Time series characterization by Information Theory based quantifiers

- ***Dr. Osvaldo A. Rosso***

- ***Instituto de Física, Universidade Federal de Alagoas (UFAL),***
  Maceió, Brazil
  &
- ***Instituto de Fisica La Plata (IFLP), UNLP***
  La Plata, Buenos Aires, Argentina

- oarosso@gmail.com

Dynamical Systems & Time Series

- Physics, as well as, other scientific disciplines like biology or finance, can be considered observational sciences, that is, they try to infer properties of an unfamiliar system from the analysis of temporal sequences of observations of it behavior, commonly called time series.

Dynamical Systems & Time Series

- Dynamical systems are systems that evolve in time.
- In practice, one may only be able to measure a scalar time series $X(t)$ which may be a function of variables $V = \{v_1, v_2, \dots., v_k\}$ describing the underlying dynamics (i.e. $\frac{dV}{dt} = f(V)$).

Dynamical Systems & Time Series

- Then, the natural question is, from $X(t)$ how much we can learn about the dynamics of the system.
- In a more formal way, given a system, be it natural or man-made, and given an observable of such system whose evolution can be tracked through time, a natural question arises:
- how much information is this observable encoding about the dynamics of the underlying system?

Dynamical Systems & Time Series

OBJETIVE:

- Given a time series:

$$X(t) = \{x_t , t = 1, \dots , M\}$$

can we said if it originated by
a chaotic low dimensional dynamics or
it is originated by a stochatics dynamics ?

Chaotic or Stochastic Dynamics ?

- If one is able to show that the system is dominated by low-dimensional deterministic chaos, then only few (nonlinear and collective) modes are required to describe the pertinent dynamics.

- If not, then the complex behavior could be modeled by a system dominated by a very large number of excited modes which are in general better described by stochastic or statistical approaches.

Chaotic or Stochastic Dynamics ?

- Stochastic & Chaotic time series sheare some characteristic which make them almost indistiguishable

- A wide-band power spectrum
- Power spectrum of type f^{-k}, with $k \geq 0$
- A delta like auto-correlation function
- An irregular behavior of the measured signals

Chaos & Information

- Chaotic systems display "sensitivity to initial conditions" which manifests instability everywhere in the phase space and leads to non-periodic motion (chaotic time series).
- They display long-term unpredictability despite the deterministic character of the temporal trajectory.
- In a system undergoing chaotic motion, two neighboring points, in phase space move away exponentially rapidly.

Chaos & Information

- Let $x_1(t)$ and $x_2(t)$ be two such points, located within a ball of radius R at time t.
- Further, assume that these two points cannot be resolved within the ball due to poor instrumental resolution.

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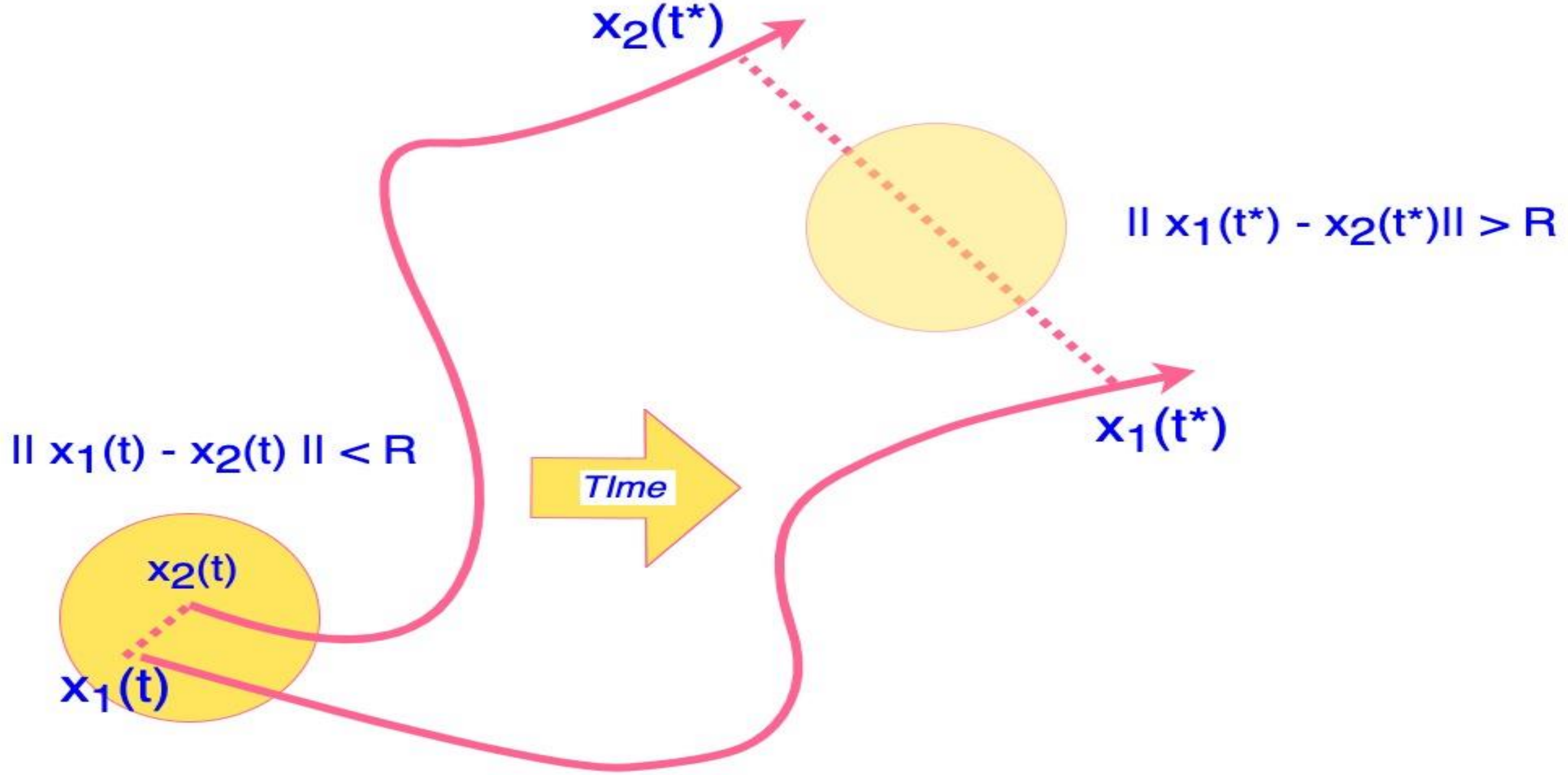
- Further, assume that these two points cannot be resolved within the ball due to poor instrumental resolution.
- At some later time t^* the distance between the points will be typically grow to

$$|x_1(t^*) - x_2(t^*)| \approx |x_1(t) - x_2(t)| \, exp(\Lambda \, |t^* - t|)$$

With $\Lambda > 0$ for a chaotic dynamics and Λ the biggest Lyapunov exponentes.

- When this distance at time t^*exceeds R, the points become experimentally distinguishable.

Chaos & Information



$\| x_1(t^*) - x_2(t^*) \| =$	$\| x_1(t) - x_2(t) \| \exp\{ L\,(t^* - t) \}$

Chaos & Information

- This implies that instability reveals some information about the phase space population that was not available at earlier times.

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- We can think of chaos as an information source

We can use Information Theory based quantifiers to characterize chaotics systems.

Information Theory Quantifiers

We can use Information Theory based quantifiers to characterize chaotic systems !!!

- ***Entropic Measures***
  - ***Shannon,***
  - ***Tsallis,***
  - ***Renyi***

- ***Fisher's Information Measure***

- ***Generalized Statistical Complexity***

Information Theory Quantifiers

❖ Given a continuos probability distribution function (PDF) $f(x)$ with $x \in \Omega \in \mathbb{R}$ and $\int_{\Omega} f(x)\,dx = 1$,

it associated Shannon Entropy is defined by

$$S[f] = -\int_{\Omega} f(x)\,\ln f(x)\,dx$$

a measure of "global character" that it is not too sensitive to strong changes in the distribution taking place on small-sized región.

Information Theory Quantifiers

- Such is not the case with Fisher's Information Measure (FIM), which constitutes a measure of the gradient content of the distribution $f(x)$, thus being quite sensitive even to tiny localized perturbations, then is a measure of "local character".

$$F[f] = \int_{\Omega} \frac{1}{f(x)} \left[\frac{df(x)}{dx}\right]^2 dx = 4 \int_{\Omega} \left[\frac{d\psi(x)}{dx}\right]^2 dx$$

where $f(x) = \psi(x)^2$ is the real probability amplitude.

Information Theory Quantifiers

- Let now $\boldsymbol{P} = \{\boldsymbol{p_i} \geq \mathbf{0}; \boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{N}\}$ be a discrete probability distribution, with $\boldsymbol{N}$ the number of possible states of the system under study and $\sum_{i=1}^{N} p_i = 1$, then:

- The associated Shannon Entropy is given by

$$S[P] = -\sum_{i=1}^{N} p_i \ln p_i$$

and the normalized Shannon entropy, is given by

$$H[P] = S[P]/S_{max}$$

with $S_{max} = \ln N$.

Information Theory Quantifiers

- The *discrete normalized Fisher's Information Measure* is given by

$$F[P] = F_0 \sum_{i=1}^{N-1} \{(p_{i+1})^{1/2} - (p_i)^{1/2}\}^2$$

where

$$F_0 = \begin{cases} 1 & \text{if } p_{i^*} \text{ for } i^* = 1, \text{or } i^* = N \text{ and } p_i = 0 \ \forall\, i^* \neq i \\ 1/2 & \text{otherwise} \end{cases}$$

F. Olivares, A. Plastino, O. A. Rosso

Contrasting chaos with noise via local versus global information quantifiers

Physics Letters A 376 (2012) 1577–1583

Information Theory Quantifiers

- *Complexity?*

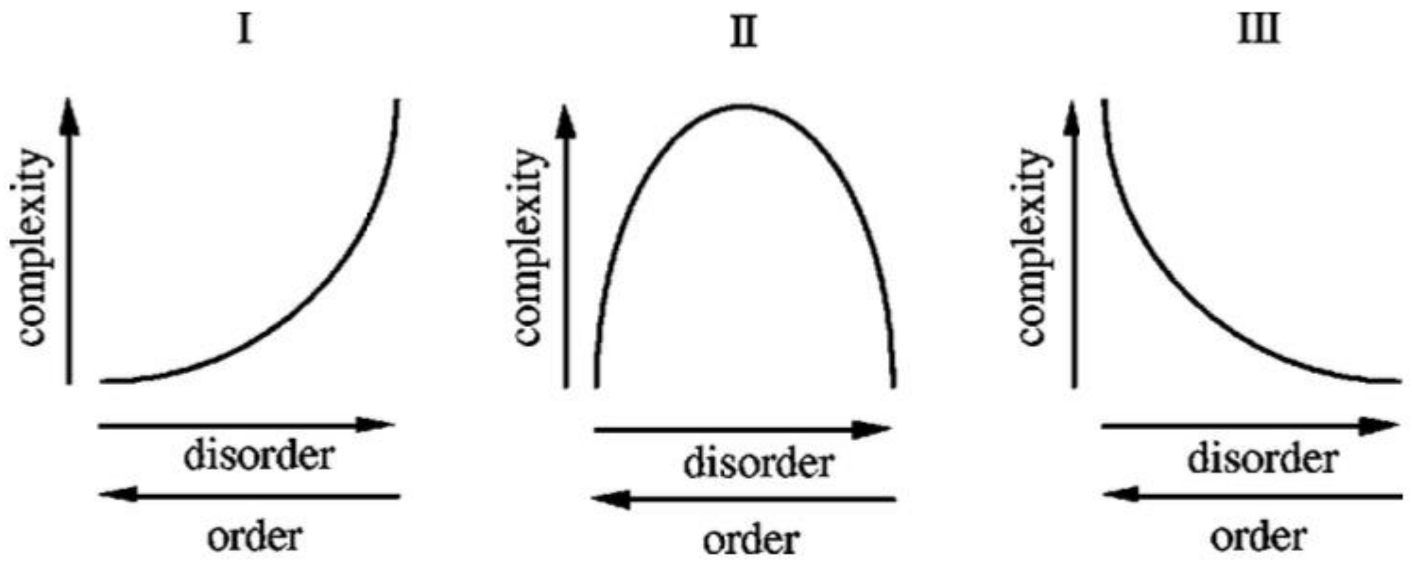
Measures for Complexity

Some options for measure of Complexity C[P] are:

* **SHINER, DAVISON, LANSBERG**

Physical Review 59 (1999) 1459, *Simple measure for complexity.*

$$C_{SDL}[P] = H[P] \cdot (1 - H[P])$$

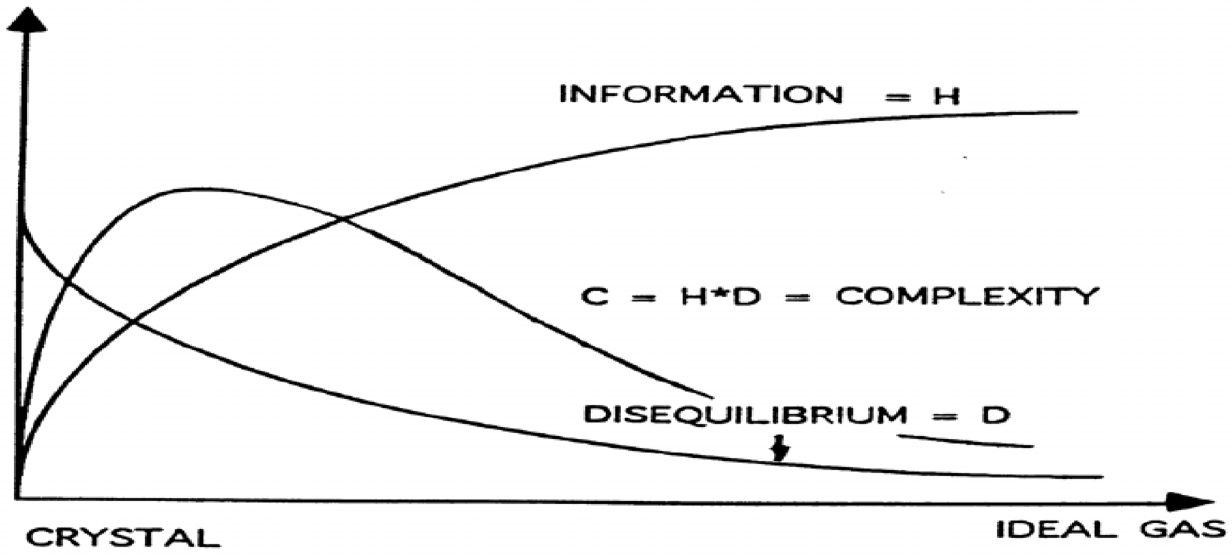


Measures for Complexity

* **LOPEZ-RUIZ, MANCINI, CALVET**

Physical Letters A 209 (1995) 321, *A statistical measure of complexity*.

$$C_{LMP}[\,P\,] = H[\,P\,]\,.\,Q_E(\,[\,P, P_e\,]\,) \quad \text{with} \quad Q_E = \|\,P - Pe\,\|$$

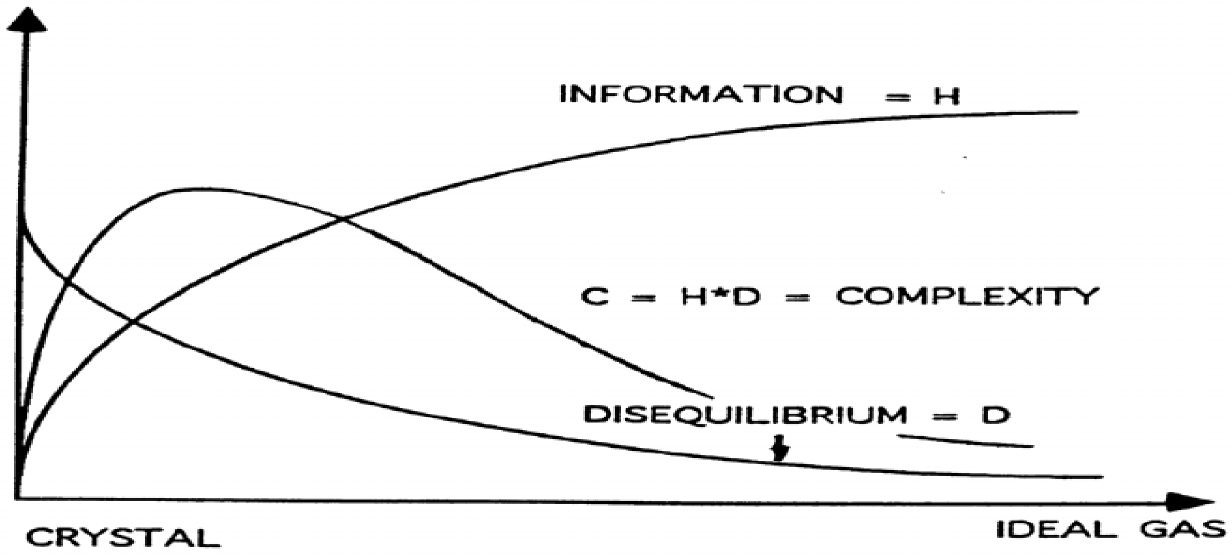


Measures for Complexity

* **MARTIN, PLASTINO, ROSSO**

Physica A 209 (2006) 439, *Generalized statistical complexity measures*.

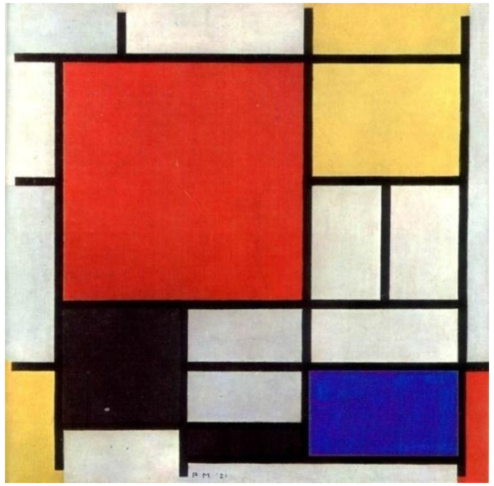
$$C_{MPR}[\,P\,] = H[\,P\,]\,.\,Q_{JS}([\,P, P_e\,]) \text{ with } Q_{JS} = JS\{\,P\,, Pe\,\}$$



Information Theory Quantifiers

- *Complexity ?*

The Simple & The Complex

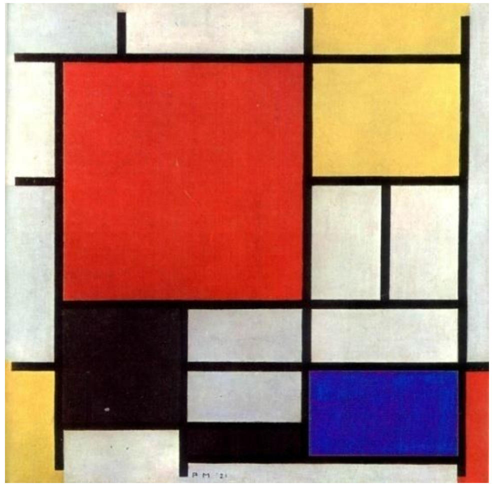
		
H = 0	H ≠ 0	H = 1

Complexity

The COMPLEXITY has to do with intricate structures hidden in the dynamics, emerging from a system which itself is much simpler than its dynamics. Complexity is characterized by the paradoxical situation of complicated dynamics of simple systems.

- Periodic motion it is not complex.
- White noise it is not complex.

The Simple & The Complex

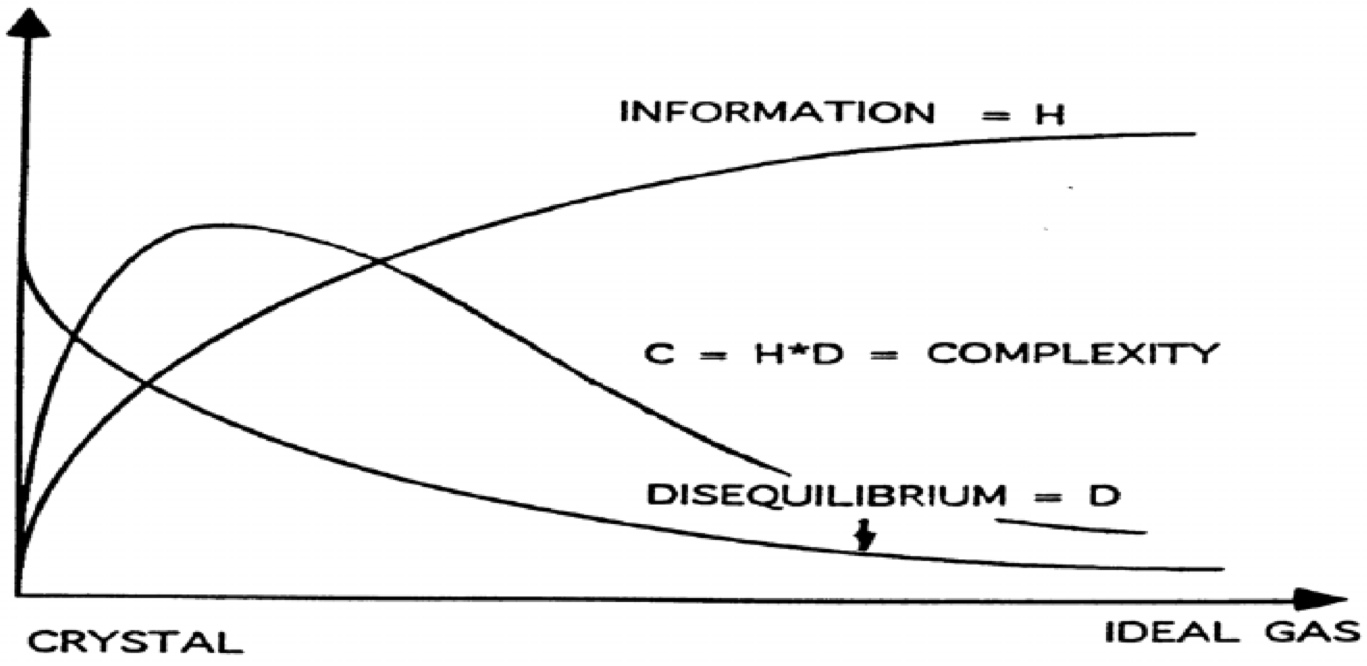
		
$H = 0$ $C = 0$	$H \neq 0$ $C \neq 0$	$H = 1$ $C = 0$

Crystal & Ideal Gas

CRYSTAL	IDEAL GAS
• High ordered system	• Completely disordered system
• Minimal information stored in the system	• Maximal information stored in the system
• Probability Distribution Function *P* in phase space: $p_j = 1$ for $j = k$; $p_j = 0$ for $j \neq k$	• Probability Distribution Function *P* in phase space: $p_j = 1/N$ for $j = 1, \dots, N$ (equiprobability distribution)
• Maximum $D(P, P_e)$	• Minimum $D(P, P_e)$

Statistical Complexity

COMPLEXITY
$C = H \cdot Q$



R. López-Ruiz, H. L. Mancini, and X. Calbet.
A statistical measure of complexity.
Physics Letter A 209 (1995) 321-326.

Disorder *H*

- We define for a given probability distribution

$$P = \{p_j,\ j = 1, \cdots, N\} \in \Omega \subset \mathbb{R}^N$$

and its associate information measure $\mathcal{I}[P]$, an amount of "disorder" H in the fashion

$$H[P] = \mathcal{I}[P] \,/\, \mathcal{I}_{max} \ ,$$

where $\mathcal{I}_{max} = \mathcal{I}[P_e]$ and P_e is the probability distribution which maximize the information measure, where P_e is the equilibrium probability distribution. Then $0 \le H \le 1$.

Disequilibrium *Q*

- We define the "disequilibrium" adopting some kind of distance from the equilibrium distribution P_e of the accessible states of the system.

$$Q[P] = Q_0 \, \mathcal{D}[P, P_e] \, ,$$

where Q_0 is a normalization constant and $0 \leq Q \leq 1$.
The disequilibrium Q would reflect on the systems's "architecture", being different from zero if there are "privileged", or more likely states among the accessible ones.

Selection of the information measure J

- **Shannon Entropy**

$$S_s[P] = -\sum_j p_j \ln[p_j]$$

- **Tsallis Entropy**

$$S_T^{(q)}[P] = \frac{1}{(q-1)} \left\{ 1 - \sum_j (p_j)^q \right\}$$

- **Escort Tsallis Entropy**

$$S_G^{(q)}[P] = \frac{1}{(q-1)} \left\{ 1 - \sum_j (p_j)^{1/q} \right\}^{-q}$$

- **Renyi Entropy**

$$S_R^{(q)}[P] = \frac{1}{(q-1)} \ln \left\{ \sum_j (p_j)^q \right\}$$

Selection of Distance D

- **Euclidean distance**

$$D_E[P_1,P_2] = \|P_1 - P_2\|_E = \sum_{j=1}^{N}\left[p_j^{(1)} - p_j^{(2)}\right]^2$$

- **Wooter distance**

$$D_W[P_1,P_2] = \cos^{-1}\left\{\sum_j \left(p_j^{(1)}\right)^{1/2}\left(p_j^{(2)}\right)^{1/2}\right\}$$

- **Relative Kullback entropy**

$$D_{K_q^{(\kappa)}}[P_1,P_2] = K_q^{(\kappa)}[P_1|\, P_2]$$

- **Jensen divergence**

$$D_{J_q^{(\kappa)}}[P_1,P_2] = J_q^{(\kappa)}[P_1|P_2] = \frac{1}{2}\left\{K_q^{(\kappa)}\left[P_1|\,\frac{P_1+P_2}{2}\right] + K_q^{(\kappa)}\left[P_2|\,\frac{P_1+P_2}{2}\right]\right\}$$

where $\kappa = S, T, G, R$, indicate Shannon, Tsallis, generalized escort Tsallis, Rényi entropic functional forms.

Generalized Statistical Complexity Measures

The family of *Statistical Complexity Measures*, $C^{(\kappa)}_{\nu,q}$, is defined by

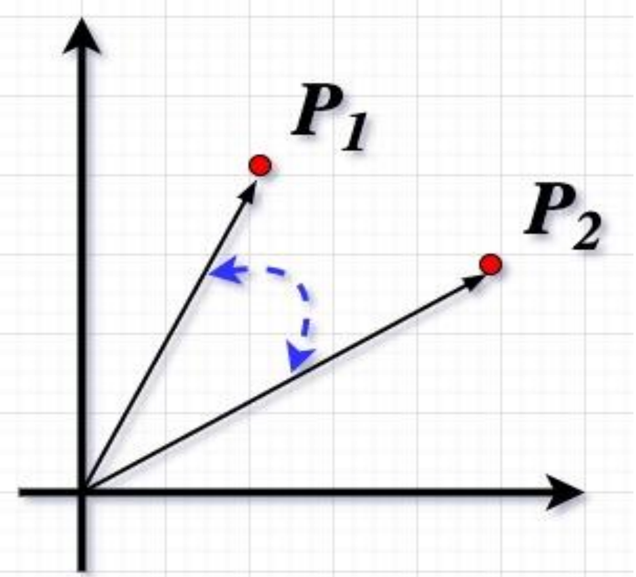
$$\boxed{C^{(\kappa)}_{\nu,q}[P] \;=\; H^{(\kappa)}_{q}[P] \;\cdot\; Q^{(\nu)}_{q}[P]}$$

This quantity reflects on the interplay between the amount of information stored in the system and its disequilibrium.

- $\kappa = S,\ T,\ G,\ R$: Shannon, Tsallis, Generalized Escort-Tsallis, Rényi, for a fixed q.
  In Shannons instance $(\kappa = S)$ we have, of course, $q = 1$.
- $\nu = E,\ W,\ K,\ J$: Euclidea, Wootters, Kullback, Jensen.

O. A. Rosso, M. T. Martín, A. Figliola, K. Keller, and A. Plastino.
EEG analysis using wavelet-based information tools.
Journal Neuroscience Methods 153 (2006) 163-182.

Distances between two PDF



- *Euclidean Distance*

$$D_E(P_1, P_2) = \sum_{j=1}^{N} \left\{p_j^{(1)} - p_j^{(2)}\right\}^2$$

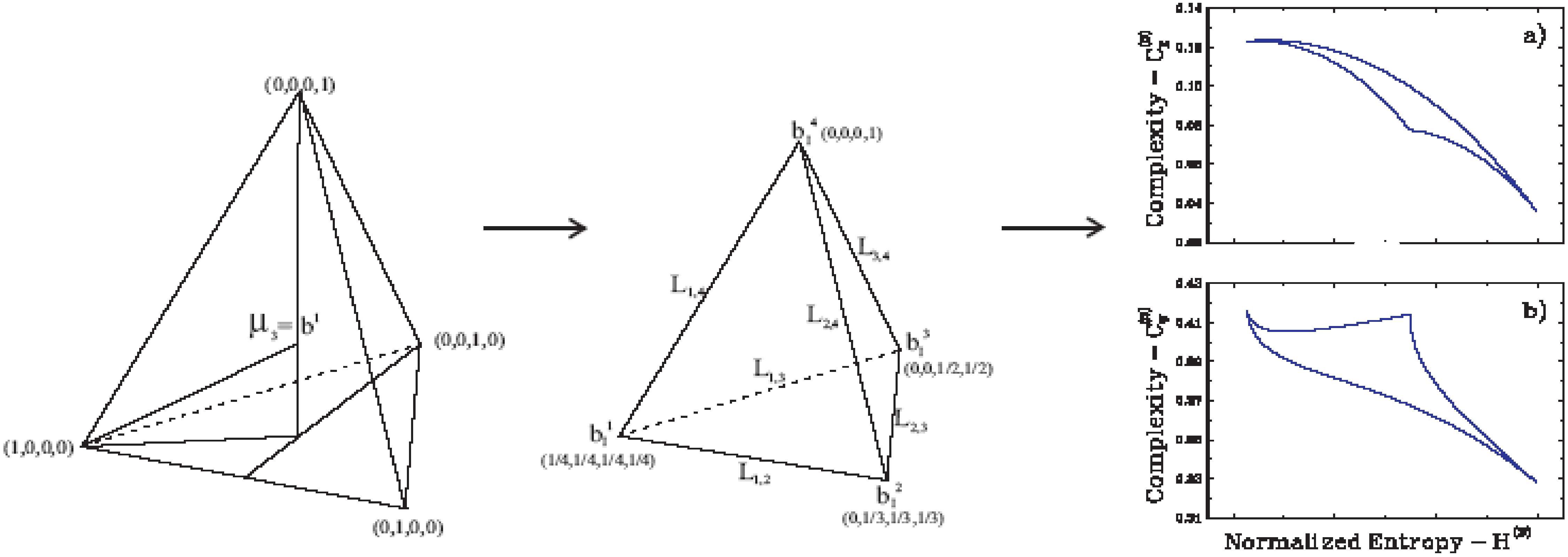
- *Wootters Distance*

$$D_W(P_1, P_2) = \cos^{-1}\left\{\sum_{j=1}^{N} \left(p_j^{(1)}\right)^{1/2} \left(p_j^{(2)}\right)^{1/2}\right\}$$

- *Kullback Relative Probability*

- $D_K(P_1, P_2) = K(P_1|P_2) = \sum_{j=1}^{N} p_j^{(1)} \ln\left(\frac{p_j^{(1)}}{p_j^{(2)}}\right)$

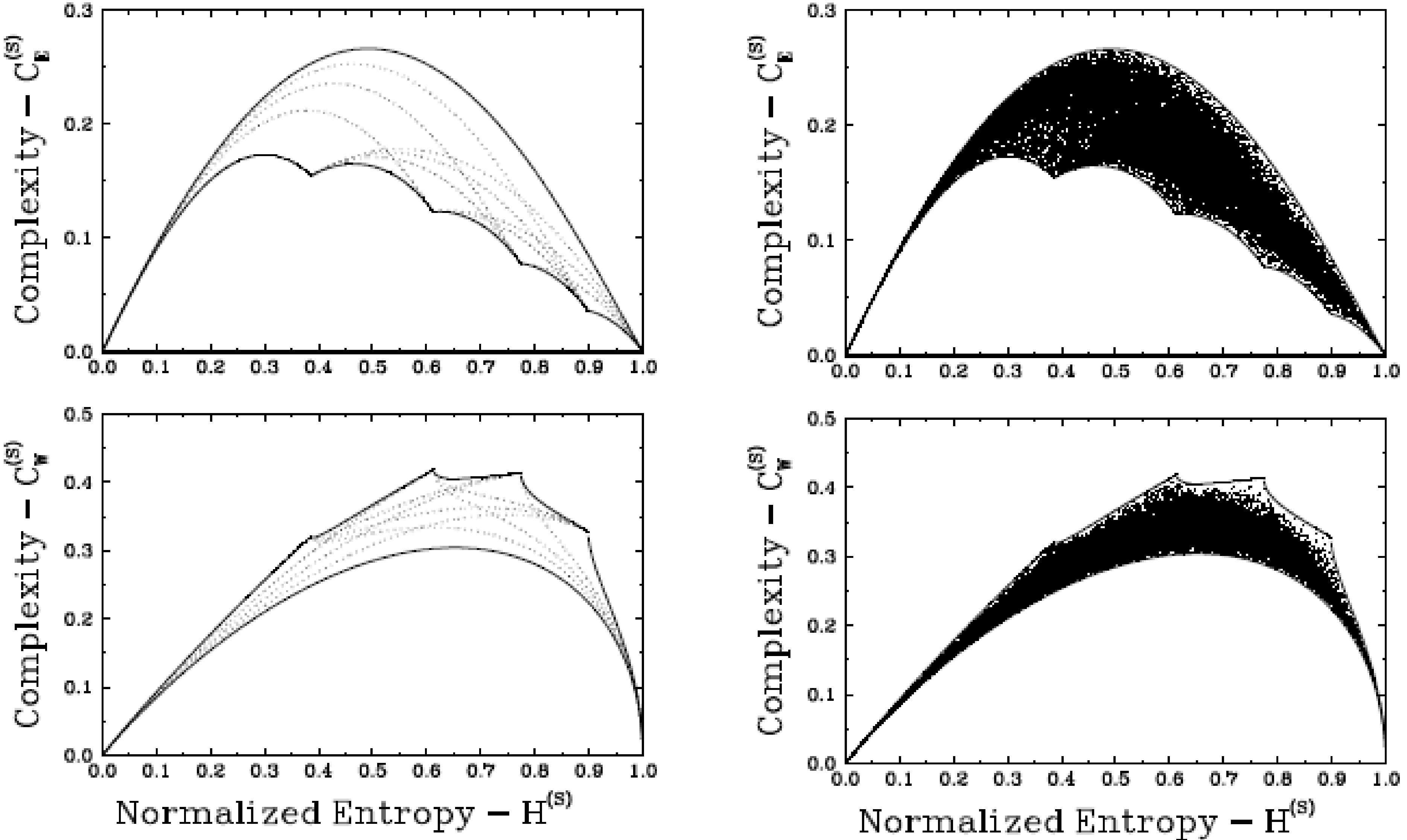
Maximum and Minimum of Generalized Statistical Complexity Measures



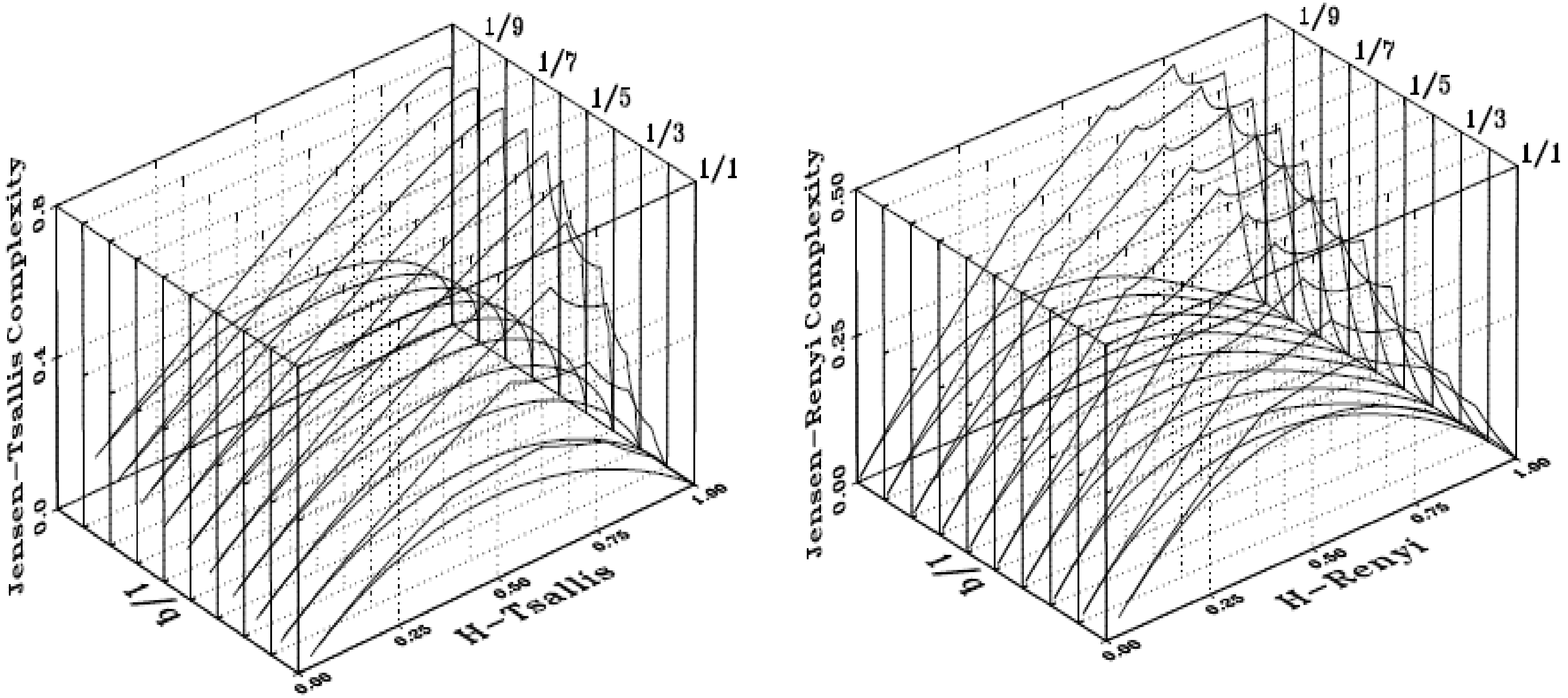
(i) Probability subspace Ω for $N = 4$: $\Omega \equiv \Delta^3$ (3-*simplex*) in an hyperplane of dimension 3. Dotted lines effect the barycentric subdivision with μ_3 the Ω-barycenter.
(ii) Sub-simplex Δ_I^3.
(iii) Maximum and minimum of complexity as function of H obtained by consecutive borders of the sub-simplex Δ_I^3.

M. T. Martín, A. Plastino, and O. A. Rosso, *Generalized statistical complexity measures: Geometrical and analytical properties.* Physica A 369 (2006) 439-462.

Maximum and Minimum of Generalized Statistical Complexity Measures

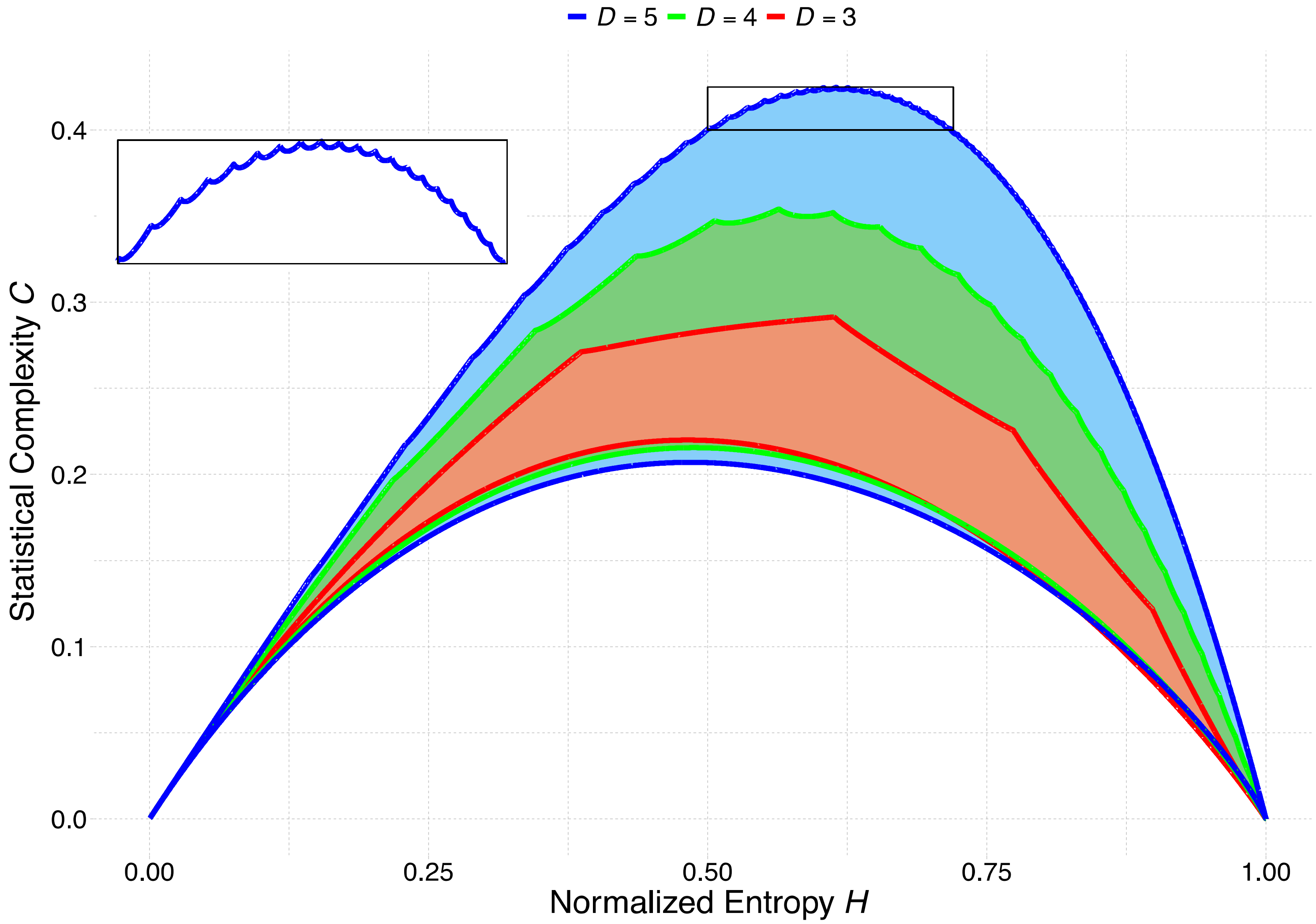


Maximum and Minimum of Generalized Statistical Complexity Measures



N = 6

Maximum and Minimum of Generalized Statistical Complexity Measures



Generalized Statistical Complexity

- Statistical Complexity Measure (SCM), $C[P]$ is able to detect essential details of the dynamical processes underlying the dataset.

- SCM depends on two different probability distributions: one associated with the system under analysis, P, and the other the uniform distribution, P_e.

- Furthermore, it was shown that for a given value of H, the range of possible C values varies between a minimum C_{min} and a maximum C_{max}, restricting the possible values of the SCM.

Generalized Statistical Complexity

- Thus, it is clear that important additional information related to the correlational structure between the components of the physical system is provided by evaluating the statistical complexity measure.

Generalized Statistical Complexity

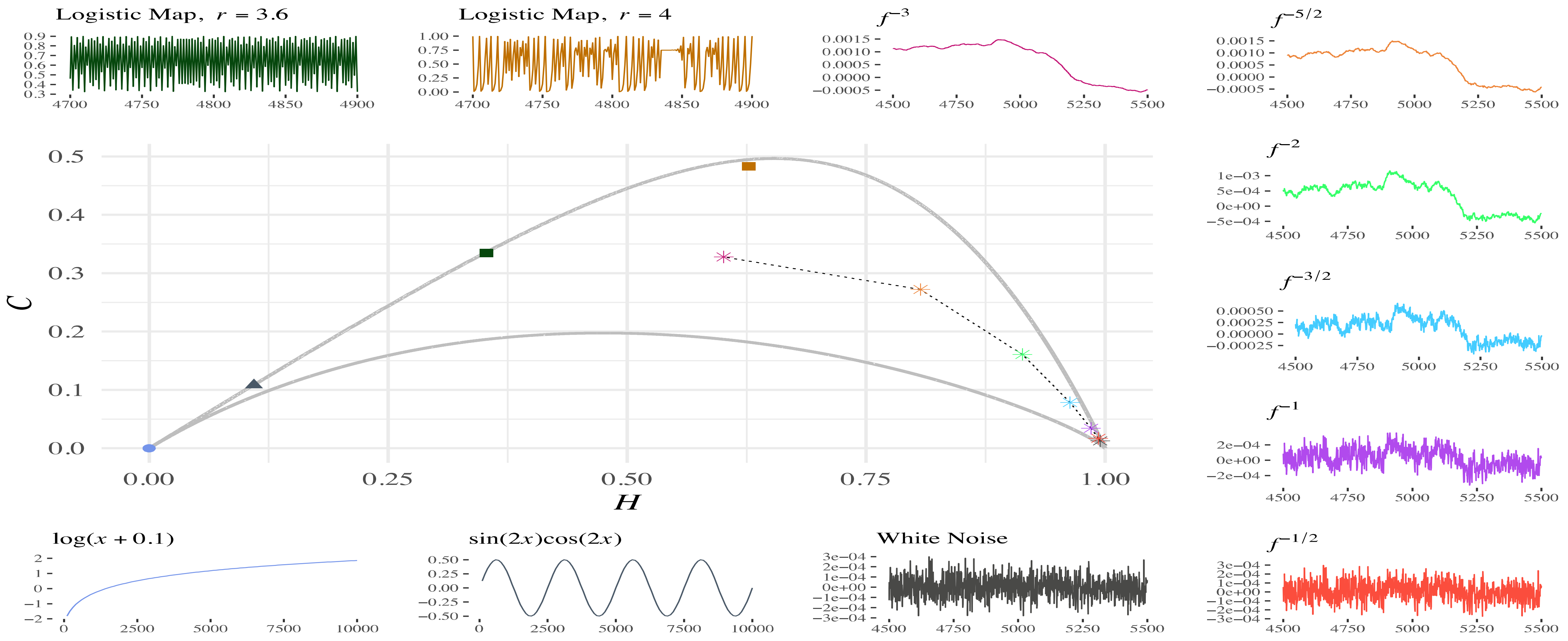


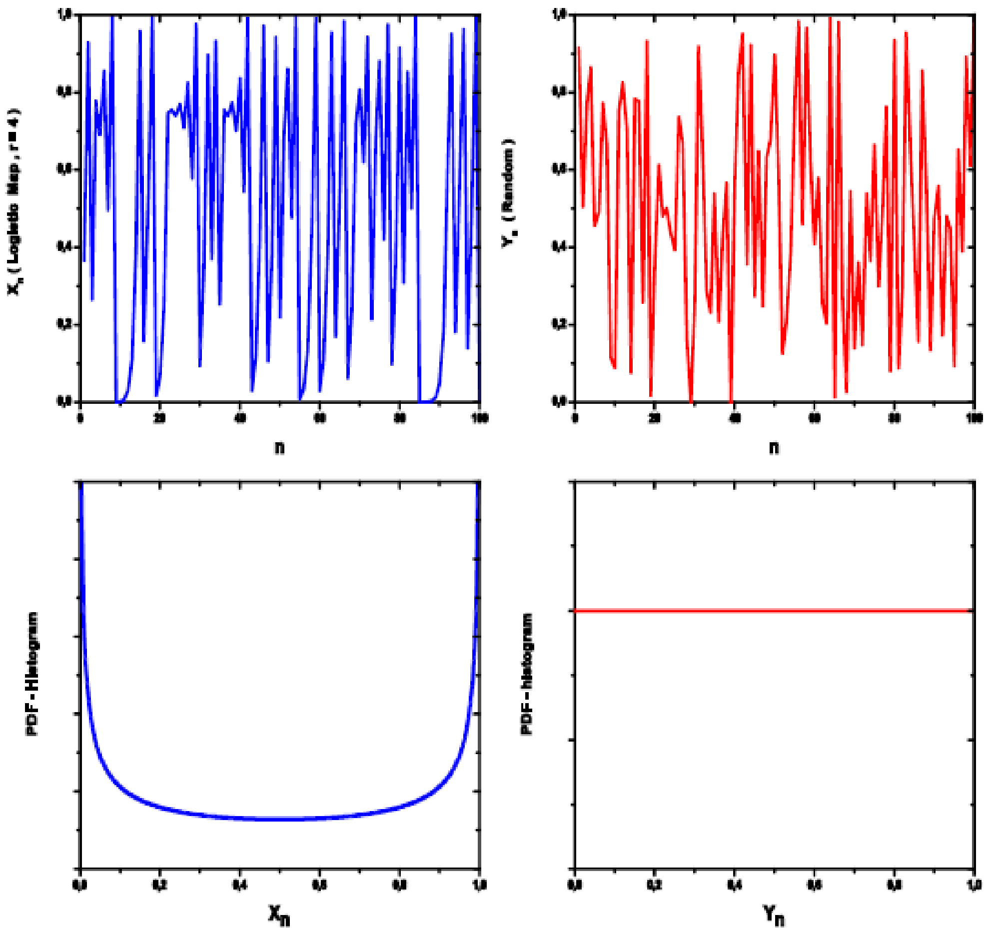
FIGURE 4 Eleven systems and their points in the $H \times C$ plane for dimension embedding $D = 6$, time delay $\tau = 1$, and sequence length $T = 10^4$

Chaotic or Stochastic Dynamics

Logistic Map
R = 4

White noise
K = 0

PDF-Histogram



PDF-Histogram

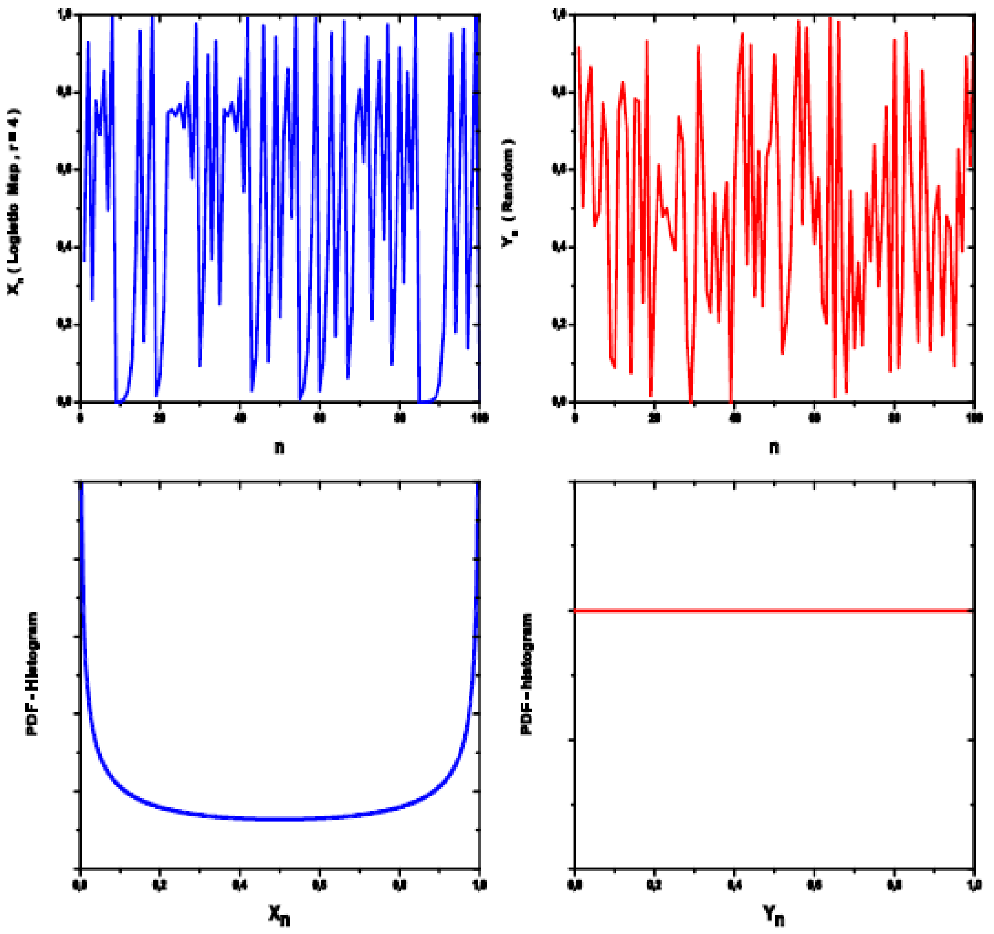
$$x_{n+1} = r\, x_n\, (1 - x_n)$$

Noise with f^{-k} PS

Chaotic or Stochastic Dynamics

Logistic Map
R = 4

$$H\left[P^{(Hist)}\right] \cong 1$$
$$C\left[P^{(Hist)}\right] \cong 0$$
$$F\left[P^{(Hist)}\right] \cong 0$$



White noise
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$$H\left[P^{(Hist)}\right] = 1$$
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Chaotic or Stochastic Dynamics

Logistic Map
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White noise
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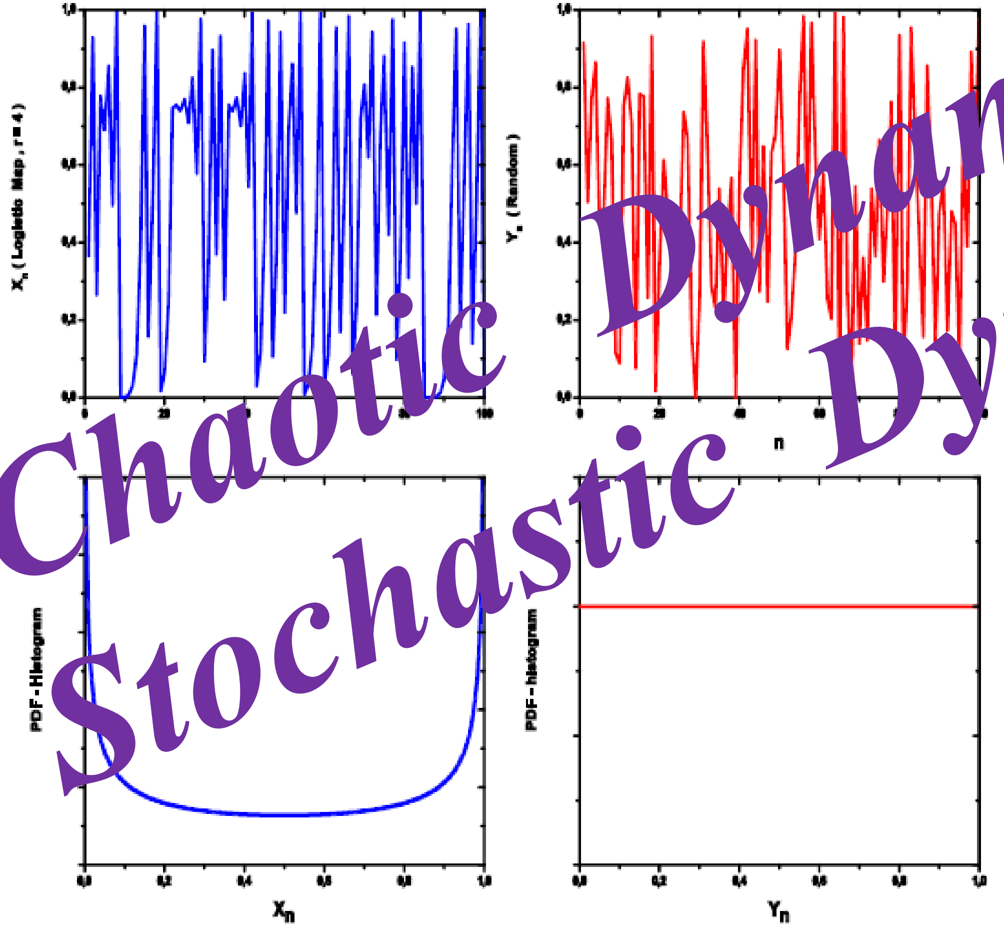


$H[P^{(Hist)}] \cong 1$
$C[P^{(Hist)}] \cong 0$
$F[P^{(Hist)}] \cong 0$

$H[P^{(Hist)}] = 1$
$C[P^{(Hist)}] = 0$
$F[P^{(Hist)}] = 0$

But Chaotic Dynamics
is NOT Stochastic Dynamics !!!

Chaotic or Stochastic Dynamics

- *The PDF must to include the time causality*

Chaotic or Stochastic Dynamics

- *The PDF must to include the time causality*

⇓

- *Bandt and Pompe Methodology*

  Bandt C, Pompe B

  Permutation entropy: a natural complexity measure for time series. Phys. Rev. Lett. 88 (2002) 174102.

- *Visibility Graph and Horizontal Visibility Grapph*

  Lacasa L, Luque B, Ballesteros F, Luque J, Nuno JC

  From time series to complex networks: The visibility graph.

  Proc. Natl. Acad. Sci. USA 105 (2008) 4972–4975.

Probability distribution *P*

Given a time series

$$X(t) = \{x_t, t = 1, \dots, M\}; \; x_t \in \mathbb{R}$$

we can define the associate probability distribution function based on:

- *Frequency counting*
- *Histogram of amplitudes*
- *Binary distribution*
- *Frequency representation (Fourier Transform)*
- *Frequency bands representation (Wavelet Transform)*
- *Ordinal patterns (Bandt-Pompe methodology)*
- *Horizontal Visibility Graph / Visibility Graph*

Application to Logistic Map

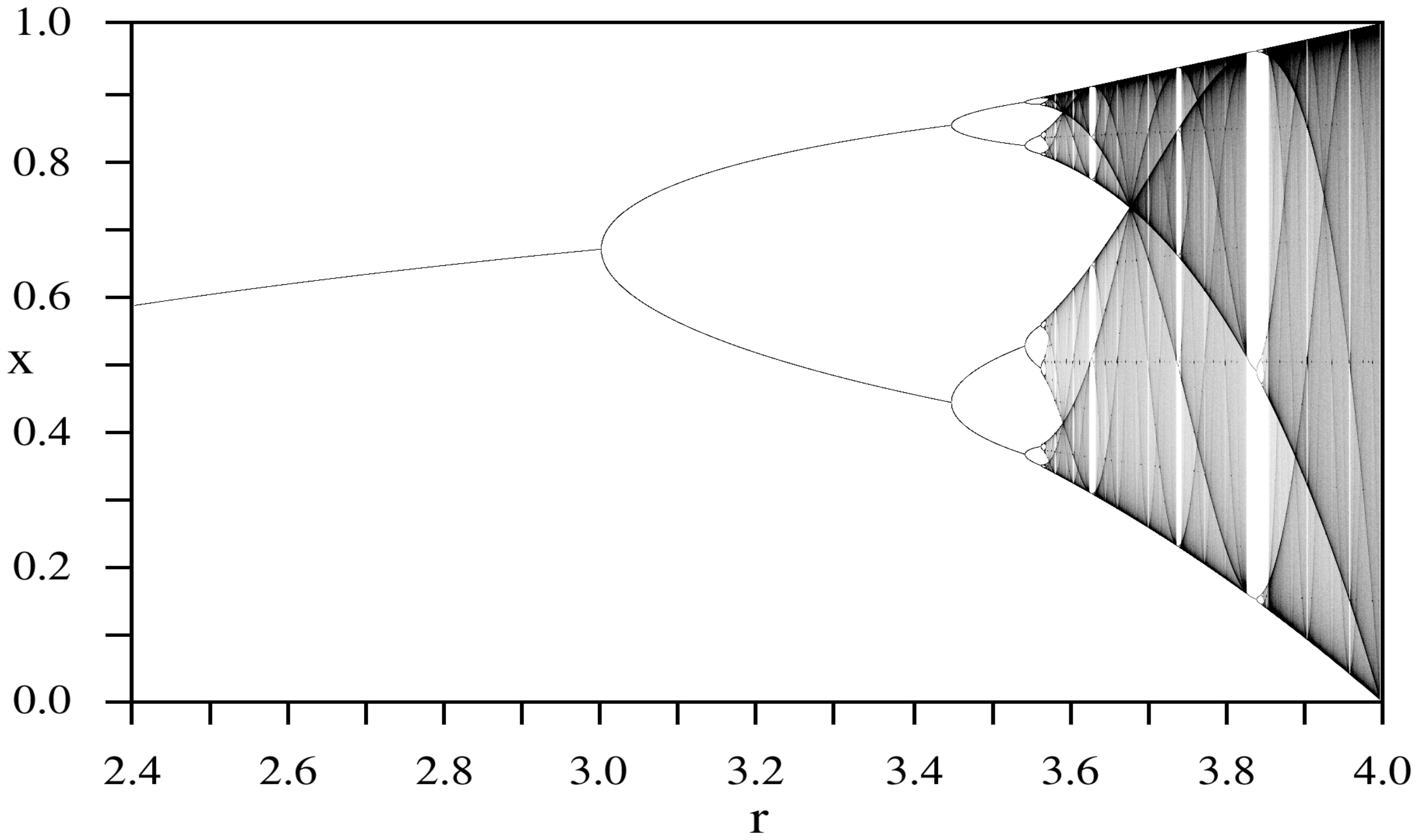
$$x_{n+1} = r\, x_n\, (1 - x_n)$$

Robert May 1976. "Simple mathematical models with very complicated dynamics." Nature 261(5560):459-467

discrete-time demographic model

x_n = population at year n,
x_o = initial population

-Reproduction (~x_n)
-Starvation (~1-x_n)



Application to Logistic Map

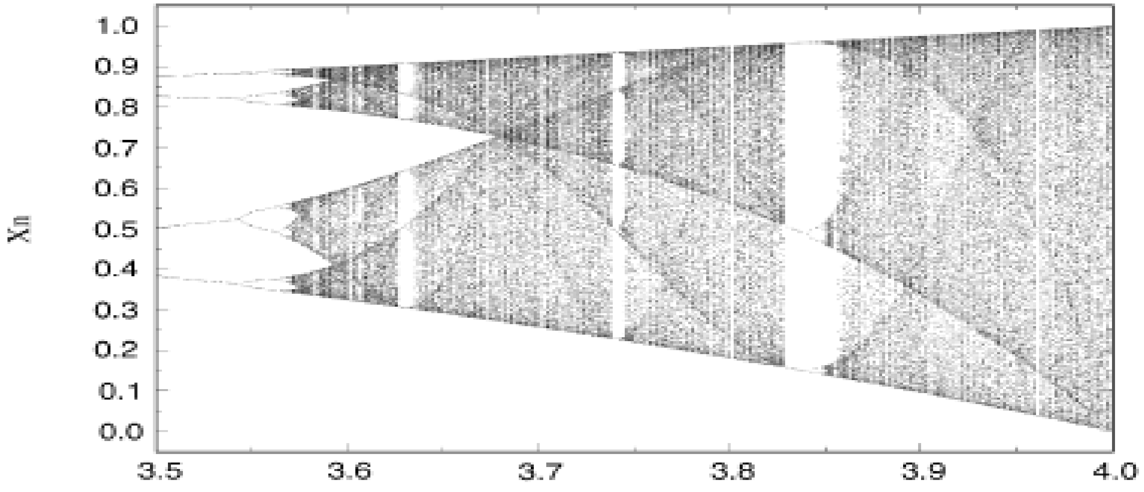
The Logistic Map, $F : x_n \to x_{n+1}$ is described by the ecologically motivated, dissipative system described by the first order difference equation

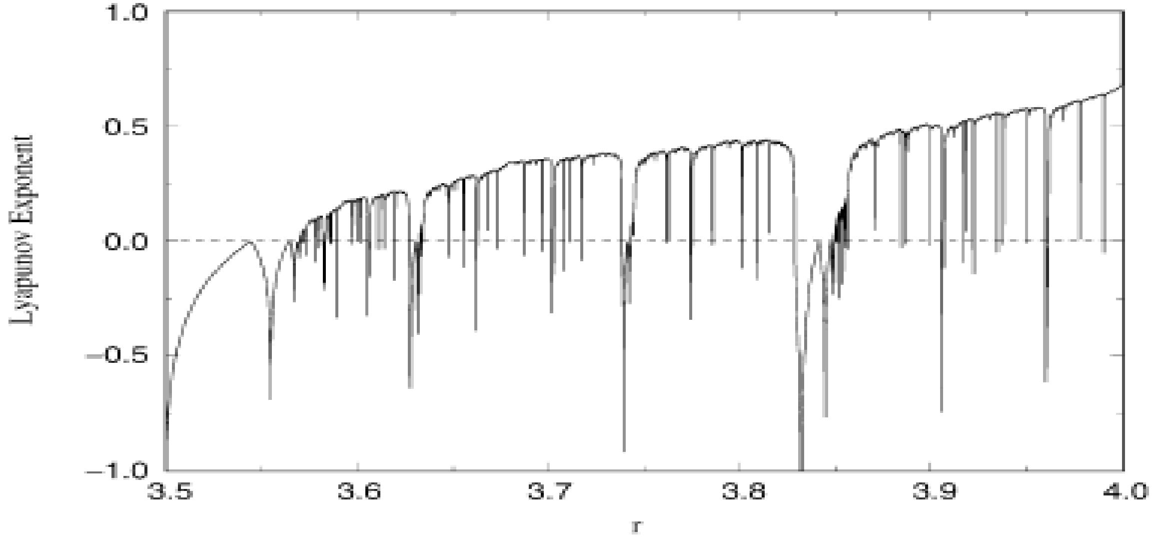
$$x_{n+1} \; = \; r \cdot x_n \cdot (1 - x_n)$$

with $0 \leq x_n \leq 1$ and $0 < r \leq 4$.

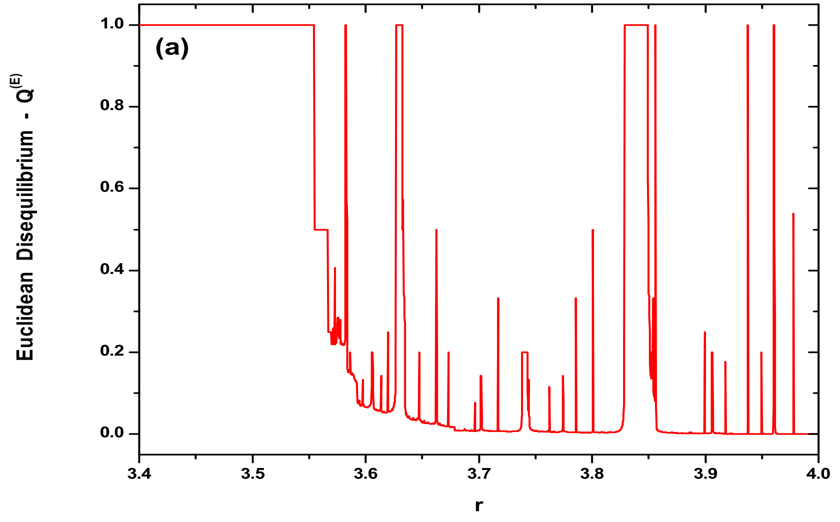
Binary treatment (symbolic dynamics) of the logistic map:

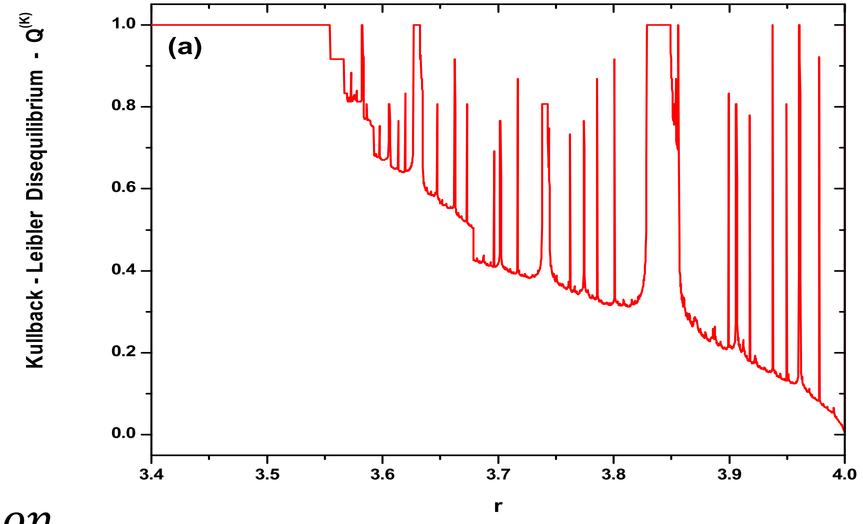
For each parameter value, r, the dynamics of the logistic map was reduced to a binary sequence (0 if $x \leq \frac{1}{2}$; 1 if $x > \frac{1}{2}$) and binary strings of length 12 were considered as states of the system. The concomitant probabilities are assigned according to the frequency of occurrence after running over at least 2^{22} iterations.



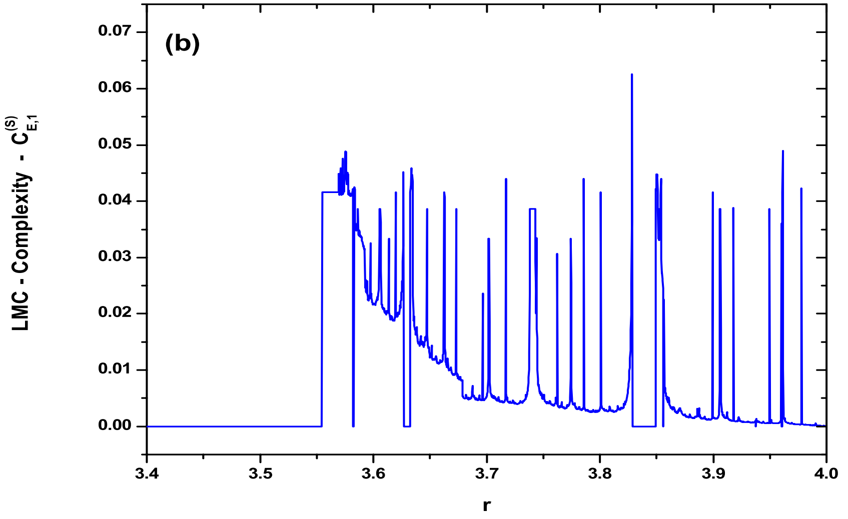


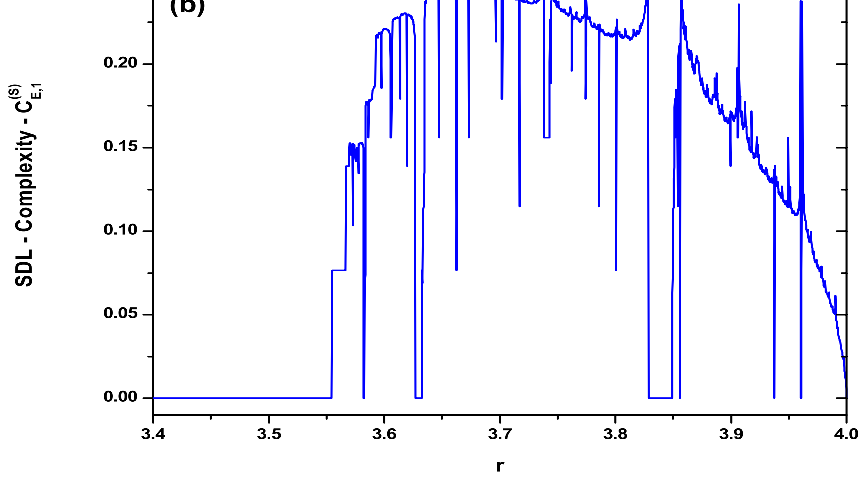
Application to Logistic Map



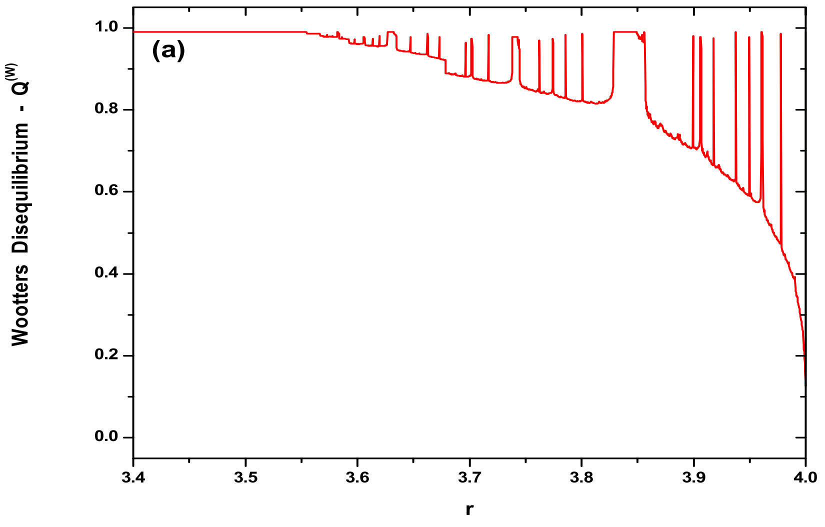


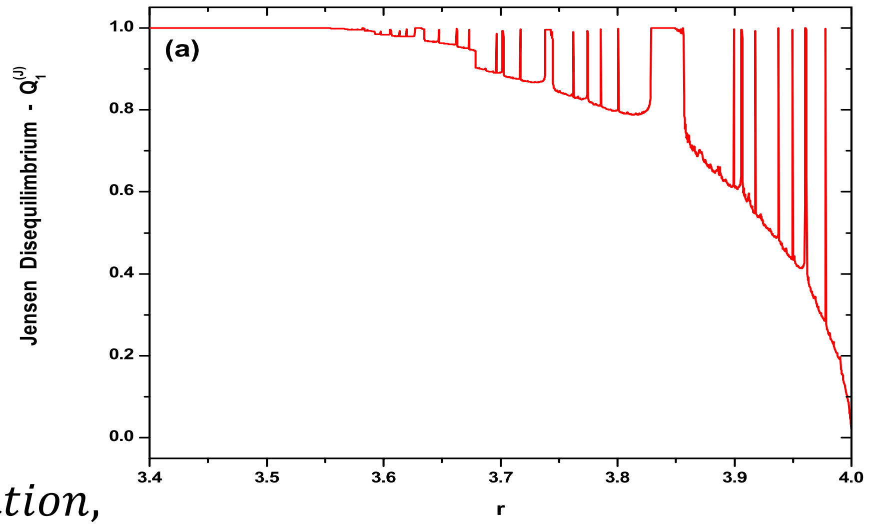
$$PDF - binay\ evalution,$$
$$3.4 \leq r \leq 4.0;\ \Delta r = 0.0003$$





Application to Logistic Map





$PDF - binay\ evalution,$
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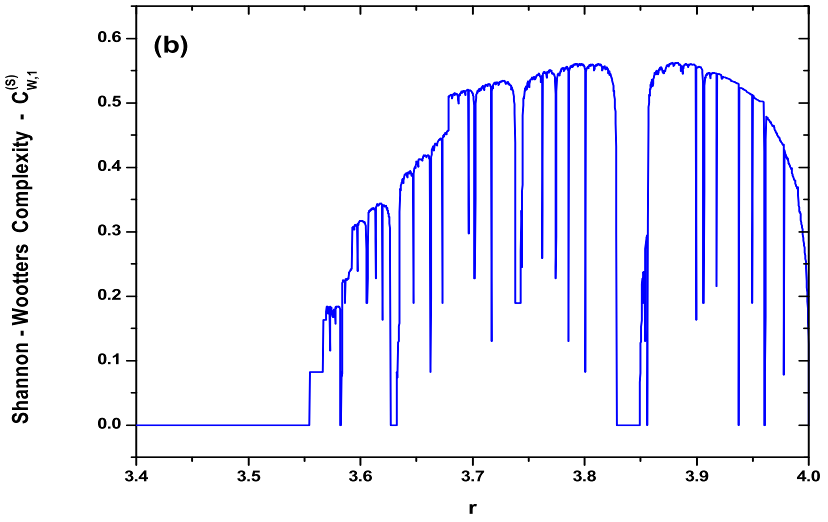
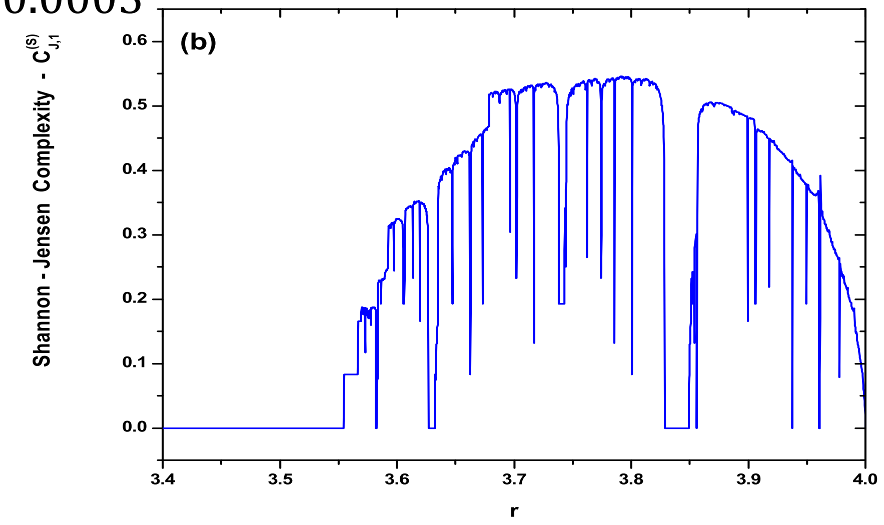
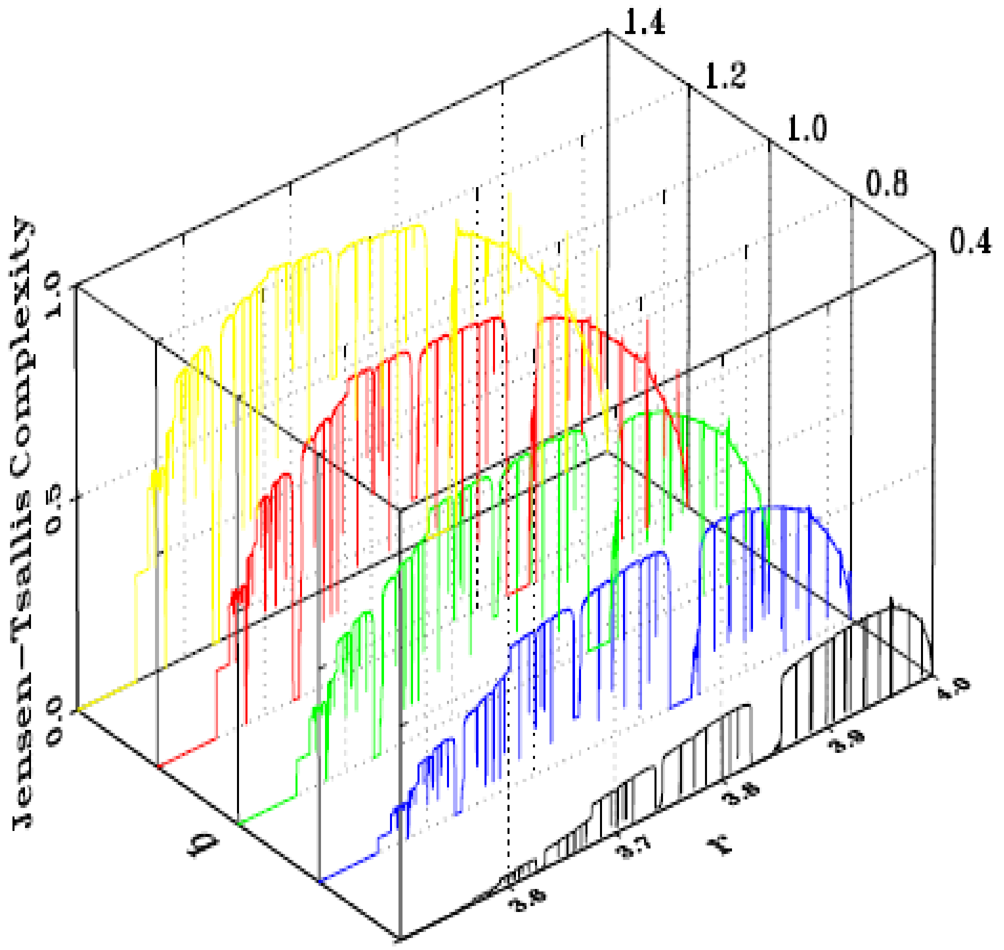
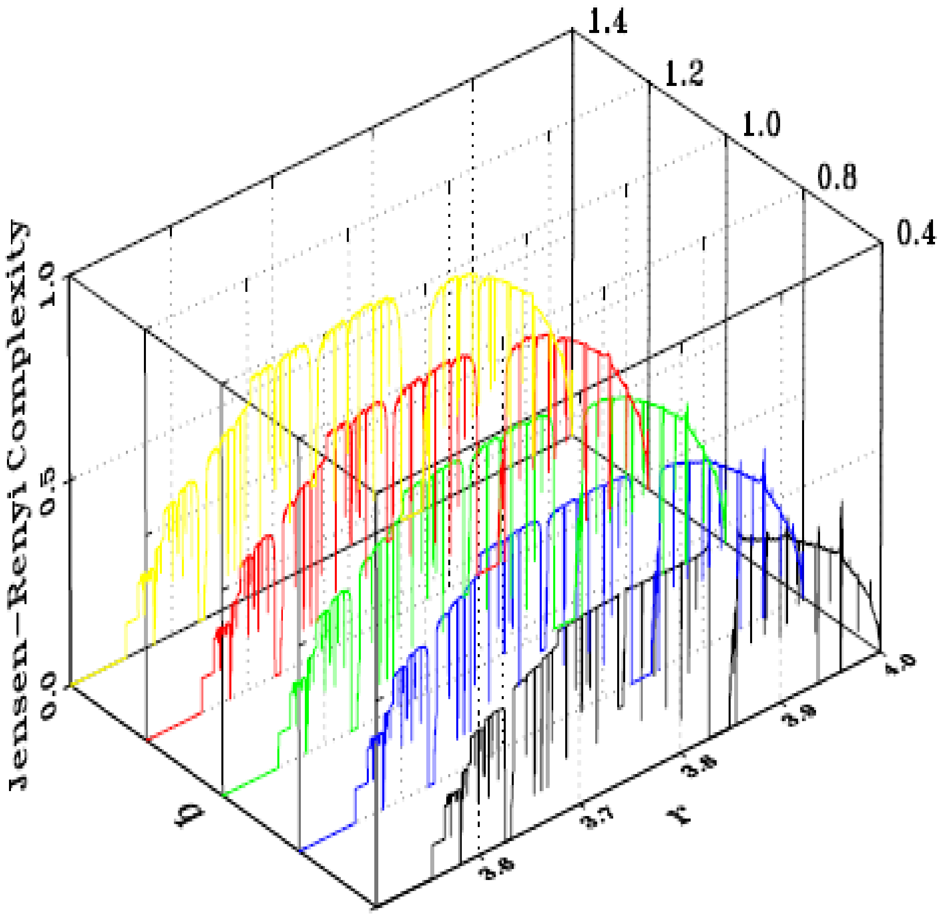




Application to Logistic Map

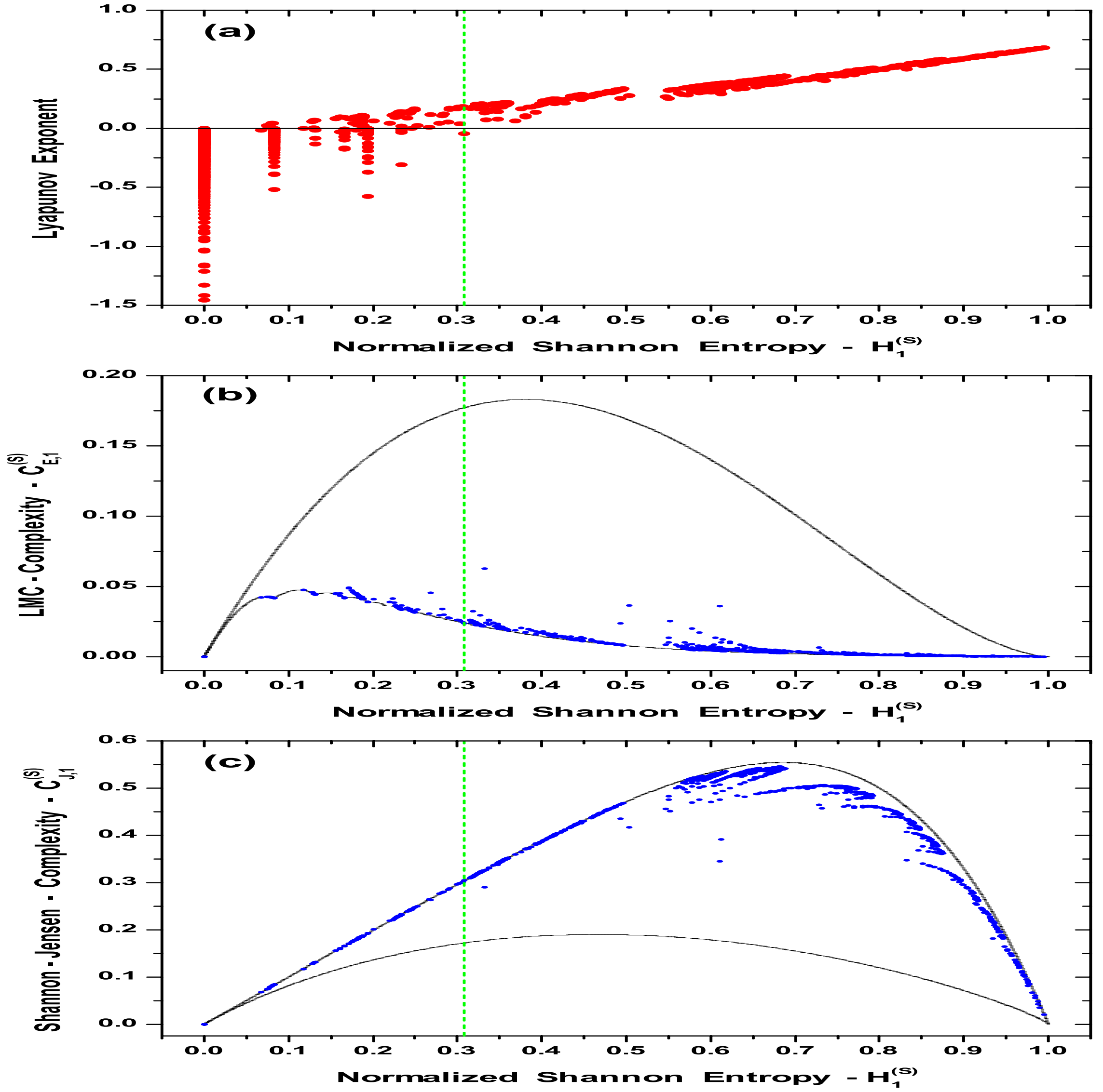




$PDF - binay\ evalution, 3.4 \leq r \leq 4.0;\ \Delta r = 0.0003$

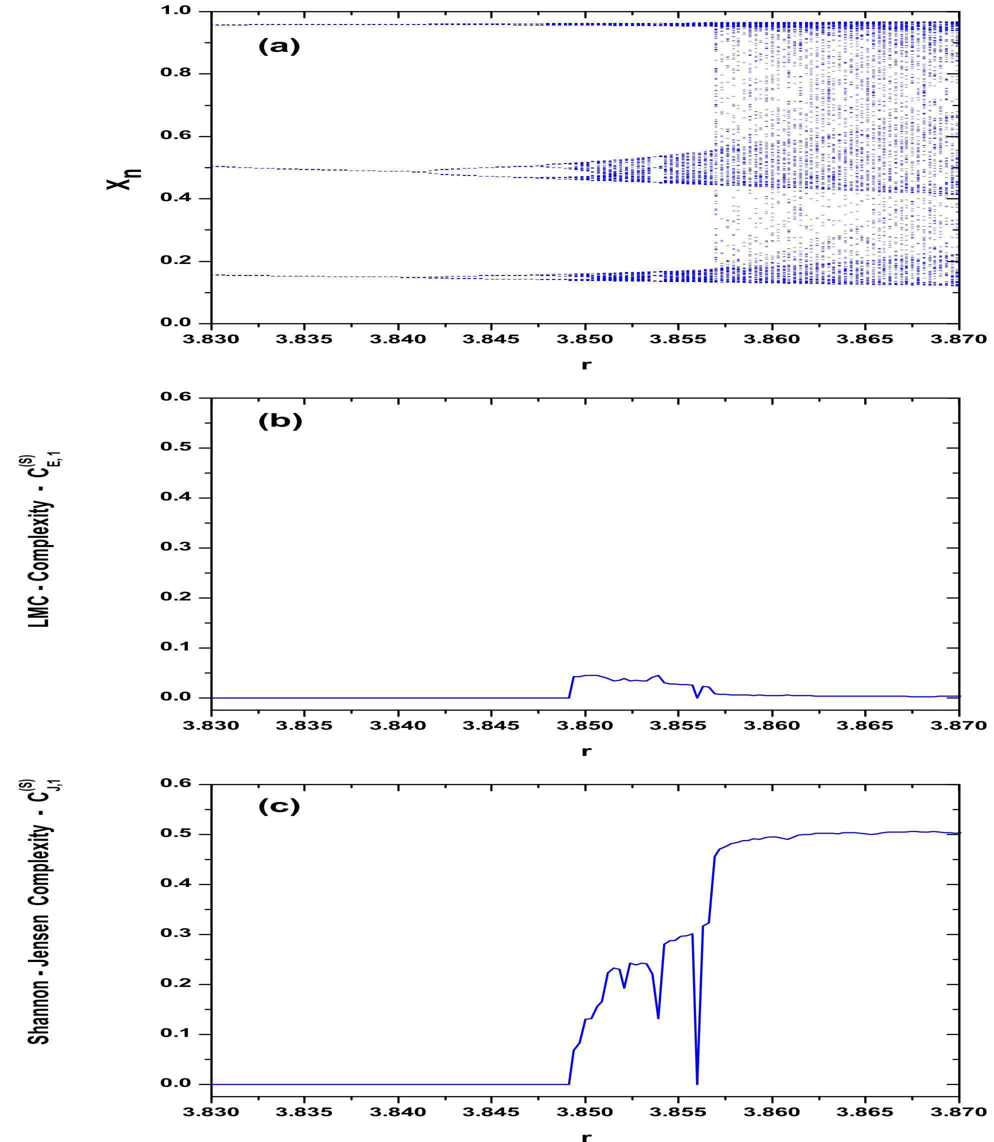
Application to Logistic Map

Notice that, for the case of periodic windows, if $H < \mathcal{H} \approx 0.3$, we can ascertain that $\Lambda < 0$, while for $H > \mathcal{H}$ we see that $\Lambda > 0$, which entails chaotic behavior. The LMC statistical complexity is larger for periodic than for chaotic motion, which is wrong!. The Jensen-Shannon statistical complexity measure, C_{JS}, on the other hand, behaves in opposite manner, and is also different for distinct degrees of periodicity.



Application to Logistic Map

Summing up: the Jensen-Shannon statistical complexity measure *i)* becomes intensive, *ii)* is able to distinguish among distinct degrees of periodicity, and *iii)* yields a better description of dynamical features (a better grasp of dynamical details).



P. W. Lamberti, M. T. Martín, A. Plastino, and O. A. Rosso.
Intensive entropic nontriviality measure.
Physica A 334 (2004) 119-131.

Bandt & Pompe - PDF: Summary

Given a time series $\boldsymbol{X(t)} = \{\boldsymbol{x_t}, \boldsymbol{t} = \mathbf{1}, \dots, \boldsymbol{M}\}\,;\ \boldsymbol{x_t} \in \mathbb{R}$

we map it on $\boldsymbol{X(t)} \mapsto \vec{\boldsymbol{Y}}_s^{(D,\tau)} = \left(x_s, x_{s+\tau}, x_{s+2\tau}, \dots, x_{s+(D-1)\tau}\right)$

ordering the observations $x_s \in \vec{\boldsymbol{Y}}_s^{(D,\tau)}$ in increasing order

$$\vec{\boldsymbol{Y}}_s^{(D,\tau)} \mapsto \vec{\boldsymbol{\pi}}_s^{(D)} = (r_0, r_1, r_2, \dots, r_{D-1})$$

the permutation pattern is given by

$$\vec{\boldsymbol{\pi}}_s^{(D)} \mapsto \pi_s^{(D)} = [0, 1, 2, \dots (D-1)]$$

such that

$$x_{s+r_0} < x_{s+r_1} < x_{s+r_2} < \dots < x_{s+r_{D-1}}$$

Bandt & Pompe - PDF: Summary

- The Bandt & Pompe PDF $\boldsymbol{\Pi}^{(\boldsymbol{D},\boldsymbol{\tau})} \equiv \left\{p\left(\pi_s^{(D)}\right)\right\}$ incorporate in natural way the time causality.

- The methodology can be applied to any kind of time series.

- The only condition for applicability of BP methodology is a very weak stationary assumption: for $\boldsymbol{k} \leq \boldsymbol{D}$ the probability for $\boldsymbol{x}_t < \boldsymbol{x}_{t+k}$ should not depend of t.

- The amplitude values of $\boldsymbol{x}_t$ are not taken into account, only its sequential ordering.

Bandt & Pompe - PDF: Summary

- The Bandt & Pompe PDF $\Pi^{(D,\tau)}$ is invariant under monotonic transform

- The time series length M must be M >> $D!$ in order to have a good statistics

- The causal $H \times C - plane$ is a good diagnostic tool for discriminate chaotic and stochastic nature of the time series, since the quantifiers have distinctive behaviors for different type of dynamics.

Bandt & Pompe - PDF: Summary

- *Chaotic maps have intermedia entropy $\mathbf{H}$ while the complexity $\mathbf{C}$ reaches larger values close to those of a limit value $\mathbf{C}_{max}$.*

- *Similar behavior is still observed when the time series is contaminated with small or moderate amount of uncorrelated or correlated noise.*

- *Pure Uncorrelated stochastic time series are localized quite close to extreme value $(\mathbf{H}, \mathbf{C}) \cong (\mathbf{1}, \mathbf{0})$. Pure correlated stochastic time series present decreasing values of entropy $\mathbf{H}$ with the increasing correlation value, and associate increase of the complexity $\mathbf{C}$ value at intermediate value between $\mathbf{C}_{min}$ and $\mathbf{C}_{max}$.*

Bandt-Pompe PDF

BANDT AND POMPE SIMBOLIZATION TECHNIQUE
(PRL 88 (2002) 174102)

The pertinent symbolic data are:

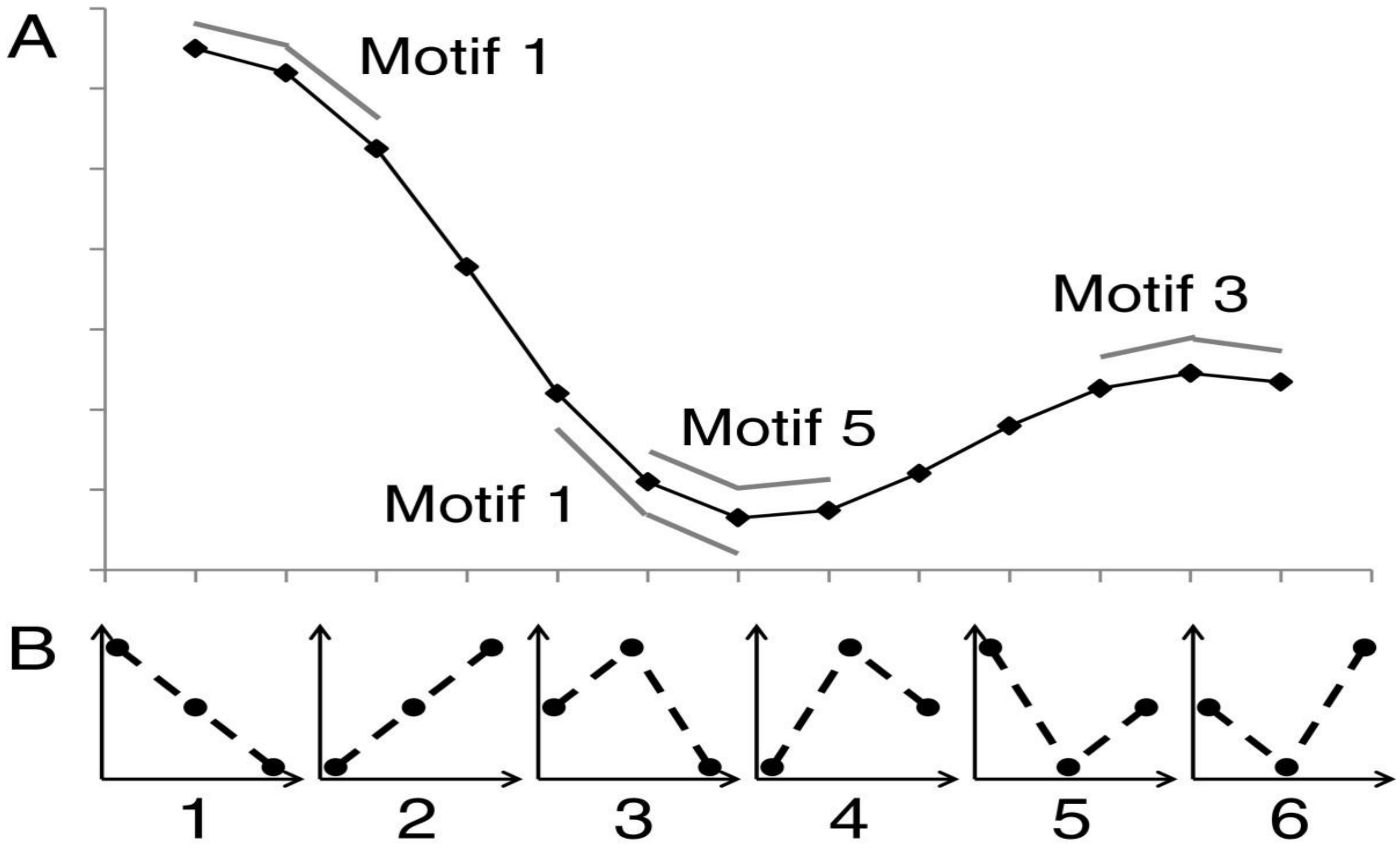
- created by ranking the values of the series
- defined by reordering the embedded data in ascending order,

which is tantamount to a phase space reconstruction with embedding dimension (pattern length) D and time lag τ. In this way, it is possible to quantify the diversity of the ordering symbols (patterns) derived from a scalar time series.

Characteristic of Bandt-Pompe PDF:

- no model-based assumptions are needed.
- "partitions" are devised by comparing the order of neighboring relative values rather than by apportioning amplitudes according to different levels
- time causality is naturally incorporate
- give information about correlations
- few parameters: pattern length/embedding D and time lag τ
- extremely fast nature of the calculation process

Bandt-Pompe PDF



Bandt-Pompe PDF

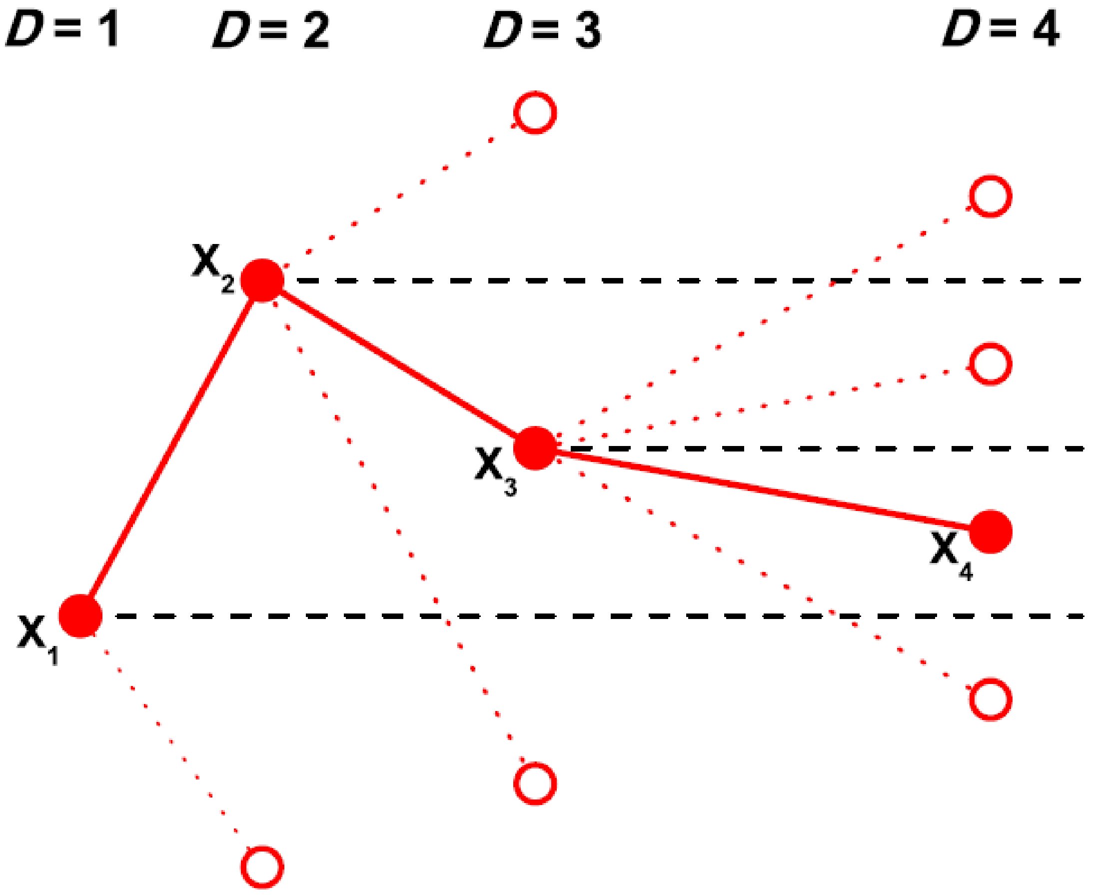
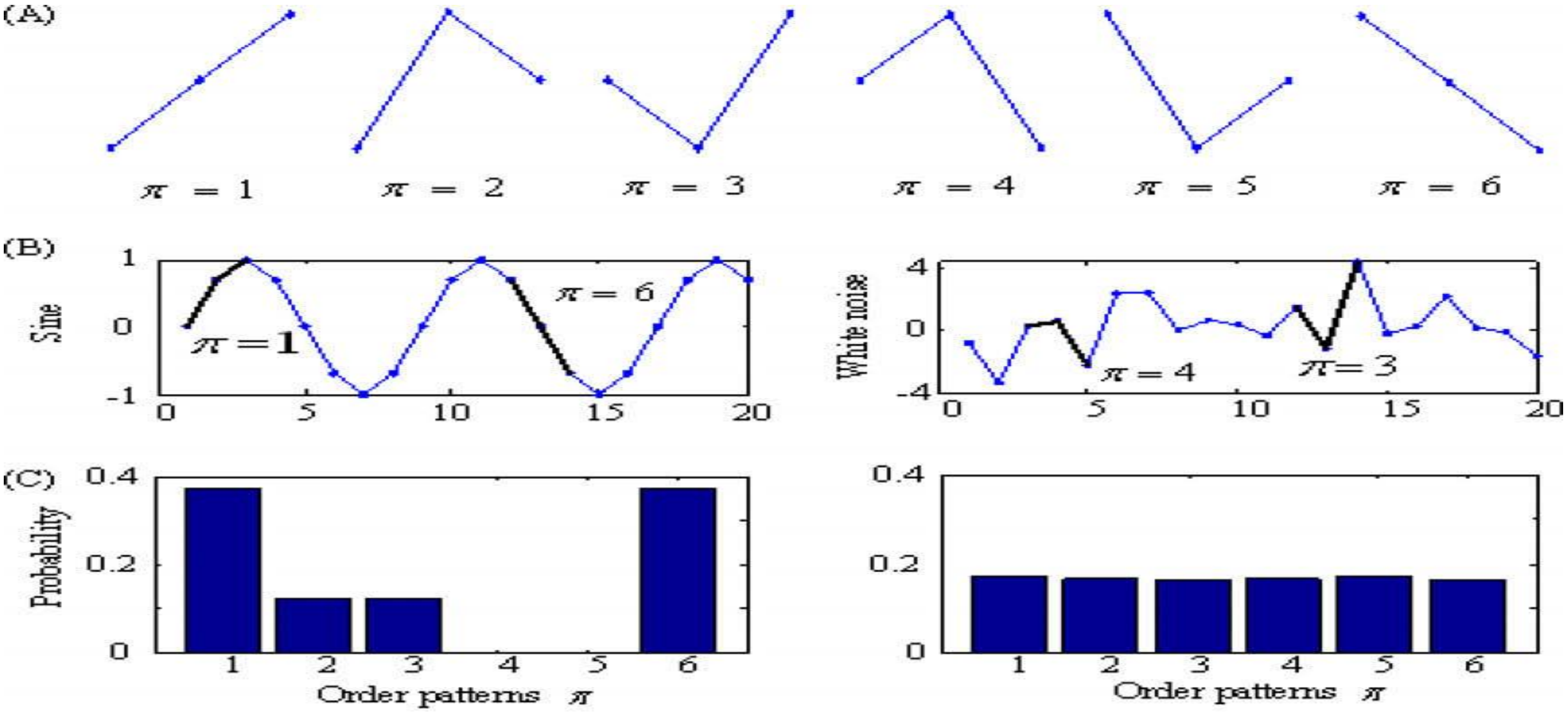


Illustration of the construction principle for ordinal patterns of length D. If $D = 4$ and $\tau = 1$, full circles and continuous lines represent the sequence values $x_1 < x_2 > x_3 > x_4$ which lead to the pattern $\pi = [1432]$.

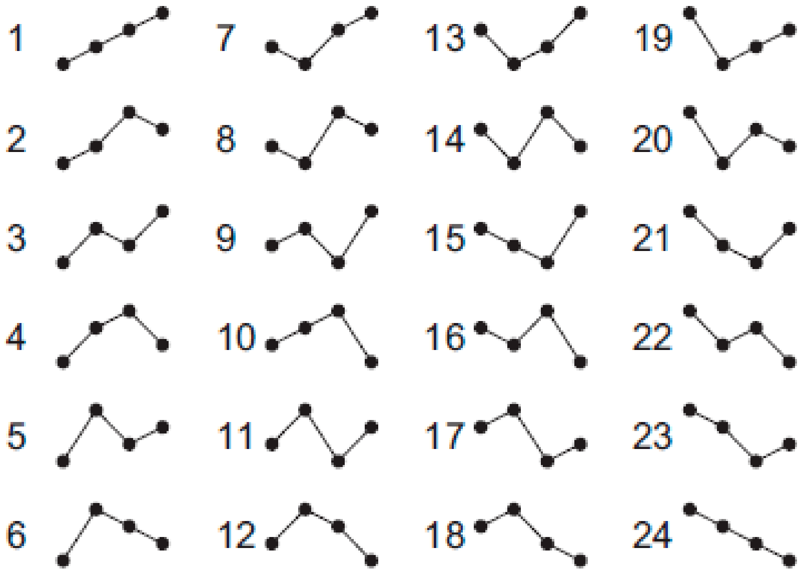
Bandt-Pompe PDF



Ordinal patterns in a simple time series. (A) Ordinal patterns at the dimension d = 3. (B) Illustration of the ordinal procedure for d = 3 for sine and white noise time series. (C) Probability distribution of ordinal patterns π.

Bandt-Pompe PDF

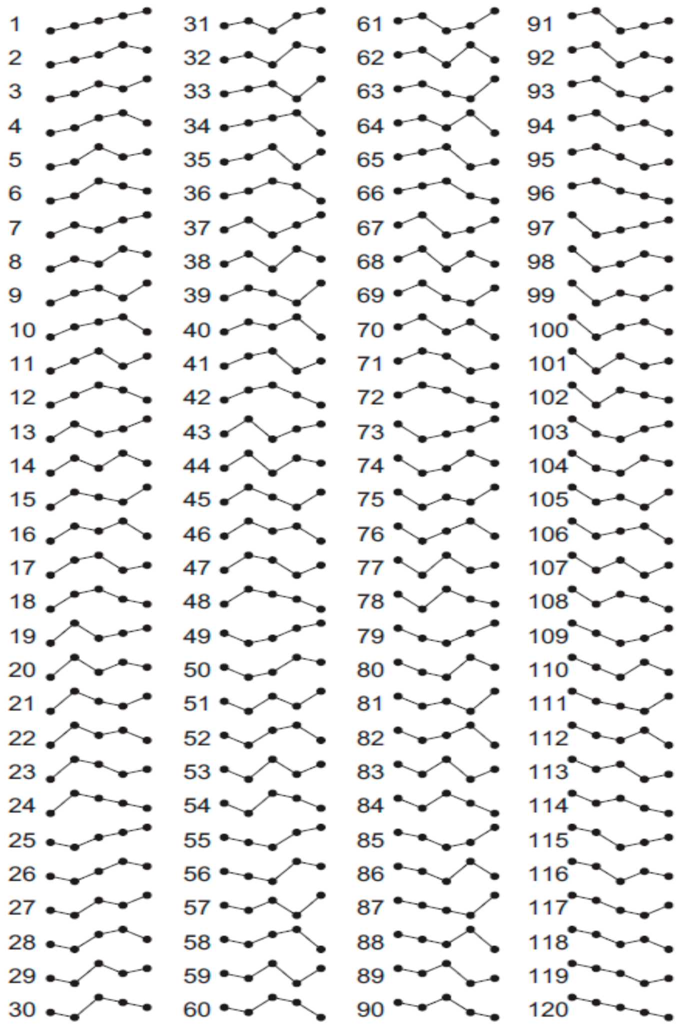
The number of patterns increases as D!
a drawback and an advantage;
is a problem for short data sets.



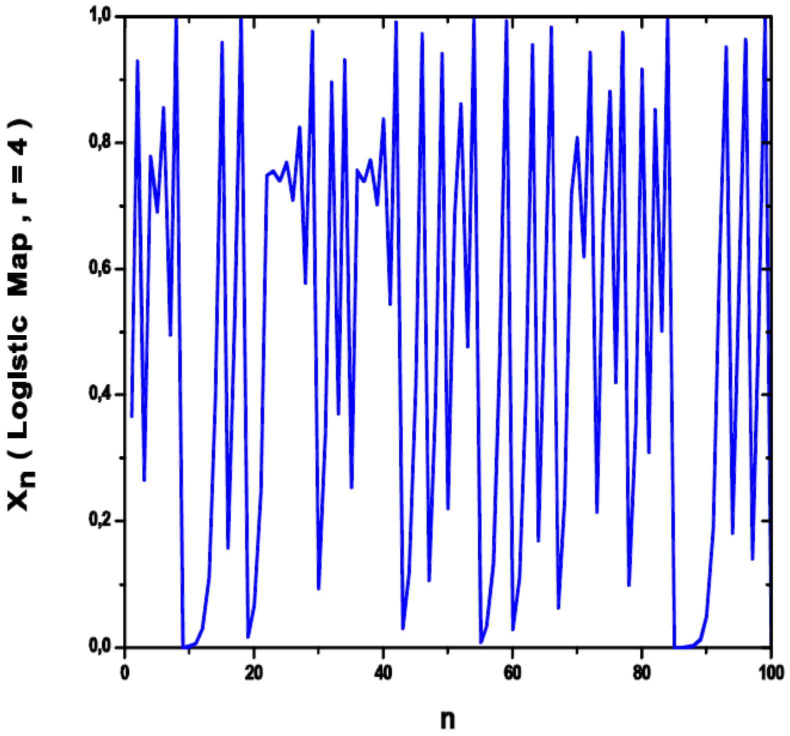
D = 4 ; D! = 24

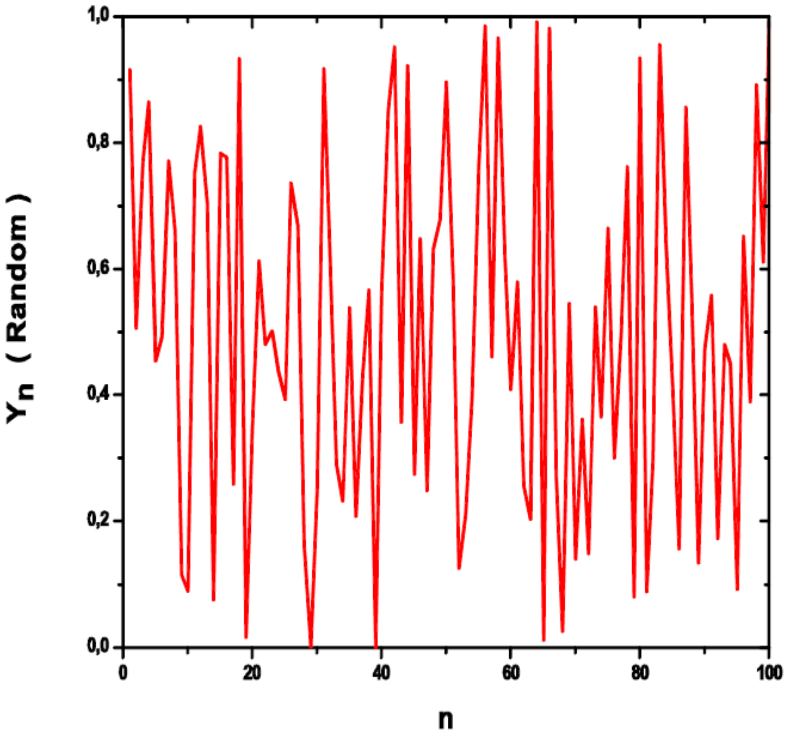
U. Parlitz et al. / Computers in Biology and Medicine 42 (2012) 319–327

D = 5 ; D! = 120



Bandt-Pompe PDF

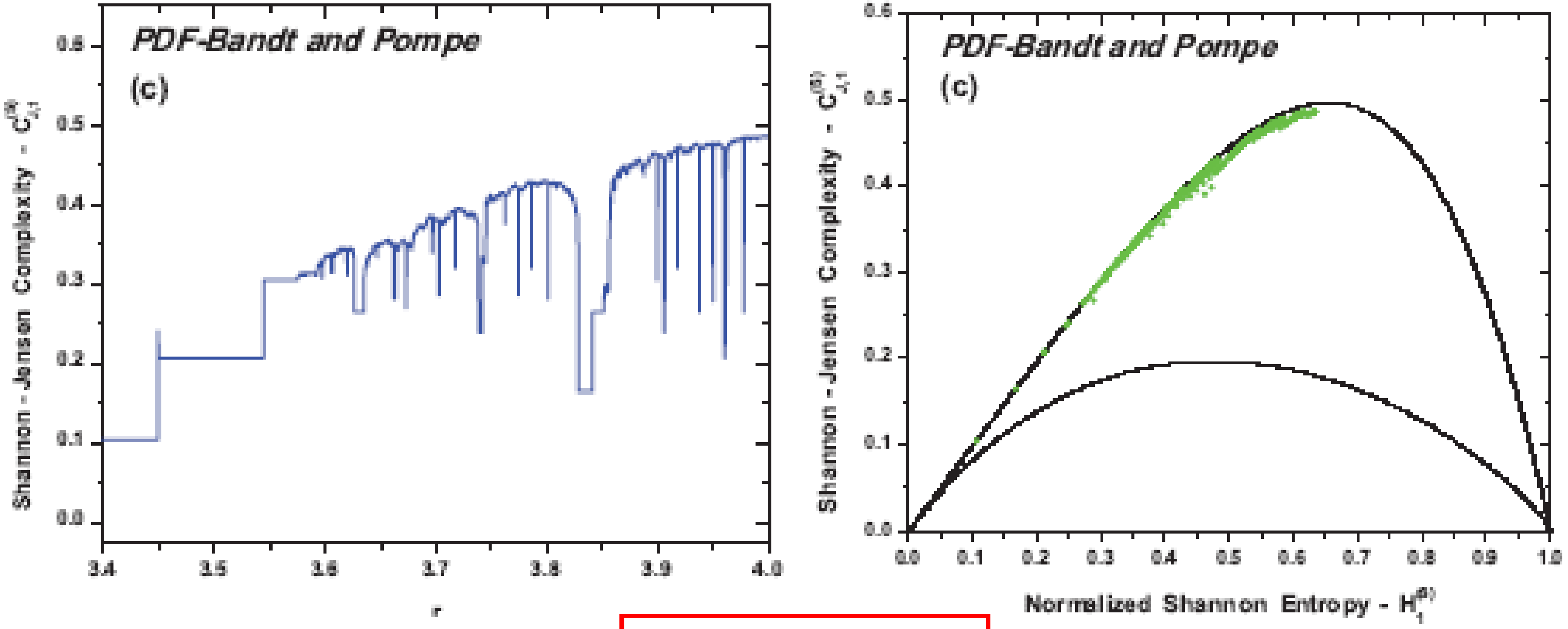




$$H\left[P^{(BP)}\right] = 0.629$$
$$C\left[P^{(BP)}\right] = 0.484$$
$$L = 6\,,\tau = 1$$

$$H\left[P^{(BP)}\right] = 0.99994$$
$$C\left[P^{(BP)}\right] = 0.00013$$
$$L = 6\,,\tau = 1$$

Bandt-Pompe PDF



$$x_{n+1} = r\, x_n\, (1 - x_n)$$

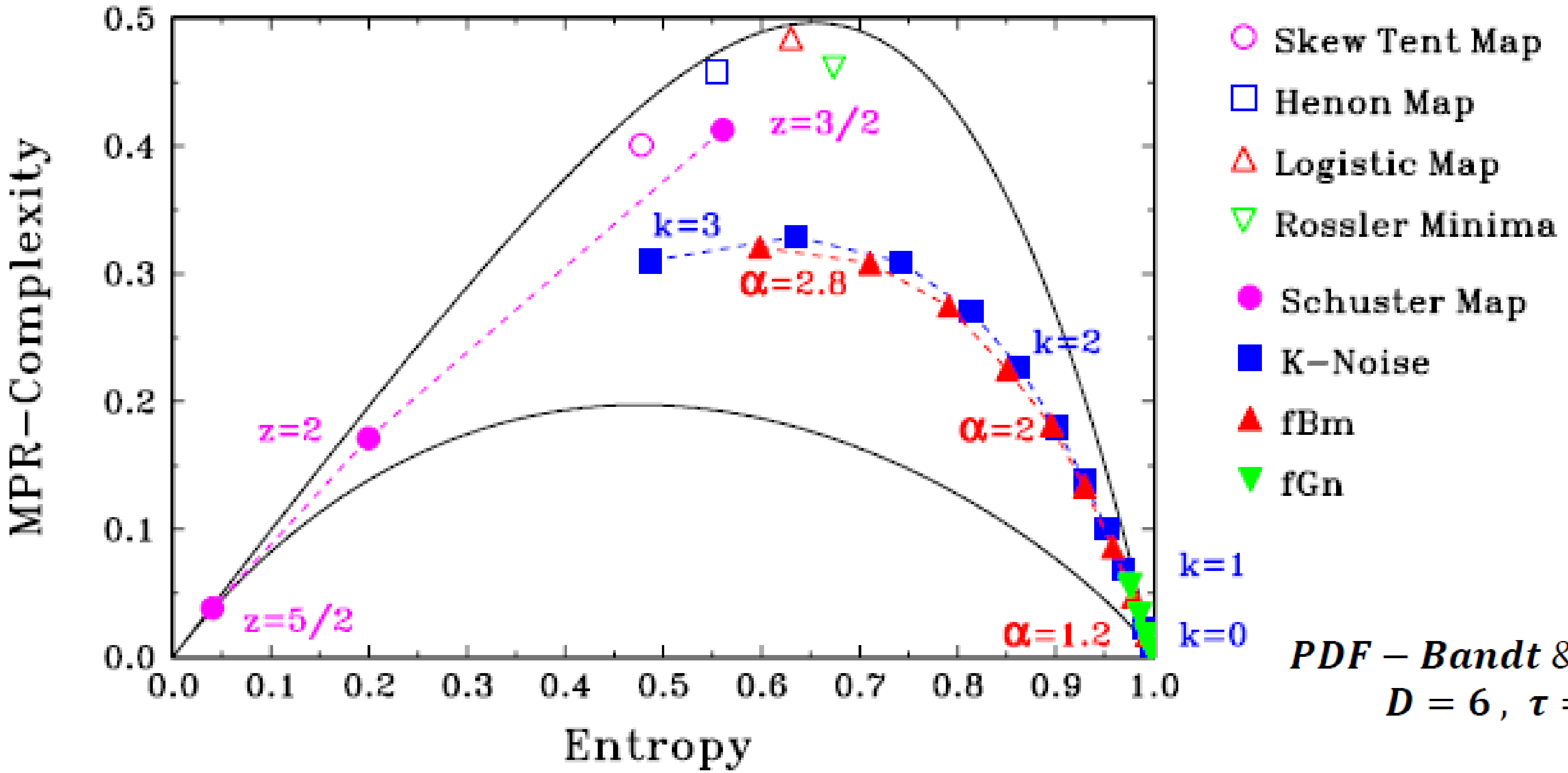
Entropy, Entropy-complexity plane for logistic map (parameter $3.4 \leq r \leq 4.0$, $\Delta r = 0.0003$) for: Band and Pompe-PDF (D=6, N=720).

Chaos & Noise

PRL **99,** 154102 (2007) PHYSICAL REVIEW LETTERS week ending 12 OCTOBER 2007

Distinguishing Noise from Chaos

O. A. Rosso,[1,2] H. A. Larrondo,[3] M. T. Martin,[4] A. Plastino,[4] and M. A. Fuentes[5,6]



PDF – Bandt & Pompe
$D = 6\,,\ \tau = 1$

Chaos & Noise

STOCHASTIC DYNAMICS:

- Nises with power spectrum

  $k = 0, 0.5, 1.00, 1.25, 1.50, 1.75, 2.00, 2.25, 2.5$

- fractional Brawnian motion (fBm):

  generalized power spectrum
  $\alpha = (2h+1) = 1.2, 1.4, 1.6, 1.8, 2.0, 2.2, 2.4, 2.6, 2.8$

- fractional Gaussian noise (fGn):

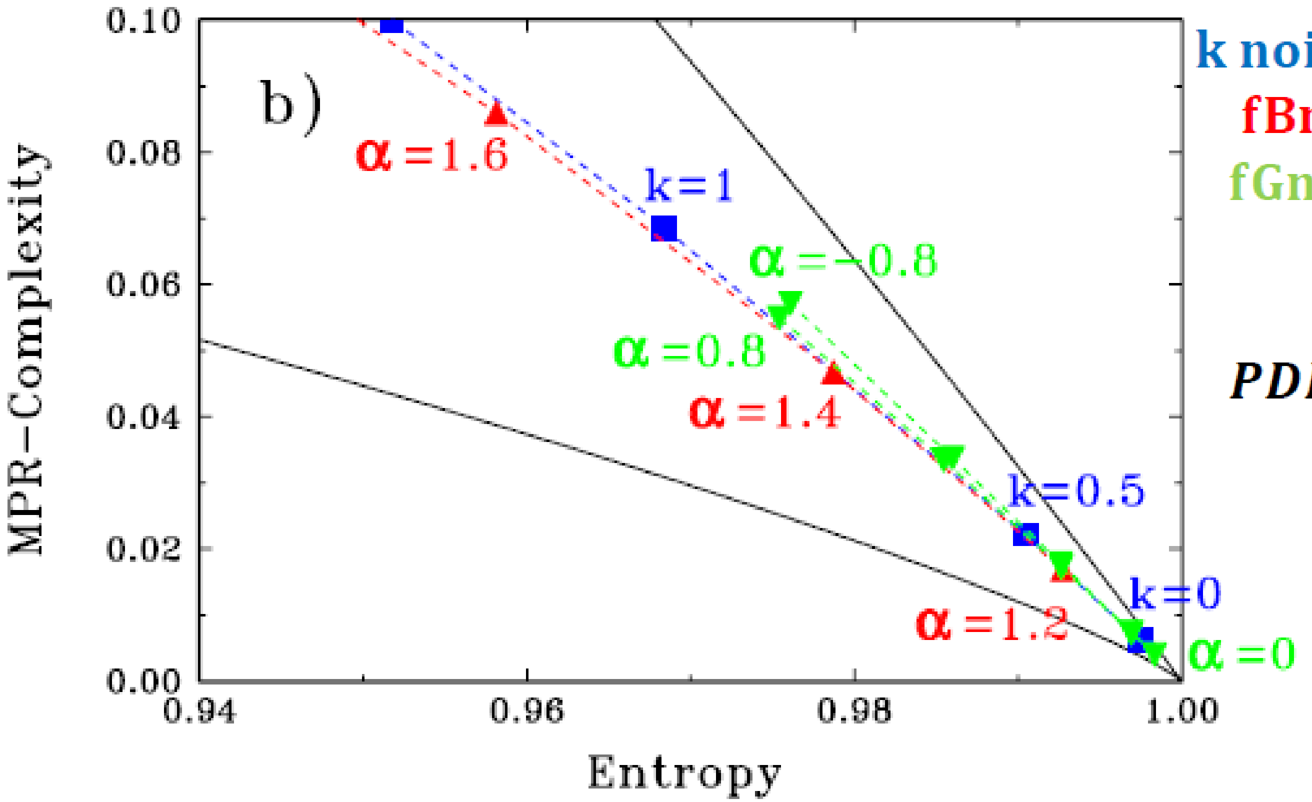
  generalized power spectrum
  $\beta = (2h-1) = -0.8, -0.6, -0.4, -0.2, 0.0, 0.2, 0.4, 0.6, 0.8$

Chaos & Noise

PRL 99, 154102 (2007) PHYSICAL REVIEW LETTERS week ending 12 OCTOBER 2007

Distinguishing Noise from Chaos

O. A. Rosso,[1,2] H. A. Larrondo,[3] M. T. Martin,[4] A. Plastino,[4] and M. A. Fuentes[5,6]



k noise : PS $f^{-k}, 0 \le k \le 3$

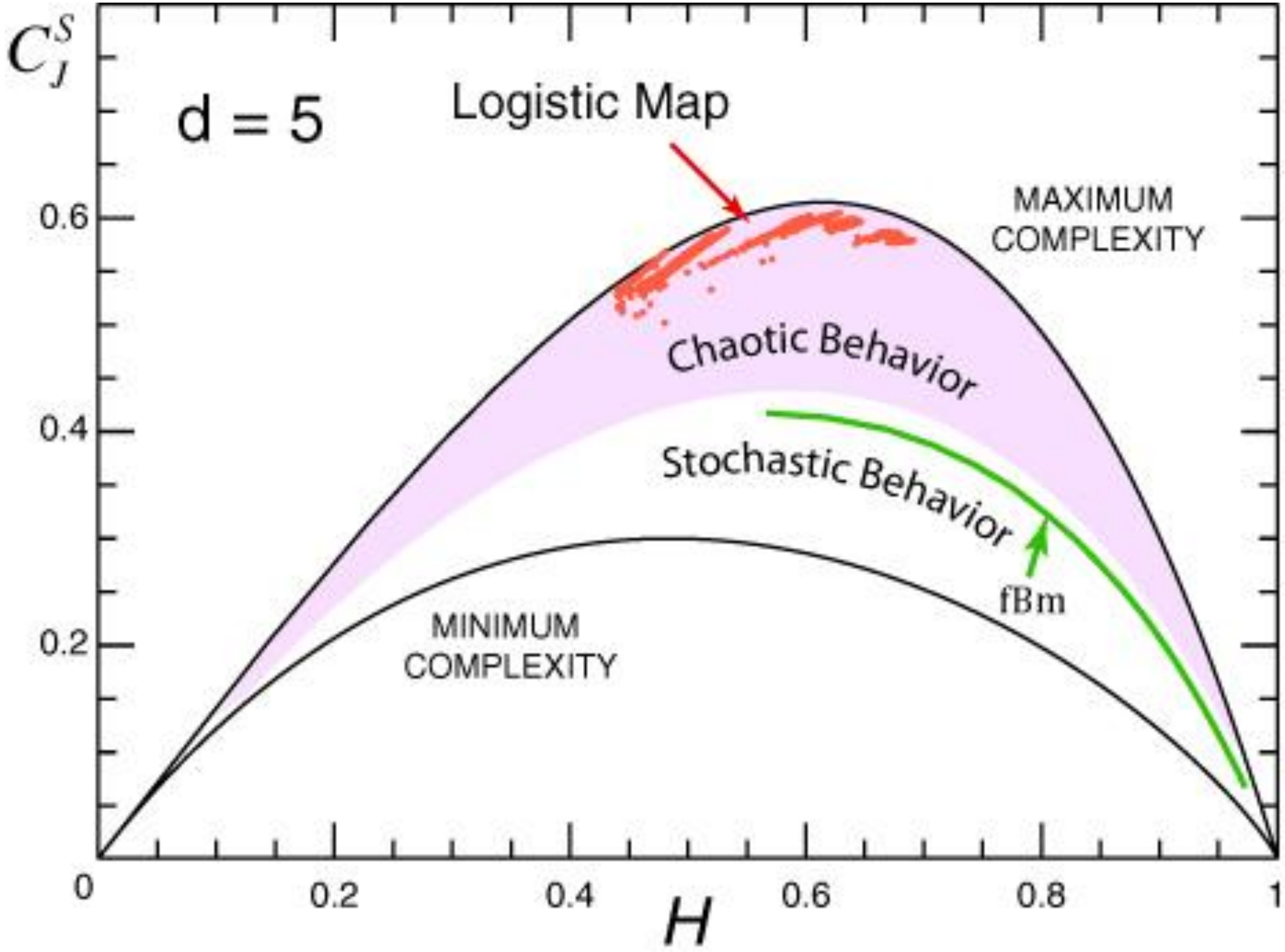
fBm : PS $f^{-\alpha}, 1 < \alpha < 3$

fGn : PS $f^{-\alpha}, -1 < \alpha < 1$

$PDF - Bandt \,\&\, Pompe$

$D = 6\,, \;\; \tau = 1$

Bandt & Pompe - PDF: Summary



Amigó Paradigm: forbidden / missing patterns

- *For deterministic one dimensional maps, Amigó et al. [Europhys. Lett. 79 (2007) 50001] have conclusively shown that not all the possible ordinal patterns can be effectively materialized into orbits, which in a sense makes these patterns "forbidden".*

- *This is an established fact, not a conjecture !!!*

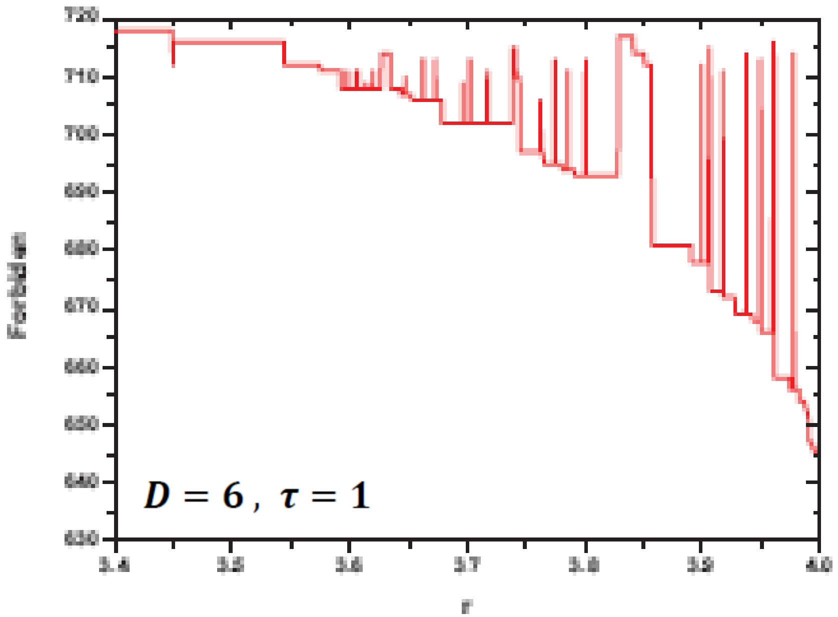
- *The existence of these forbidden ordinal patterns becomes a persistent feature, a "new" dynamical property.*

- *For a fixed pattern-length the number of forbidden patterns of a time series (unobserved patterns) is independent of the series length N.*

Amigó Paradigm: forbidden / missing patterns

- *Remark that this independence does not characterize other properties of the series such as proximity and correlation, which die out with time.*

- *For example, in the time series generated by the logistic map if we consider patterns of length D=3, the pattern {2,1,0} is forbidden.*
  That is, the pattern
  $x_{k+2} < x_{k+1} < x_k$
  never appears !!!!



Amigó Paradigm: forbidden / missing patterns

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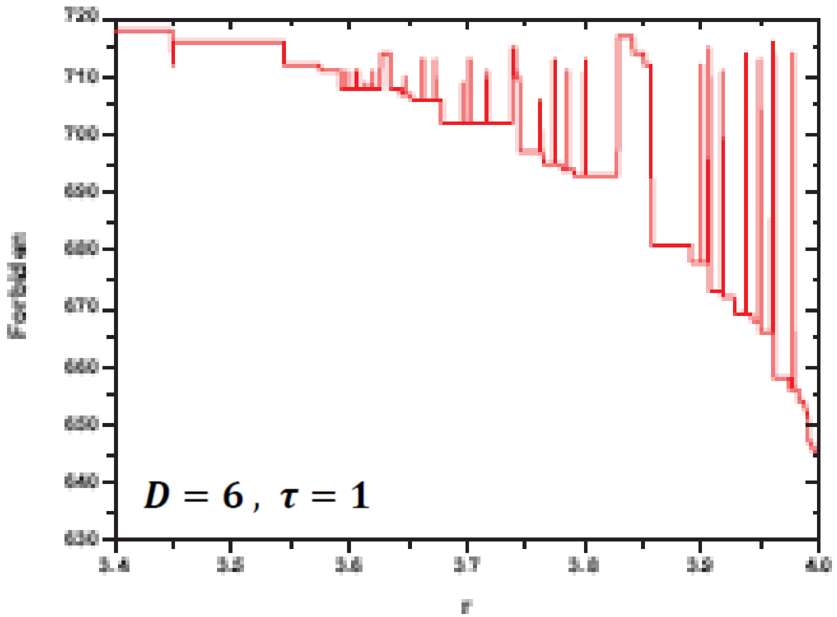
  That is, the pattern
  $x_{k+2} < x_{k+1} < x_k$
  never appears !!!!

- *Stochastic process COULD ALSO PRESENT forbidden patterns !!!.*



Amigó Paradigm: forbidden / missing patterns

- *However, in the case of either uncorrelated (white noise) or correlated stochastic processes (noise with f^{-k} PS, oBm, fBm, fGn) it can be numerically ascertained that no forbidden patterns emerge.*

- *If the data set is large enough, all the ordinal patterns should eventually appear. The PDF is the uniform.*

- *For correlated stochastic processes the probability of observing individual pattern depends not only on the time series length N but also on the correlation structure.*

- *The existence of a non-observed ordinal pattern does not qualify it as "forbidden", only as "missing" and is due to the finite length of the time series.*

Permutation Min-Entropy

- *Consider the ordinal pattern PDF with embedding dimension D and embedding lag τ, for the time series $x(t)$*

$$\mathrm{P}(x, D, \tau) \equiv \left\{ p^{(\tau)}(\pi_k),\, k = 1, \ldots, D! \right\}$$

Permutation Min-Entropy

- *Consider the ordinal pattern PDF with embedding dimension D and embedding lag τ, for the time series $x(t)$*

$$\mathrm{P}(x, D, \tau) \equiv \{p^{(\tau)}(\pi_k), k = 1, \ldots, D!\}$$

- *The Normalized Permutation Shannon Entropy is given by:*

$$\mathrm{PE}(x, D, \tau) \equiv - \frac{1}{\ln D!} \sum_{k=1}^{D!} p^{(\tau)}(\pi_k) \ln\{p^{(\tau)}(\pi_k)\}$$

Permutation Min-Entropy

- *Consider the ordinal pattern PDF with embedding dimension D and embedding lag τ, for the time series $x(t)$*

$$\mathrm{P}(x, D, \tau) \equiv \{p^{(\tau)}(\pi_k)\,, k = 1, \ldots, D!\}$$

- *The Normalized Permutation Shannon Entropy is given by:*

$$\mathrm{PE}(x, D, \tau) \equiv -\frac{1}{\ln D!} \sum_{k=1}^{D!} p^{(\tau)}(\pi_k)\, \ln\{p^{(\tau)}(\pi_k)\}$$

- *The Normalized Permutation Renyi Entropy is given by:*

$$\mathrm{RPE}(x, D, \tau, q) \equiv -\frac{1}{\ln D!}\frac{1}{1-q} \ln\left\{\sum_{k=1}^{D!} \left(p^{(\tau)}(\pi_k)\right)^q\right\}$$

Where the parameter $q \geq 0$, and $q \neq 1$

Permutation Min-Entropy

Note that:

- $q < 1 \Longrightarrow$ *benefits rare events*
- $q > 1 \Longrightarrow$ *prinvileges salient events*
- $q = 1 \Longrightarrow$ *Shannon Entropy*

Permutation Min-Entropy

Note that:

- $q < 1 \Rightarrow$ benefits rare events
- $q > 1 \Rightarrow$ prinvileges salient events
- $q = 1 \Rightarrow$ Shannon Entropy

When $q \rightarrow \infty$ the RPE converge to the Permutation Min-Entropy given by by

$$\mathbf{PME}(x, D, \tau) \equiv - \frac{1}{\ln D!} \ln \left(\underset{k = 1, \ldots, D!}{Max} \left[p^{(\tau)}(\pi_k) \right] \right)$$

- We retain the main adventages of PE:

a) Simplicity
b) Low computational cost
c) Noise robustness
d) Invariance with respect to monotonus transformations

Chaos + Noise

Logistic Map + Observational Noise (Correlated Noise)

$$Y_n = X_n + A\,\eta_n^{(k)}$$

$$X_{n+1} = R\,X_n(1 - X_n)$$

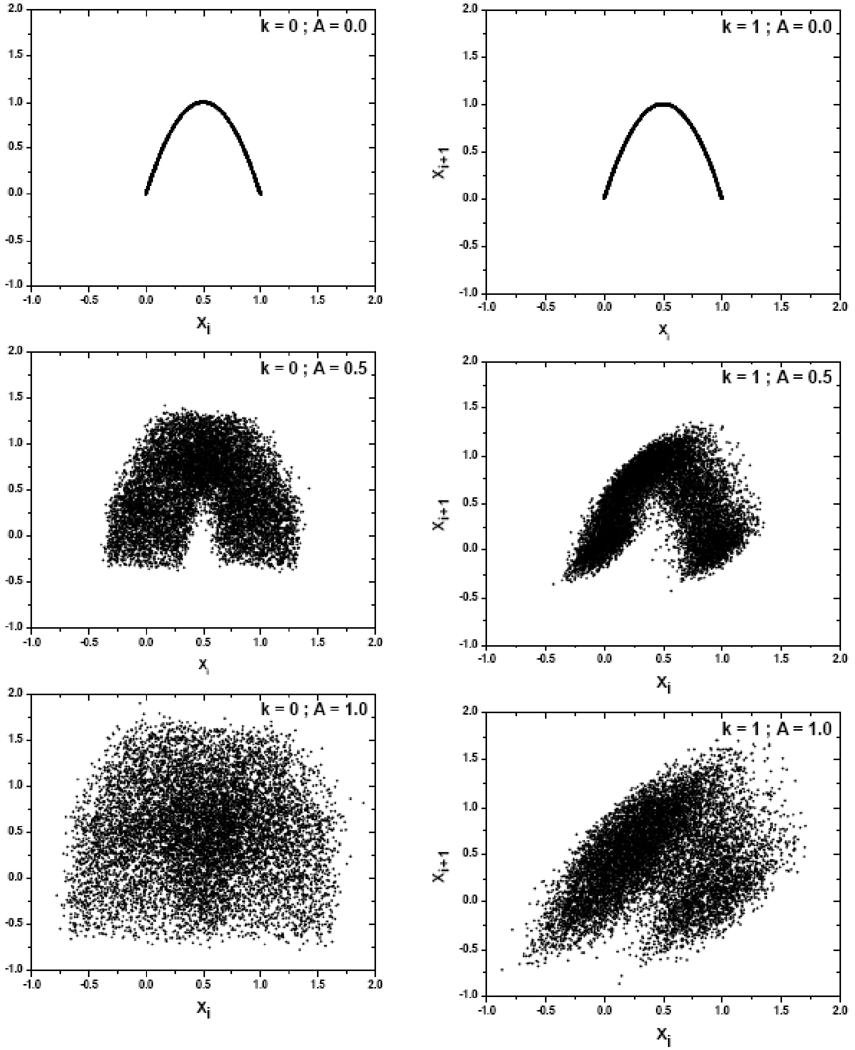
$$R = 4$$

$\eta_n^{(k)}$: Colored Noise with

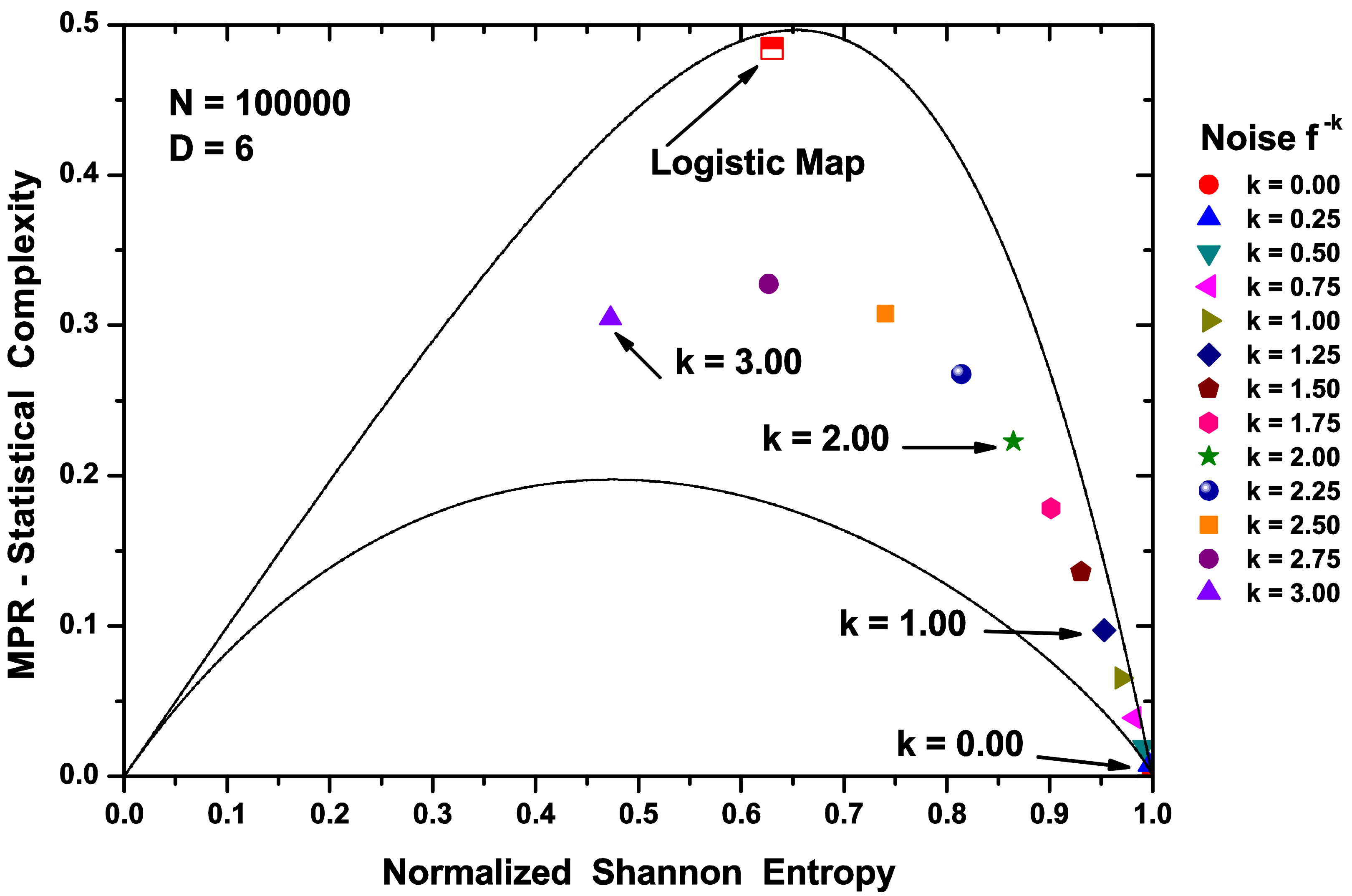
Power Spectrum f^{-k}

$$0 \leq k \leq 2$$

$A \geq 0$ Noise Amplitude

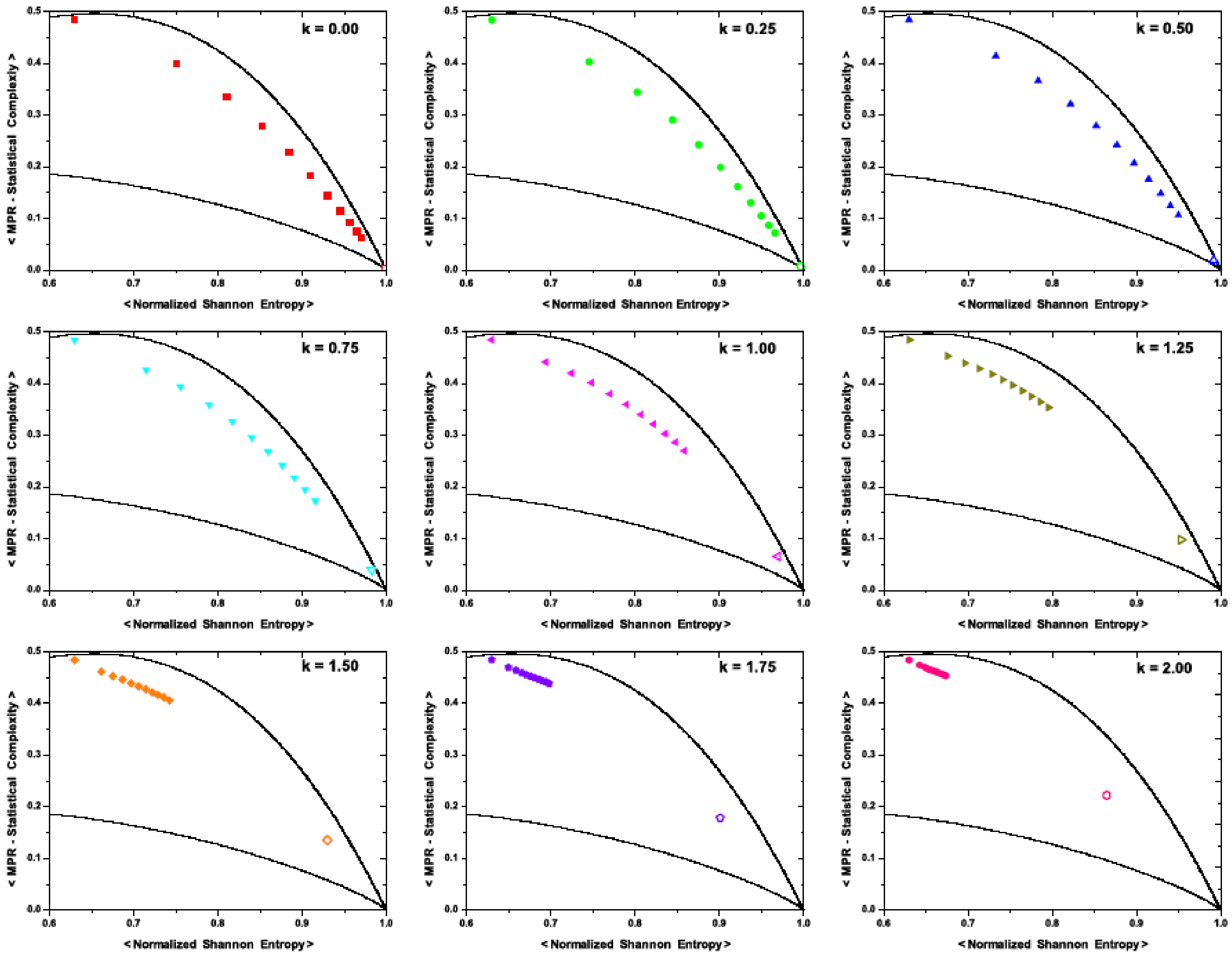


Chaos + Noise

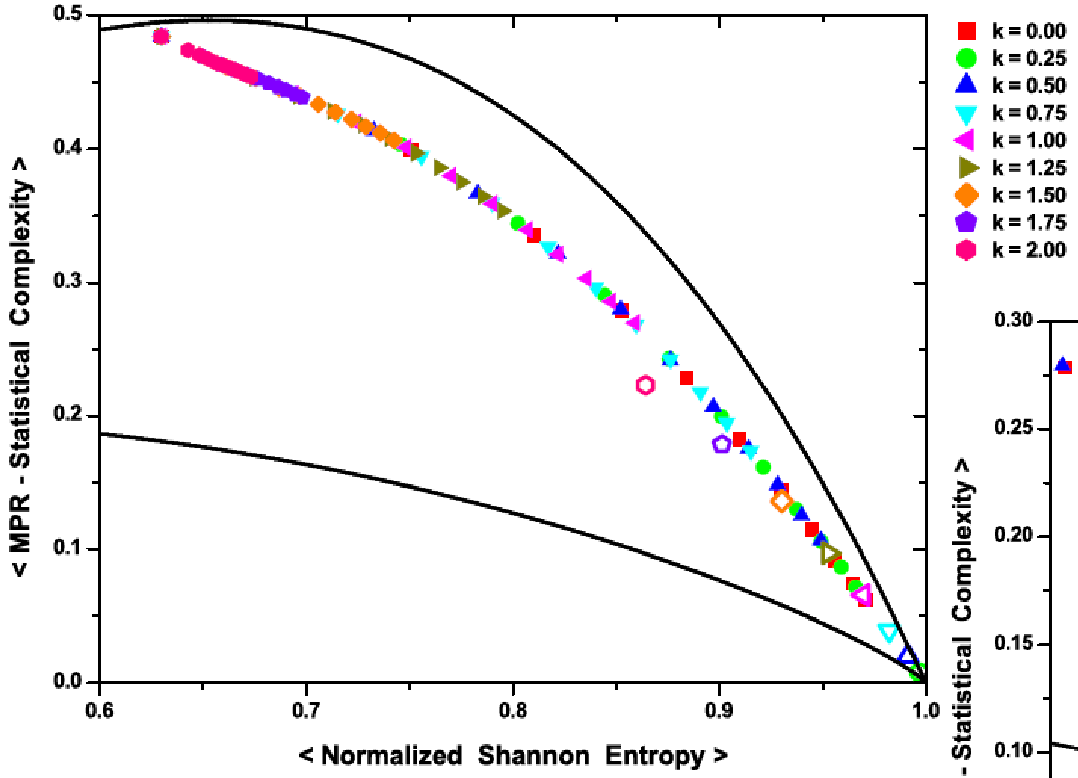


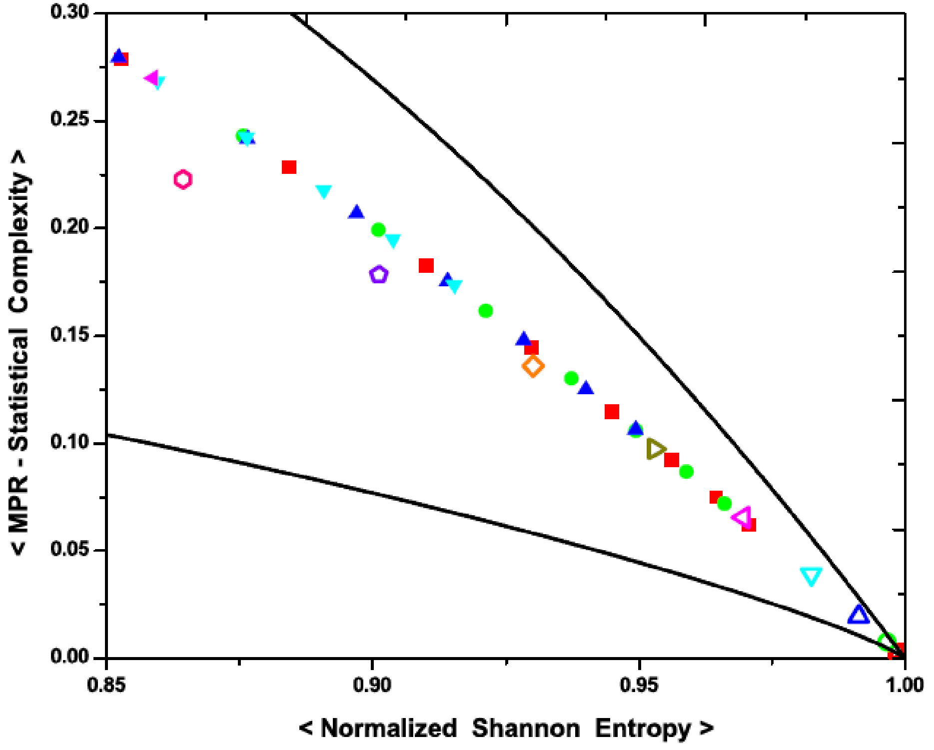
N = 100000
D = 6
Logistic Map
k = 3.00
k = 2.00
k = 1.00
k = 0.00
MPR - Statistical Complexity
Normalized Shannon Entropy
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Noise f -k
k = 0.00
k = 0.25
k = 0.50
k = 0.75
k = 1.00
k = 1.25
k = 1.50
k = 1.75
k = 2.00
k = 2.25
k = 2.50
k = 2.75
k = 3.00

Chaos + Noise


k = 0.00
k = 0.25
k = 0.50
k = 0.75
k = 1.00
k = 1.25
k = 1.50
k = 1.75
k = 2.00
< MPR - Statistical Complexity >
<Normalized Shannon Entropy>

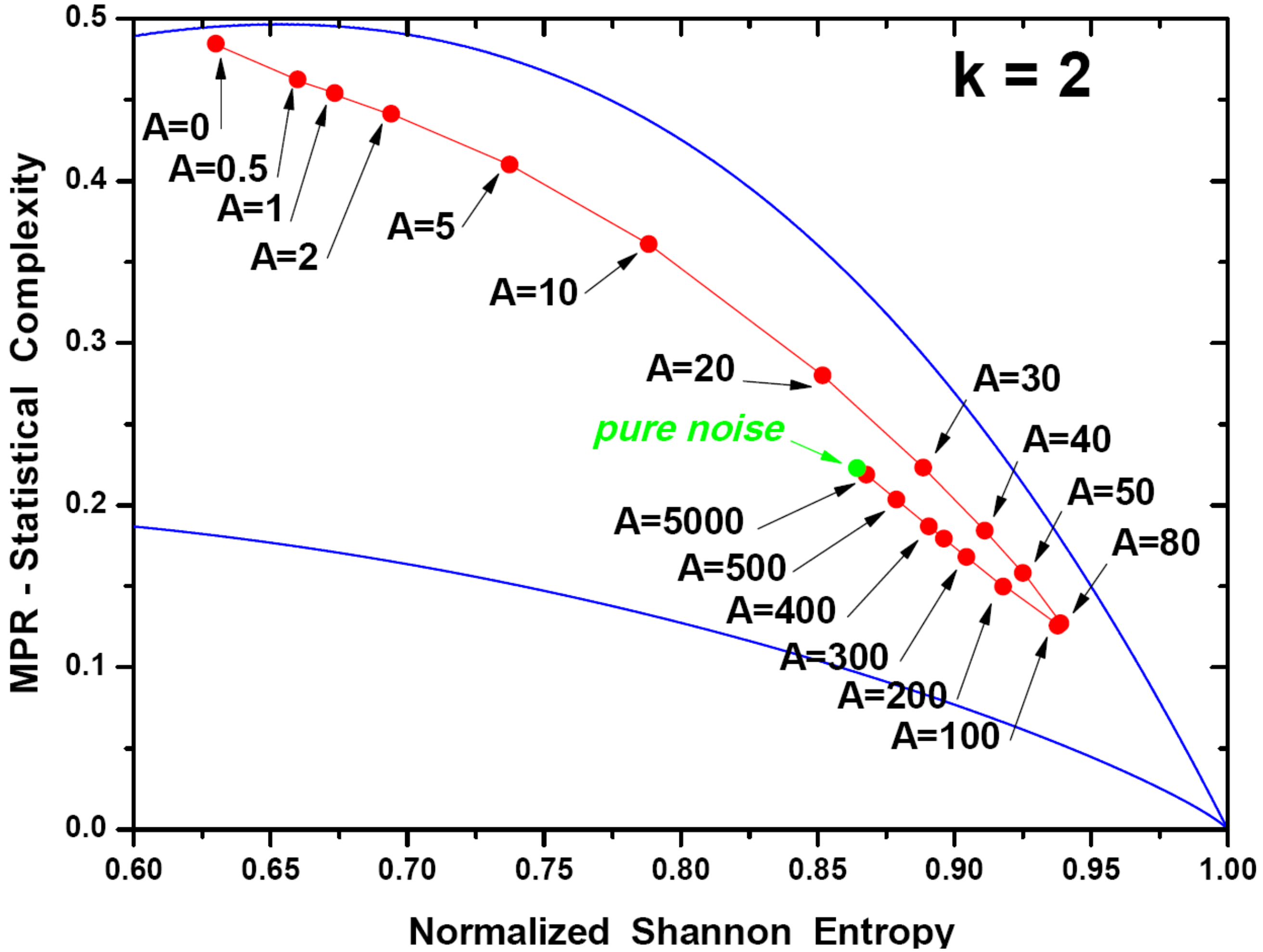
Chaos + Noise





O. A. Rosso, L. C. Carpi, P. M. Saco, M. Gómez Ravetti, A. Plastino, H. A. Larrondo
Causality and the Entropy-Complexity Plane: Robustness and Missing Ordinal Patterns ,
Physica A 391 (2012) 42 – 55

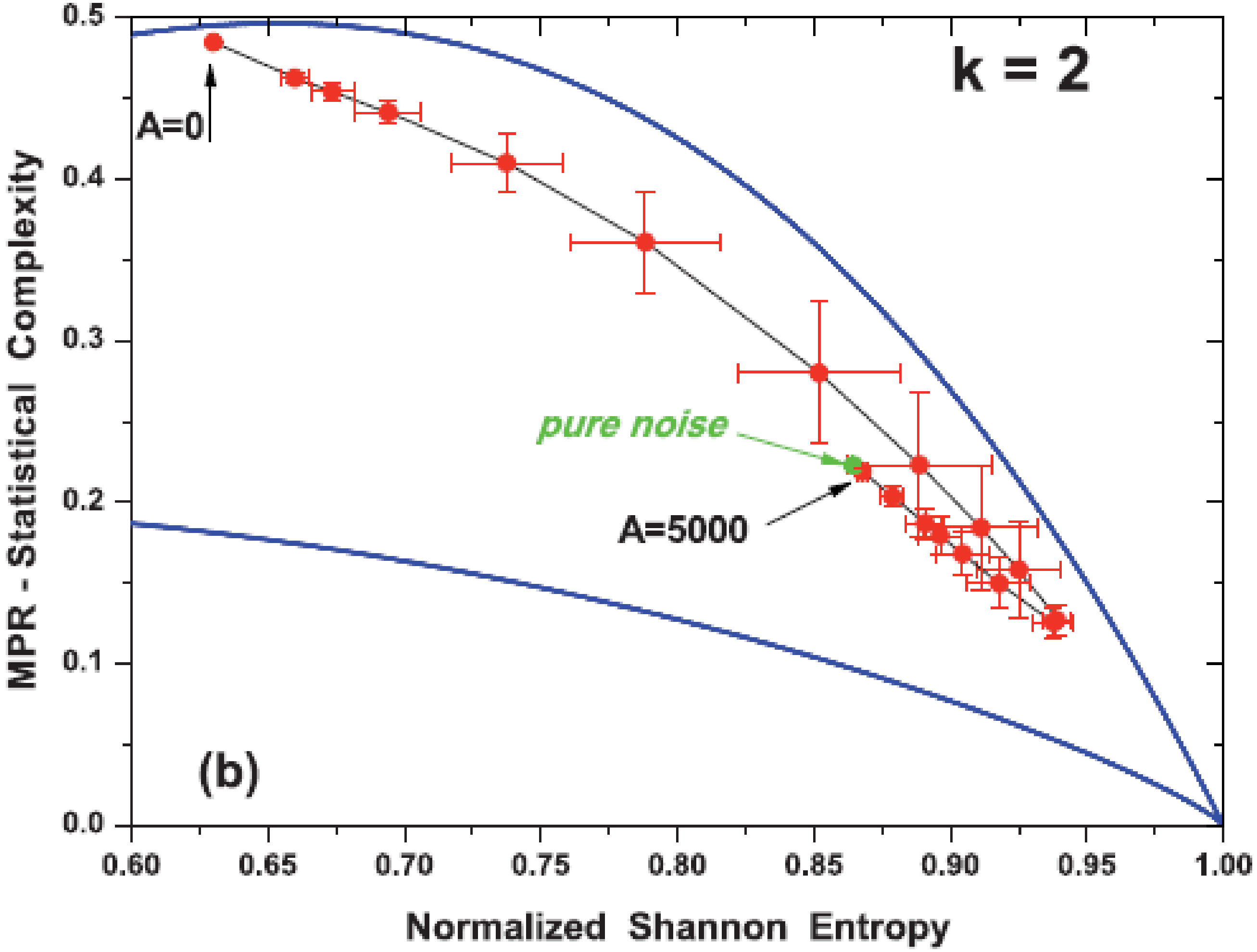
Chaos + Noise



N = 10**5 ;
D = 6

O. A. Rosso, L. C. Carpi, P. M. Saco, M. Gómez Ravetti, H. A. Larrondo, A. Plastino
The Amigó paradigm of forbidden/missing patterns: a detailed analysis, The European Physics Journal B 85 (2012) 419 – 430

Chaos + Noise



N = 10**5 ;
D = 6

O. A. Rosso, L. C. Carpi, P. M. Saco, M. Gómez Ravetti, H. A. Larrondo, A. Plastino
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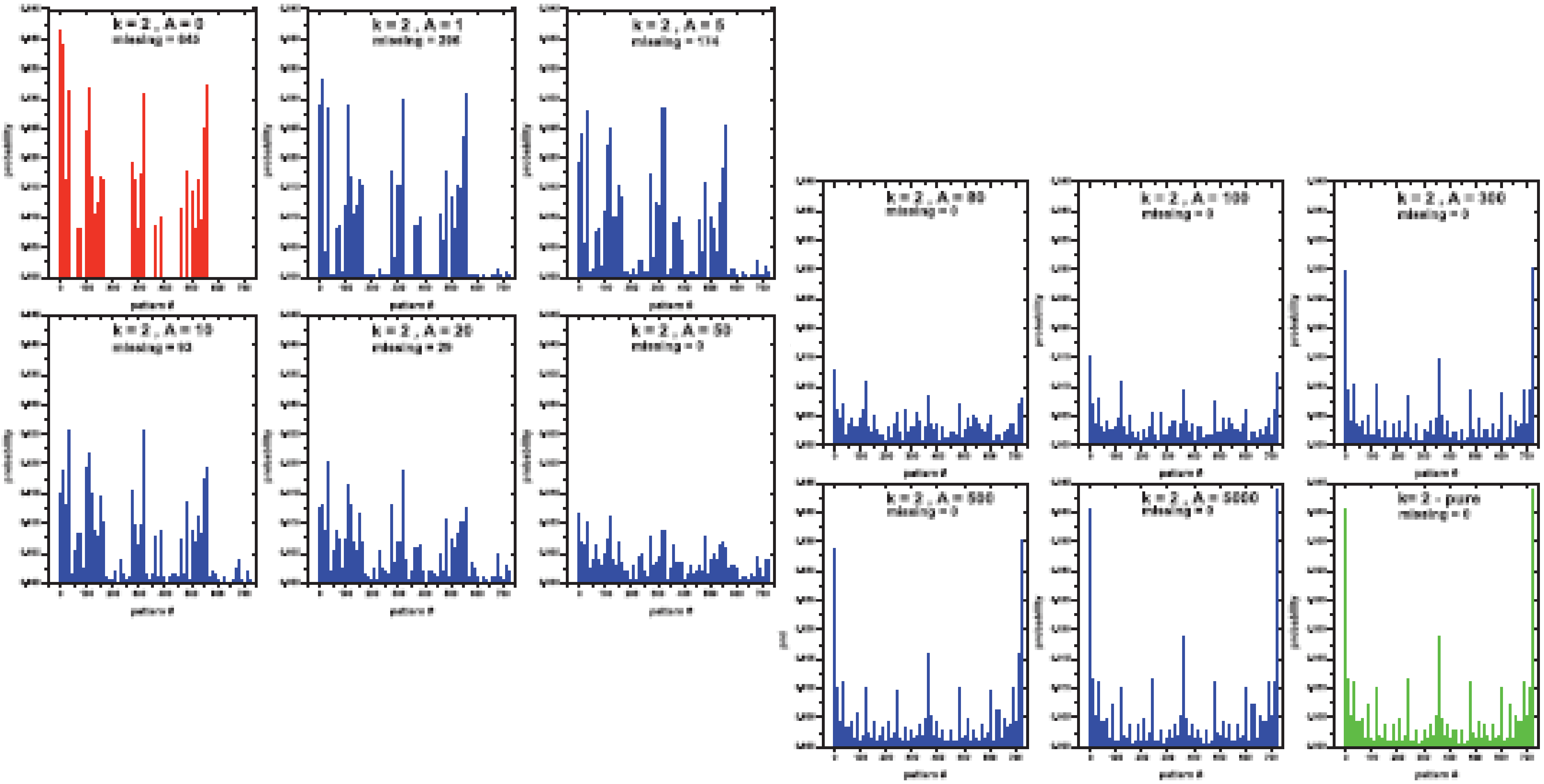
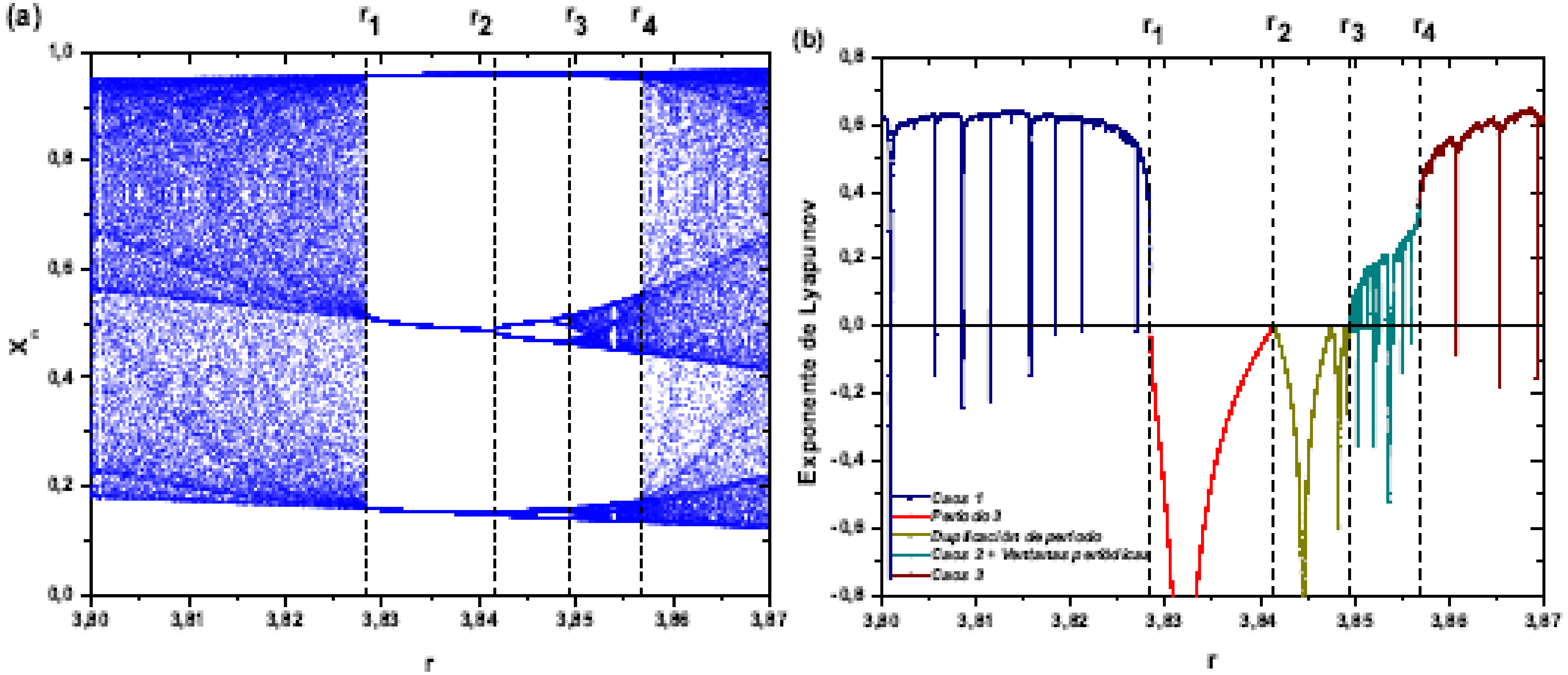
Chaos + Noise



O. A. Rosso, L. C. Carpi, P. M. Saco, M. Gómez Ravetti, H. A. Larrondo, A. Plastino
The Amigó paradigm of forbidden/missing patterns: a detailed analysis, The European Physics Journal B 85 (2012) 419 – 430

Fisher Information Measure

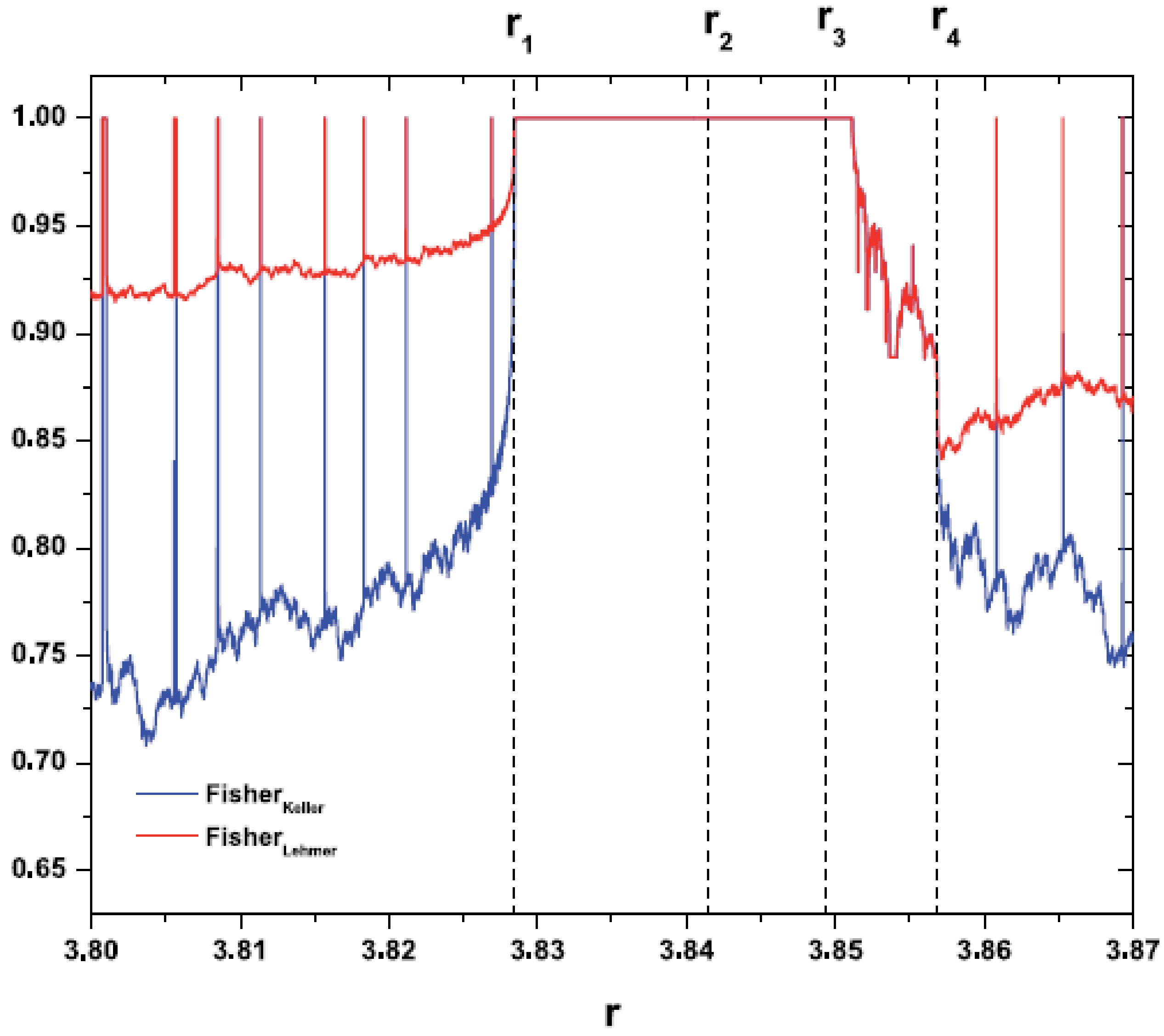


Logistic Map: $X_{n+1} = R\, X_n(\,1\; -\; X_n)$

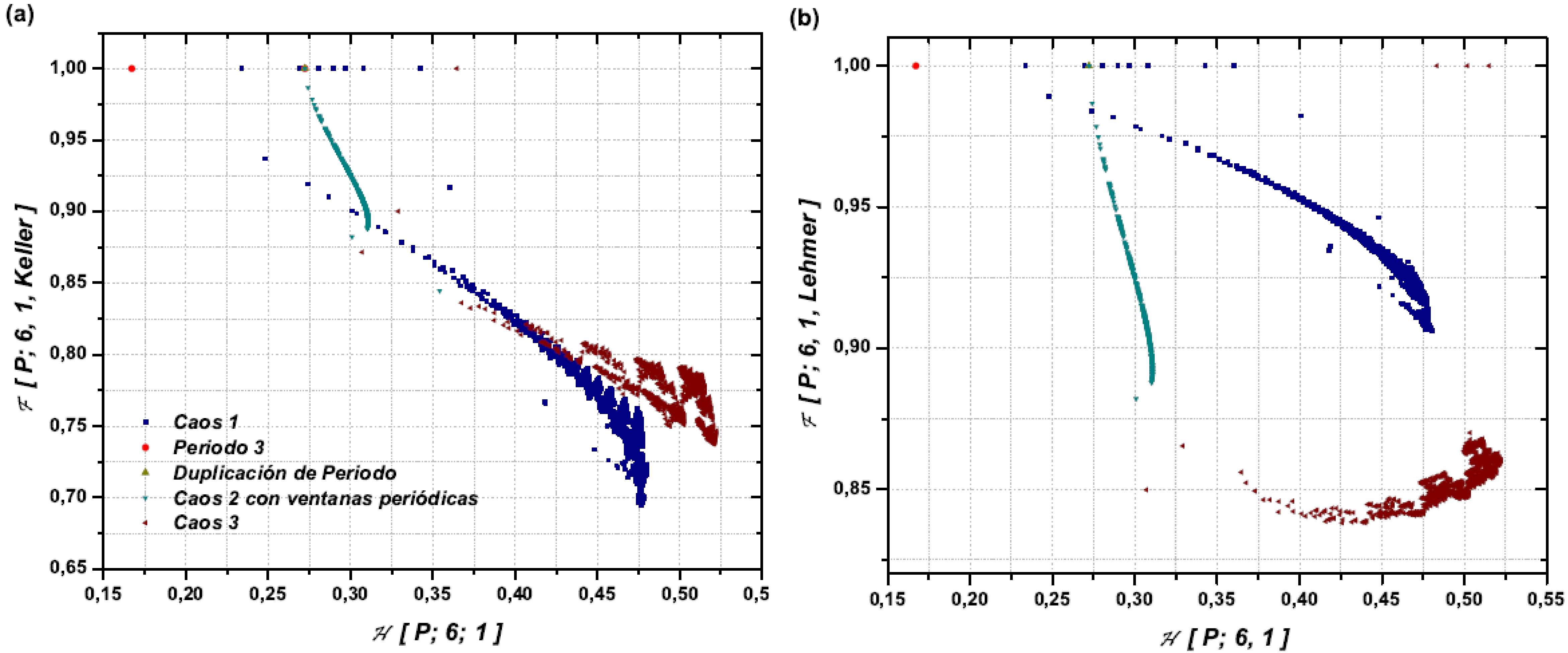
Fisher Information Measure

D = 2			D = 3			D = 4		
{i}	*Keller*	*Lehmer*	{i}	*Keller*	*Lehmer*	{i}	*Keller*	*Lehmer*
1	{01}	{01}	1	{012}	{012}	1	{0123}	{0123}
						2	{0132}	{0132}
						3	{0312}	{0213}
						4	{3012}	{0231}
			2	{021}	{021}	5	{0213}	{0312}
						6	{0231}	{0321}
						7	{0321}	{1023}
						8	{3021}	{1032}
			3	{201}	{102}	9	{2013}	{1203}
						10	{2031}	{1230}
						11	{2301}	{1302}
						12	{3201}	{1320}
2	{10}	{10}	4	{102}	{120}	13	{1023}	{2013}
						14	{1032}	{2031}
						15	{1302}	{2103}
						16	{3102}	{2130}
			5	{120}	{201}	17	{1203}	{2301}
						18	{1230}	{2310}
						19	{1320}	{3012}
						20	{3120}	{3021}
			6	{210}	{210}	21	{2103}	{3102}
						22	{2130}	{3120}
						23	{2310}	{3201}
						24	{3210}	{3210}

Tabla 3.1: Secuencia de patrones ordinales generados por los algoritmos de Keller [76] y Lehmer [77], para longitudes de patrones $D = 2, 3$ y 4.



Fisher Information Measure

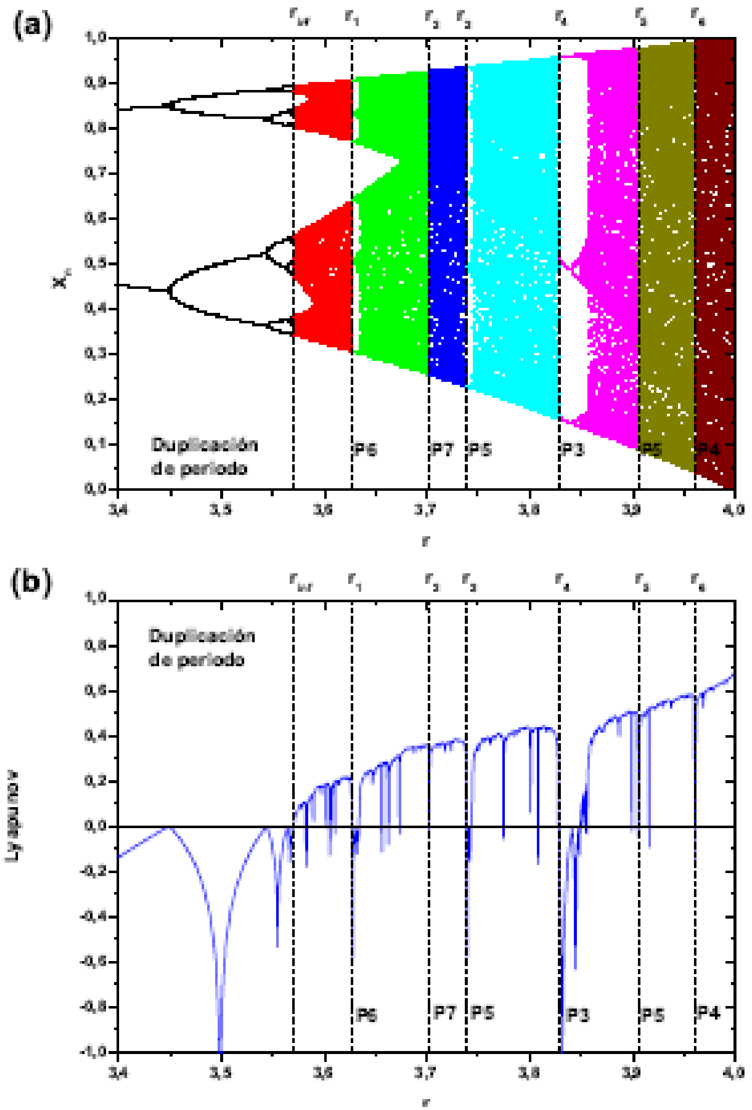


F. Olivares, A. Plastino, O. A. Rosso
Contrasting chaos with noise via local versus global information quantifiers
Physics Letters A 376 (2012) 1577–1583

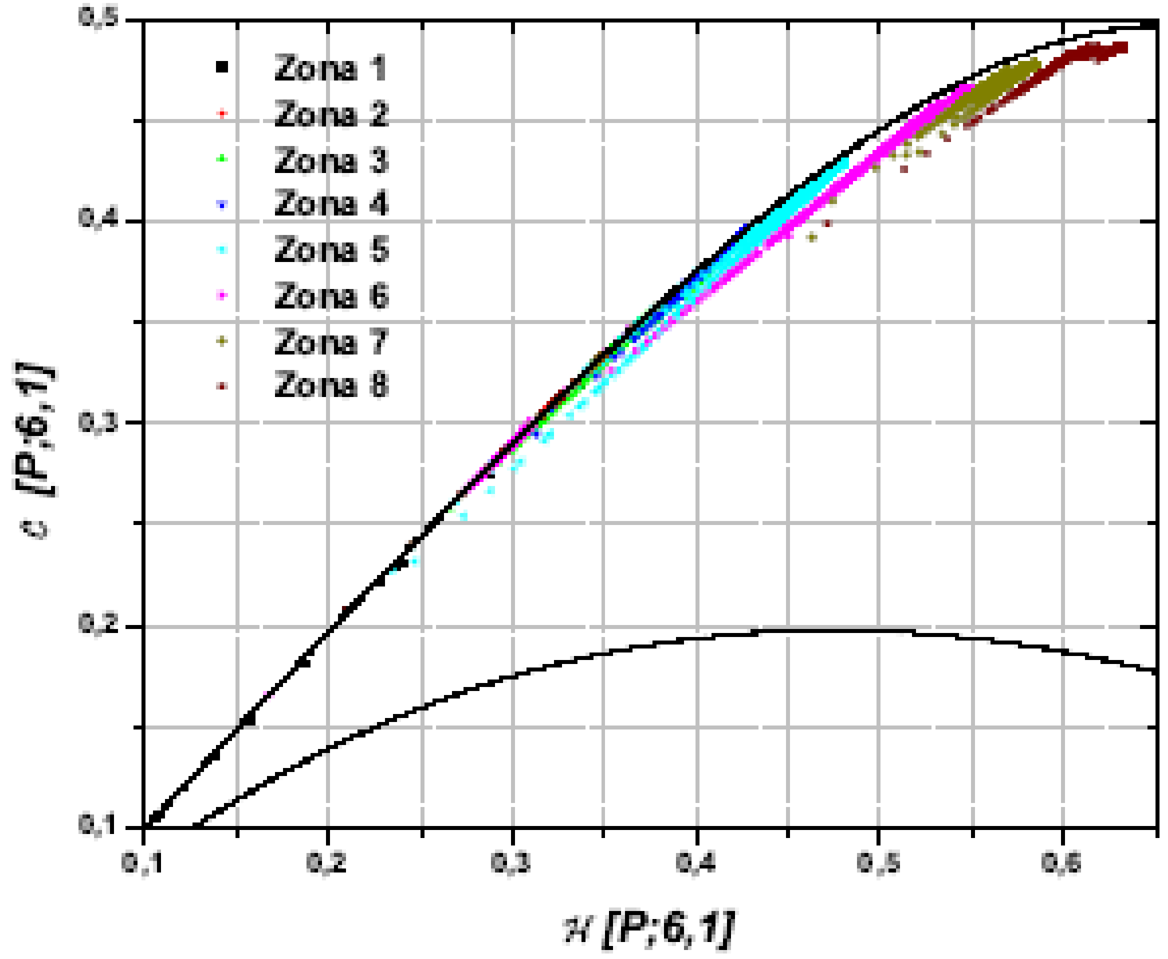
Logistic Map: $X_{n+1} = R\, X_n(\,1\, -\, X_n)$

Fisher Information Measure

Logistic Map: $X_{n+1} = R\, X_n (1 - X_n)$



C & Fisher Information Measure



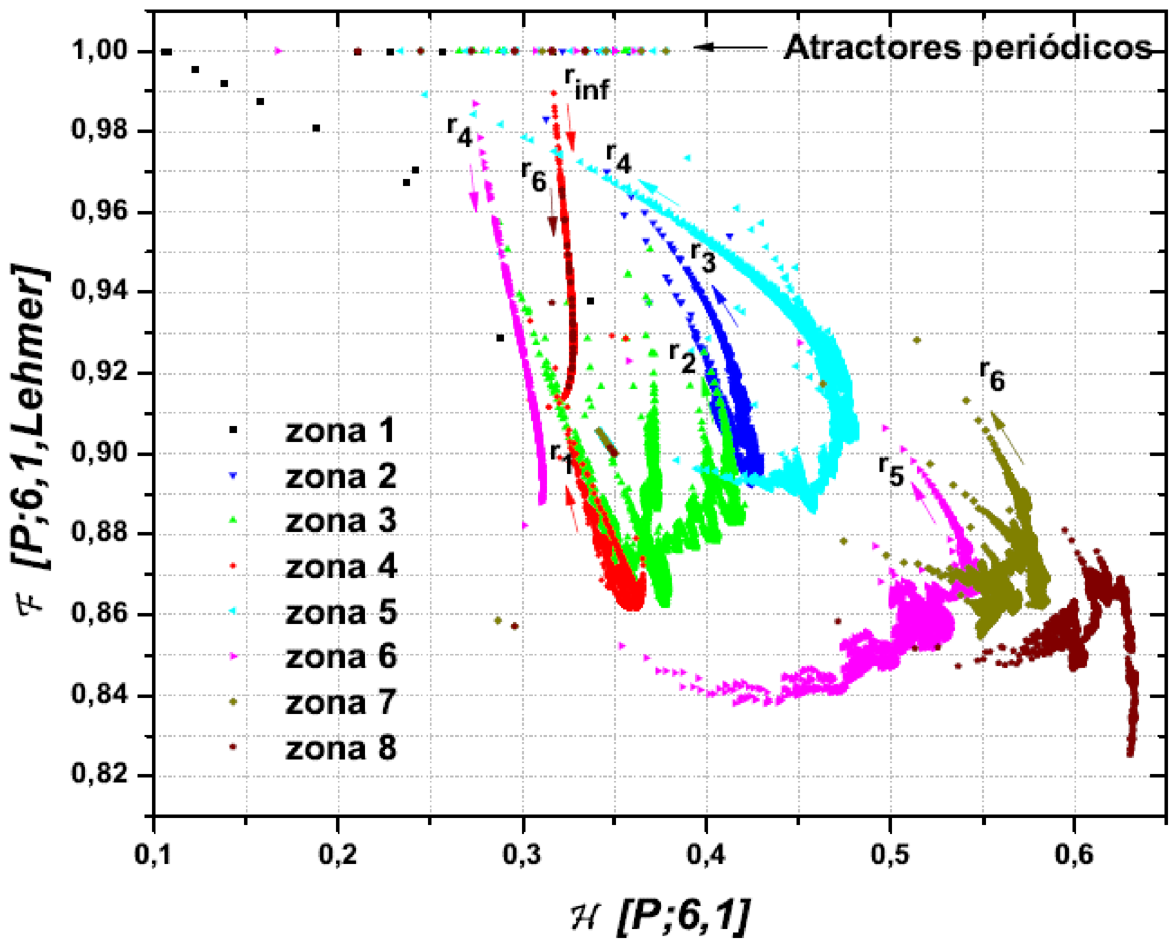
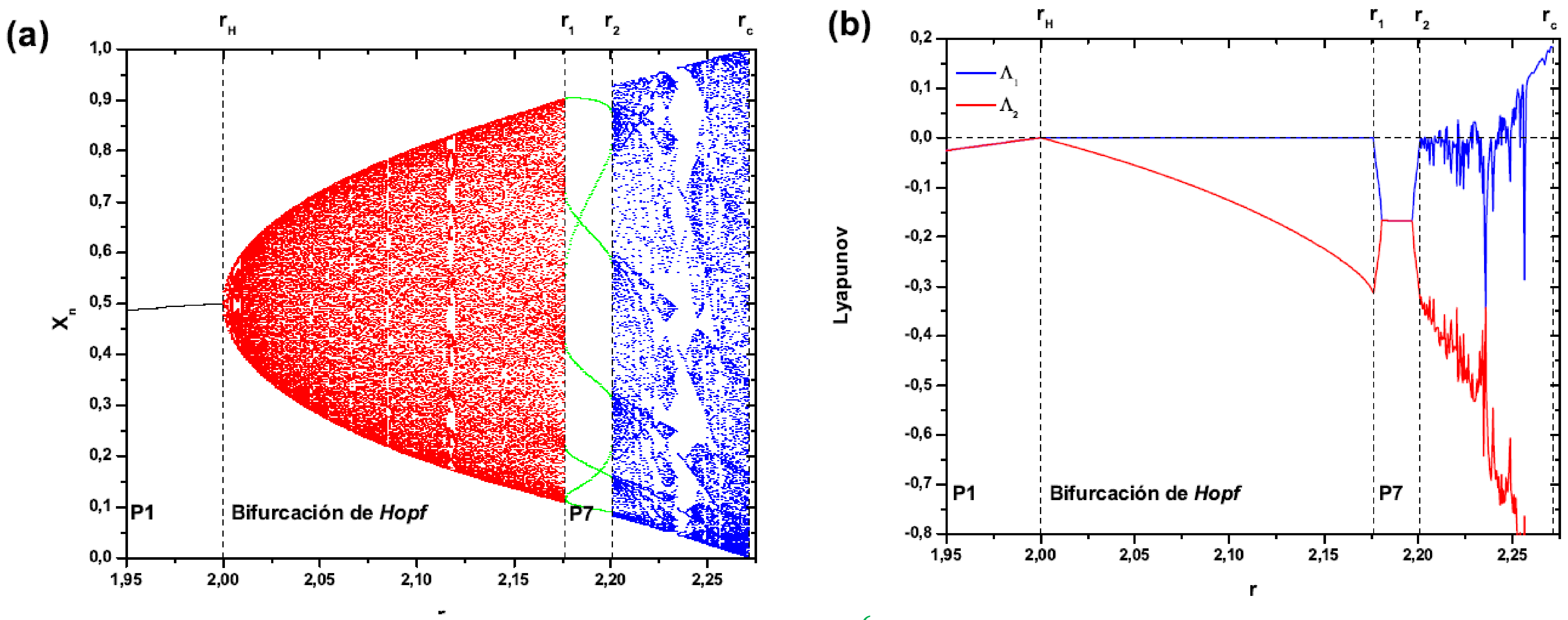


Figura 6.5: Plano causal $\mathcal{H} \times \mathcal{F}$ para el mapa logístico en el rango $3{,}4 < r < 4{,}0$ ($\Delta r = 0{,}00001$). Para la evaluación de la PDF de BP se usó $D = 6$, $\tau = 1$ y series temporales con $M = 10^7$ datos.

Logistic Map: $X_{n+1} = R\, X_n(\,1 - X_n)$

Fisher Information Measure



Delay Logistic Map: $\begin{cases} X_{n+1} = R\,X_n(1 - Y_n) \\ Y_{n+1} = X_n \end{cases}$

Fisher Information Measure

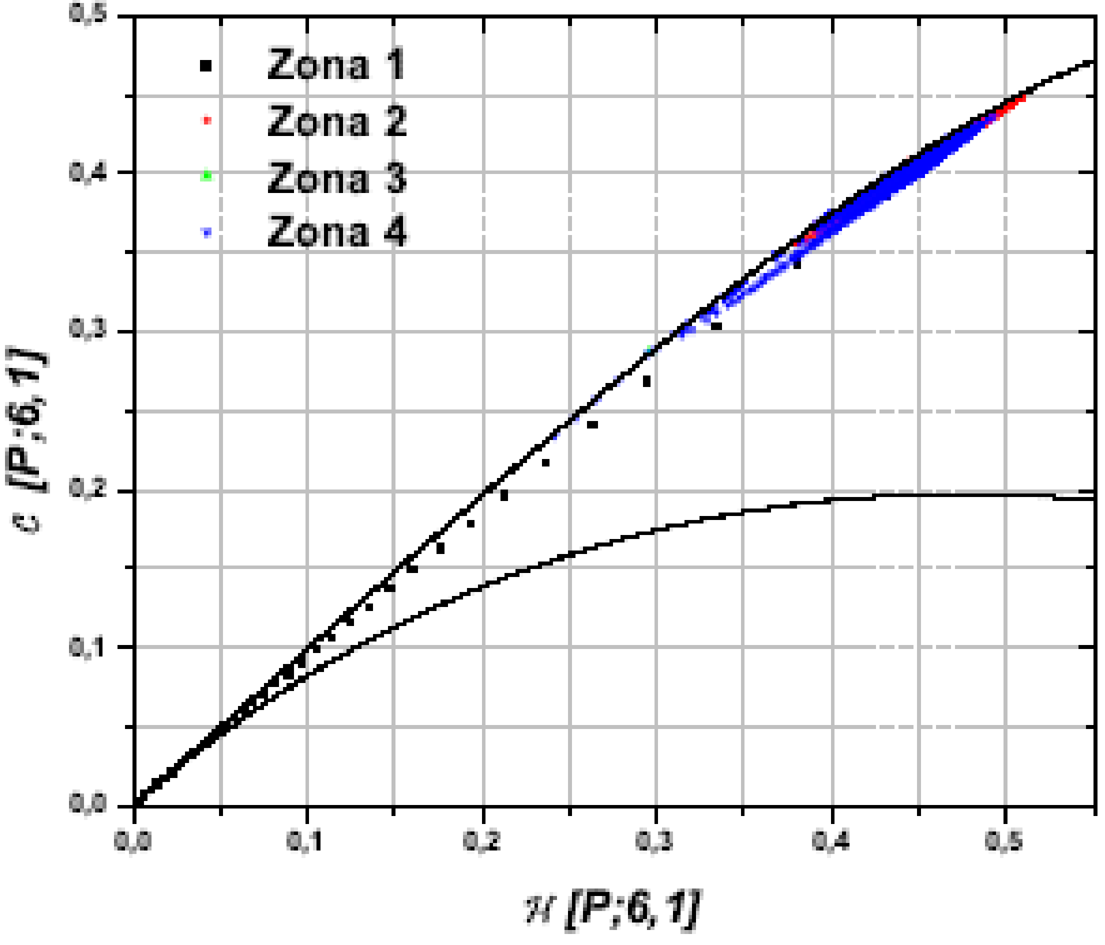


Figura 6.12: Plano causal $\mathcal{H} \times \mathcal{C}$ para el mapa logístico con retraso en el rango $1{,}95 < r < 2{,}271$ ($\Delta r = 0{,}00001$). Las líneas continuas identifican la curva de máxima y mínima complejidad). Para la evaluación de la PDF de BP se usó $D = 6$ y $\tau = 1$.

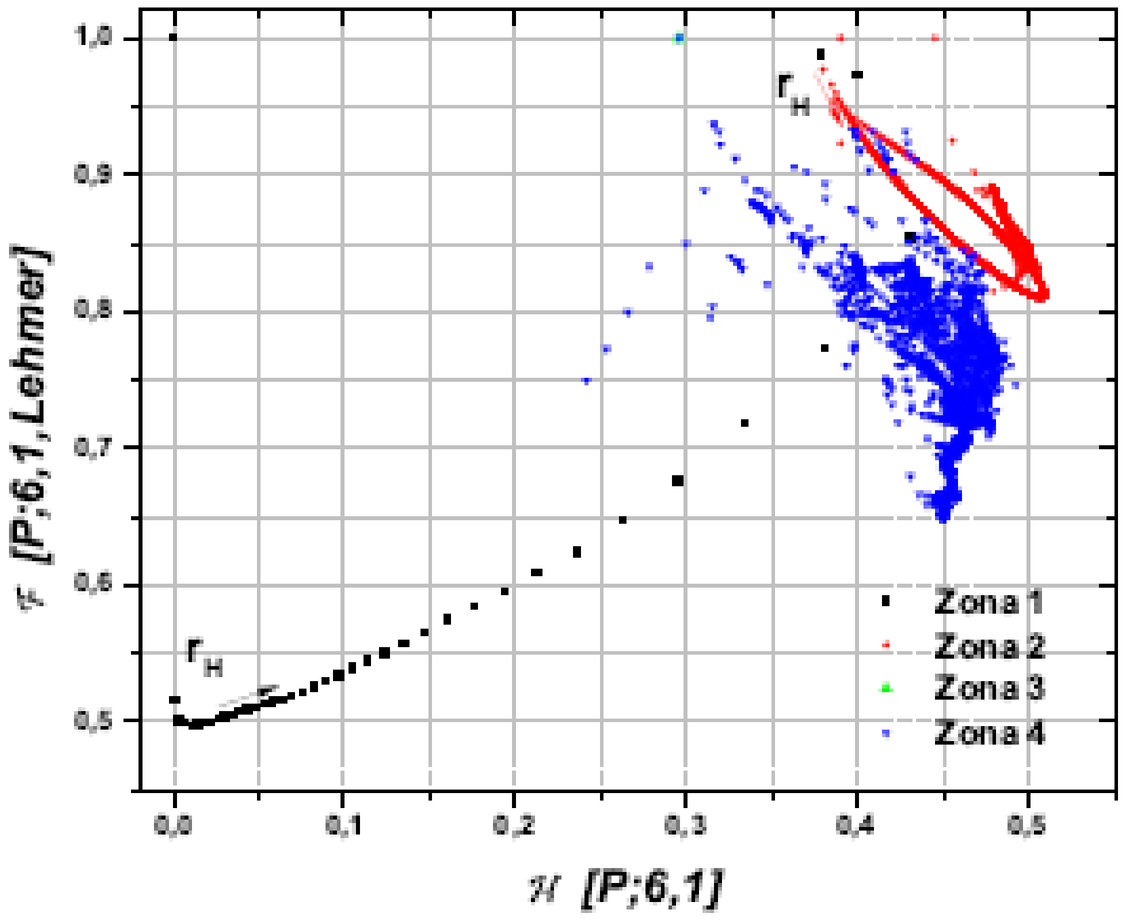
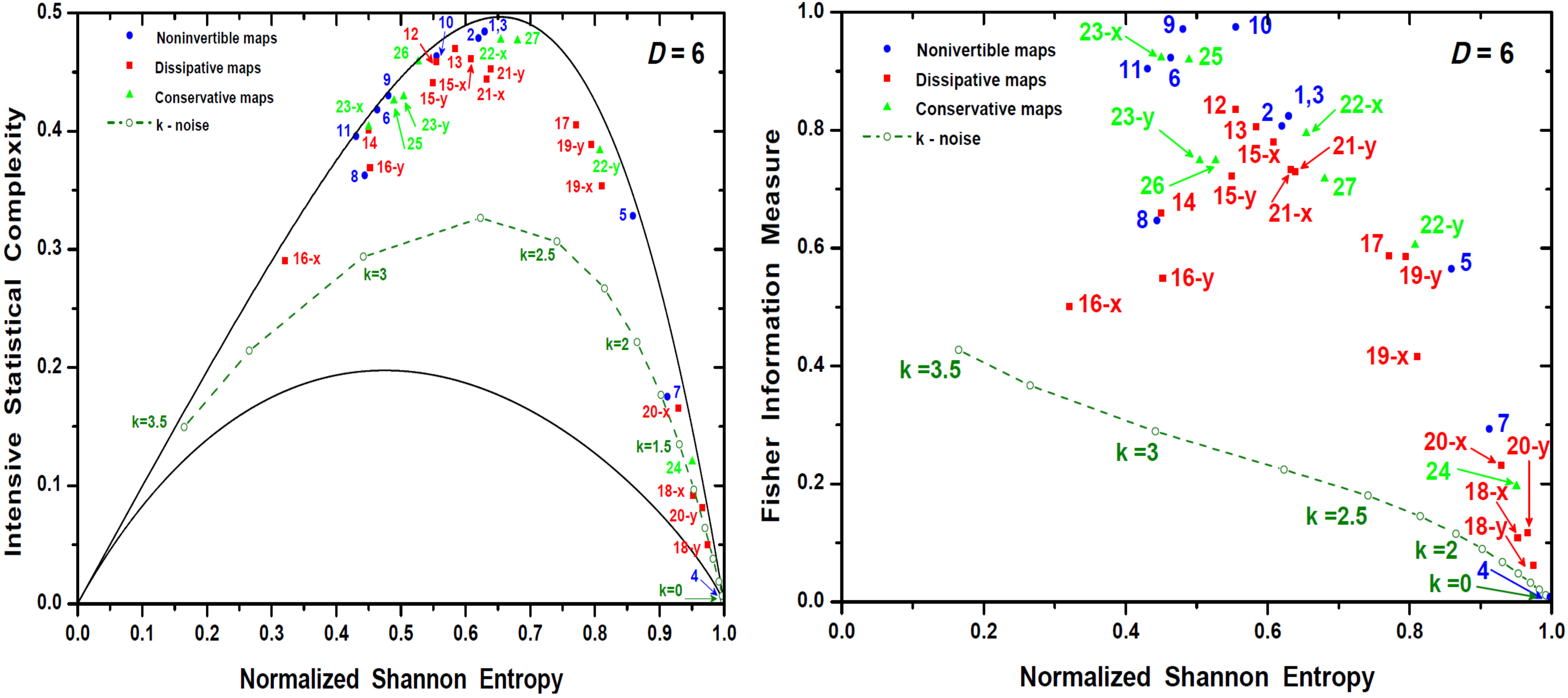
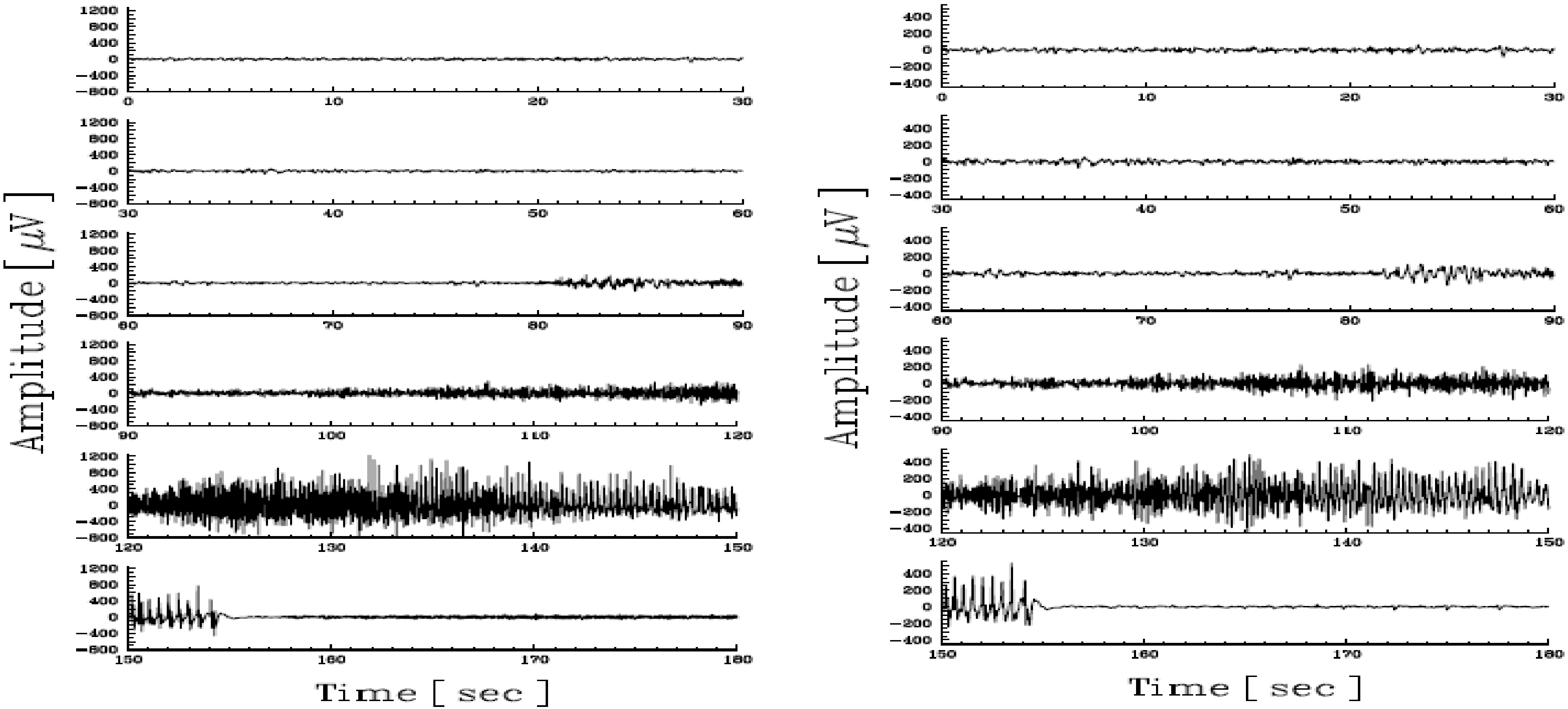


Figura 6.13: Plano causal $\mathcal{H} \times \mathcal{F}$ para el mapa logístico con retraso en el rango $1{,}95 < r < 2{,}271$ ($\Delta r = 0{,}00001$). Para la evaluación de la PDF de BP se usó $D = 6$ y $\tau = 1$.

Fisher Information Measure

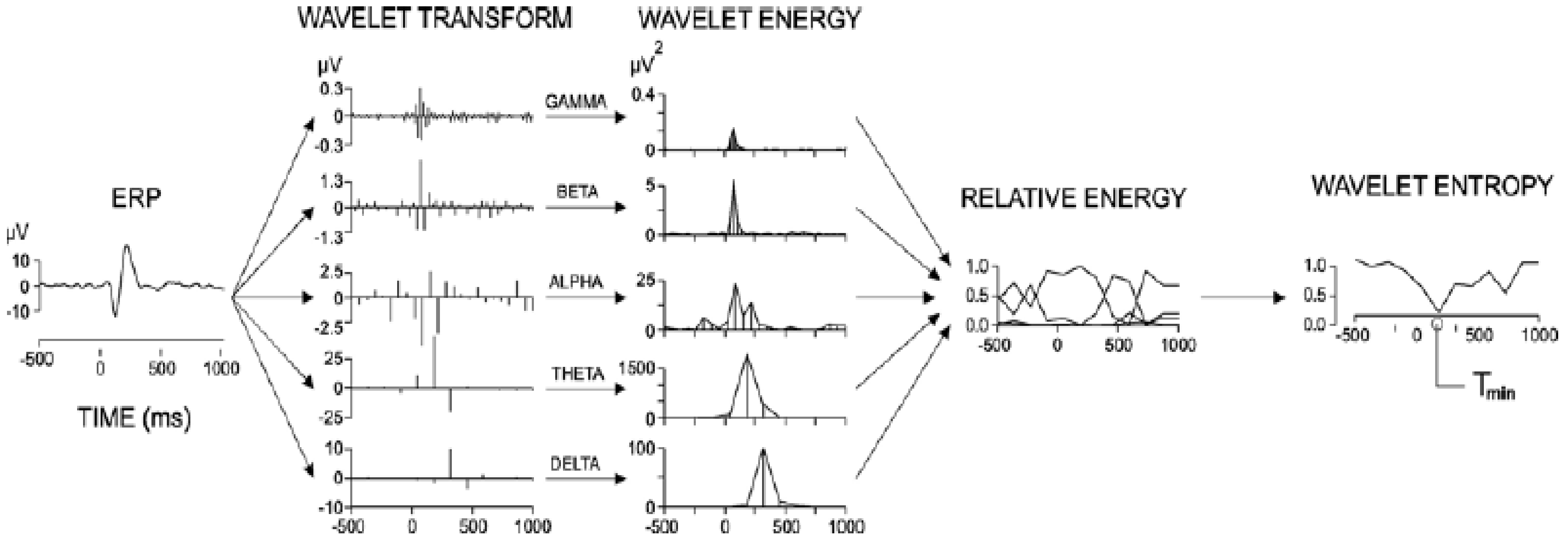


Application to EEG Tonic-Clonic Epileptic records



Scalp EEG signal for an epileptic tonic–clonic seizure, recorded at the central right location C4. Original (left) and without muscular contributions.

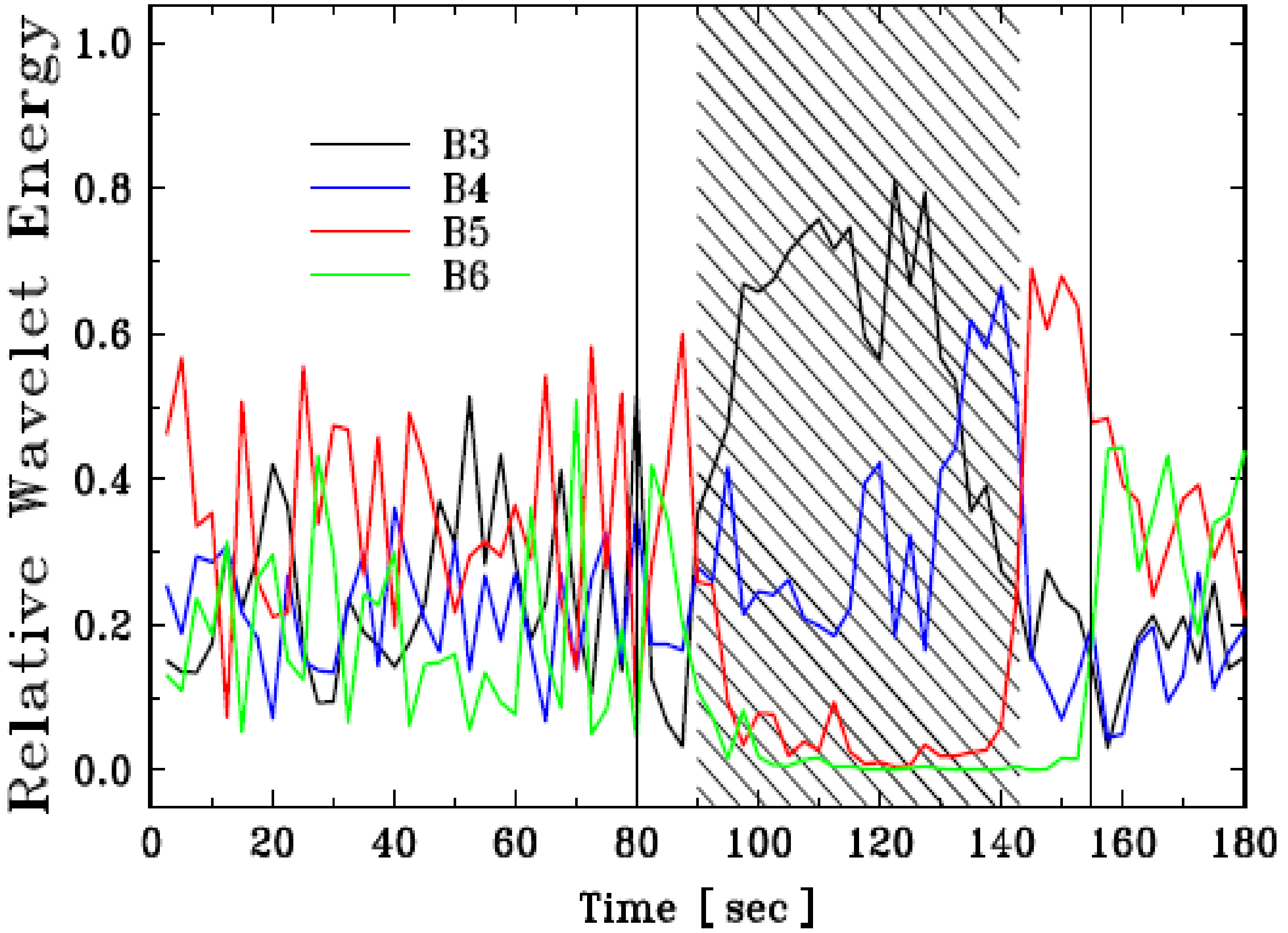
Application to EEG Tonic-Clonic Epileptic records



Schematic illustration of the method: the EEG is transformed to the time–frequency domain by the ODWT. Wavelet coefficients for each resolution level are obtained and used for calculation of the wavelet energy, from which the Relative Wavelet Energy is computed (RWE). RWE are further used for the calculation of Normalized Total Wavelet Entropy and Wavelet Statistical Complexity.

Application to EEG Tonic-Clonic Epileptic records

Time-evolution of RWE energy corresponding to EEG noise-free signal (without contribution of frequency bands B1and B2, representing frequency contributions
> 12.8Hz corresponding mainly to muscular activity that blur the EEG signal. It is clear that the seizure is dominated by the middle frequency bands B3 and B4 (12.8-3.2 Hz), with a corresponding abrupt activity decreases in the low frequency bands B5 and B6 (3.2-0.8Hz). This behavior can be associated with the *epileptic recruiting rhythm* – 10 Hz (shadowed area).



O. A. Rosso, M. T. Martín, A. Figliola, K. Keller, and A. Plastino.
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Journal Neuroscience Methods 153 (2006) 163-182.

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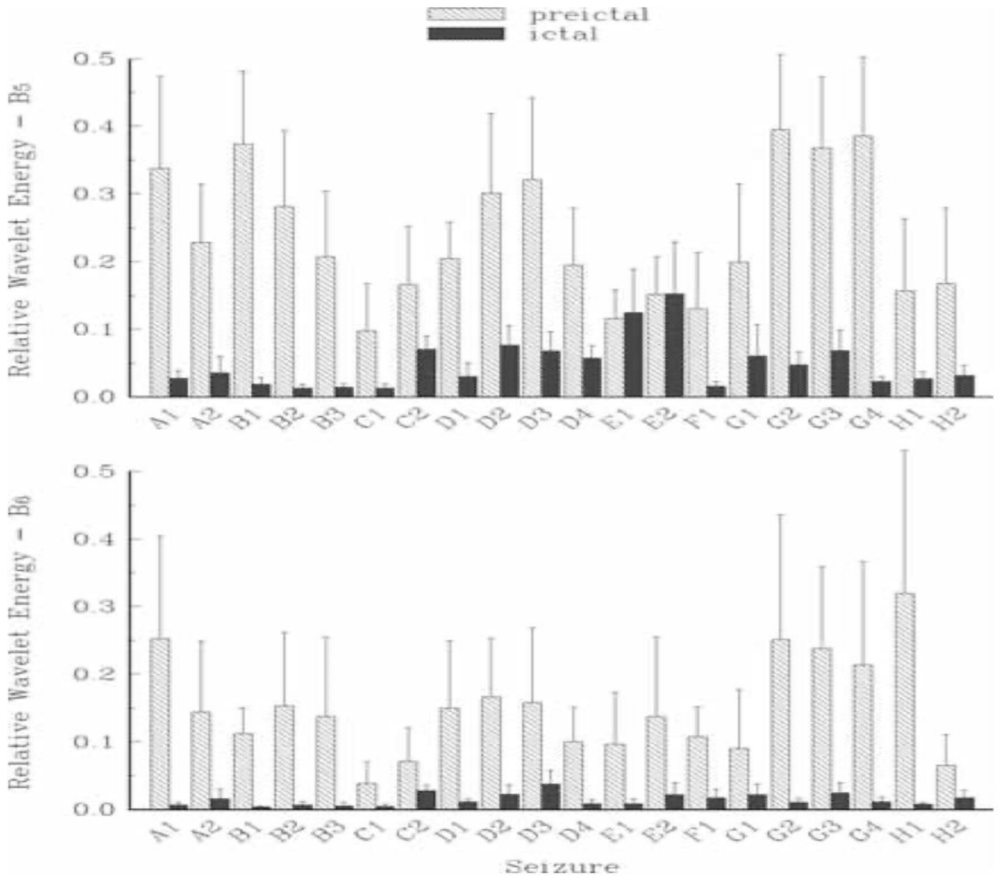

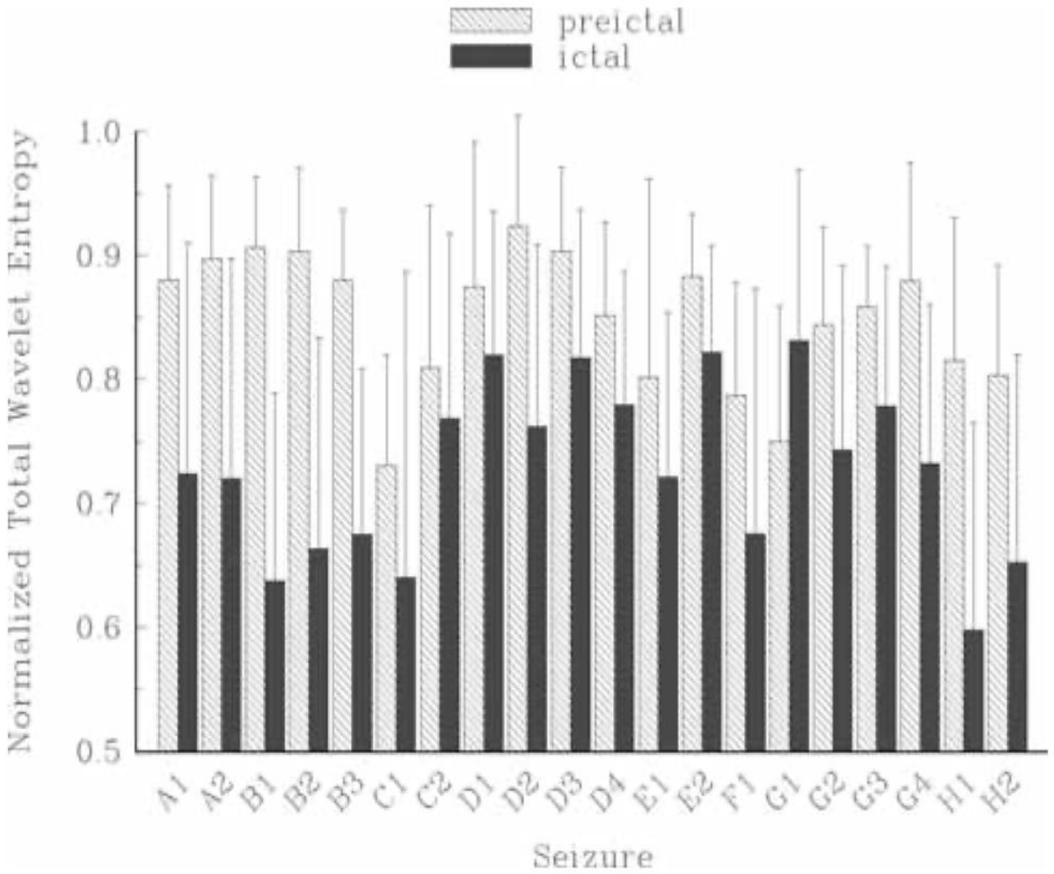

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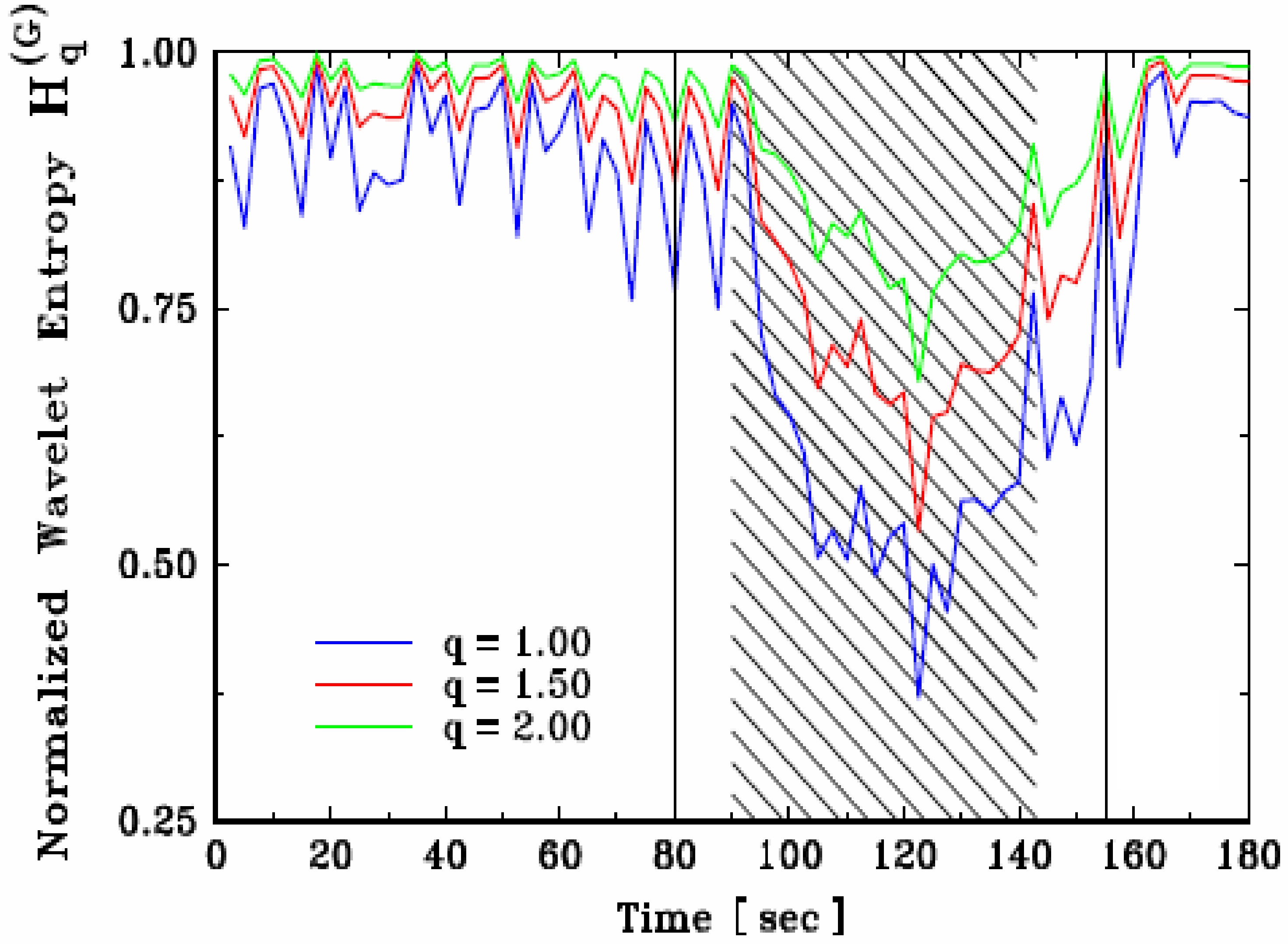

Temporal evolution of the normalized escort-Tsallis wavelet entropy (GWS) corresponding to an EEG noise-free signal.
The behavior of the GWS clearly varies with q in the temporal domain.
During the pre- and post-ictal stages, these normalized GWS-values acquire a rather regular, constant behavior, with a dispersion that diminishes as q grows.
For all q >1, the normalized GWS values during the ictal stage are much smaller than those pertaining to the pre-ictal stage.
This difference is better appreciate in the time range corresponding to the ``epileptic recruiting rhythm" (shadowed area in the figure).

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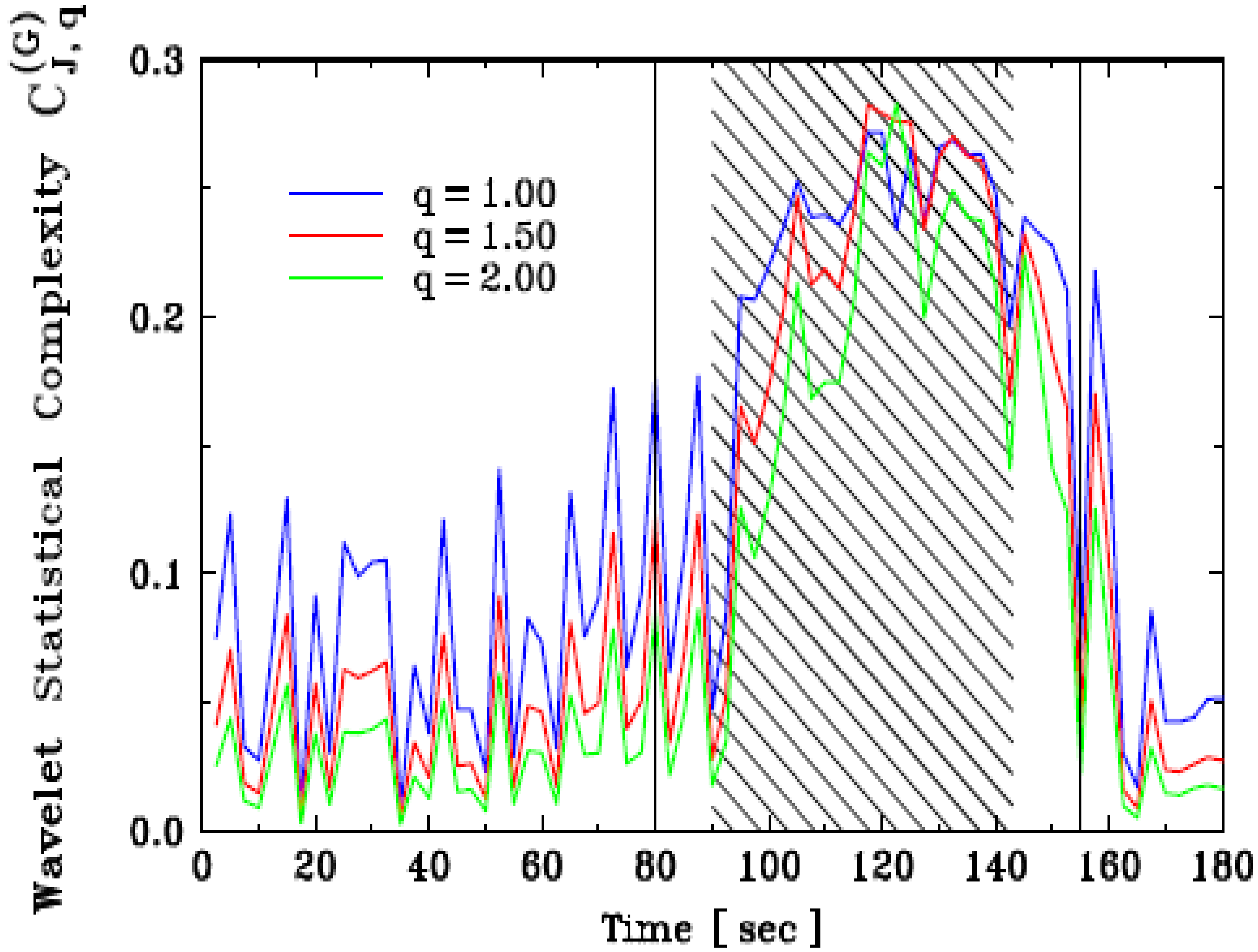

Temporal evolution of the Jensen escort-Tsallis wavelet statistical compplexity (JGWC) corresponding to an EEG noise-free signal.
The behavior of the JGWC clearly varies with q in the temporal domain. For all q >1, the JGWC values during the ictal stage are much grater than those pertaining to the pre-ictal stage. This difference is better appreciate in the time range corresponding to the ``epileptic recruiting rhythm" (shadowed area in the figure).

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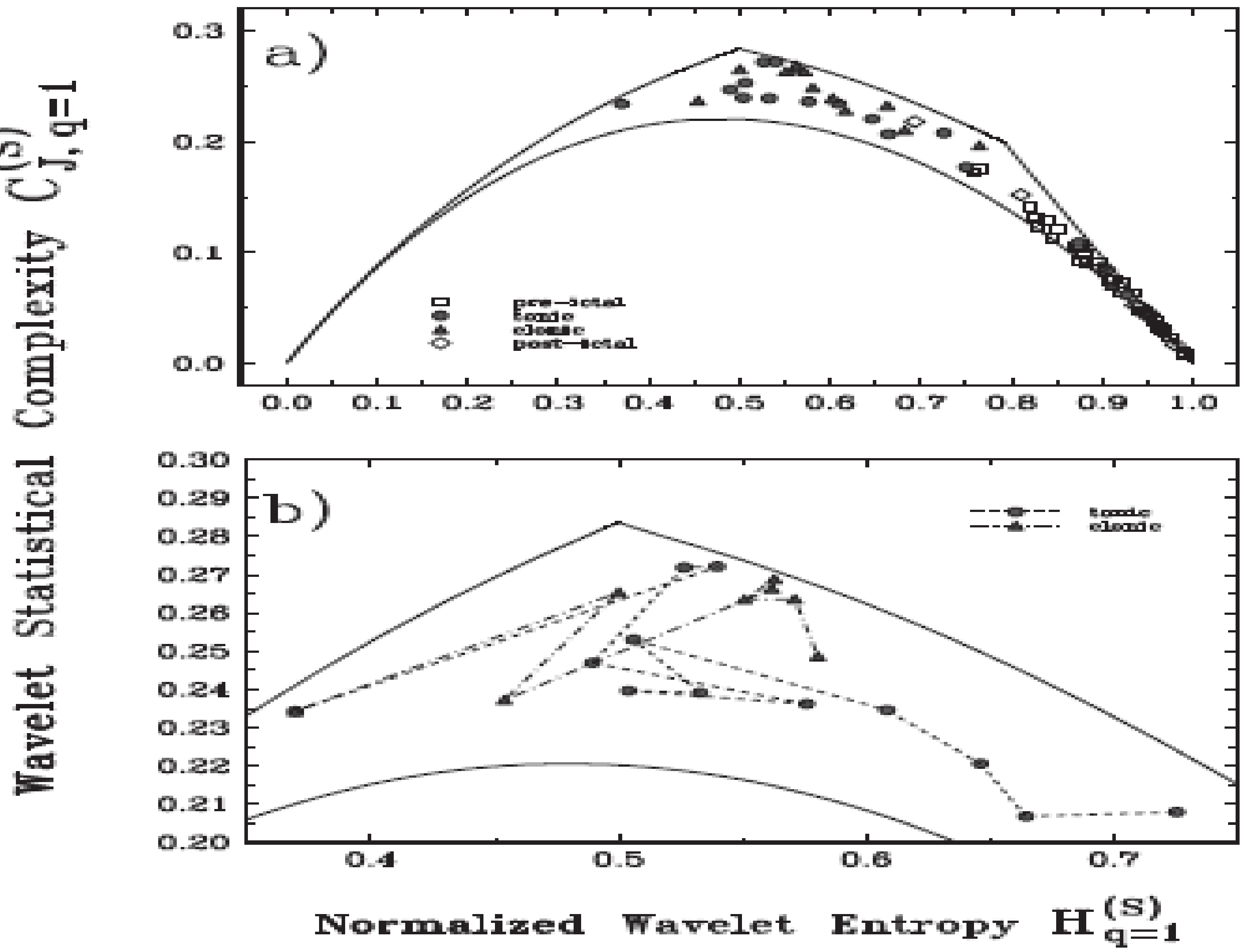

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Information Theory & Time Series

- *Thank a lot !!!*
- *Questions ?*

- *Gracias !!!*
- *Preguntas ?*