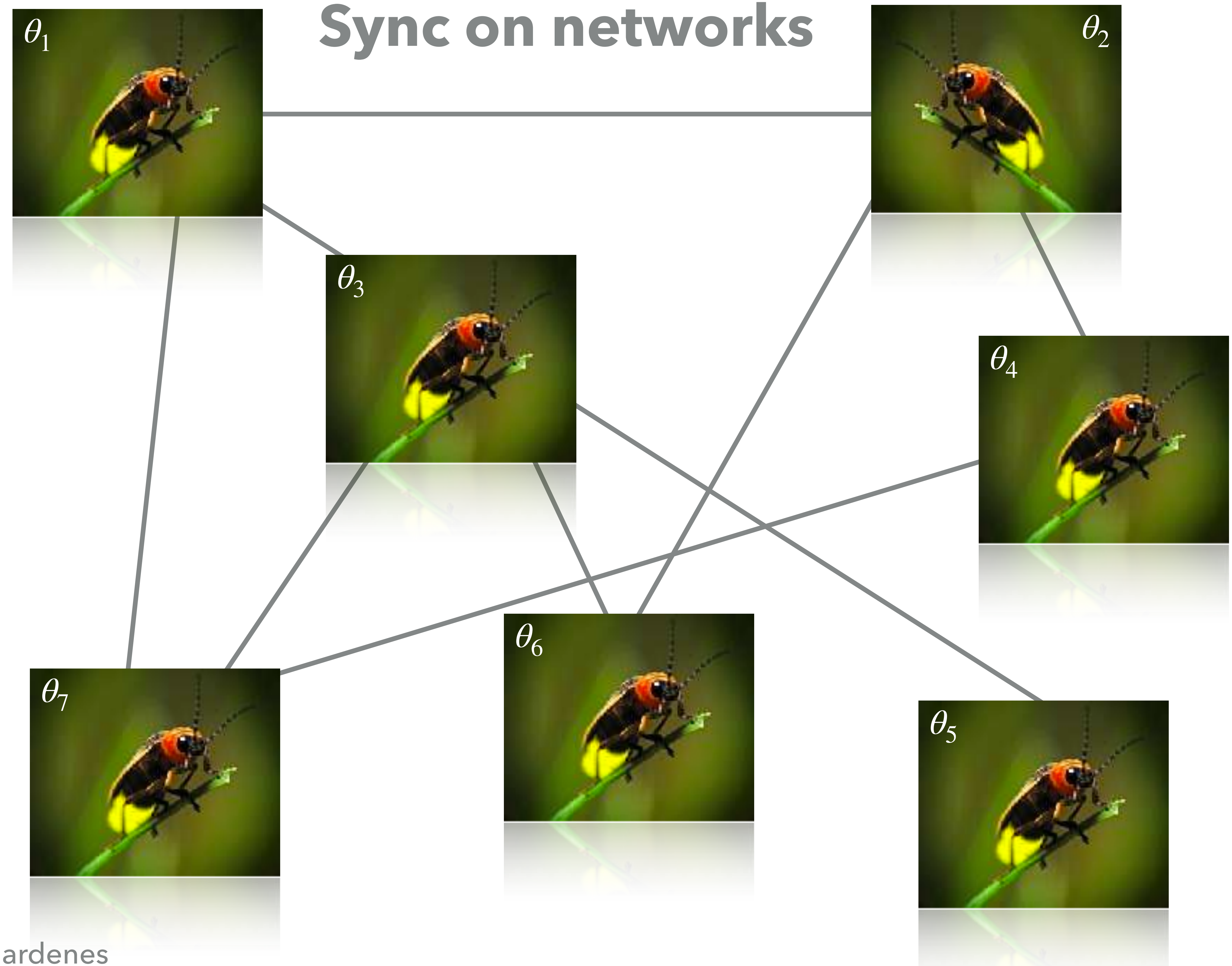
$$K_c = \frac{2}{\pi g(0)}$$ **Critical Coupling**



MOTIVATION



Sync on networks



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- KURAMOTO MODEL IN COMPLEX NETWORKS
- SYNC. TRANSITION: THE CRITICAL COUPLING
- PLAYING WITH THE TRANSITION: EXPLOSIVE SYNC.

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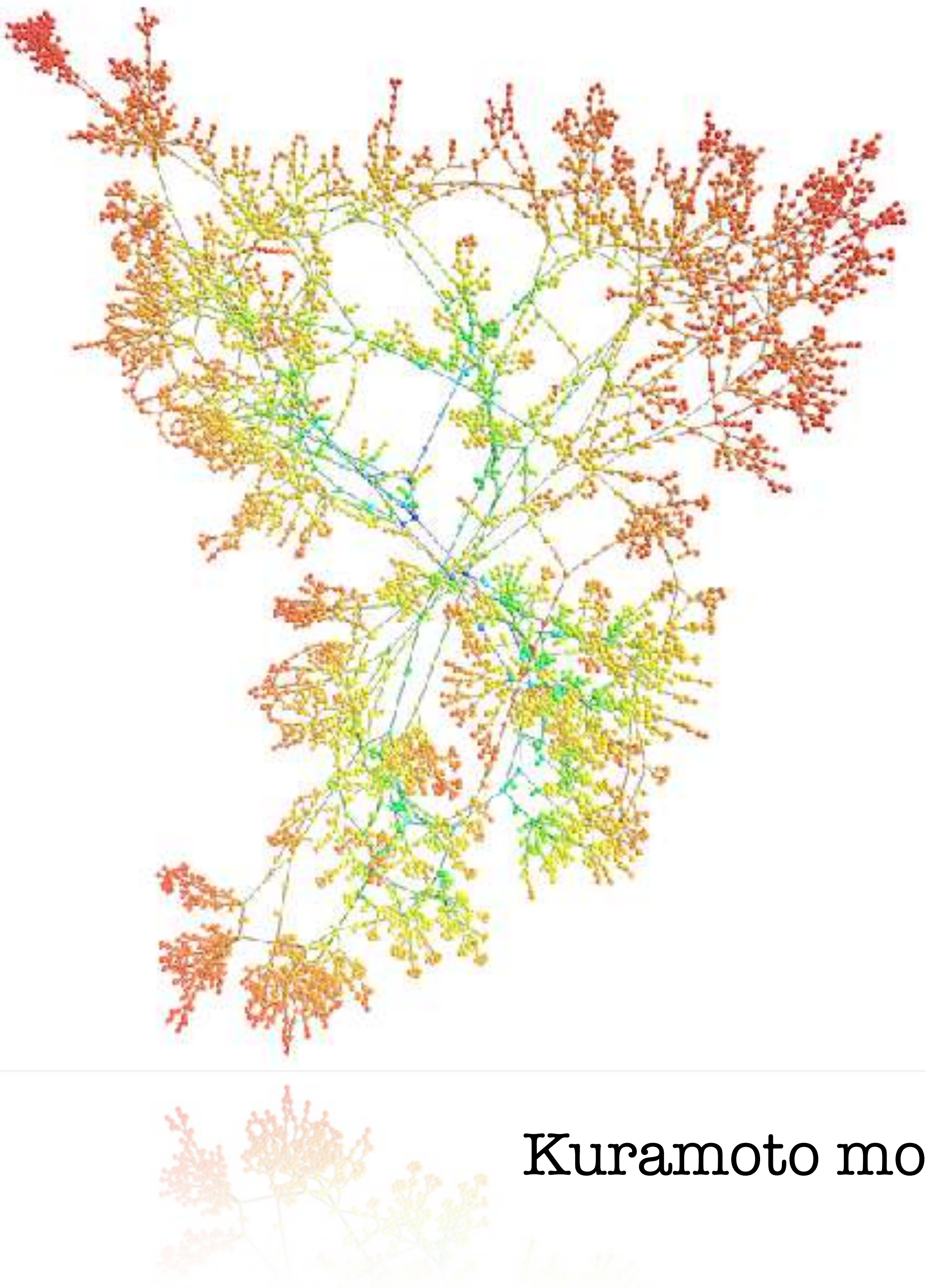
KM in Complex Networks



N nodes connected by L links

Adjacency Matrix

$$\mathbf{A} = \begin{pmatrix} 0 & A_{12} & A_{13} & \cdots\cdot & A_{1N} \\ A_{21} & 0 & A_{23} & \cdots\cdots & A_{2N} \\ \vdots & \vdots & \vdots & \vdots & \\ A_{N1} & A_{N2} & \cdots & A_{N(N-1)} & 0 \end{pmatrix}$$

Undirected networks $A_{ij} = A_{ji}$

Non-weighted networks $A_{ij} \in \{1,0\}$

Kuramoto model in a Network

$$\dot{\theta}_i(t) = \omega_i + \lambda \sum_{j=1}^{N} \boxed{A_{ij}} \sin(\theta_j(t) - \theta_i(t))$$

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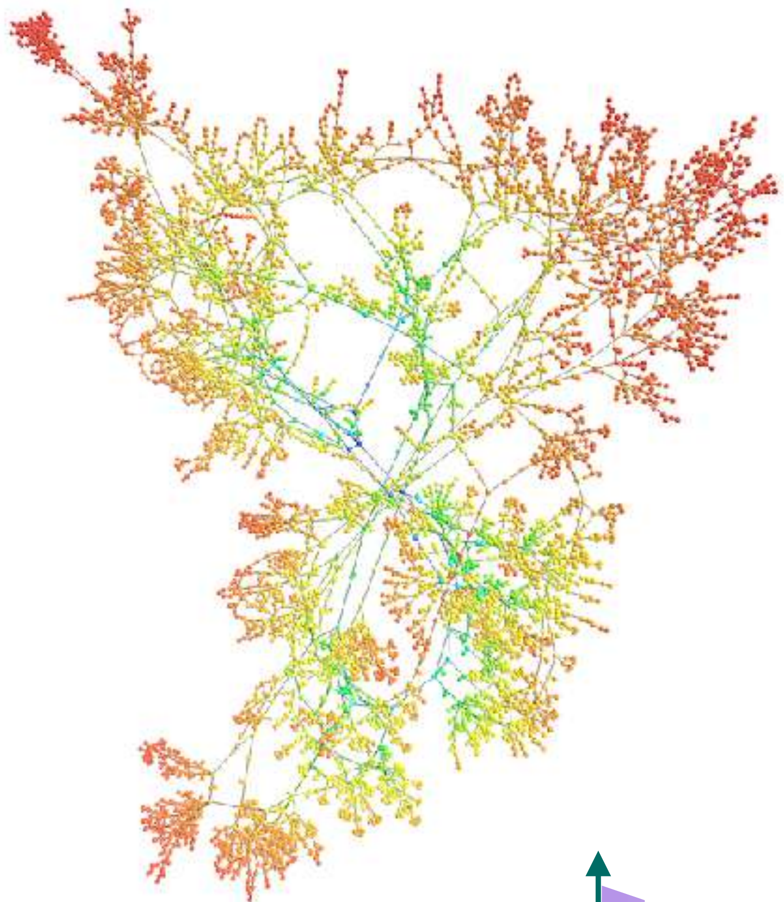

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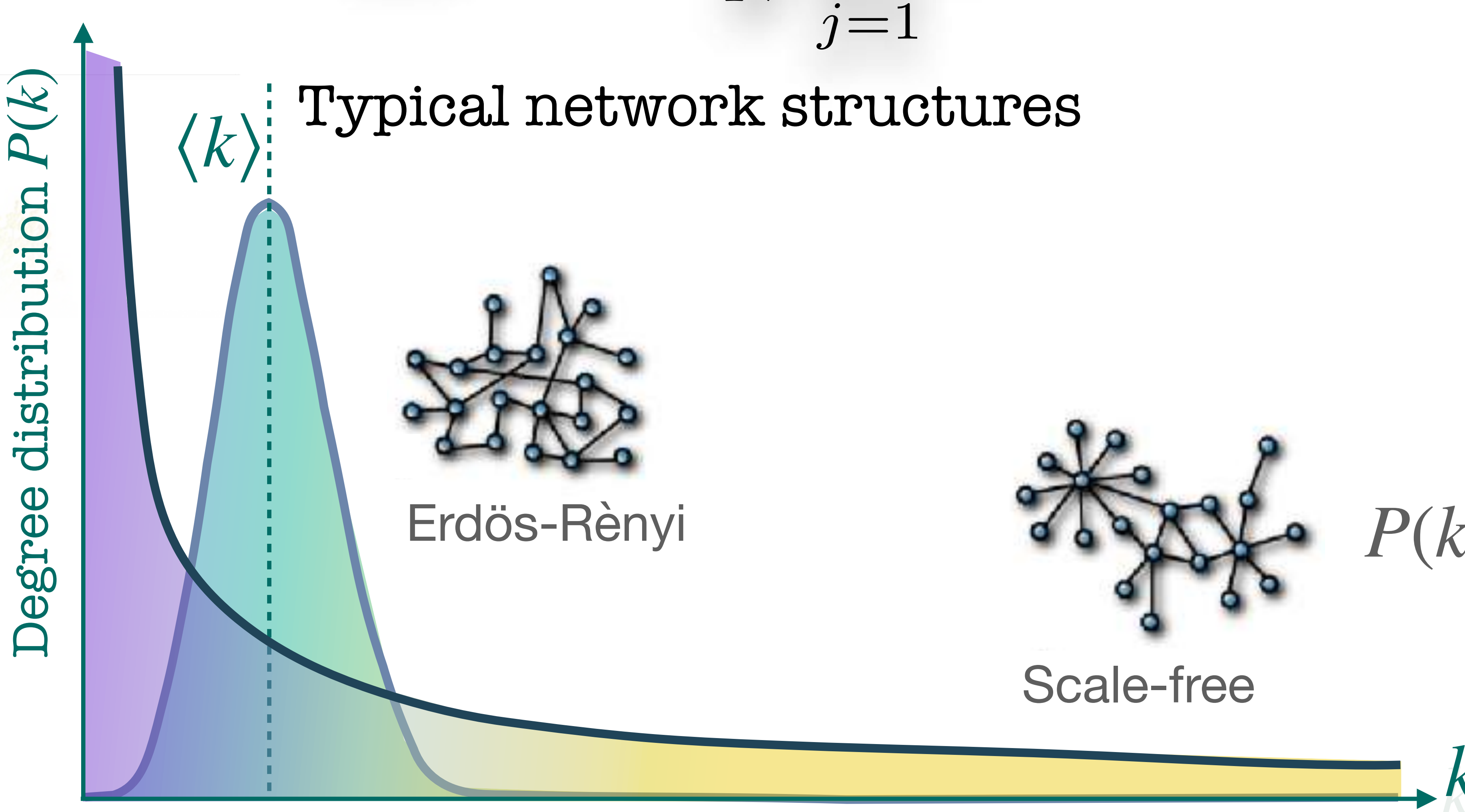
Kuramoto Order Parameter

$$r(t)\mathrm{e}^{\mathrm{i}\Psi(t)} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)}$$

Typical network structures



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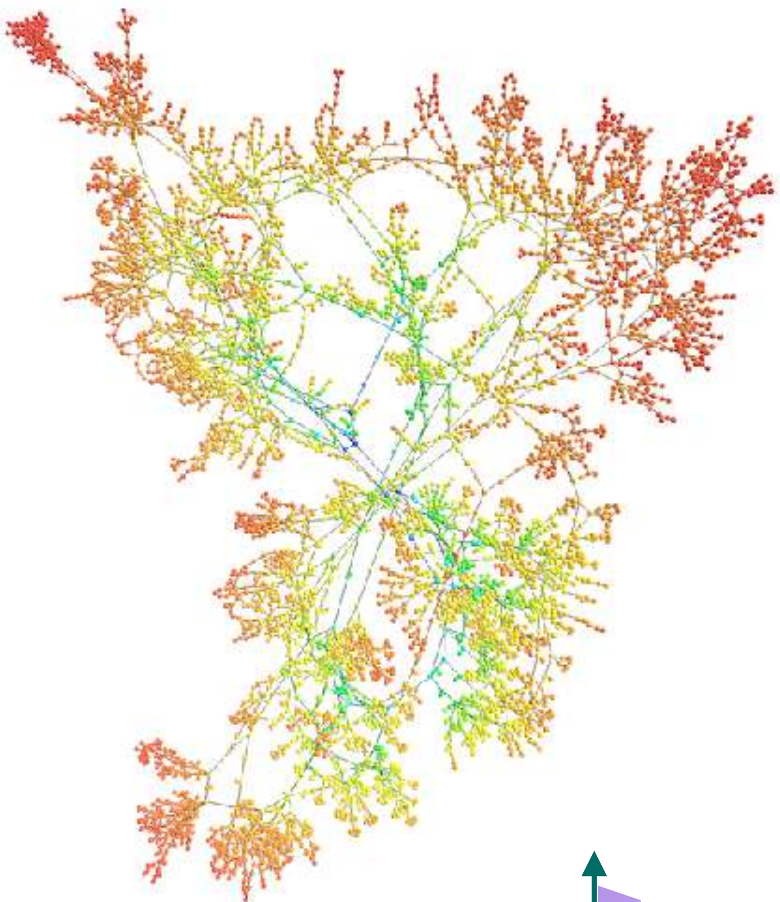

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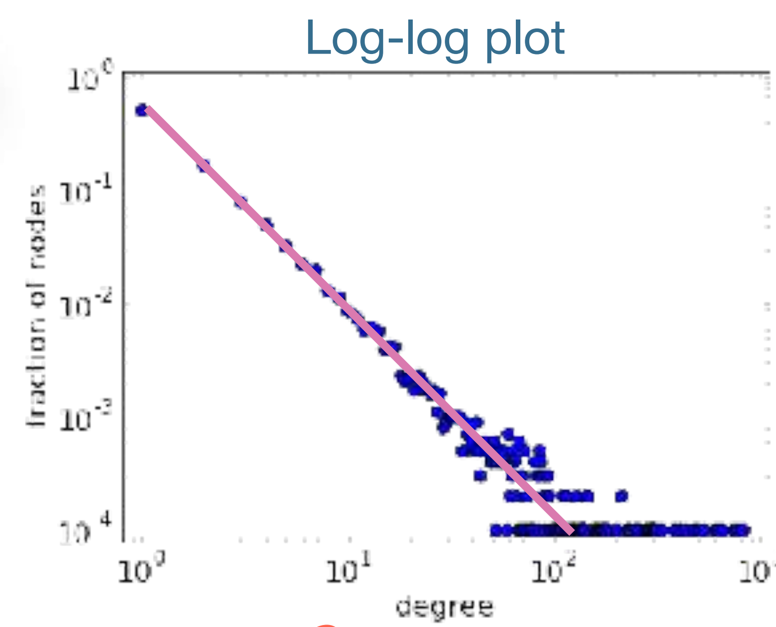
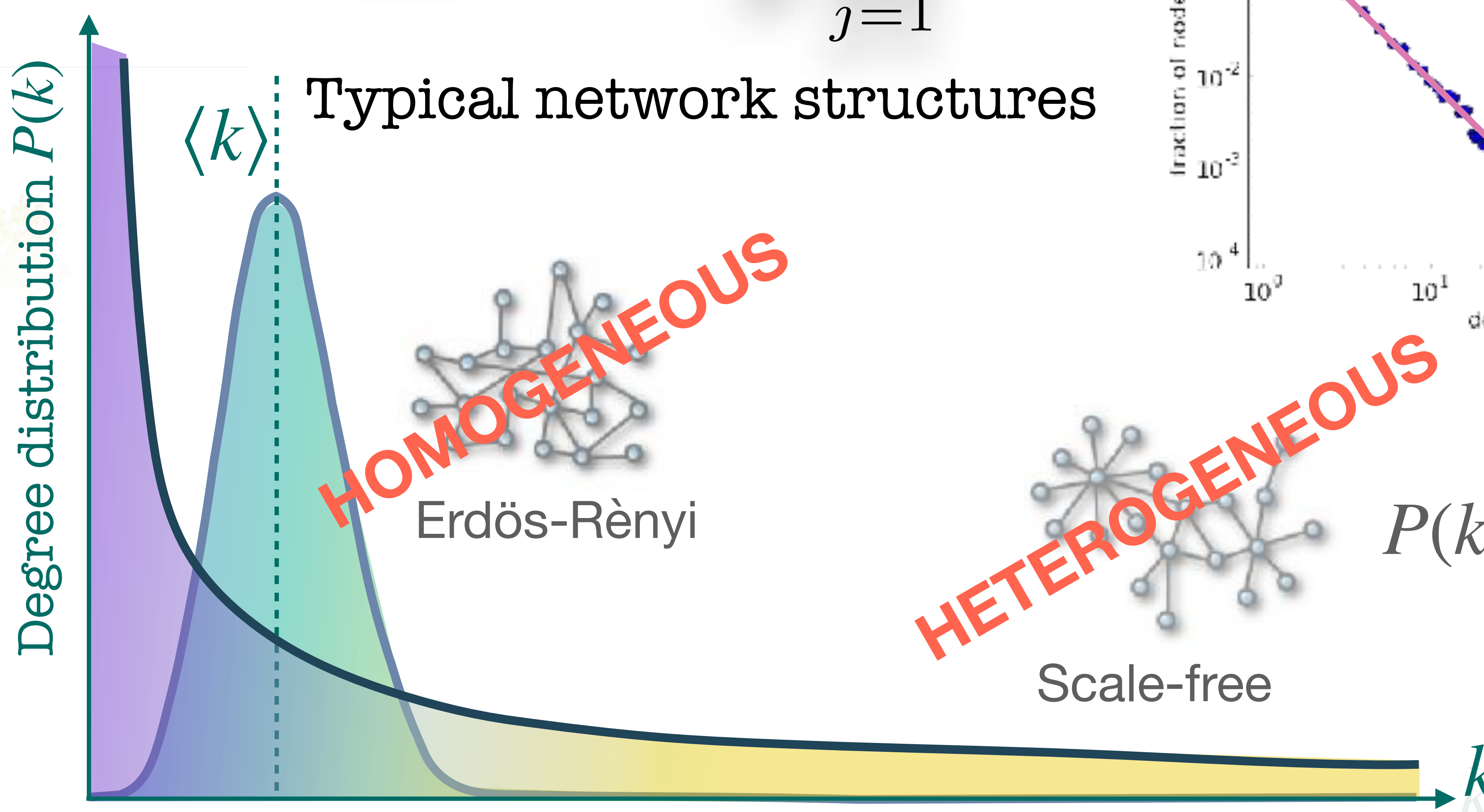


Typical network structures



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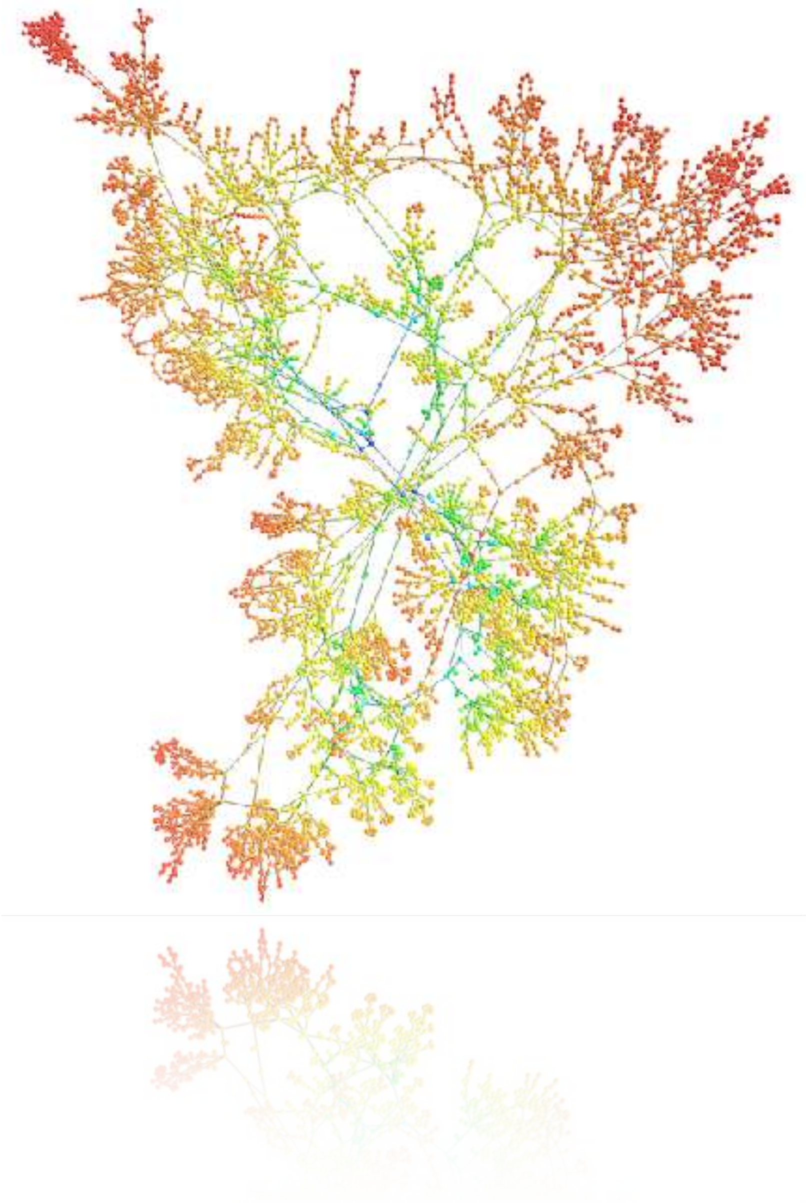
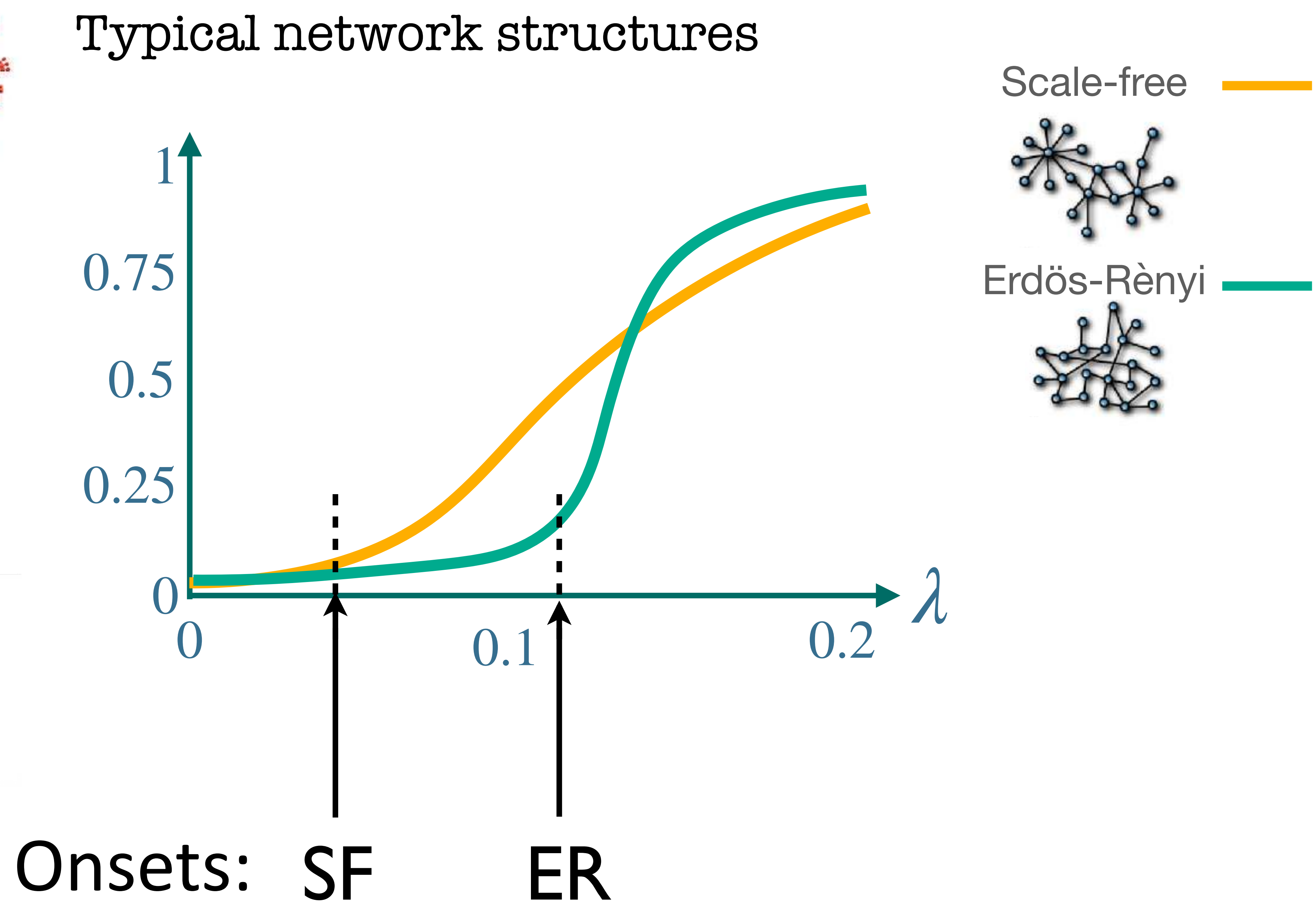
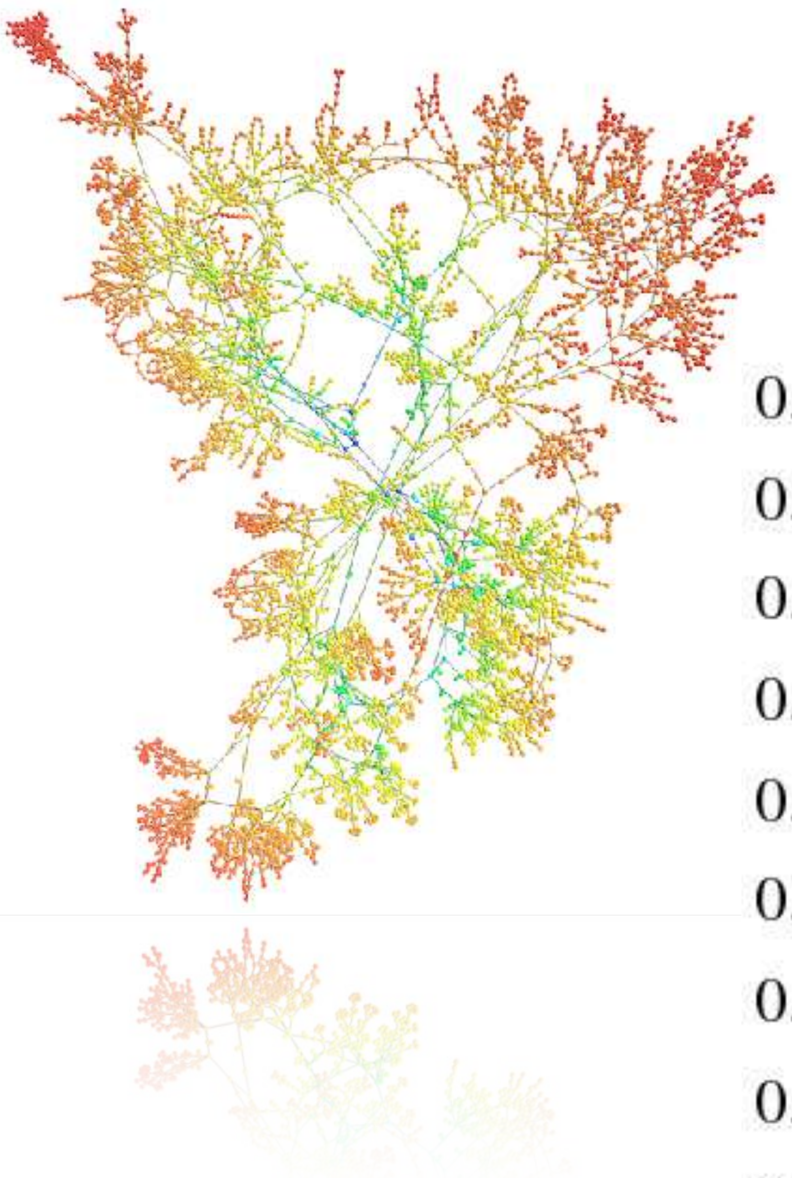
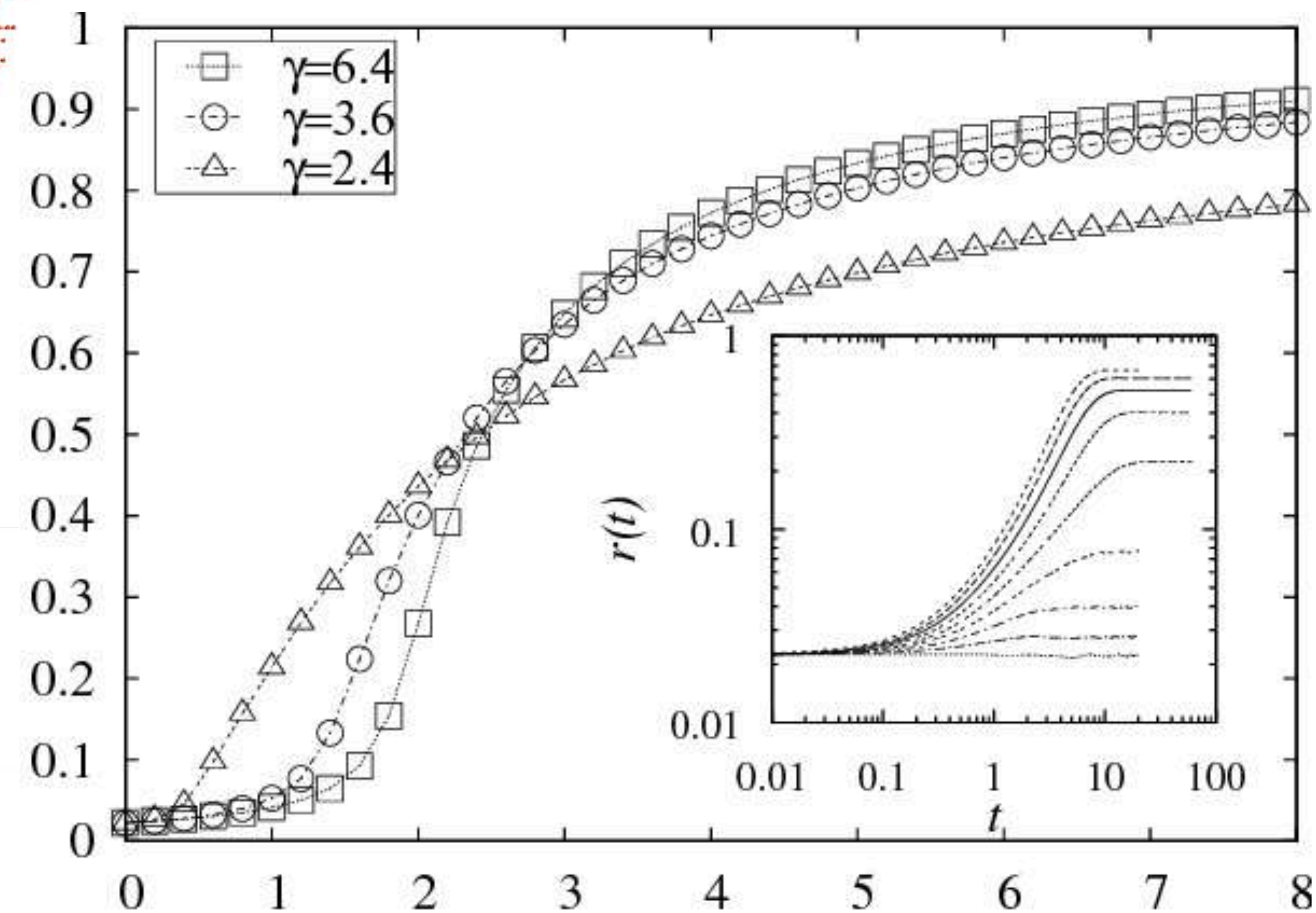




SF topologies anticipate (favor) the onset of Sync



Typical network structures



The more heterogeneous the degree distribution $P(k) \sim k^{-\gamma}$ of the SF graph, the lower the synchronization threshold

Can we connect in a precise way the critical coupling with the topology of the network?

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$$K_c = \frac{2}{\pi g(0)}$$ **Critical Coupling**

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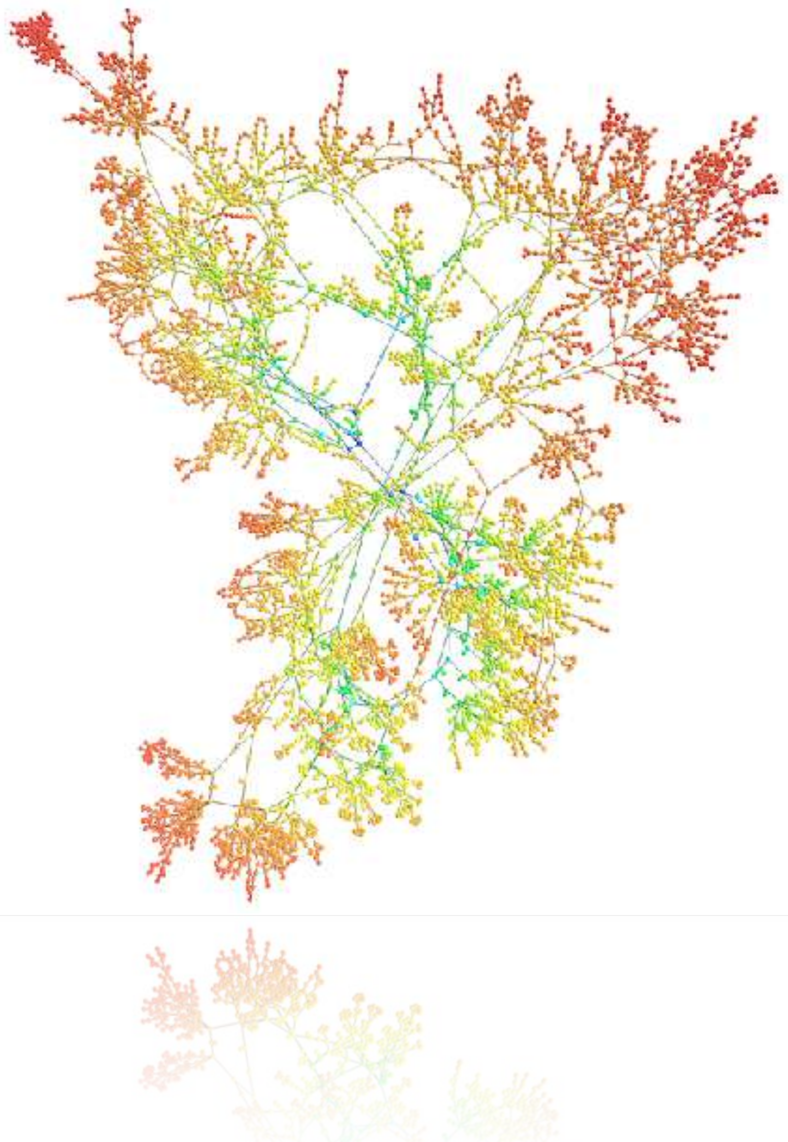


Kuramoto model in a Network

$$\dot{\theta}_i(t) = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin\Big(\theta_j(t) - \theta_i(t)\Big)$$

Kuramoto Order Parameter

$$r(t)e^{i\Psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}$$

Main Question:

What is the influence of network structure ($\mathbf{A}$) in the synchronization transition?

$$\mathbf{A} \rightarrow \lambda_c \;\&\; r(\lambda)$$

Main Question: $$\dot{\theta}_i(t) = \omega_i + \lambda \underline{\sum_{j=1}^{N} A_{ij} \sin\Big(\theta_j(t) - \theta_i(t)\Big)}$$

What is the influence of network structure ($\mathbf{A}$) in the synchronization transition?

$$\mathbf{A} \rightarrow \lambda_c \; \& \; r(\lambda)$$

Main difficulty: we lose the MF behavior of the model equations

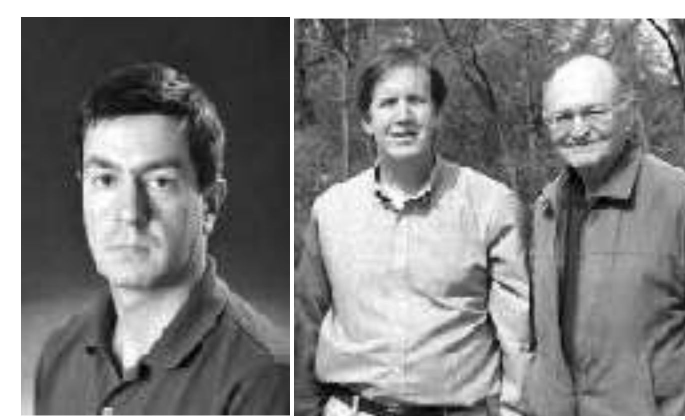
Approaches:

- *Continuum limit approach*

$$\rho_{\omega,k}(\theta, t)$$

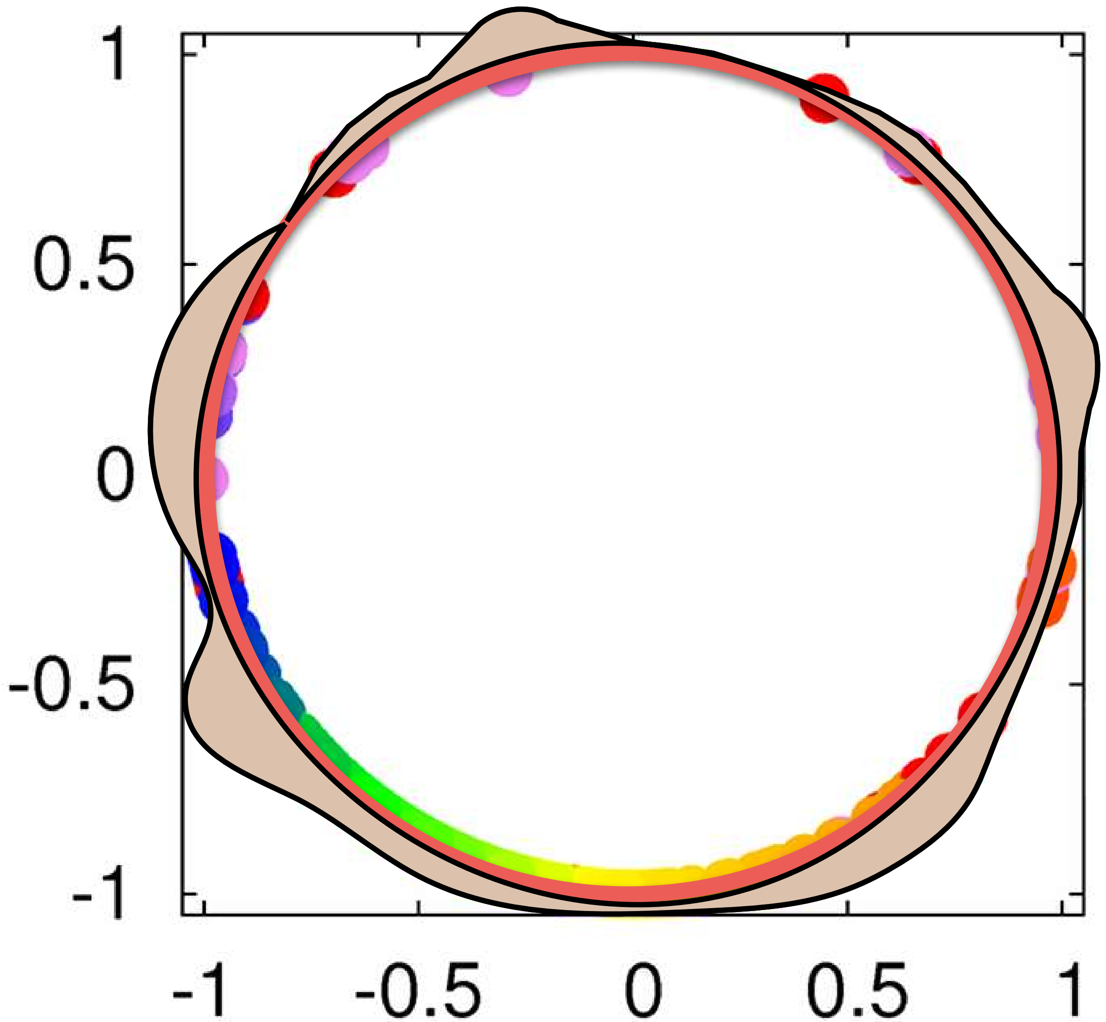
- *Time-average Theory* (TAT)

Ichininomiya (2004)



Restrepo, Hunt & Ott (2005)

CRITICAL COUPLING



Oscillator density functional $\rho_{\omega,k}(\theta,t)$ **:**

Density of oscillators with degree k and natural frequency ω located at angle θ at time t

$$g(\omega) \quad \& \quad P(k)$$

Approaches:

- *Continuum limit approach*

$$\rho_{\omega,k}(\theta,t)$$

Ichininomiya (2004)

- *Time-average Theory* (TAT)

$$r_i e^{i\Psi_i} = \sum_{j=1}^{N} A_{ij} \langle e^{i\theta_j(t)} \rangle_T$$

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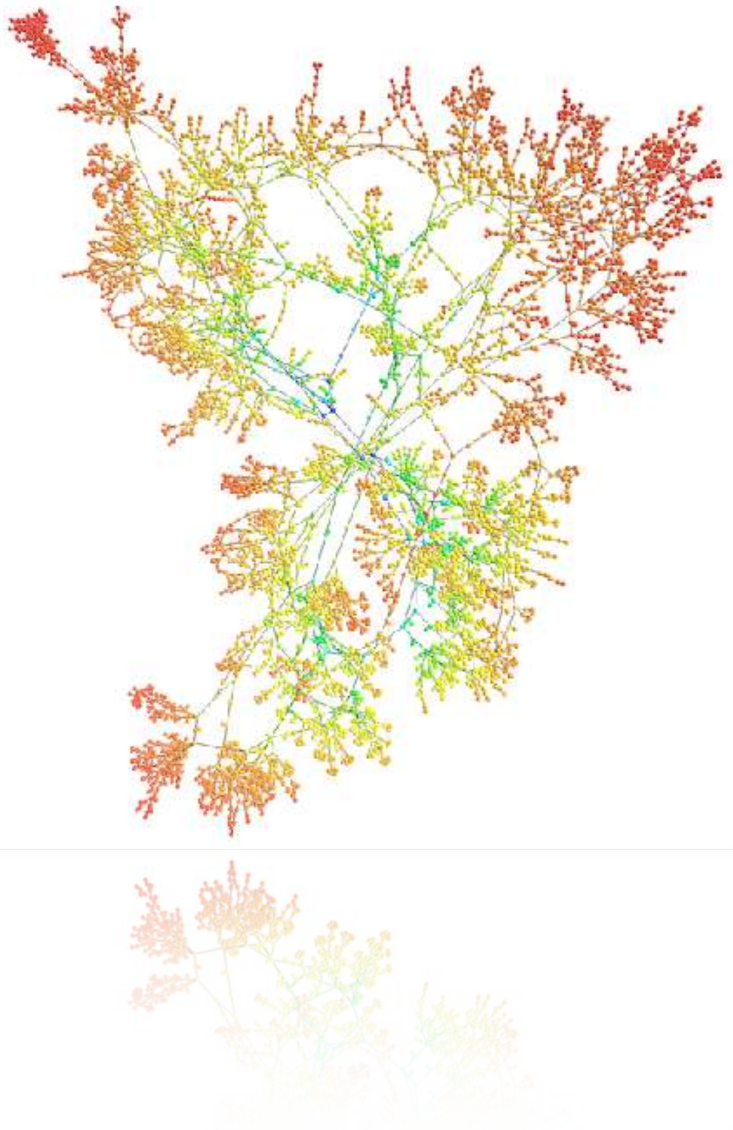
CRITICAL COUPLING



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Kuramoto Order Parameter

$$r(t)e^{i\Psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}$$

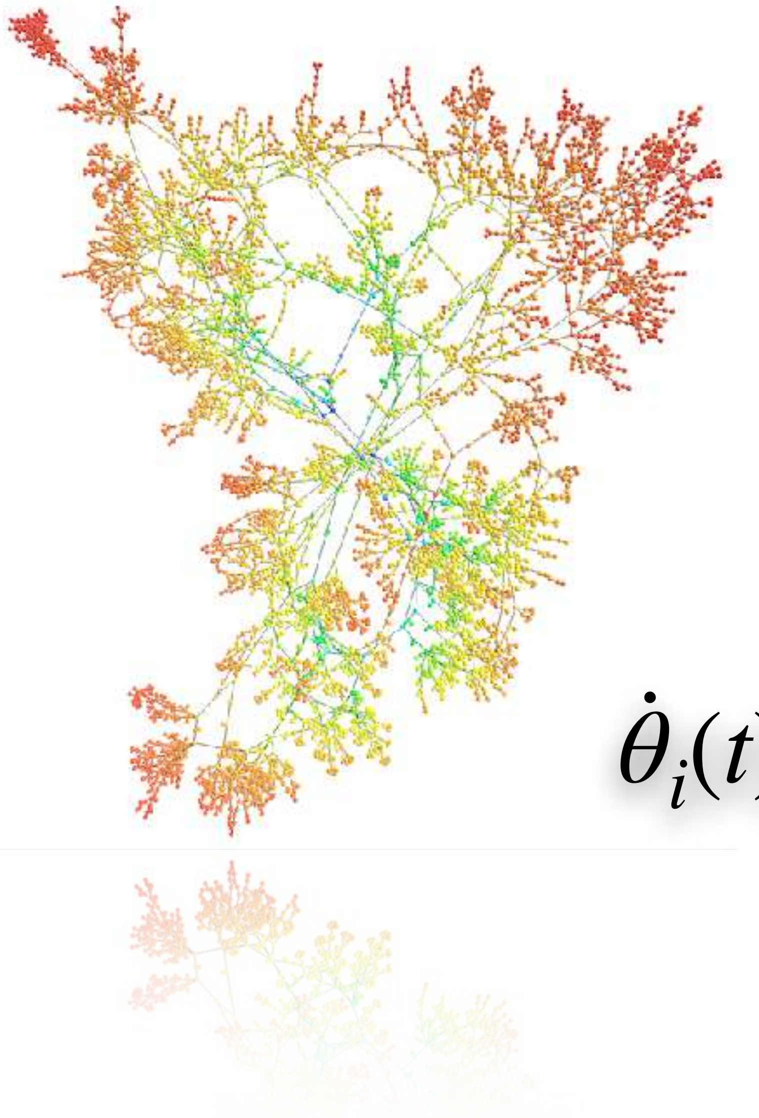
Local Order Parameter

$$r_i e^{i\Psi_i} = \sum_{j=1}^{N} A_{ij}\langle e^{i\theta_j(t)}\rangle_T$$

$\longrightarrow$

Global Order Parameter

$$\tilde{r}e^{i\tilde{\Psi}} = \frac{\sum_{j=1}^{N} r_j}{\sum_{j=1}^{N} k_j} = \frac{\sum_{j=1}^{N} r_j}{2L}$$



Kuramoto model in a Network

$$\dot{\theta}_i(t) = \omega_i + \lambda \mathrm{Im} \sum_{j=1}^{N} \left[A_{ij} e^{i(\theta_j(t)-\theta_i(t))} \right]$$

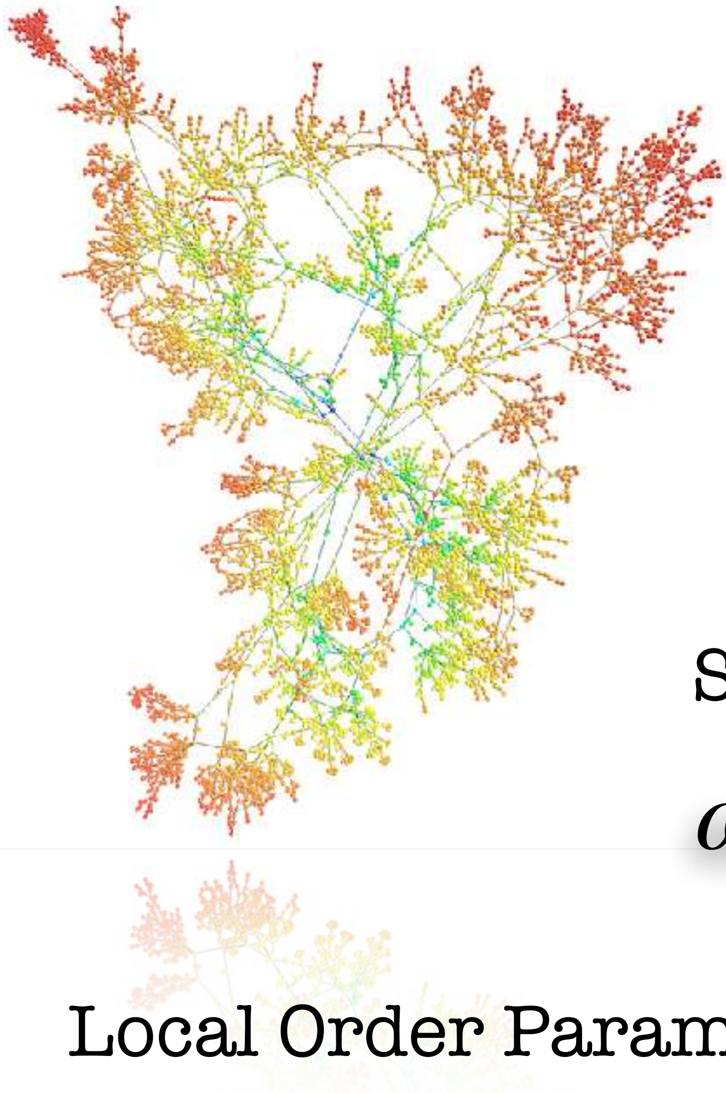
$$\dot{\theta}_i(t) = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin\left(\theta_j(t) - \theta_i(t)\right)$$

$$\dot{\theta}_i(t) = \omega_i + \lambda r_i \sin\left(\psi_i - \theta_i(t)\right) \cancel{- \lambda h_i(t)} \ ???$$

with $\cancel{h_i(t) = \mathrm{Im}\left\{ e^{-i\theta_i} \sum_{j=1}^{N} A_{ij} \left(\langle e^{i\theta_j} \rangle_T - e^{i\theta_j} \right) \right\}}$

Local Order Parameter

$$r_i e^{i\Psi_i} = \sum_{j=1}^{N} A_{ij} \langle e^{i\theta_j(t)} \rangle_T \longrightarrow r_i e^{i(\Psi_i - \theta_i(t))} = \sum_{j=1}^{N} A_{ij} e^{-i\theta_i(t)} \langle e^{i\theta_j(t)} \rangle_T$$



Kuramoto model in a Network

$$\dot{\theta}_i(t) = \omega_i + \lambda r_i \sin\left(\psi_i - \theta_i(t)\right)$$

Steady State:

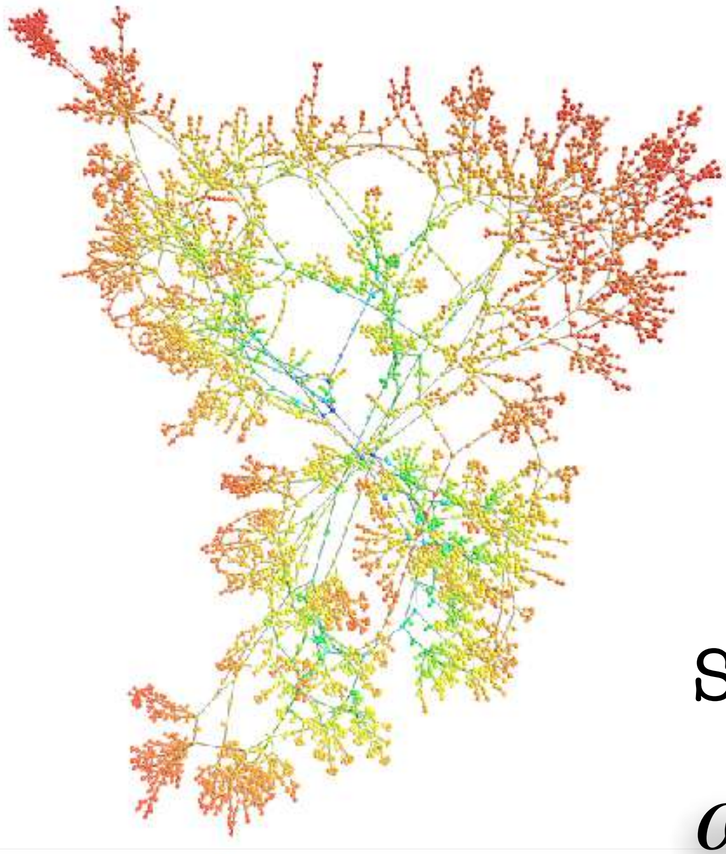
$$\omega_i = \lambda r_i \sin\left(\theta_i(t) - \psi_i\right) \longrightarrow$$

Locking condition

$$|\omega_i| \leq \lambda r_i$$

Local Order Parameter

$$r_i e^{i\Psi_i} = \sum_{j=1}^{N} A_{ij} \langle e^{i\theta_j(t)} \rangle_T \longrightarrow r_i = \sum_{j=1}^{N} A_{ij} e^{-i\Psi_i} \langle e^{i\theta_j(t)} \rangle_T$$



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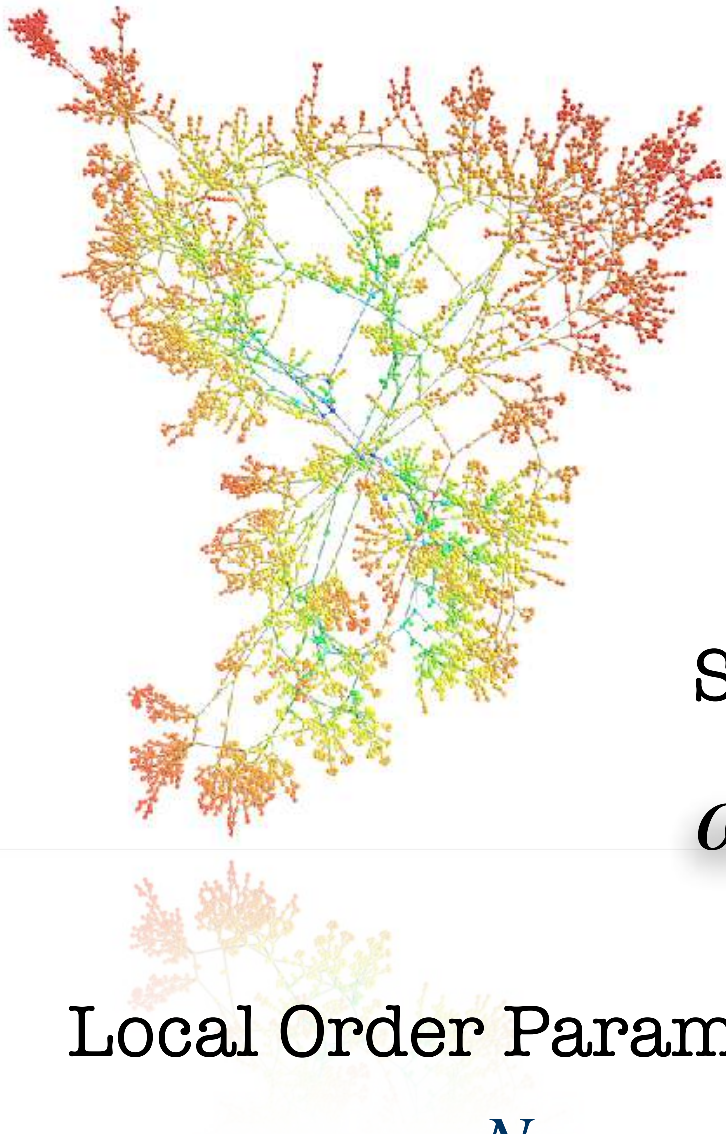
Local Order Parameter

$$\longrightarrow r_i = \sum_{j=1}^{N} A_{ij} e^{-i\Psi_i} \langle e^{i\theta_j(t)} \rangle_T = \sum_{j=1}^{N} A_{ij} \langle e^{i(\theta_j(t) - \Psi_i)} \rangle_T$$

$$= \sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} \langle e^{i(\theta_j(t) - \Psi_i)} \rangle_T + \sum_{|\omega_j| > \lambda r_j}^{N} A_{ij} \langle e^{i(\theta_j(t) - \Psi_i)} \rangle_T$$

CRITICAL COUPLING



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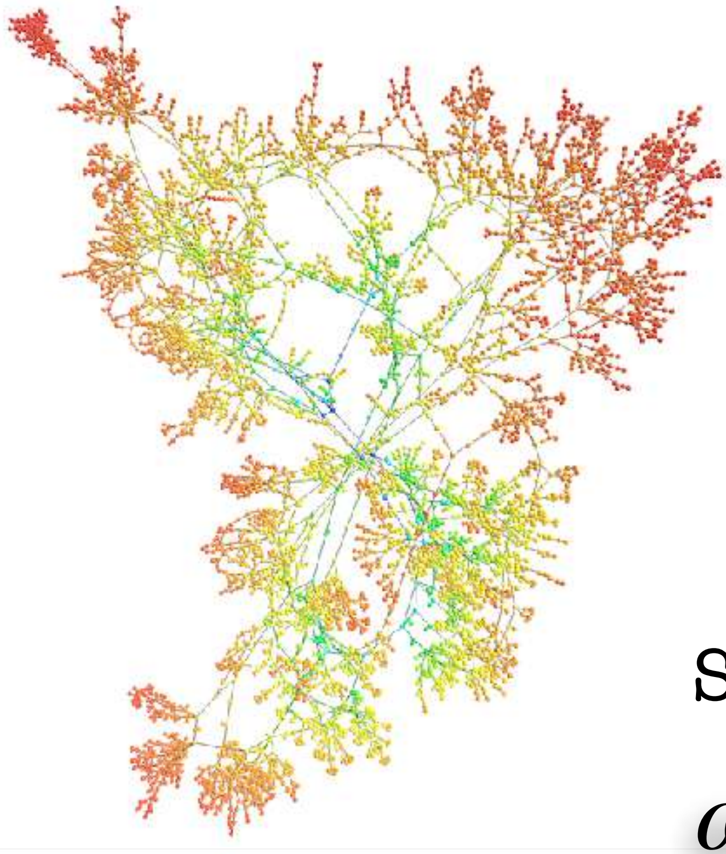
Local Order Parameter

$$\longrightarrow r_i = \sum_{j=1}^{N} A_{ij} e^{-i\Psi_i} \langle e^{i\theta_j(t)} \rangle_T = \sum_{j=1}^{N} A_{ij} \langle e^{i(\theta_j(t)-\Psi_i)} \rangle_T$$

$$= \sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} e^{i(\theta_j - \Psi_i)} + \cancel{\sum_{|\omega_j| > \lambda r_j}^{N} A_{ij} \langle e^{i(\theta_j(t)-\Psi_i)} \rangle_T}$$

CRITICAL COUPLING



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Steady State:

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Locking condition

$$|\omega_i| \leq \lambda r_i$$

Local Order Parameter

$$\longrightarrow r_i = \sum_{j=1}^{N} A_{ij} e^{-i\Psi_i} \langle e^{i\theta_j(t)} \rangle_T = \mathbf{Re} \left[\sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} e^{i(\theta_j - \Psi_i)} \right]$$

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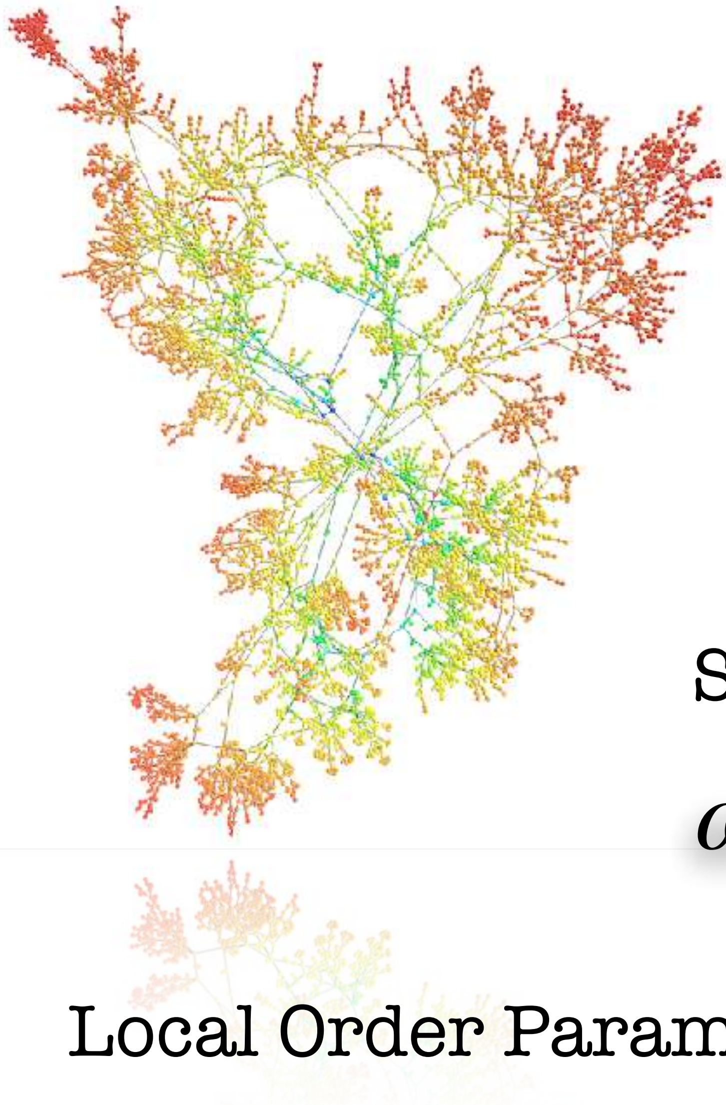
CRITICAL COUPLING



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Local Order Parameter

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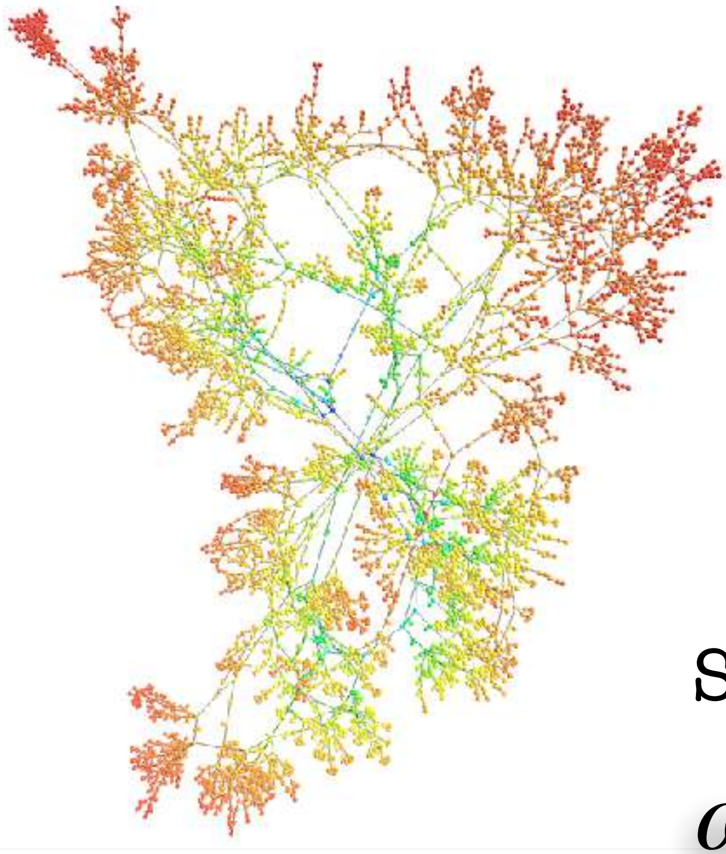
$$= \sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} \left[\cos(\theta_j - \Psi_j)\cos(\Psi_j - \Psi_i) - \sin(\theta_j - \Psi_j)\sin(\Psi_j - \Psi_i)\right]$$

CRITICAL COUPLING



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Steady State:

$$\omega_i = \lambda r_i \sin\left(\theta_i(t) - \psi_i\right) \longrightarrow |\omega_i| \leq \lambda r_i$$

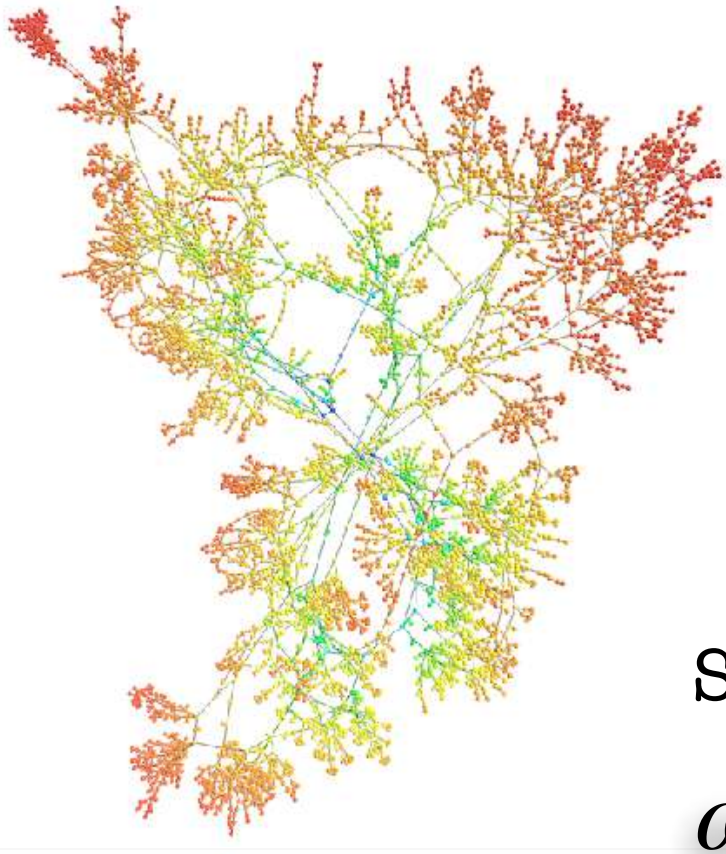
Locking condition

Local Order Parameter

$$r_i = \sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} \left[\sqrt{1 - \left(\frac{\omega_j}{\lambda r_j}\right)^2} \cos(\Psi_j - \Psi_i) - \cancel{\frac{\omega_j}{\lambda r_j} \sin(\Psi_j - \Psi_i)} \right]$$

Symmetry of $g(\omega)$

CRITICAL COUPLING



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Locking condition

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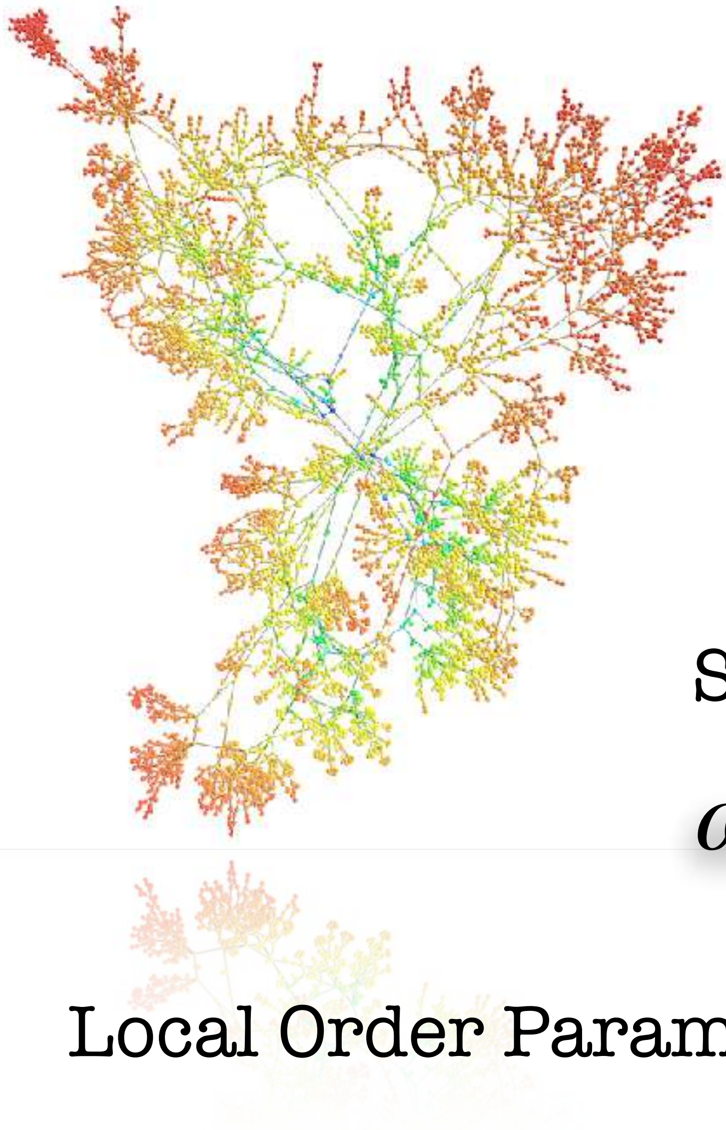
Local Order Parameter

$$r_i = \sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} \sqrt{1 - \left(\frac{\omega_j}{\lambda r_j}\right)^2} \cos(\Psi_j - \Psi_i) = \sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} \sqrt{1 - \left(\frac{\omega_j}{\lambda r_j}\right)^2}$$

Smallest possible λ

CRITICAL COUPLING



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Global Order Parameter

$$\tilde{r} e^{i\tilde{\Psi}} = \frac{\sum_{j=1}^{N} r_j}{\sum_{j=1}^{N} k_j} = \frac{\sum_{j=1}^{N} r_j}{2L}$$

TAT Auto-consistent equation for r_i

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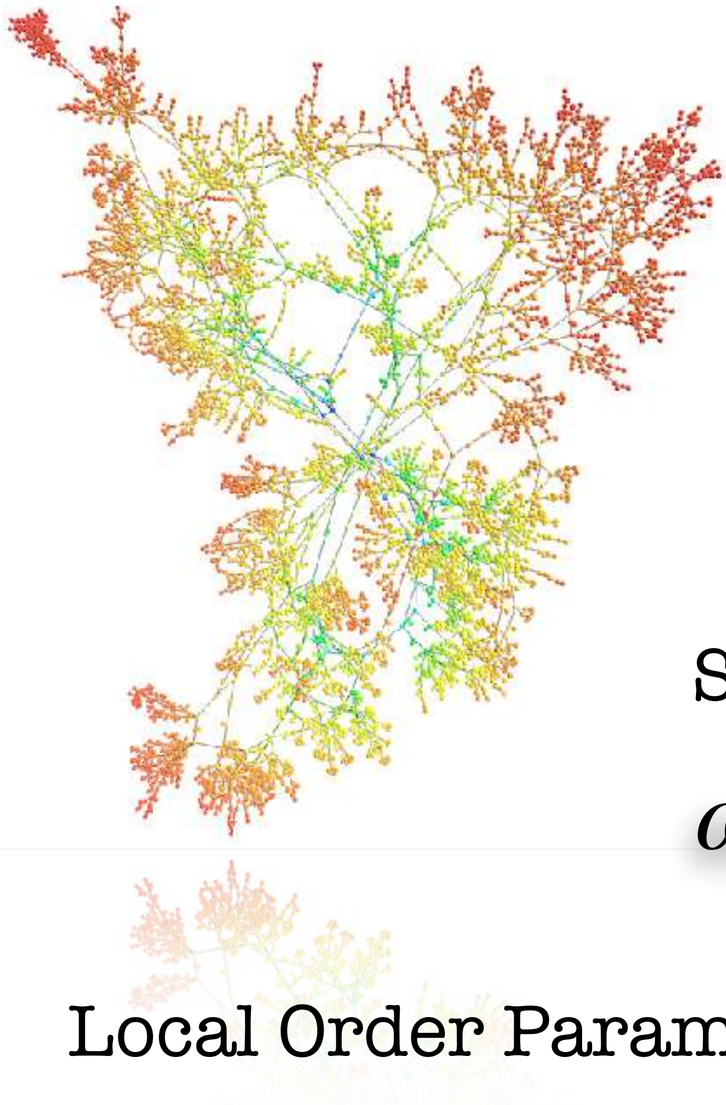


Kuramoto model in a Network

$$\dot{\theta}_i(t) = \omega_i + \lambda r_i \sin\left(\psi_i - \theta_i(t)\right)$$

Steady State:

$$\omega_i = \lambda r_i \sin\left(\theta_i(t) - \psi_i\right) \longrightarrow$$

Locking condition

$$|\omega_i| \leq \lambda r_i$$

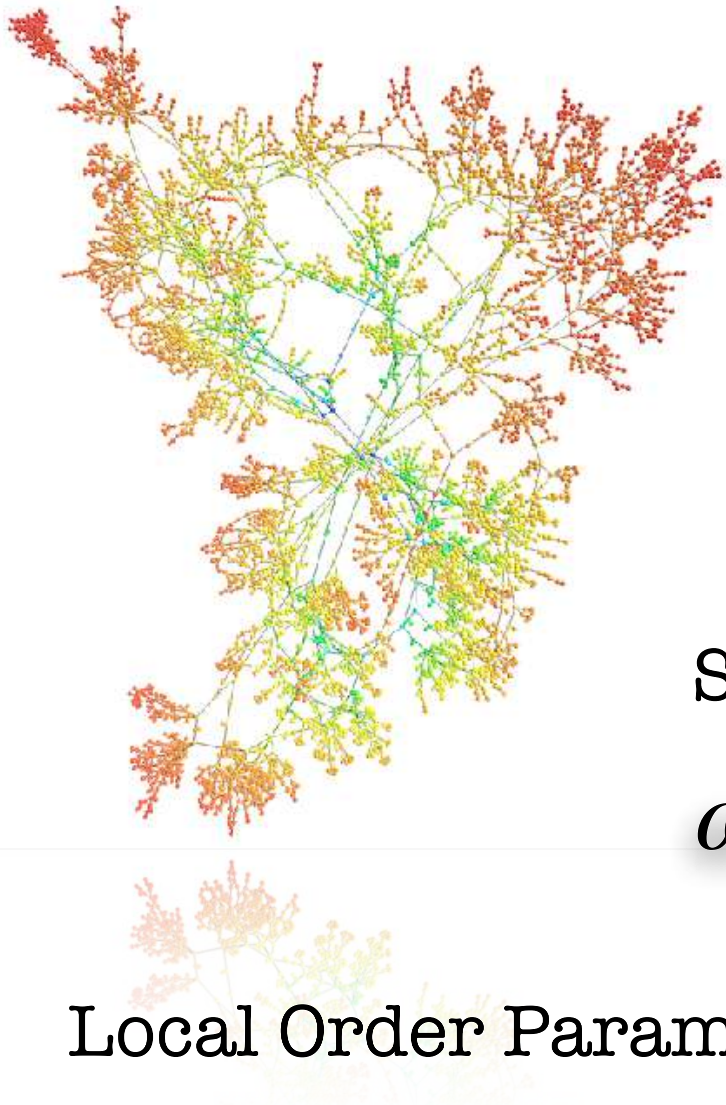
Local Order Parameter

$$r_i = \sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} \sqrt{1 - \left(\frac{\omega_j}{\lambda r_j}\right)^2}$$

Requires:

- Adjacency Matrix
- Set of nodes' frequencies

TAT Auto-consistent equation for r_i



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Locking condition

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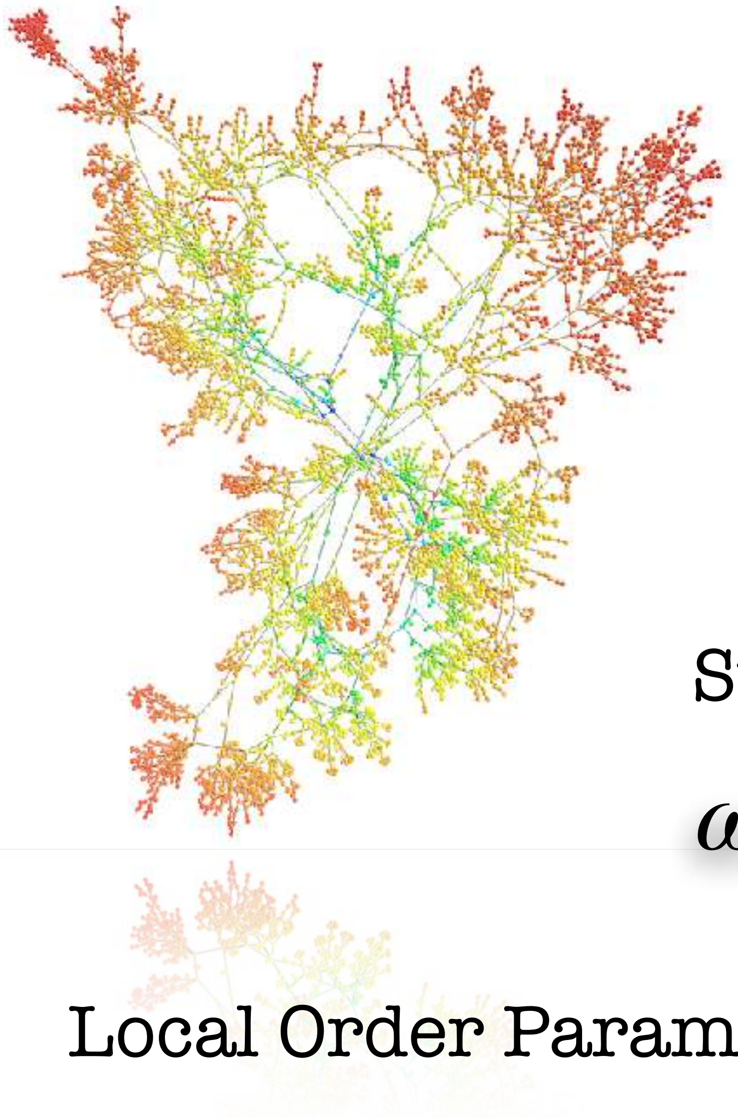
Local Order Parameter

$$r_i = \sum_{|\omega_j| \leq \lambda r_j}^{N} A_{ij} \sqrt{1 - \left(\frac{\omega_j}{\lambda r_j}\right)^2} \simeq \sum_j A_{ij} \int_{-\lambda r_j}^{\lambda r_j} g(\omega) \sqrt{1 - \left(\frac{\omega}{\lambda r_j}\right)^2} d\omega$$

Frequency distribution approximation (FDA)

@gomezgardenes



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Locking condition

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Local Order Parameter

$$r_i \simeq \sum_j A_{ij} \int_{-\lambda r_j}^{\lambda r_j} g(\omega) \sqrt{1 - \left(\frac{\omega}{\lambda r_j}\right)^2}\, d\omega \qquad x = \frac{\omega}{\lambda r_j}$$

Frequency distribution approximation (FDA)

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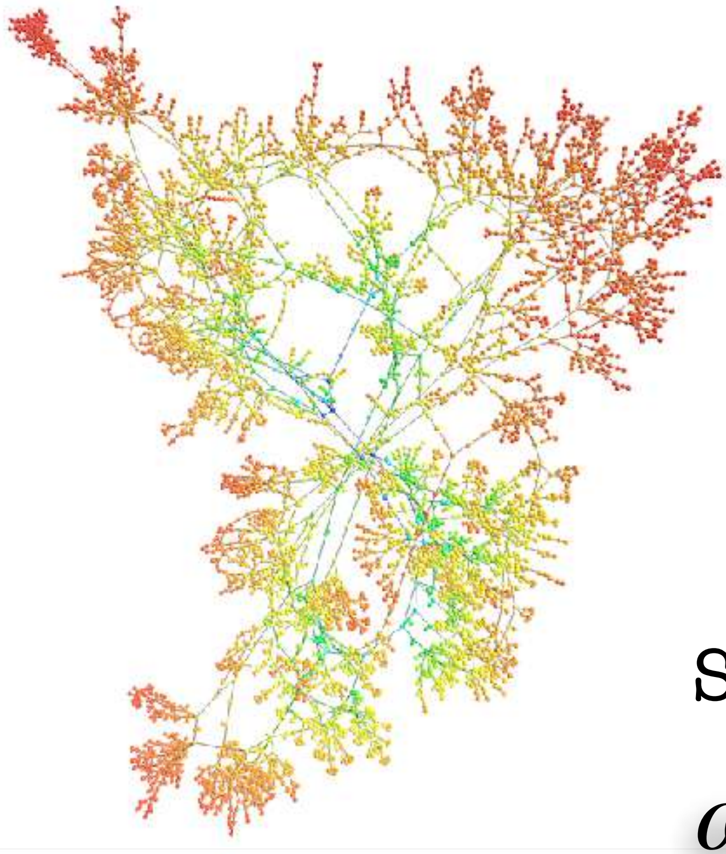
CRITICAL COUPLING



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Locking condition

$$|\omega_i| \leq \lambda r_i$$

Local Order Parameter

At λ_c we have $r_i \to 0^+$:

$$r_i \simeq \lambda \sum_j A_{ij} r_j \int_{-1}^{1} g(x\lambda r_j)\sqrt{1-x^2}dx$$

Requires:

- Adjacency Matrix
- Frequency Distribution

Frequency distribution approximation (FDA)

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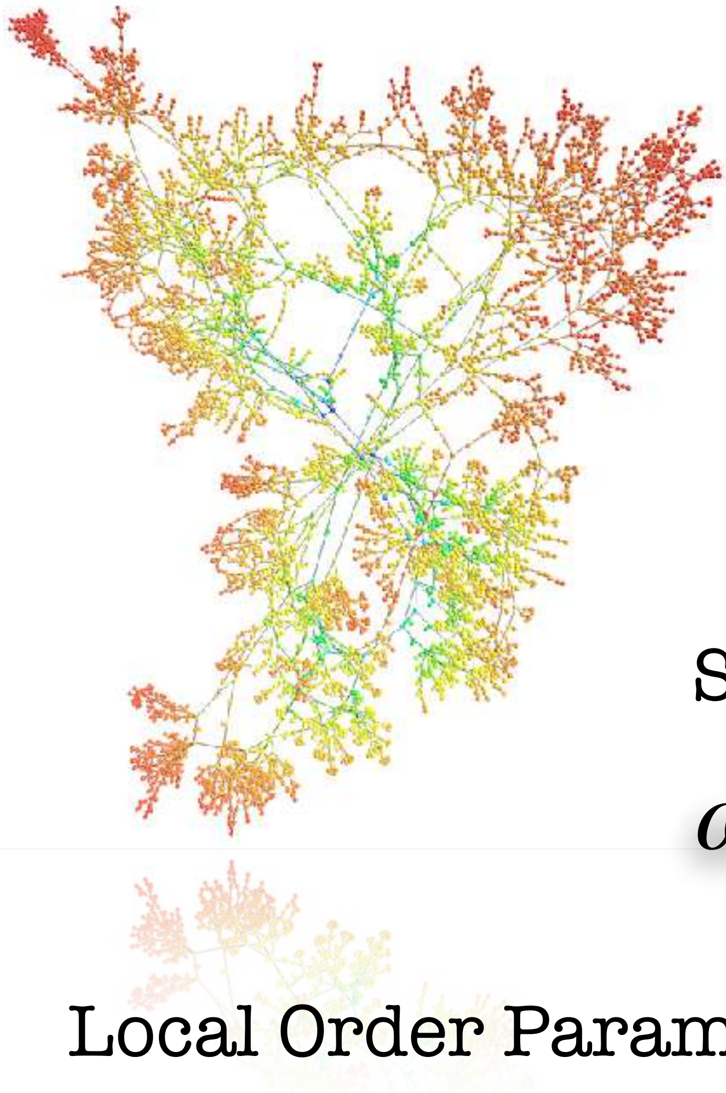

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Locking condition

$$|\omega_i| \leq \lambda r_i$$

Local Order Parameter

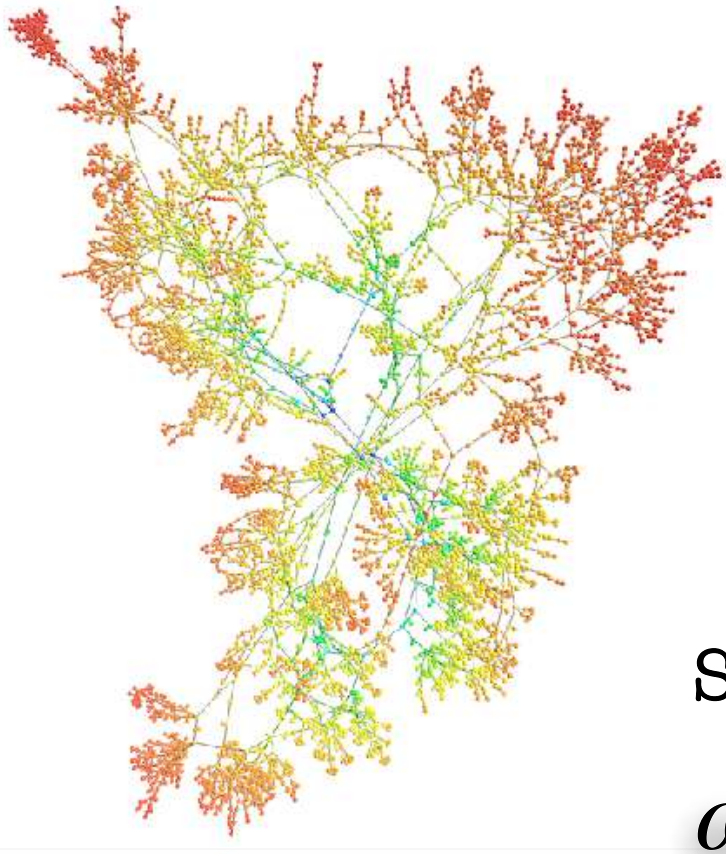
At λ_c we have $r_i \to 0^+$:

$$r_i \simeq \lambda_c \sum_j A_{ij} r_j g(0) \int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

Requires:

- Adjacency Matrix
- Frequency Distribution

Frequency distribution approximation (FDA)



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Locking condition

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Local Order Parameter

At λ_c we have $r_i \to 0^+$:

$$r_i \simeq \lambda_c \sum_j A_{ij} r_j g(0) \frac{\pi}{2}$$

Requires:

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- Frequency Distribution

Frequency distribution approximation (FDA)

@gomezgardenes

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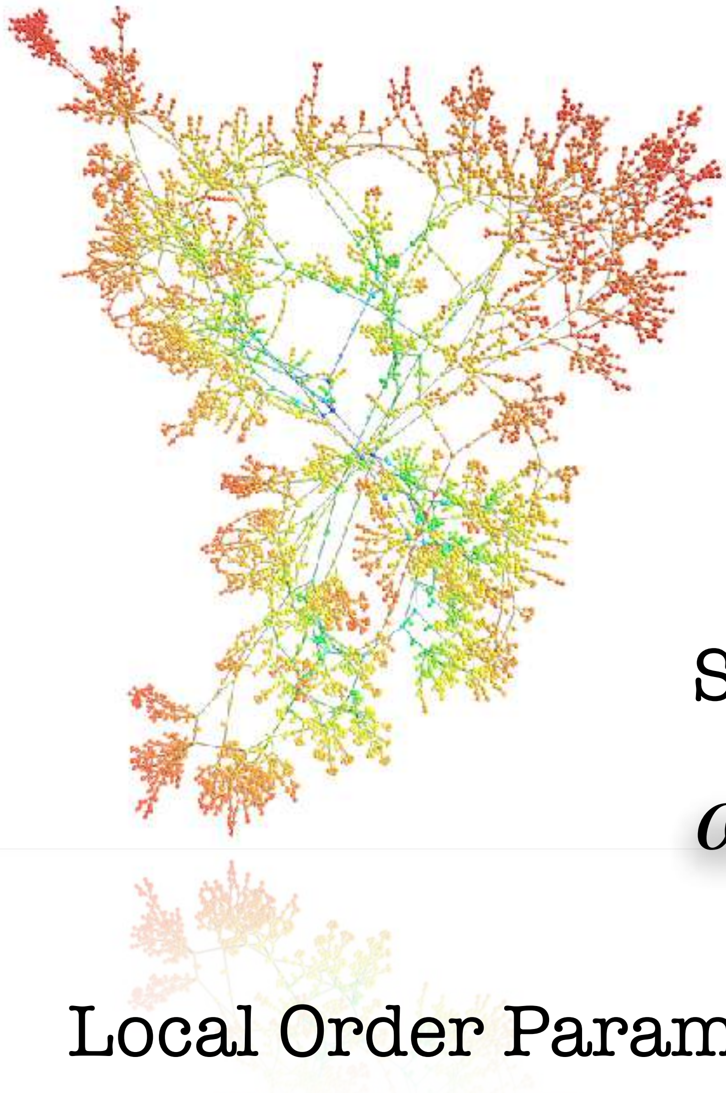

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Steady State:

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Locking condition

Local Order Parameter

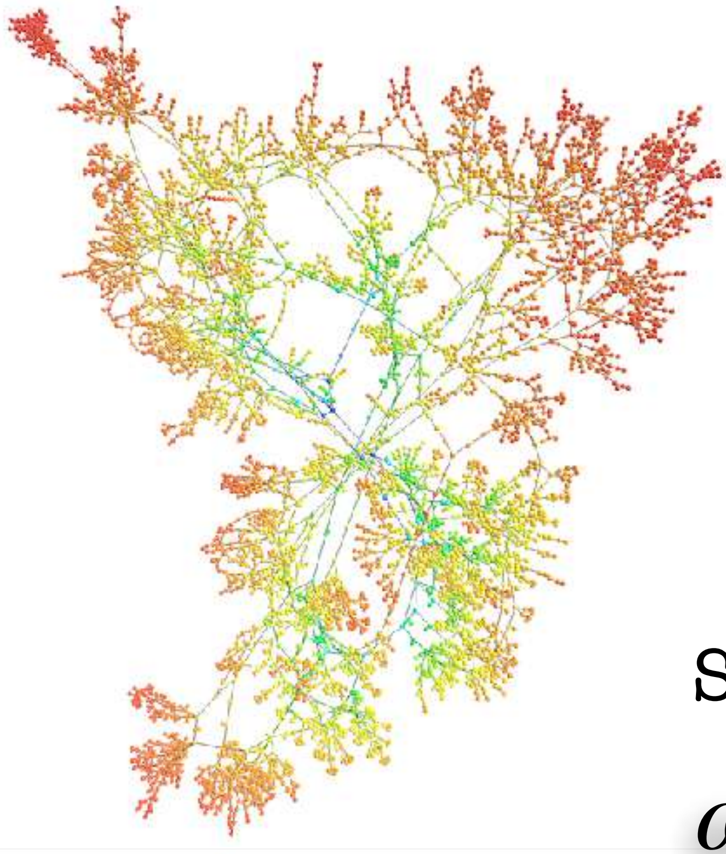
At λ_c we have $r_i \to 0^+$:

$$r_i \simeq \frac{\lambda_c g(0)\pi}{2} \sum_j A_{ij} r_j$$

Requires:

- Adjacency Matrix
- Frequency Distribution

Frequency distribution approximation (FDA)



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Locking condition

$$|\omega_i| \leq \lambda r_i$$

Local Order Parameter

At λ_c we have $r_i \to 0^+$:

$$r_i \simeq \boxed{\frac{2}{\lambda_c g(0)\pi}} \sum_j A_{ij} r_j$$

$$\vec{r} \simeq A\vec{r}$$

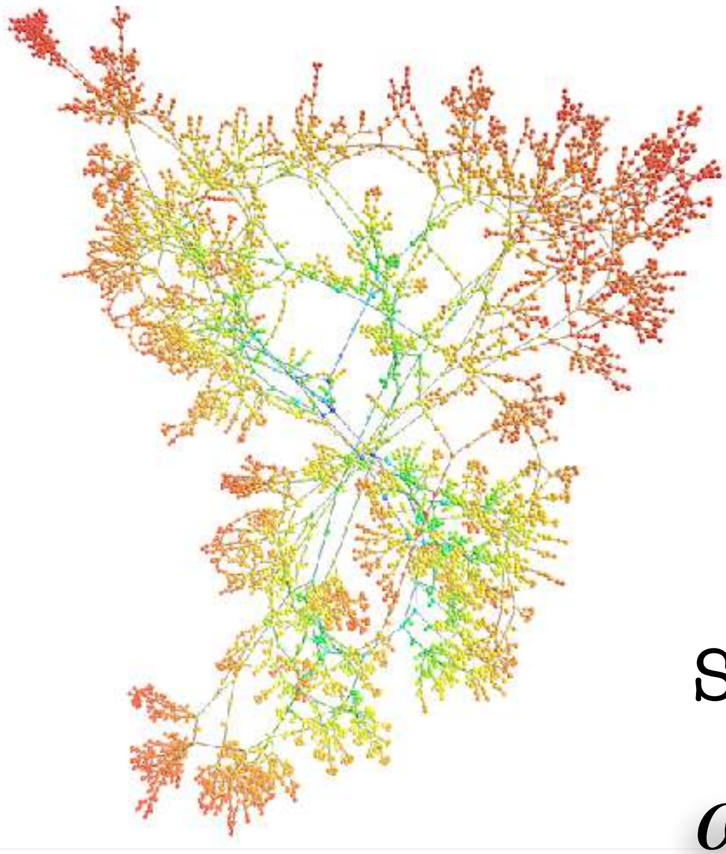
is an eigenvector of $\mathbf{A}$!!!

Requires:

- Adjacency Matrix
- Frequency Distribution

Frequency distribution approximation (FDA)

CRITICAL COUPLING



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Steady State:

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Locking condition

Local Order Parameter

At λ_c we have $r_i \to 0^+$:

$$\boxed{\frac{2}{\lambda_c g(0)\pi}} = \Lambda_{max}(A)$$

is an eigenvector of $\mathbf{A}$!!!

Requires:

- Adjacency Matrix
- Frequency Distribution

Frequency distribution approximation (FDA)

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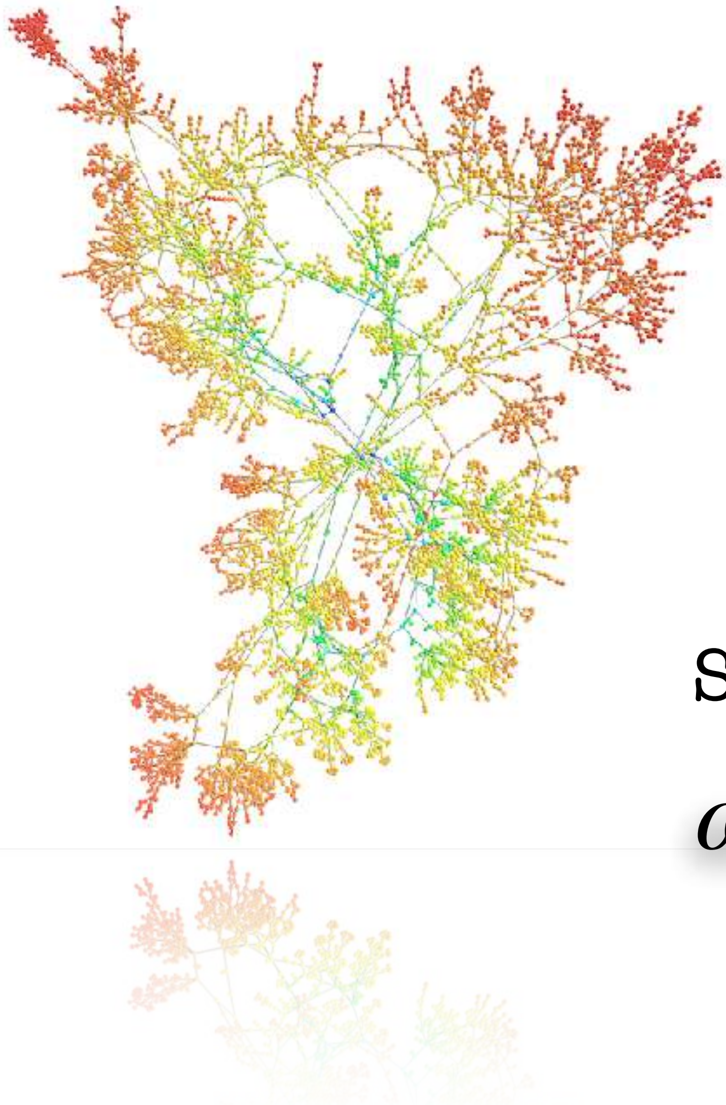

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Locking condition

$$|\omega_i| \leq \lambda r_i$$

Critical Coupling

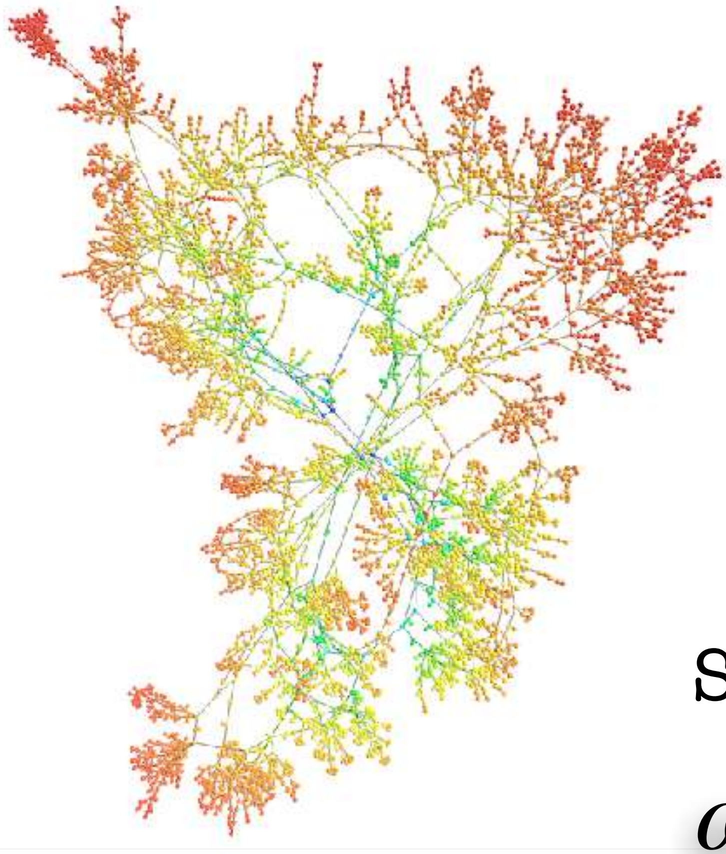
$$\lambda_c = \frac{2}{g(0)\pi} \frac{1}{\Lambda_{max}(A)}$$

Requires:

- Adjacency Matrix
- Frequency Distribution

Frequency distribution approximation (FDA)

@gomezgardenes



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Locking condition

$$|\omega_i| \leq \lambda r_i$$

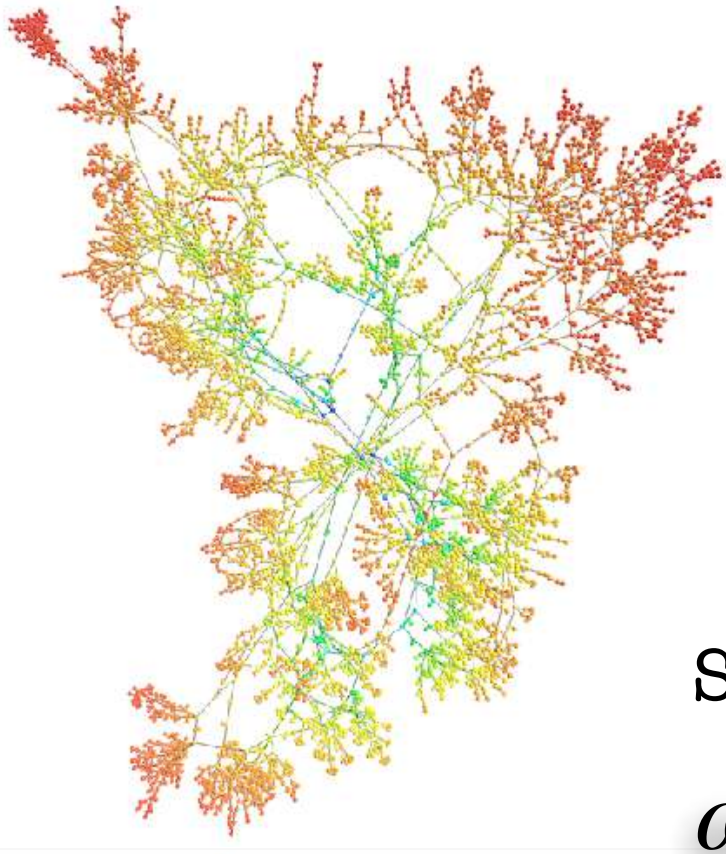
Critical Coupling

$$\lambda_c = \frac{2}{g(0)\pi}\frac{1}{\Lambda_{max}(A)} = K_c^{KM}\frac{1}{\Lambda_{max}(A)}$$

$$K_c^{KM} = \frac{2}{g(0)\pi}$$

Requires:

- Adjacency Matrix
- Frequency Distribution



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$$\lambda_c = \frac{2}{g(0)\pi}\frac{1}{\Lambda_{max}(A)} = K_c^{KM}\frac{1}{\Lambda_{max}(A)}$$

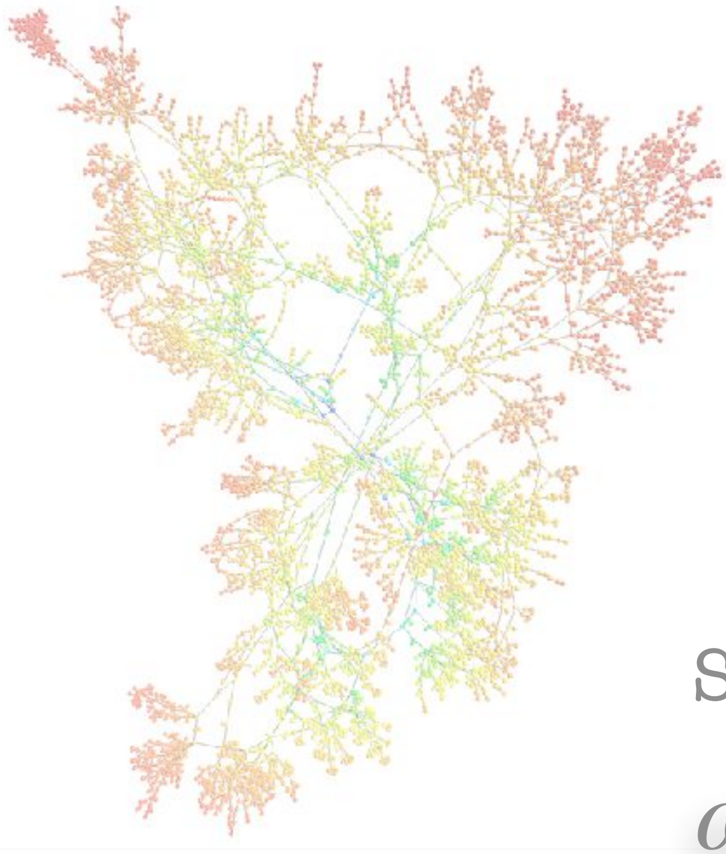
Requires:

- Adjacency Matrix
- Frequency Distribution

HMF: $r_i = \tilde{r}k_i \longrightarrow \lambda_c = \dfrac{2}{g(0)\pi}\dfrac{\langle k \rangle}{\langle k^2 \rangle}$

Requires:

- Degree Distribution
- Frequency Distribution



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Requires:

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HMF: $r_i = \tilde{r}k_i \longrightarrow \lambda_c = \frac{2}{g(0)\pi}\frac{\langle k \rangle}{\langle k^2 \rangle}$

Requires:

- Degree Distribution
- Frequency Distribution

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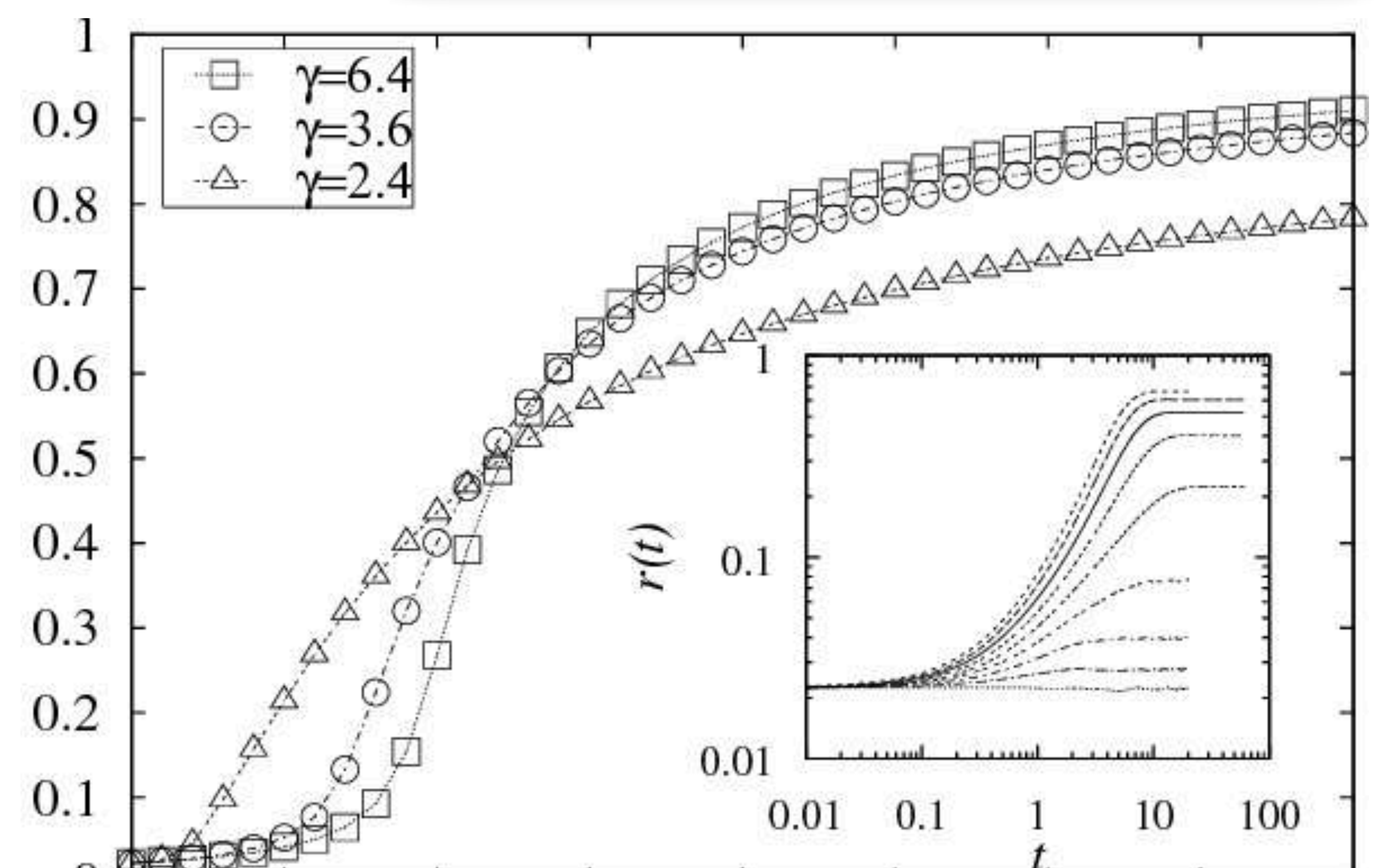
Scale-free Networks $P(k) \sim k^{-\gamma}$

$$\langle k^2 \rangle \simeq \int_0^\infty k^2 P(k) dk \sim \int_0^\infty k^{2-\gamma} dk$$

As γ decreases the network becomes more heterogeneous and the second moment increases

$$\lambda_c = \frac{2}{\pi g(0)} \frac{\langle k \rangle}{\langle k^2 \rangle}$$

The critical coupling λ_c decreases as γ decreases



Microscopic view

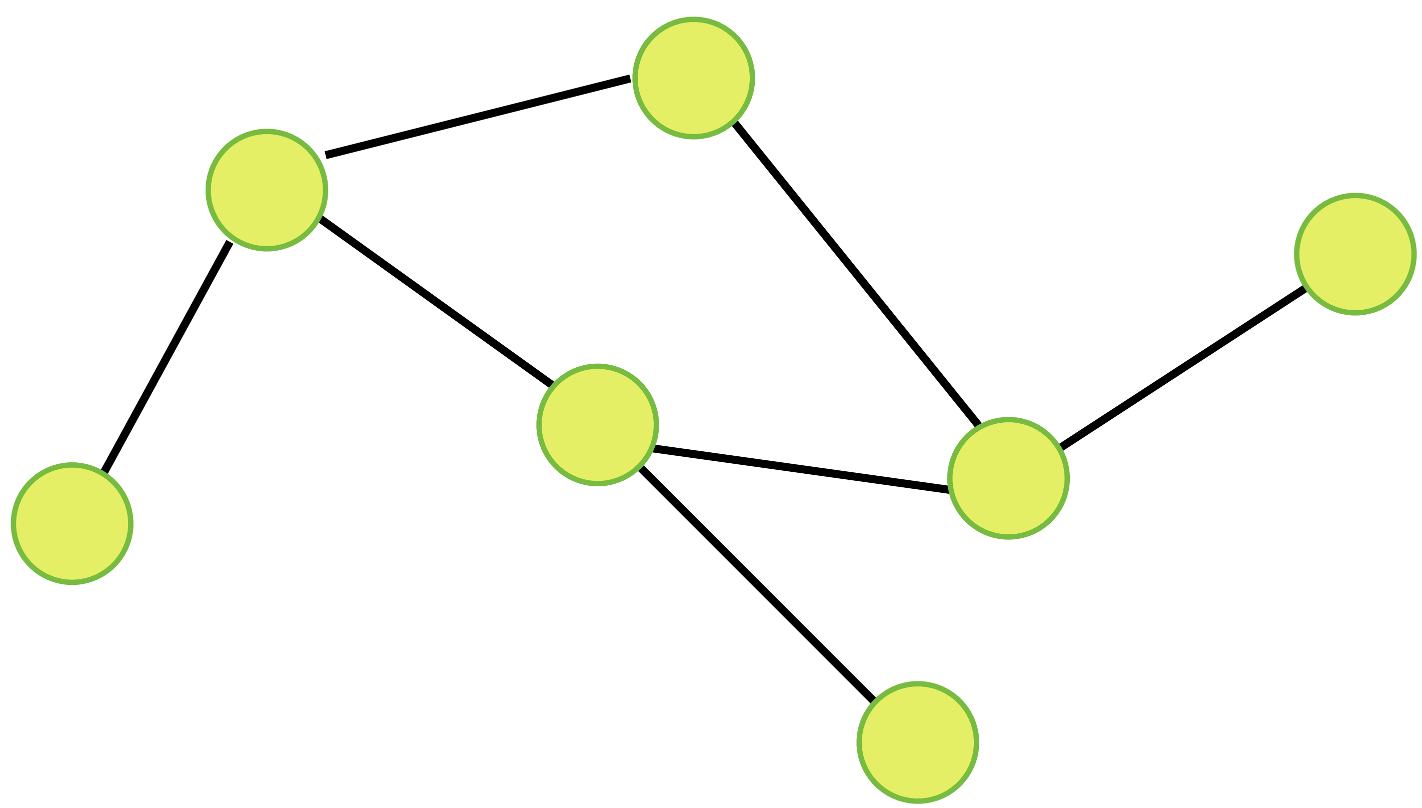
LOCAL SYNCHRONIZATION

New Measure:

- the degree of synchronization between pairs of connected nodes:

$$D_{ij} = \lim_{T\to\infty} A_{ij} \left| \frac{1}{T} \int_{\tau}^{\tau+T} e^{i[\theta_i(t)-\theta_j(t)]}\, dt \right| .$$

$D_{ij} \simeq 0 \rightarrow$ local incoherence $\quad D_{ij} = 1 \rightarrow$ local synchronization.



Gómez-Gardeñes et al. Phys. Rev. Lett. (2007)

LOCAL SYNCHRONIZATION

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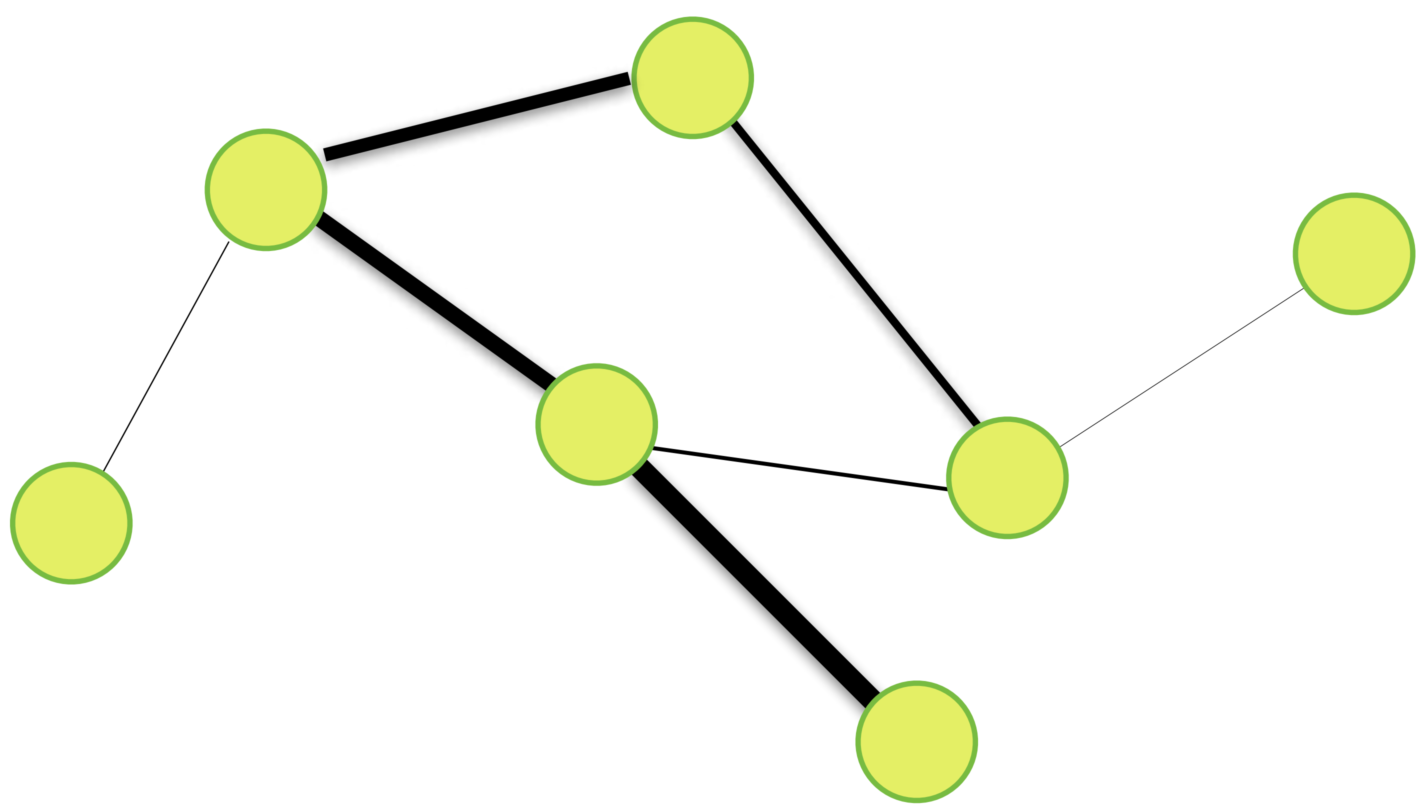

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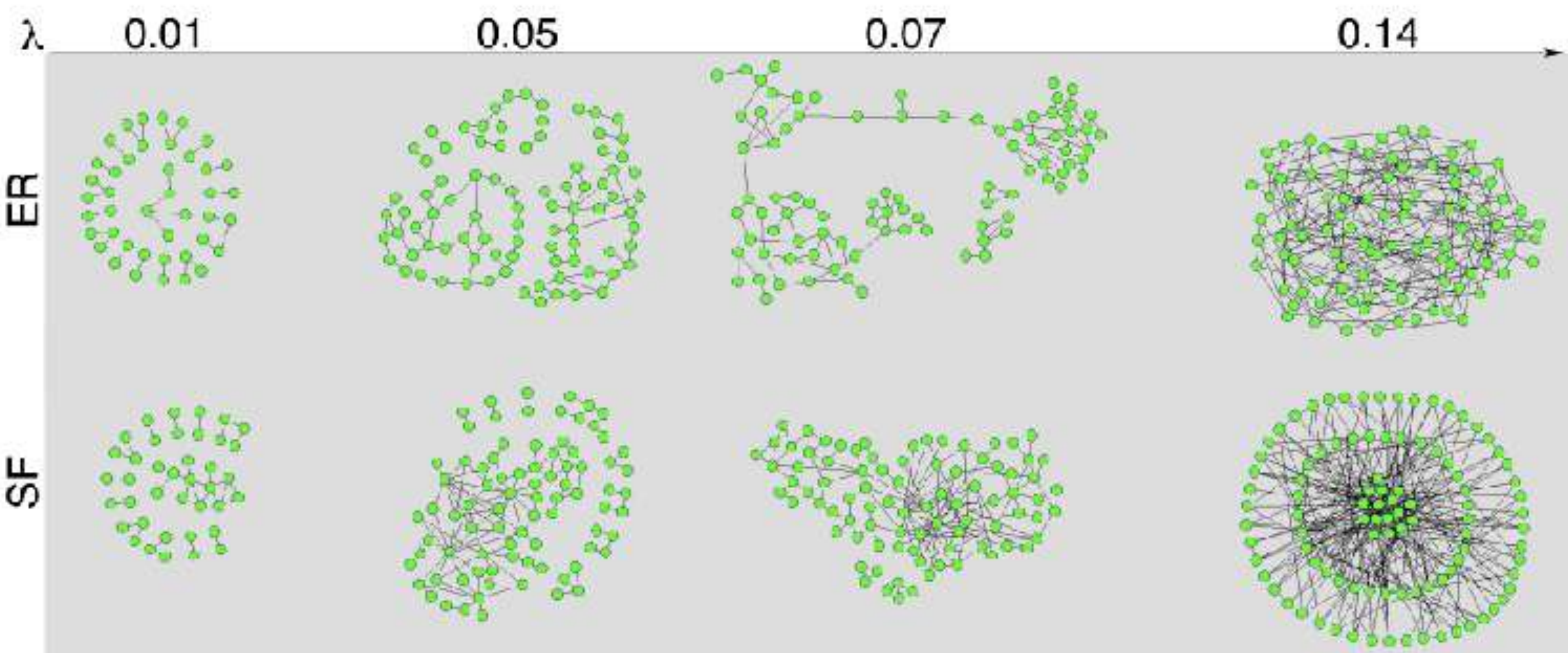
Gómez-Gardeñes et al. Phys. Rev. Lett. (2007)

LOCAL SYNCHRONIZATION

$$D_{ij} = \lim_{T \to \infty} A_{ij} \left| \frac{1}{T} \int_{\tau}^{\tau+T} e^{i[\theta_i(t) - \theta_j(t)]} \, dt \right| .$$

$D_{ij} \simeq 0 \rightarrow$ local incoherence $\quad$ $D_{ij} = 1 \rightarrow$ local synchronization.

Studying D_{ij}: We can monitor how synchronized links are created as a function of the coupling. Different paths towards synchronization:



Gómez-Gardeñes et al. Phys. Rev. Lett. (2007)

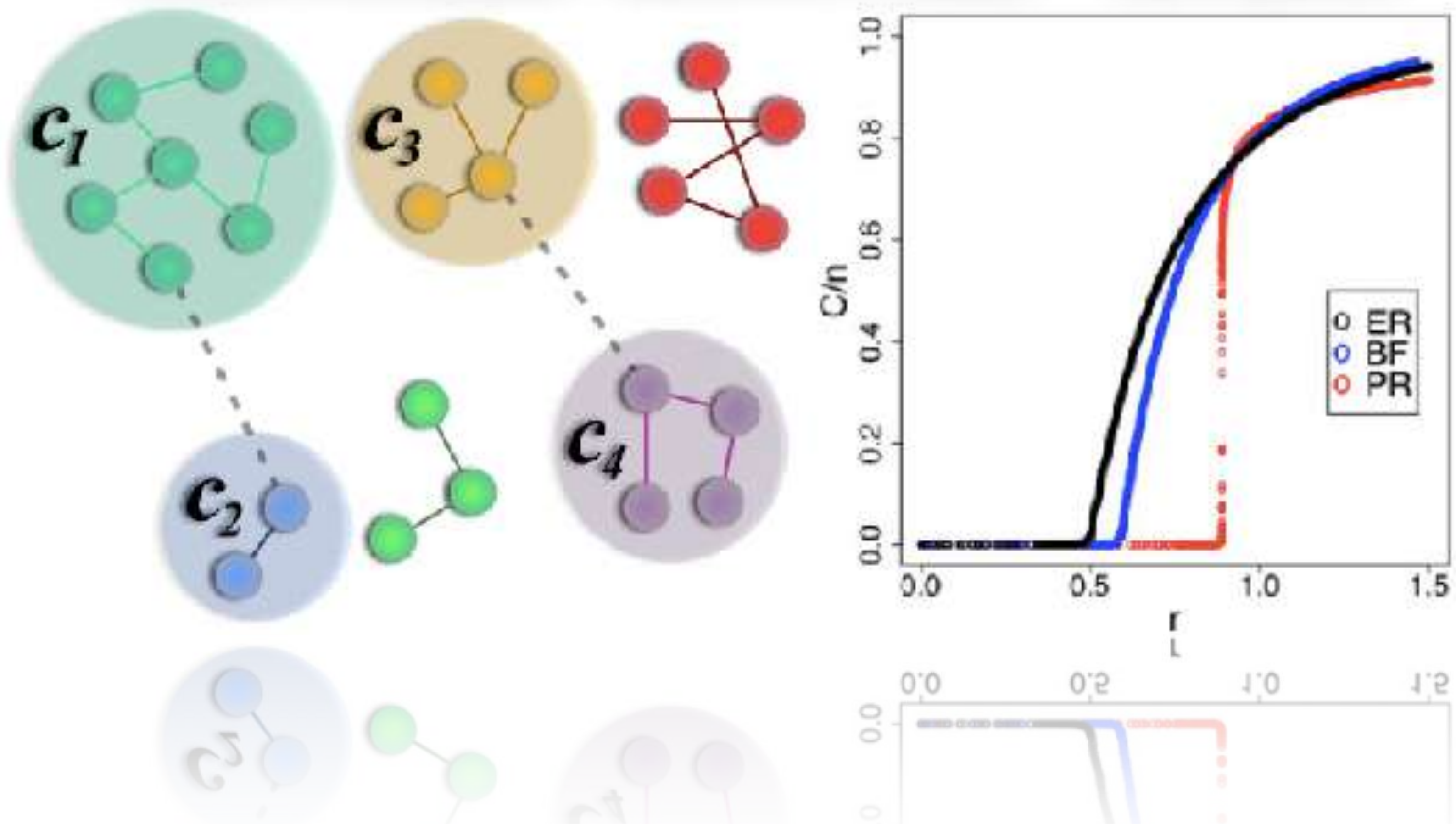
Index

- KURAMOTO MODEL IN COMPLEX NETWORKS
- SYNC. TRANSITION: THE CRITICAL COUPLING
- PLAYING WITH THE TRANSITION: EXPLOSIVE SYNC.

www.sciencemag.org SCIENCE VOL 323 13 MARCH 2009

Explosive Percolation in Random Networks

Dimitris Achlioptas,[1] Raissa M. D'Souza,[2,3]* Joel Spencer[4]



Networks in which the formation of connections is governed by a random process often undergo a percolation transition, wherein around a critical point, the addition of a small number of connections causes a sizable fraction of the network to suddenly become linked together. Typically such transitions are continuous, so that the percentage of the network linked together tends to zero right above the transition point. Whether percolation transitions could be discontinuous has been an open question. Here, we show that incorporating a limited amount of choice in the classic Erdös-Rényi network formation model causes its percolation transition to become discontinuous.

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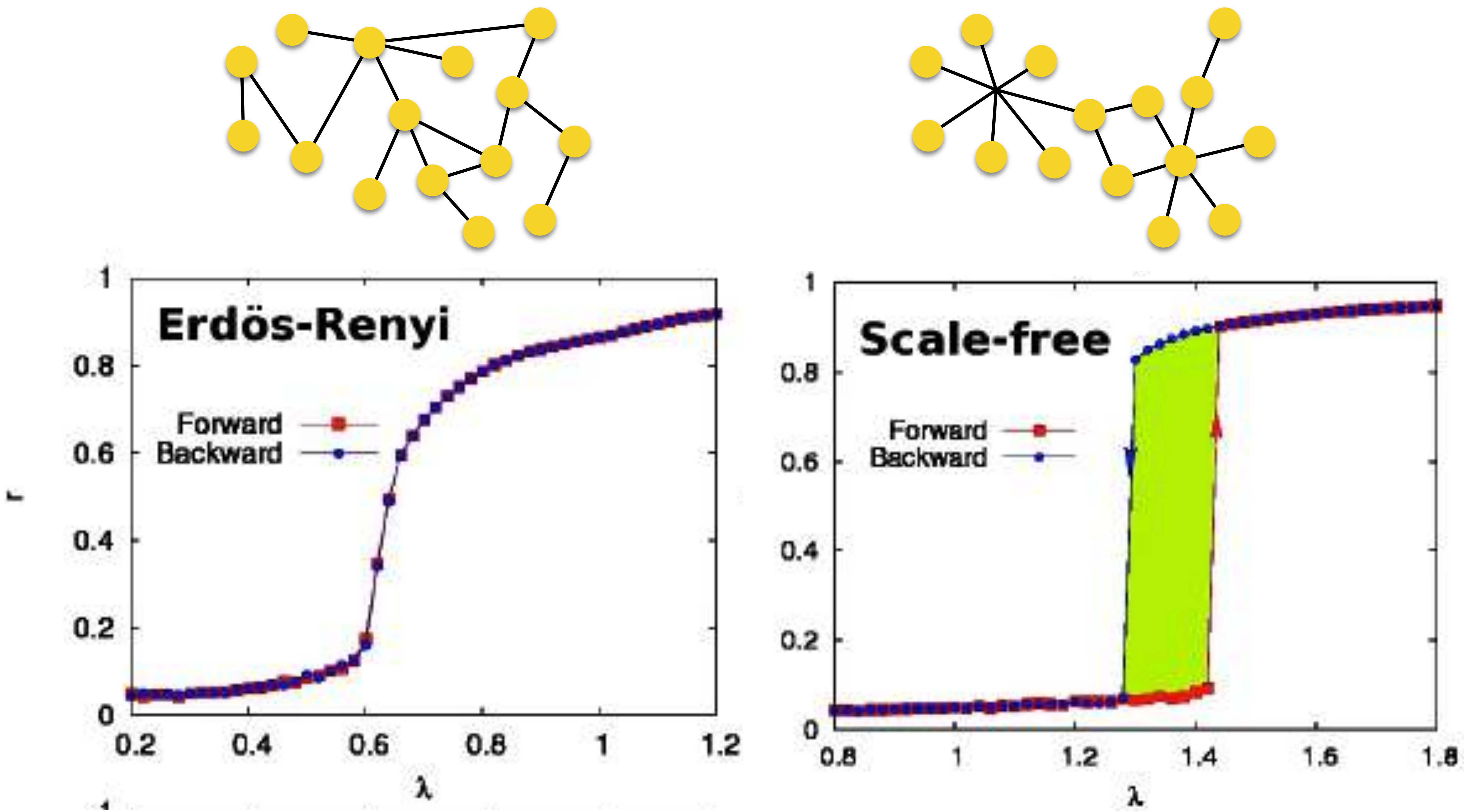
EXPLOSIVE SYNCHRONIZATION?

Correlated Degree-Frequency

Gómez-Gardeñes et al. Phys. Rev. Lett. (2011)

EXPLOSIVE SYNCHRONIZATION?

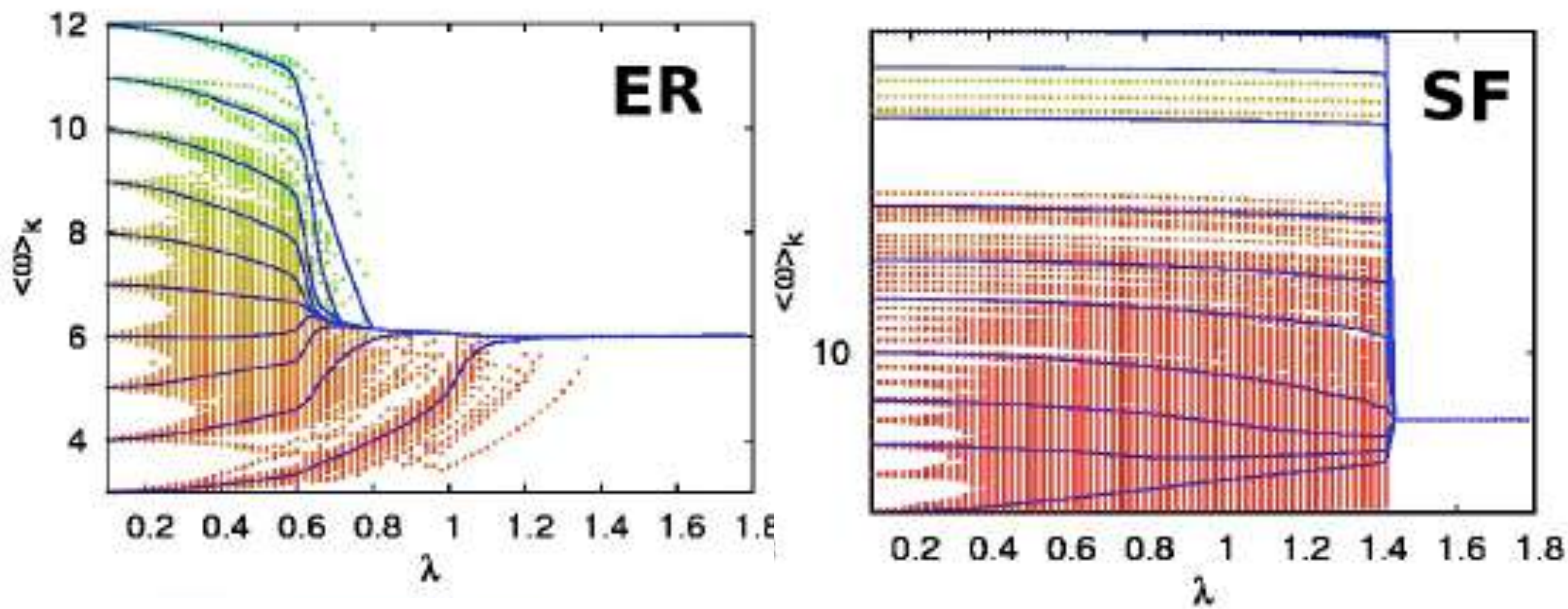


Gómez-Gardeñes et al. Phys. Rev. Lett. (2011)

EXPLOSIVE SYNCHRONIZATION?

- We compute for each value of λ the effective frequency of each node:

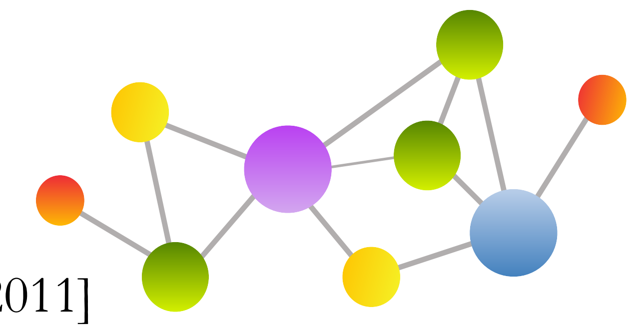
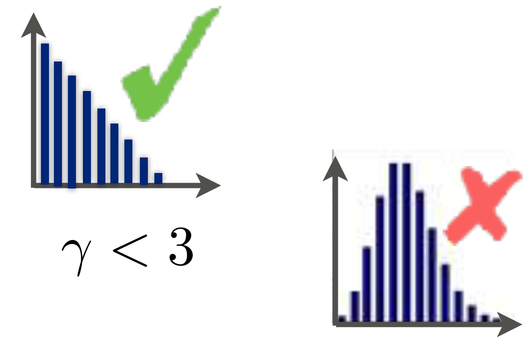
$$\omega_i^{\text{eff}} = \frac{1}{T}\int_t^{t+T} \dot{\theta}_i(\tau)\,\mathrm{d}\tau\ , \quad \text{with} \quad T \gg 1\ .$$



Gómez-Gardeñes et al. Phys. Rev. Lett. (2011)

Other possible Explosive set-ups

Other possible Explosive set-ups

Framework $\omega \in \{\;\}$	$g(\omega)$	$P(k)$	**Information**
CFD $\omega_i = k_i$ [Gómez-Gardeñes et al. 2011] 	$P(k) \to g(\omega)$		Local

Suppressing the emergence of a Sync macroscopic component

Explosive Sync in Experiments

NEXT STEP

We now explore the possibility of obtaining an explosive synchronization transition in an experimental setup.

- Possibility of using simple network configurations as the star graph.
- Robustness of results under perturbations.

Bad News

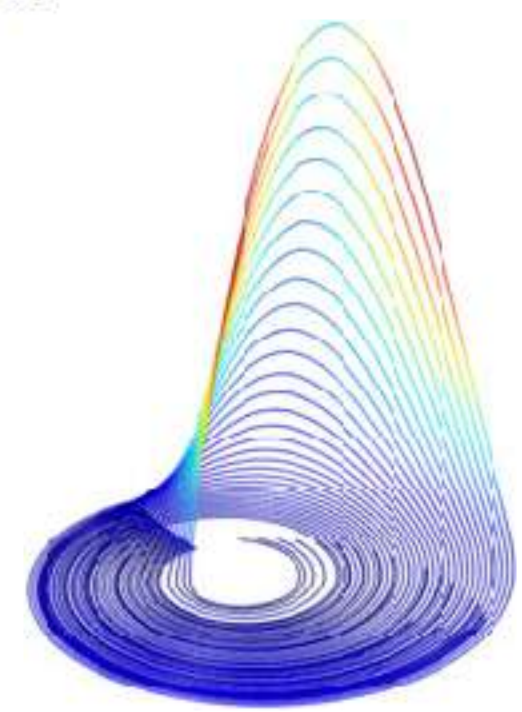
However, we have to move from the Kuramoto model to a more realistic and complicated dynamical systems.

Rossler System:

$$\dot{x} = -(y + z)$$

$$\dot{y} = \sigma y + x$$

$$\dot{z} = \beta + z(x - c)$$



ROSSLER SYSTEM

Let us consider an ensemble of N piecewise Rössler units, interacting in a network via a bidirectional diffusive-like coupling:

$$\begin{aligned}
\dot{x}_i &= -\alpha_i \left[0.05 \left(x_i - \boxed{d \sum_{j=1}^{N} A_{ij}(x_j - x_i)} \right) + 0.5 y_i + z_i \right], \\
\dot{y}_i &= -\alpha_i (-x_i + \boxed{\nu} y_i), \\
\dot{z}_i &= -\alpha_i (\boxed{-g(x_i)} + z_i),
\end{aligned}$$

where the piecewise part is:

$$g(x_i) = \begin{cases} 0 & \text{if } x_i \leq 3 \\ \mu(x_i - 3) & \text{if } x_i > 3 \end{cases}$$

The parameter $\nu = 0.02 - 10/R$ controls the dynamical state of the system. For $R \in [55, 110]$ the above system is in the chaotic phase.

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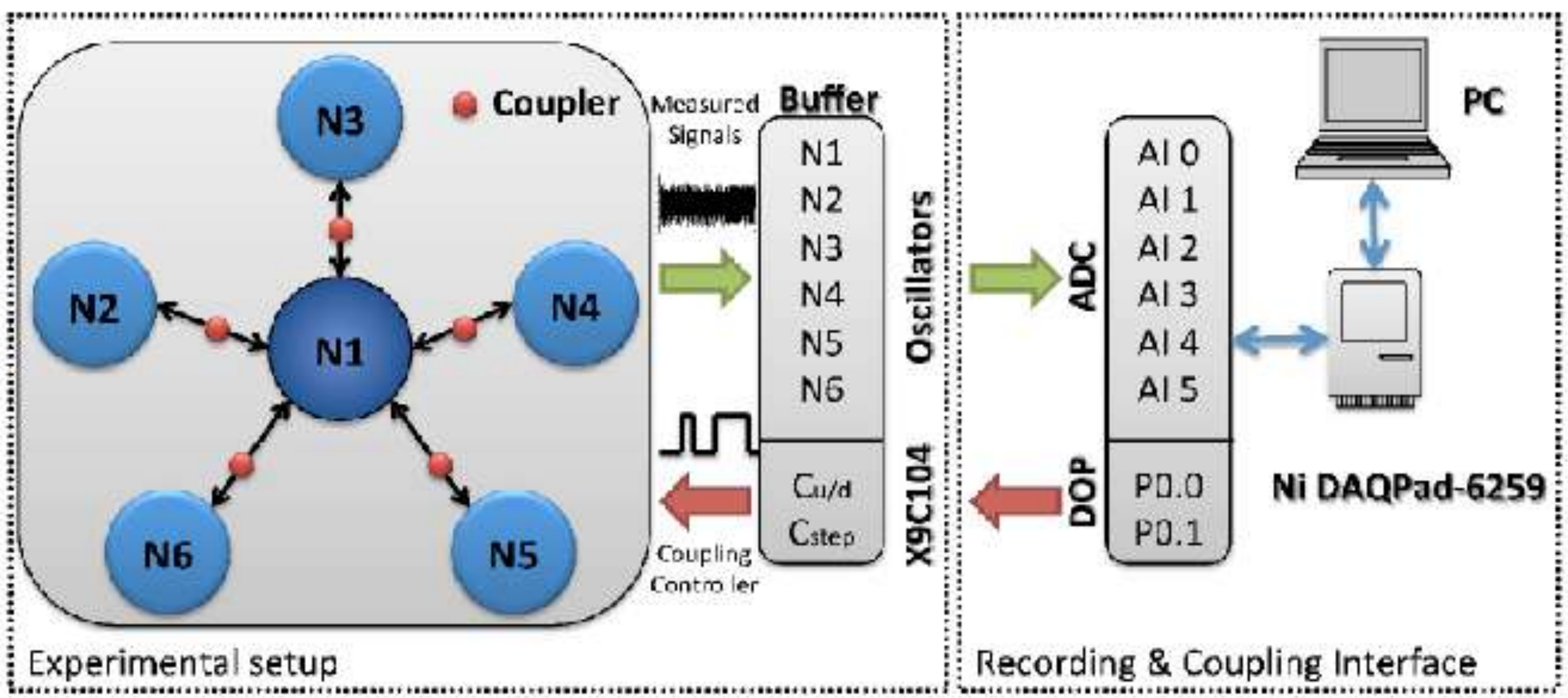
ROSSLER SYSTEM

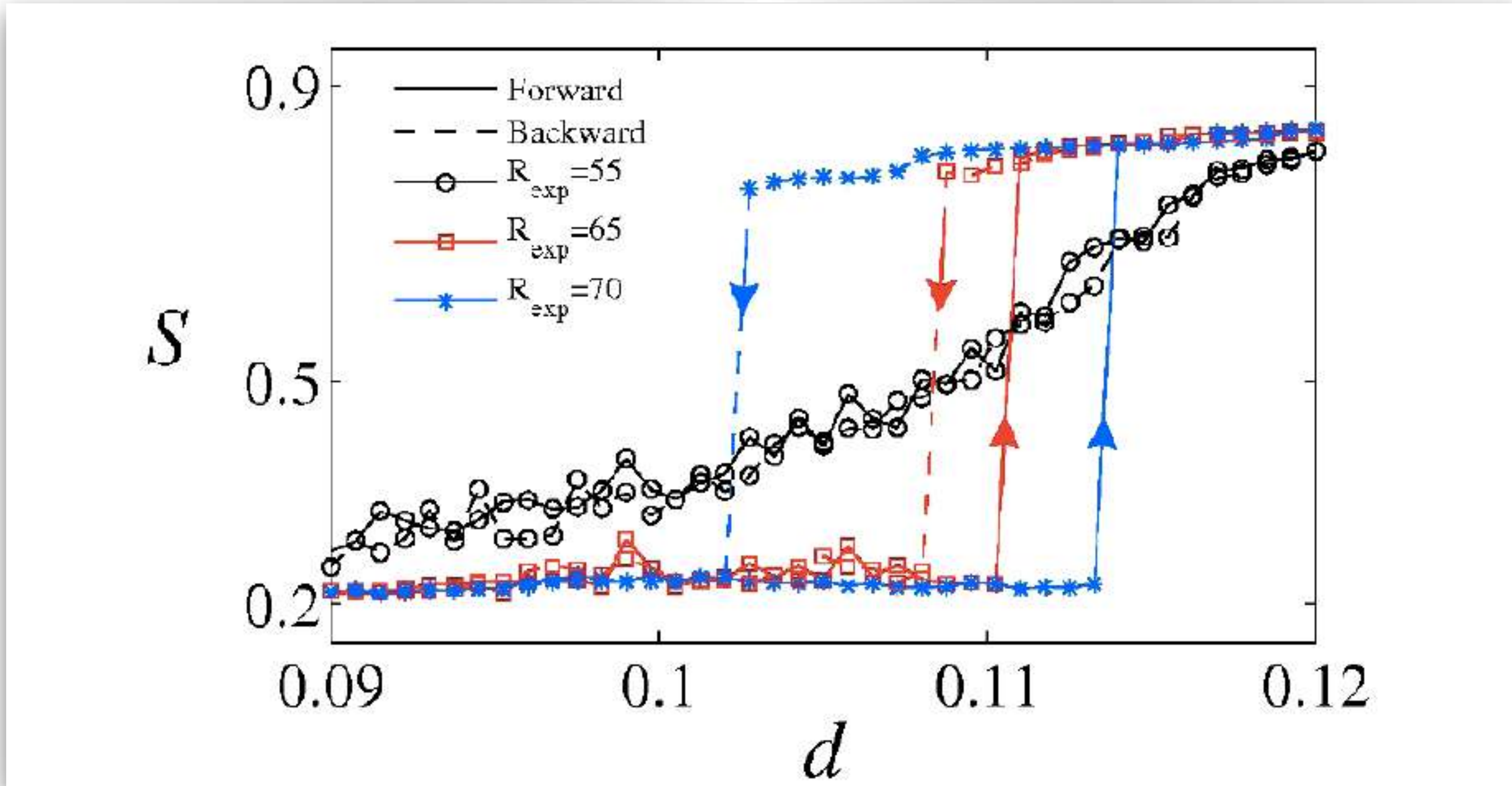
Each of the natural frequencies depends linearly on each value α_i. Thus we include the correlation between structure and dynamics by setting:

$$\alpha_i = \alpha\left(1 + \Delta\alpha\frac{k_i - 1}{N}\right), \text{ with } \alpha = 10^4$$



$N = 6$ **oscillators arranged in a star-graph topology. The coupling is controlled by digital potenciometers and the output signals are recorded and analyzed.**

ROSSLER SYSTEM



Gómez-Gardeñes et al. Phys. Rev. Lett. (2011)

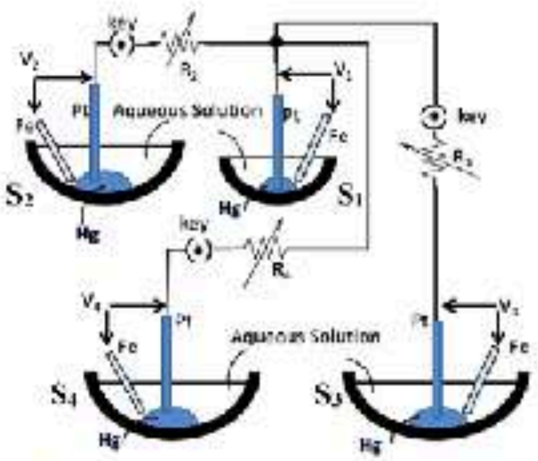
OTHER EXPERIMENTS

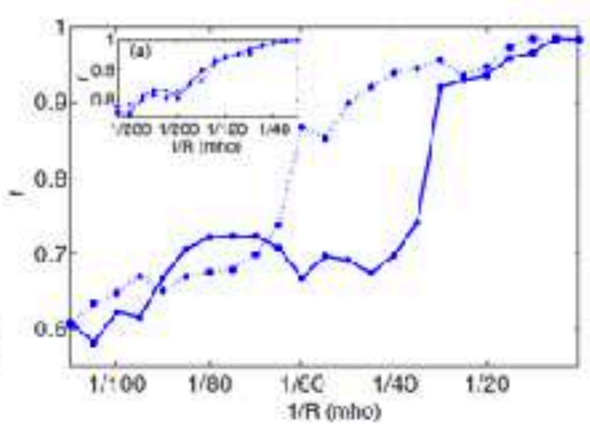
PHYSICAL REVIEW E 91, 062909 (2015)

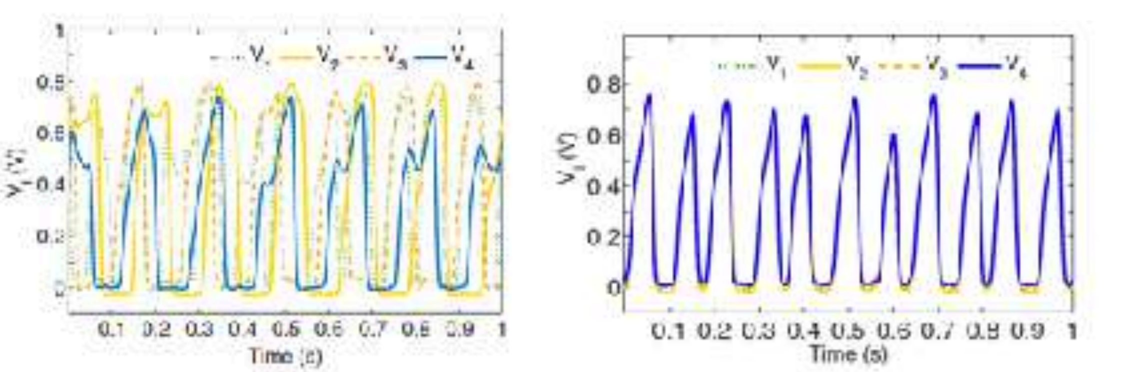
Experimental evidence of explosive synchronization in mercury beating-heart oscillators

Pawan Kumar,[1] Dinesh Kumar Verma,[1] P. Parmananda,[1] and S. Boccaletti[2,3]
[1]Department of Physics, Indian Institute of Technology Bombay, Powai, Mumbai-400 076, India
[2]CNR–Institute of Complex Systems, Via Madonna del Piano, 10, 50019 Sesto Fiorentino (FI), Italy
[3]Embassy of Italy in Israel, Trade Tower, 25 Hamered St., 68125 Tel Aviv, Israel
(Received 18 March 2015; published 9 June 2015)

We report experimental evidence of explosive synchronization in coupled chemo mechanical systems, namely in mercury beating-heart (MBH) oscillators. Connecting four MBH oscillators in a star network configuration and setting natural frequencies of each oscillator in proportion to the number of its links, a gradual increase of the coupling strength results in an abrupt and irreversible (first-order-like) transition from the system's unordered to ordered phase. On its turn, such a transition indicates the emergence of a bistable regime wherein coexisting states can be experimentally revealed. Finally, we prove how such a regime allows an experimental implementation of magneticlike states of synchronization, by the use of an external signal.







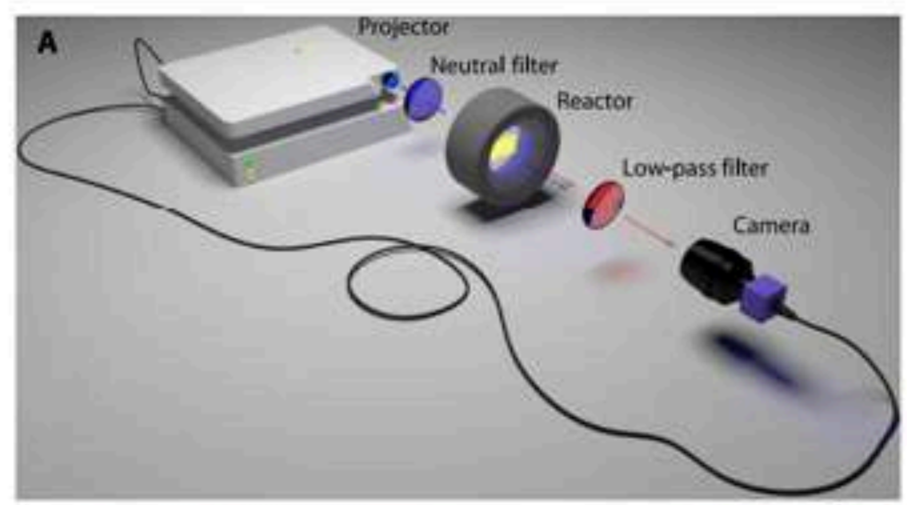
SCIENCE ADVANCES | RESEARCH ARTICLE

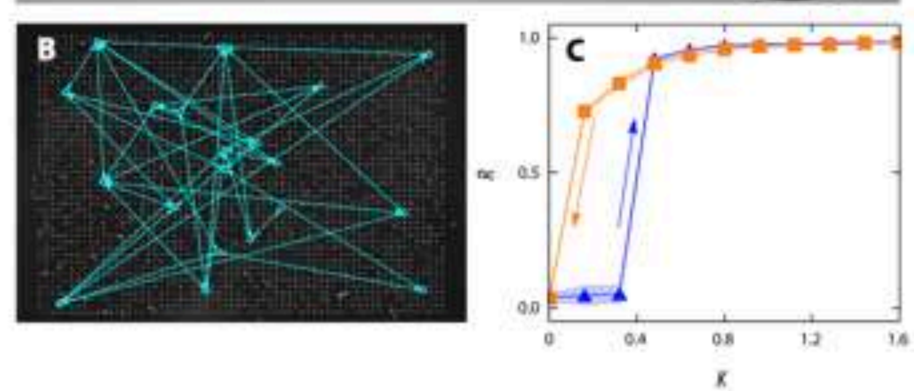
PHYSICS

First-order synchronization transition in a large population of strongly coupled relaxation oscillators

Dumitru Călugăru[1,2], Jan Frederik Totz[3,4,5]*, Erik A. Martens[6], Harald Engel[3]

Onset and loss of synchronization in coupled oscillators are of fundamental importance in understanding emergent behavior in natural and man-made systems, which range from neural networks to power grids. We report on experiments with hundreds of strongly coupled photochemical relaxation oscillators that exhibit a discontinuous synchronization transition with hysteresis, as opposed to the paradigmatic continuous transition expected from the widely used weak coupling theory. The resulting first-order transition is robust with respect to changes in network connectivity and natural frequency distribution. This allows us to identify the relaxation character of the oscillators as the essential parameter that determines the nature of the synchronization transition. We further support this hypothesis by revealing the mechanism of the transition, which cannot be accounted for by standard phase reduction techniques.





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Explosive Sync in Real World?

Explosive Sync. Evidences

* Ongoing consensus about the role of **bistability** as a natural framework for **biological switches**

* **Abrupt** transitions and **hypersensitive responses** have been analyzed from the perspective of **ES**

Conscious-Unconscious Transitions (Anesthesia)
[Kim et al. 2016, Kim et al. 2017]

Choroid Plexus & Circadian clocks
[Myung et al. 2018]

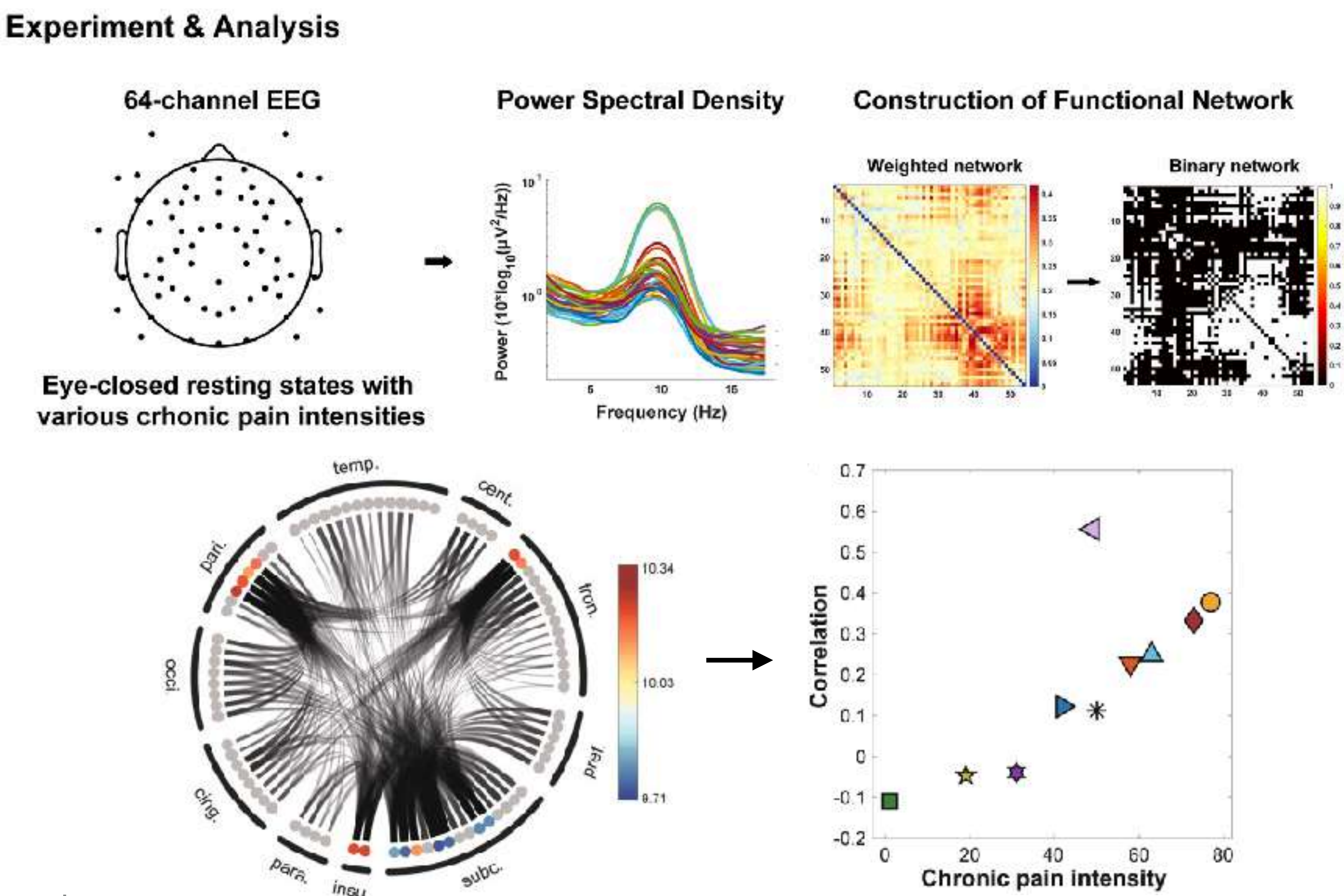
Epileptic Seizures
[Wang et al. 2017]

Frequency detection of the cochlea
[Wang et al. 2016]

Hypersensitivity of Fibromyalgia patients
[Lee et al. 2018]

EXPLOSIVE SYNC. EVIDENCES

Hypersensitivity of Fibromyalgia patients [Lee et al. 2018]



EXPLOSIVE SYNC. EVIDENCES

Hypersensitivity of Fibromyalgia patients [Lee et al. 2018]

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