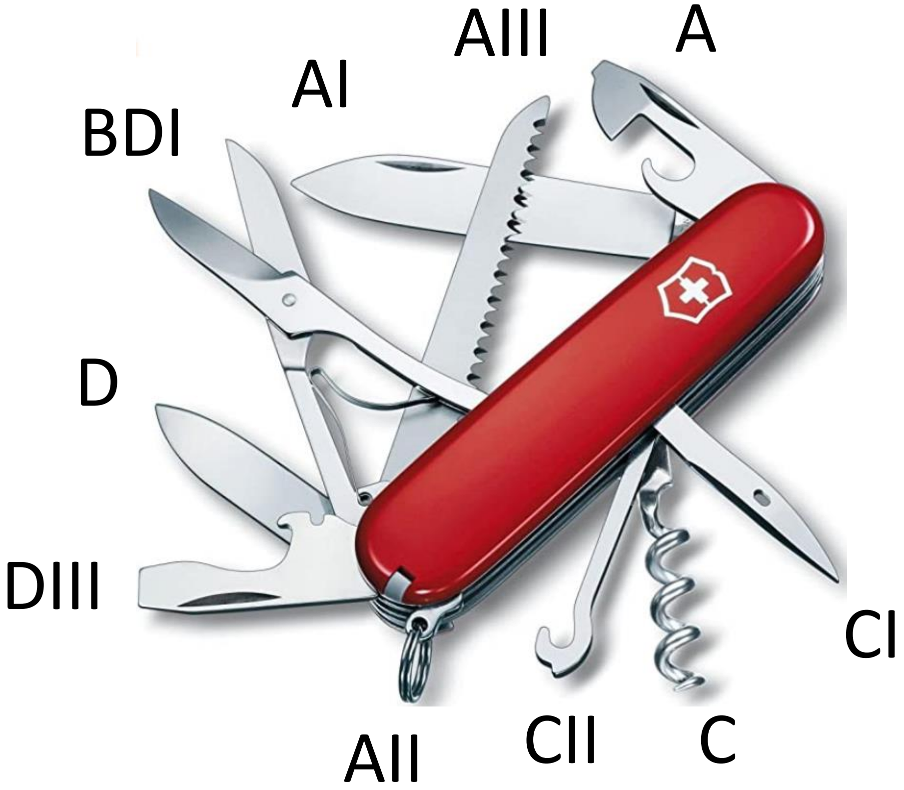
Unification of topological invariants and topological markers

Wei Chen

wchen@puc-rio.br

S. Panahiyan
U. Hamburg

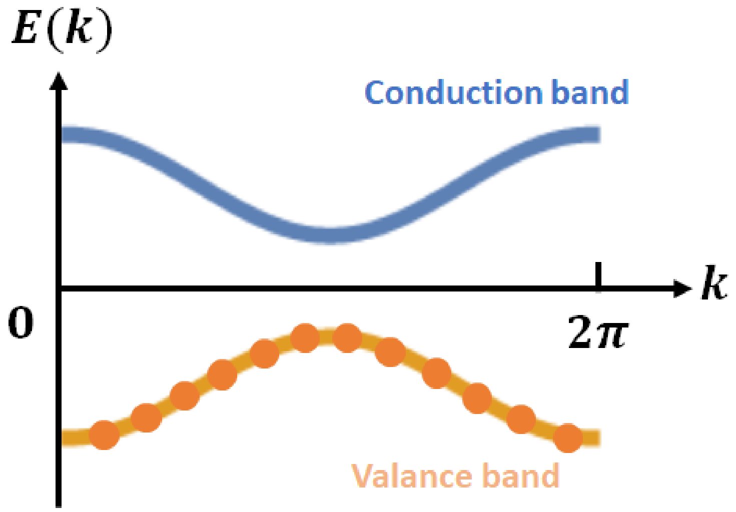
G. von Gersdorff
PUC-Rio



Outline

- ❑ Unification of topological invariants in Dirac models
- ❑ Metric-curvature correspondence and measurement of topological order
- ❑ Universal topological marker

Topological insulators and superconductors described by Dirac models



Dirac Hamiltonian $\quad H(\mathbf{k}) = \boldsymbol{d}(\mathbf{k}) \cdot \boldsymbol{\Gamma}$

$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \quad E(\mathbf{k}) = \pm|d(\mathbf{k})|$

TR, PH, and Chiral symmetries

$$TH^*(\mathbf{k})T^{-1} = H(-\mathbf{k}) \; ,$$
$$CH^*(\mathbf{k})C^{-1} = -H(-\mathbf{k})$$
$$SH(\mathbf{k})S^{-1} = -H(\mathbf{k}) \; .$$

Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Schnyder et al. PRB 2008, Kitaev, AIP 2009, Ryu et al, NJP 2010

The zoo of topological invariants

Class	1D	2D	3D
A	0	Z	0
AIII	Z	0	Z
AI	0	0	0
BDI	Z	0	0
D	Z_2	Z	0
DIII	Z_2	Z_2	Z
AII	0	Z_2	Z_2
CII	$2Z$	0	Z_2
C	0	$2Z$	0
CI	0	0	$2Z$

Zak phase (charge polarization and variations)

$$C = \int dk \, \langle u_k | i\partial_k | u_k \rangle = \int dk \, \frac{1}{2d^2}(d_1 \partial_k d_2 - d_2 \partial_k d_1)$$

Majorana number

$$(-1)^C = \prod_i \mathrm{Sgn}(\mathrm{Pf}[X_{k_0}])$$

Is there some more unified description of topological order?

Pfaffian of TR operator (QSHE)

$$(-1)^C = \prod_i \mathrm{Sgn}(\mathrm{Pf}[\langle u_\alpha | T | u_\beta \rangle])$$

Chern number (Hall conductance)

$$C = \int d^2\mathbf{k} \;\; \boldsymbol{\nabla}_{\boldsymbol{k}} \times \langle u_{\boldsymbol{k}} | i \boldsymbol{\nabla}_{\boldsymbol{k}} | u_{\boldsymbol{k}} \rangle$$

$$= \int d^2\mathbf{k} \; \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{d}^{\mathbf{3}}} \boldsymbol{d}(\mathbf{k}) \cdot (\partial_x \boldsymbol{d}(\mathbf{k}) \times \partial_y \boldsymbol{d}(\mathbf{k}))$$

3D winding number (and variations)

$$C = \int d^3\mathbf{k} \, \varepsilon^{\mu\nu\rho} \mathrm{Tr}[q^\dagger \partial_\mu q q^\dagger \partial_\nu q q^\dagger \partial_\rho q]$$

Some patterns for the topological invariants

Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Unit vector $\boldsymbol{n}(\mathbf{k}) = \boldsymbol{d}(\mathbf{k})/|\boldsymbol{d}(\mathbf{k})|$

Spec. Flat. $\tilde{Q}(\mathbf{k}) = \boldsymbol{n}(\mathbf{k}) \cdot \boldsymbol{\Gamma}$

Odd-D and chiral: Winding number

$$\nu_D = \frac{(\frac{D-1}{2})!}{2D!}\left(\frac{i}{2\pi}\right)^{\frac{D+1}{2}} \int \mathrm{tr}\, S(\widetilde{Q} d\widetilde{Q})^D$$

Even-D and nonchiral: Chern number

$$\mathrm{Ch}_{\frac{D}{2}} = \frac{1}{V_D D!}\left(-\frac{i}{2}\right)^{\frac{D}{2}} \int \mathrm{tr}\, \widetilde{Q}(d\widetilde{Q})^D$$

First and second descendent

$Z_2 = (-1)^{\nu_D}$ or $Z_2 = (-1)^{\mathrm{Ch}_{D/2}}$

Can we unify them further?

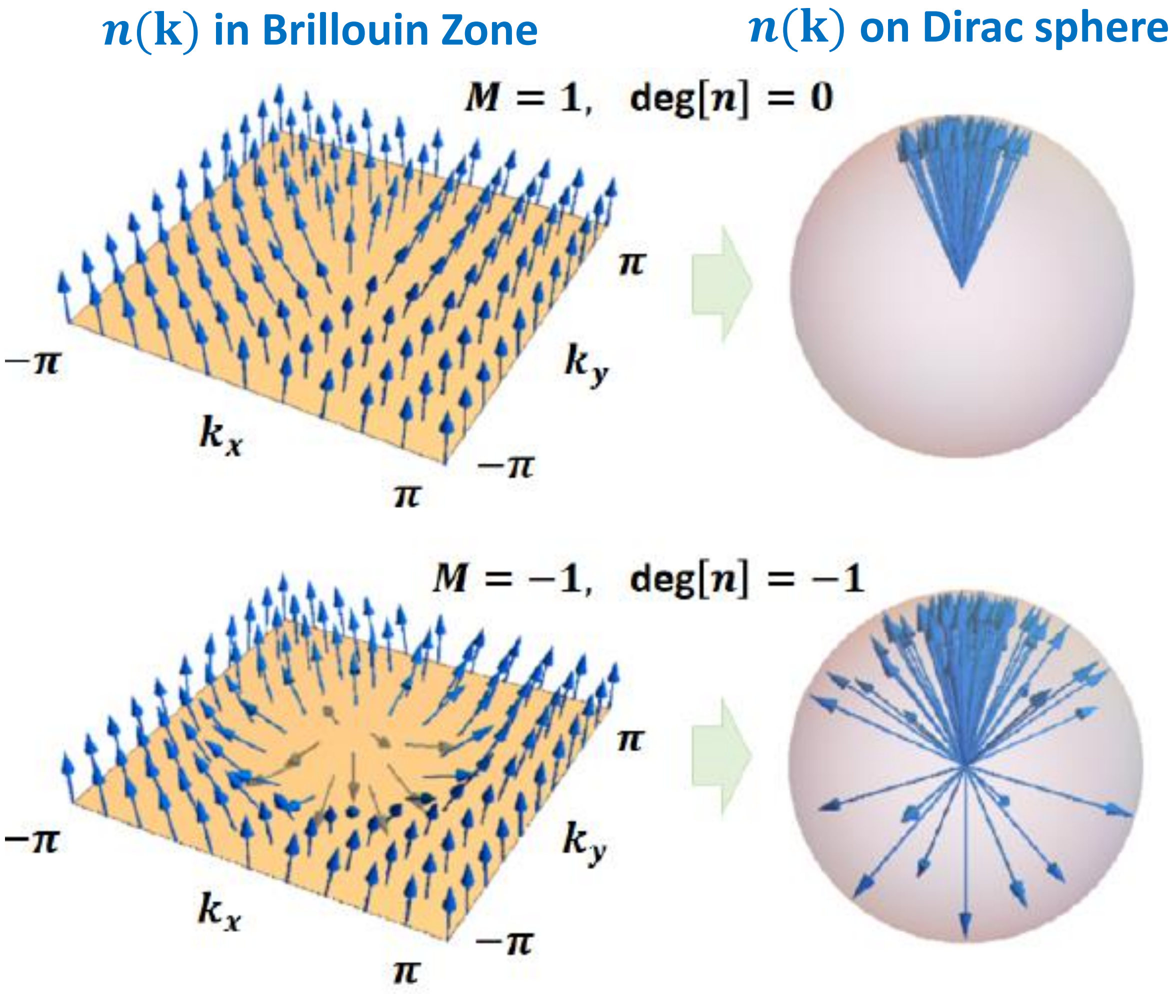
Chiu, Teo, Schnyder, Ryu, RMP 88, 035005 (2016)

Introducing the wrapping number (a.k.a. degree of the map)

Dirac Hamiltonian $H(\mathbf{k}) = \boldsymbol{d}(\mathbf{k}) \cdot \boldsymbol{\Gamma}$ Unit vector $\boldsymbol{n}(\mathbf{k}) = \boldsymbol{d}(\mathbf{k})/|\boldsymbol{d}(\mathbf{k})|$

Degree of the map between T^D Brillouin zone and the "Dirac sphere" where $\boldsymbol{n}(\mathbf{k})$ forms

$$\deg[\boldsymbol{n}] = \frac{1}{V_D}\int d^D k \, \underbrace{\epsilon_{i_0 \dots i_D} n^{i_0} \partial_1 n^{i_1} \dots \partial_D n^{i_D}}_{\text{Cyclic derivative of } \boldsymbol{n}(\mathbf{k})}$$



Example: Lattice model of 2D Chern insulator

$$H(\mathbf{k}) = d_1\sigma_1 + d_2\sigma_2 + d_3\sigma_3$$

$$d_1 = \sin k_x \qquad d_2 = \sin k_y$$

$$d_3 = M + \cos k_x + \cos k_y$$

All topological invariants can be expressed by wrapping number

Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Odd-D and chiral: Winding number

$$\nu_D = \text{... some algebra ...} = \deg[\boldsymbol{n}(\mathbf{k})]$$

$$\begin{aligned} &\operatorname{tr}\left[S\widetilde{Q}(d\widetilde{Q})^D\right] \\ &= \operatorname{tr}\left[S\Gamma_{i_0}\cdots\Gamma_{i_D}\right] n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D} \\ &= \operatorname{tr}\left[S\Gamma_0\cdots\Gamma_D\right] \epsilon_{i_0\ldots i_D} n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D} \\ &= -i^{\frac{D+1}{2}} \operatorname{tr}[\mathbb{1}]\, \epsilon_{i_0\ldots i_D} n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D} \end{aligned}$$

Even-D and nonchiral: Chern number

$$\mathrm{Ch}_{D/2} = \text{... some algebra ...} = \deg[\boldsymbol{n}(\mathbf{k})]$$

$$\begin{aligned} &\operatorname{tr}\left[\widetilde{Q}(d\widetilde{Q})^D\right] \\ &= \operatorname{tr}\left[\Gamma_{i_0}\cdots\Gamma_{i_D}\right] n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D} \\ &= \operatorname{tr}\left[\Gamma_0\cdots\Gamma_D\right] \epsilon_{i_0\ldots i_D} n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D} \\ &= i^{\frac{D}{2}} \operatorname{tr}[\mathbb{1}]\epsilon_{i_0\ldots i_D} n^{i_0} dn^{i_1} \wedge \cdots \wedge dn^{i_D}, \end{aligned}$$

First and second descendent

$$Z_2 = (-1)^{\nu_D \text{ or } \mathrm{Ch}_{D/2}} = (-1)^{\deg[\boldsymbol{n}(\mathbf{k})]}$$

$$= P[\boldsymbol{n}(\mathbf{k})] \equiv \prod_{\bar{\mathbf{k}}} n_0(\bar{\mathbf{k}})$$

Unification of topological invariants

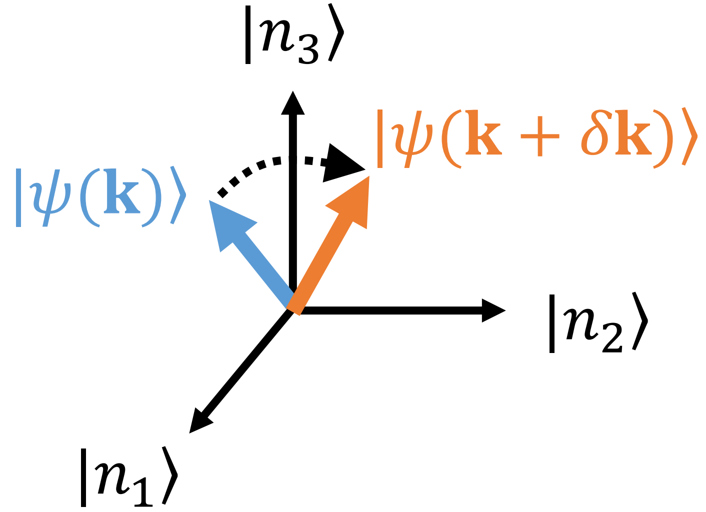
	$D=0$	$D=1$	$D=2$	$D=3$	$D=4$	$D=5$	$D=6$	$D=7$
A	$\deg \mathbf{n}[\mathbf{k}]$		$\deg \mathbf{n}[\mathbf{k}]$		$\deg \mathbf{n}[\mathbf{k}]$		$\deg \mathbf{n}[\mathbf{k}]$	
AIII		$\deg \mathbf{n}[\mathbf{k}]$		$\deg \mathbf{n}[\mathbf{k}]$		$\deg \mathbf{n}[\mathbf{k}]$		$\deg \mathbf{n}[\mathbf{k}]$
AI	$\deg \mathbf{n}[\mathbf{k}]$				$2\deg \mathbf{n}[\mathbf{k}]$		$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$
BDI	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$\deg \mathbf{n}[\mathbf{k}]$				$2\deg \mathbf{n}[\mathbf{k}]$		$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$
D	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$\deg \mathbf{n}[\mathbf{k}]$				$2\deg \mathbf{n}[\mathbf{k}]$	
DIII		$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$\deg \mathbf{n}[\mathbf{k}]$				$2\deg \mathbf{n}[\mathbf{k}]$
AII	$2\deg \mathbf{n}[\mathbf{k}]$		$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$\deg \mathbf{n}[\mathbf{k}]$			
CII		$2\deg \mathbf{n}[\mathbf{k}]$		$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$\deg \mathbf{n}[\mathbf{k}]$		
C			$2\deg \mathbf{n}[\mathbf{k}]$		$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$\deg \mathbf{n}[\mathbf{k}]$	
CI				$2\deg \mathbf{n}[\mathbf{k}]$		$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$(-1)^{\deg \mathbf{n}[\mathbf{k}]}$	$\deg \mathbf{n}[\mathbf{k}]$

von Gersdorff, Panahiyan, WC, PRB 103, 245146 (2021)

Quantum metric of (degenerate) filled band states

The N_{-}-particle filled band state of Dirac models

$$|\psi(\mathbf{k})\rangle = \frac{1}{\sqrt{N_{-}!}} \varepsilon^{a_1 \ldots a_{N_-}} |u^{-}_{a_1}\rangle |u^{-}_{a_2}\rangle \ldots |u^{-}_{a_{N_-}}\rangle$$



Quantum metric (aka fidelity susceptibility) on the BZ manifold

$$|\langle\psi(\mathbf{k})|\psi(\mathbf{k}+\delta\mathbf{k})\rangle| = 1 - \frac{1}{2} g_{\mu\nu} \delta k^{\mu} \delta k^{\nu}$$

$$g_{\mu\nu} = \frac{1}{2}\langle\partial_\mu\psi|\partial_\nu\psi\rangle + \frac{1}{2}\langle\partial_\nu\psi|\partial_\mu\psi\rangle - \langle\partial_\mu\psi|\psi\rangle\langle\psi|\partial_\nu\psi\rangle$$

For Dirac models

$$g_{\mu\nu} = \frac{N}{8} \partial_\mu \boldsymbol{n} \cdot \partial_\nu \boldsymbol{n}$$

Vielbein

Provost and Vallee, Comm. Math. Phys. 76, 289 (1980)
Ma et al, EPL 103, 10008 (2013)

Metric-curvature correspondence

The wrapping number can be written as the integration of a Jacobian

$$\deg[\boldsymbol{n}] = \frac{1}{V_D}\int d^D k\ J_{\boldsymbol{n}}(\mathbf{k}) \qquad J_{\boldsymbol{n}}(\mathbf{k}) = \det(\boldsymbol{n}, \partial_1\boldsymbol{n}, \partial_2\boldsymbol{n} \dots \partial_D\boldsymbol{n}) = \det(\boldsymbol{n}, \partial_\mu\boldsymbol{n})$$

The square of the Jacobian is also given by the vielbein form

$$J_{\boldsymbol{n}}^2 = \det\begin{pmatrix} \boldsymbol{n}\cdot\boldsymbol{n} & \boldsymbol{n}\cdot\partial_\nu\boldsymbol{n} \\ \partial_\mu\boldsymbol{n}\cdot\boldsymbol{n} & \partial_\mu\boldsymbol{n}\cdot\partial_\nu\boldsymbol{n} \end{pmatrix} = \det \partial_\mu\boldsymbol{n}\cdot\partial_\nu\boldsymbol{n}$$

Thus the module of the Jacobian is equal to the square of $\det g_{\mu\nu}$

$$|J_{\boldsymbol{n}}| = \left(\frac{8}{N}\right)^{\frac{D}{2}} \sqrt{\det g_{\mu\nu}}$$

"Metric-curvature correspondence"

von Gersdorff and WC, PRB 104, 095113 (2021)

This means topological order can be measured via measuring the quantum metric!!!

Time-dependent perturbation theory of measuring quantum metric

Applying a pulse electric field of profile $g(t)$ and polarization $\hat{\boldsymbol{\mu}}$

$$\boldsymbol{E}(t) = \hat{\boldsymbol{\mu}}\, E^0 h(t) = \; i\, E^0 h(t) \partial_\mu$$

Causes a transition of electrons from filled bands a to empty bands b, which is equal to quantum metric

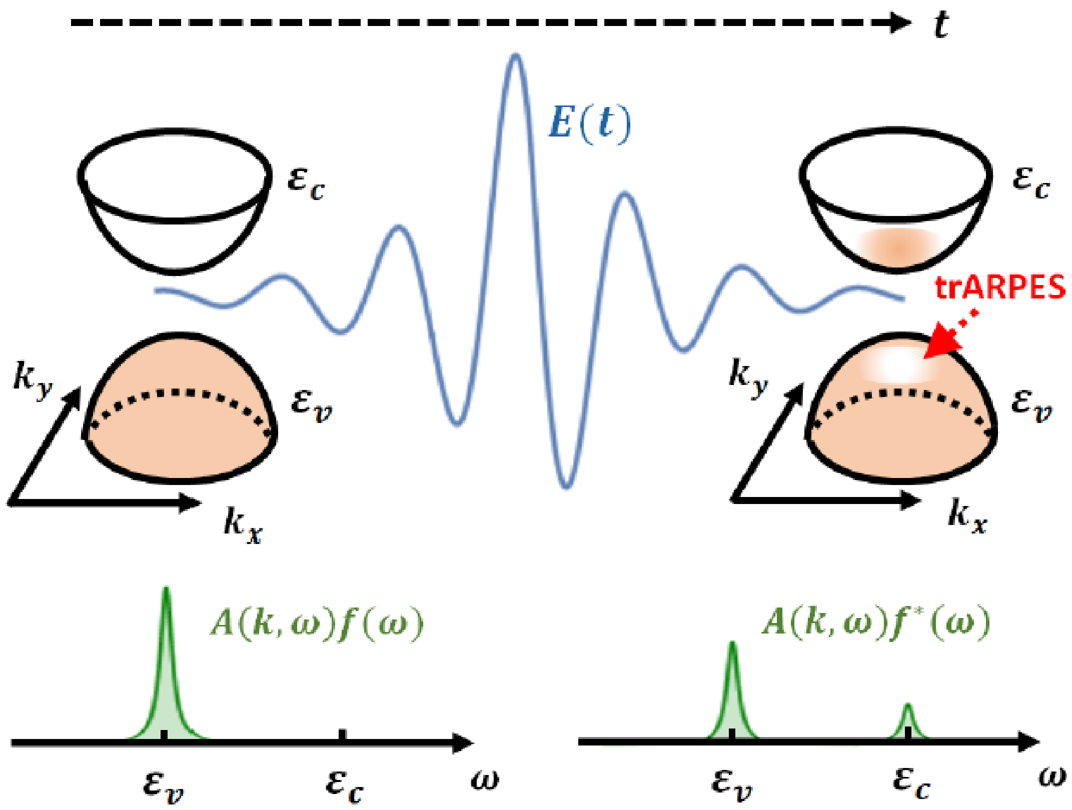
$$\nu(\mathbf{k}) = \left(\frac{eE^0}{\hbar}\right)^2 |\tilde{h}(\omega(\mathbf{k}))|^2 \sum_{a,b} |\langle u_b(\boldsymbol{k})|\partial_\mu u_a(\mathbf{k})\rangle|^2$$

$$= \left(\frac{eE^0}{\hbar}\right)^2 |\tilde{h}(\omega(\mathbf{k}))|^2 \, g_{\mu\mu}(\mathbf{k})$$

Which is equal to the loss of spectral weight in ARPES

$$\nu(\mathbf{k}) = \frac{N}{2} - \int_{-\infty}^{\infty} d\omega \;\, A(\mathbf{k},\omega) f^*(\omega)$$

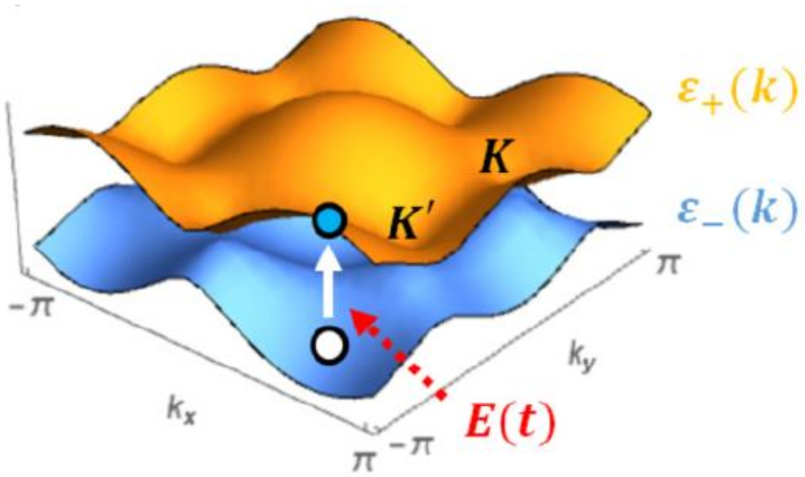
Ozawa and Goldman, PRB 97, 201117 (2018)

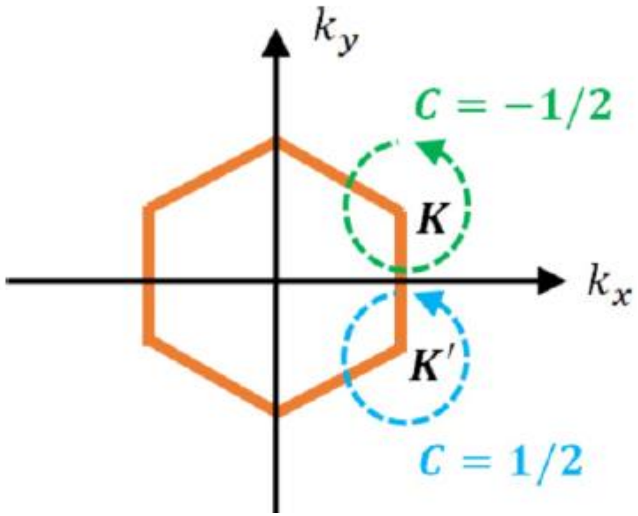


"Loss-fluence spectroscopy"

von Gersdorff and WC, PRB 104, 095113 (2021)

Estimation for graphene





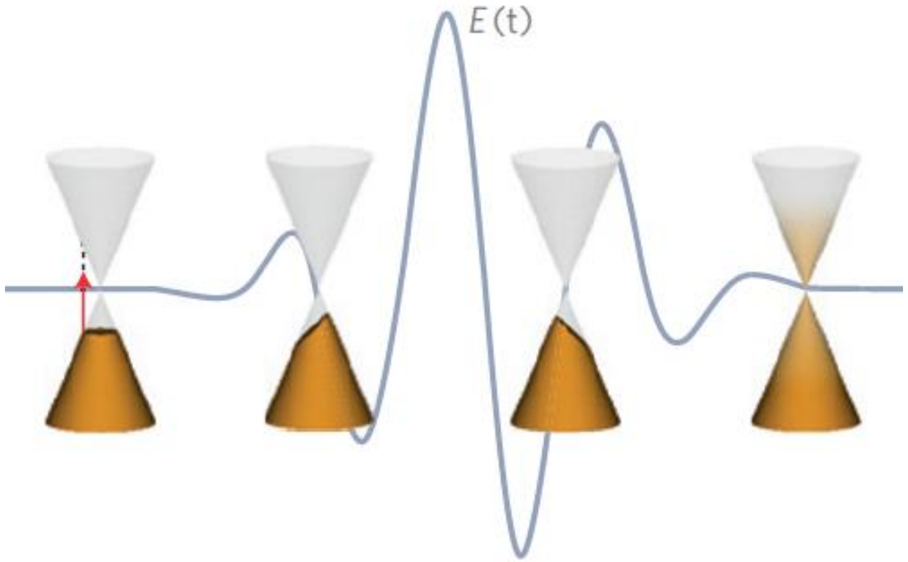
$$\frac{1}{2\pi}\oint d\phi \langle u^{\mathbf{K}}_{\mathbf{k}-}|i\partial_\phi|u^{\mathbf{K}}_{\mathbf{k}-}\rangle = -\frac{1}{2\pi}\oint d\phi \langle u^{\mathbf{K}'}_{\mathbf{k}-}|i\partial_\phi|u^{\mathbf{K}'}_{\mathbf{k}-}\rangle = -1/2$$

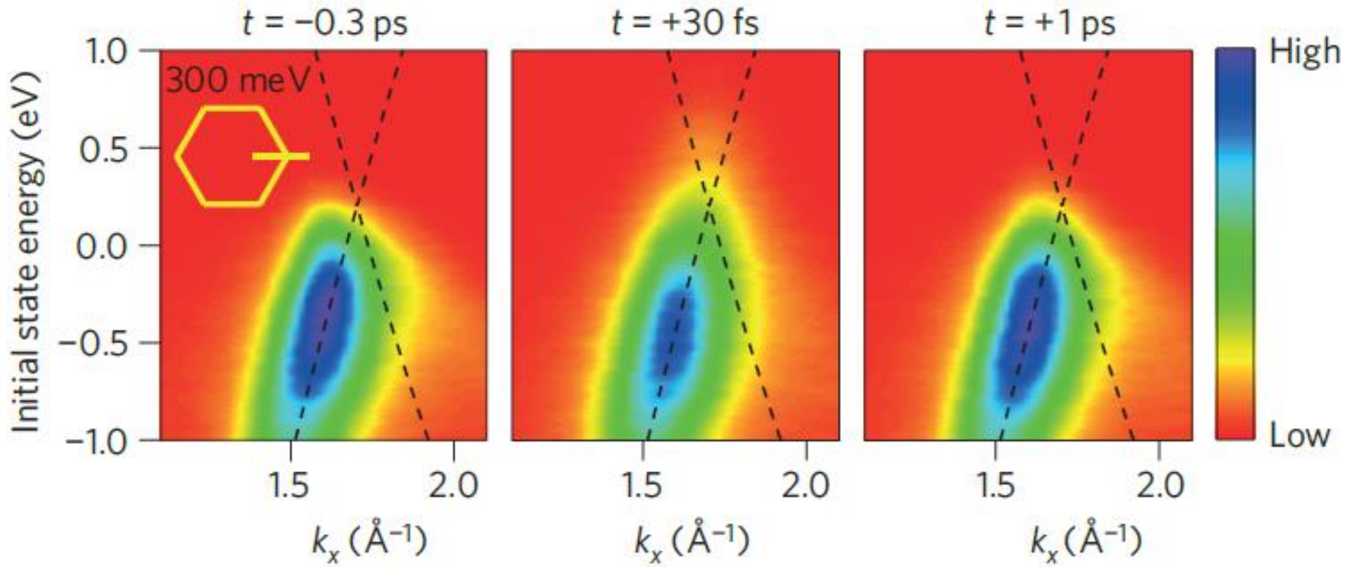
Metric-curvature correspondence $g^u_{\phi\phi} = |\langle u|i\partial_\phi|u\rangle|^2 = 1/4$

The fluence $F \sim \frac{c\,\varepsilon_0}{2}|E^0|^2 T \sim 10^{-4}\,\frac{\mathrm{mJ}}{\mathrm{cm}^2}$

Can cause an excitation $\nu(\mathbf{k}) \sim \left(\frac{eE^0 v_F T}{\hbar\omega}\right)^2 \sim 10\%$

Pump-probe on graphene





Message to experimentalists: Use polarized light!!

Gierz et al, Mat. Mater. 12, 1119 (2013)

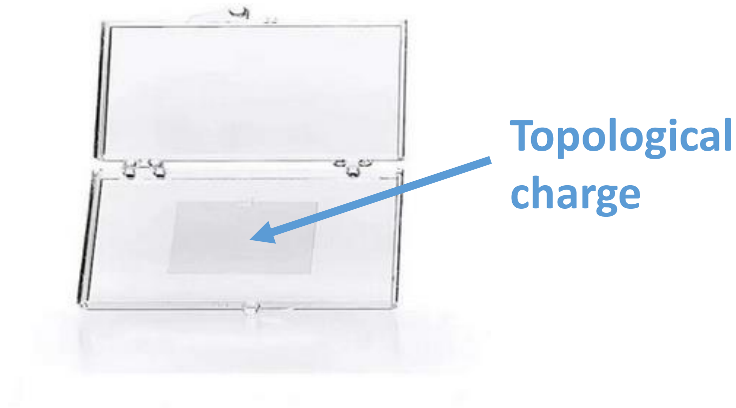
Other intriguing applications of quantum metric

Opacity of grahpene: M. de Sousa talk

de Sousa, Cruz, and Chen, arXiv:2303.14549

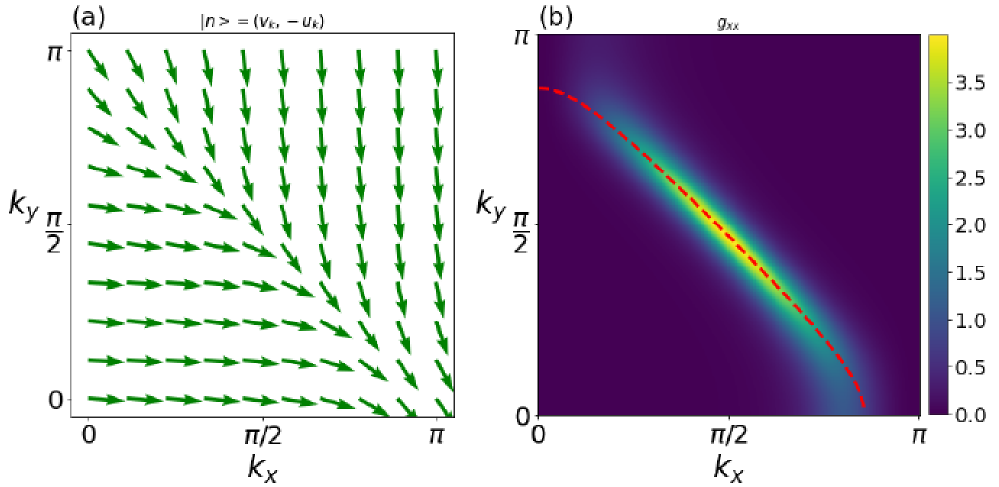
$$\lim_{T\to 0} \mathcal{O}(\omega) = \pi\alpha \times 4\mathcal{C}^2 = \pi\alpha \approx 2.3\%$$

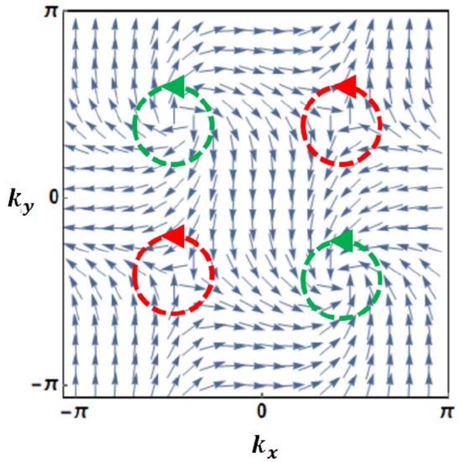
(1) One can literally see the topological charge by naked eyes
(2) The fine-structure constant is topologically protected
(3) 3D TI thin films of any thickness has the same opacity



CVD Graphene PET/Glass Base Film

s-wave and d-wave superconductors have quantum metric: D. Porlles poster





Topological charge and metric-curvature correspondence in d-wave SC

Twisting of Bogoliubov coefficients gives quantum metric

Expression in terms of spectrally flattened Hamiltonian

The wrapping number can be expressed in terms of spectrally flattened Hamiltonian $\tilde{Q}$

$$\deg[\boldsymbol{n}] = \frac{(2\pi)^D}{2^n c V_D} \int \frac{d^D k}{(2\pi)^D} \,\mathrm{Tr}[W \tilde{Q} \partial_1 \tilde{Q} \partial_2 \tilde{Q} \dots \partial_D \tilde{Q}] \qquad \tilde{Q} = \boldsymbol{n} \cdot \boldsymbol{\Gamma} = q - p$$

Projector to valence band $p = \Sigma_n |n\rangle\langle n|$ Projector to conduction band $q = \Sigma_m |m\rangle\langle m|$

This is because the trace of all the Dirac matrices multiplied together is a constant $c = \{\pm 1, \pm i\}$

$$\mathrm{Tr}\left[W \tilde{Q} \partial_1 \tilde{Q} \partial_2 \tilde{Q} \dots \partial_D \tilde{Q}\right]$$

$$= \mathrm{Tr}[\Gamma_{D+1} \Gamma_{D+2} \dots \Gamma_{2n} \Gamma_{i_0} \Gamma_{i_1} \dots \Gamma_{i_D}]\, n^{i_0} \partial_1 n^{i_1} \dots \partial_D n^{i_D}$$

$$= 2^n c \epsilon_{i_0 \dots i_D} n^{i_0} \partial_1 n^{i_1} \dots \partial_D n^{i_D}$$

The derivative on $\tilde{Q}$ is equivalently derivatives on p or on q

$$\partial_i \tilde{Q} = 2 \partial_i q = -2 \partial_i p$$

Projector formalism

Take odd dimension as an example $D = \{1,3,5 \ldots\}$, we can further write

Bianco and Resta, PRB 84, 241106 (2011)

$$W\tilde{Q}\partial_1\tilde{Q}\partial_2\tilde{Q} \ldots \partial_D\tilde{Q}$$

$$\propto W\{q\partial_1 p\partial_2 q \ldots \partial_D p + p\partial_1 q\partial_2 p \ldots \partial_D q\}$$

$$\propto W\{|m_1\rangle\langle m_1|\partial_1 n_1\rangle\langle n_1|\partial_2 m_2\rangle\langle m_2| \ldots (m \leftrightarrow n)\}$$

The reason to write it in this form is to use the expression $\langle m|\partial_x n\rangle = -i\langle\psi_m|\hat{x}|\psi_n\rangle$

Such that the $\boldsymbol{k}$-integration of the full Bloch state becomes the projector to filled and empty lattice engenstates,

$$\int \frac{d^D k}{(2\pi)^D} \sum_m |\psi_m\rangle\langle\psi_m| = \sum_m |E_m\rangle\langle E_m| \equiv Q$$

$$\int \frac{d^D k}{(2\pi)^D} \sum_n |\psi_n\rangle\langle\psi_n| = \sum_n |E_n\rangle\langle E_n| \equiv P$$

And the topological invariant becomes:

$$\int \frac{d^D\mathbf{k}}{(2\pi)^D} \mathrm{Tr}[W\tilde{Q}(d\tilde{Q})^D]_{D\in odd} = 2^D i \mathrm{Tr}\left[WQ\,\hat{i}_1 P\,\hat{i}_2 \ldots Q\,\hat{i}_D P + WP\,\hat{i}_1 Q\,\hat{i}_2 \ldots P\,\hat{i}_D Q\right]$$

Local and nonlocal topological marker

Introducing the topological operator

$$\hat{\mathcal{C}} = N_D W \left[Q\,\hat{i}_1 P\,\hat{i}_2 ... \hat{i}_D \mathcal{O} + (-1)^{D+1} P\,\hat{i}_1 Q\,\hat{i}_2 ... \hat{i}_D \overline{\mathcal{O}} \right]$$

Unused Γ matrices   Alternating Q and P Insert position operators

Diagonal elements: Local marker quantized to integer

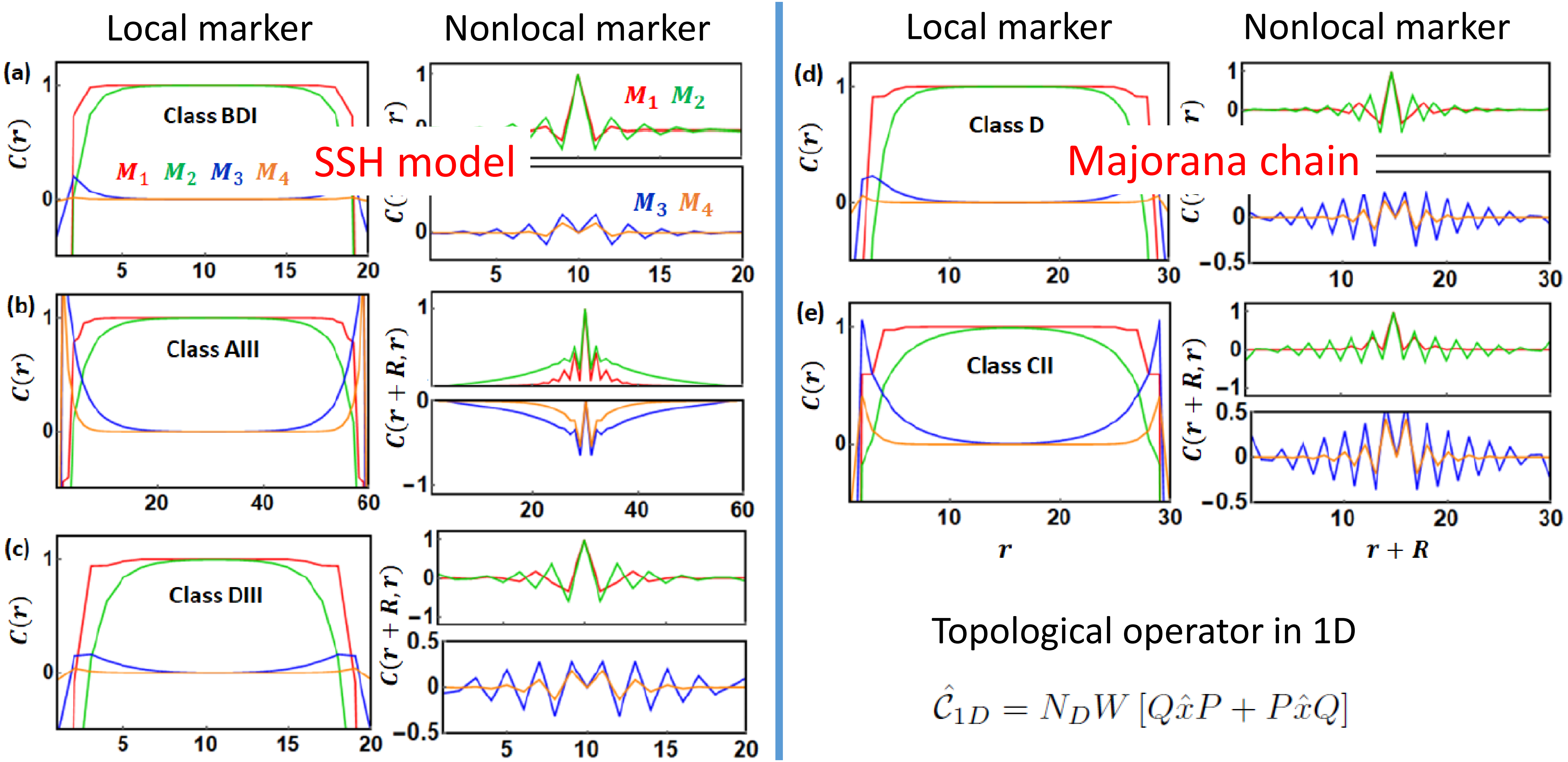
$$\mathcal{C}(\mathbf{r}) = \langle \mathbf{r} | \hat{\mathcal{C}} | \mathbf{r} \rangle = \sum_{\sigma} \langle \mathbf{r}\sigma | \hat{\mathcal{C}} | \mathbf{r}\sigma \rangle$$

Off-diagonal elements: Nonlocal marker
That becomes more long ranged as the system approaches topological phase transitions

$$\mathcal{C}(\mathbf{r}+\mathbf{R}, \mathbf{r}) = \langle \mathbf{r}+\mathbf{R} | \hat{\mathcal{C}} | \mathbf{r} \rangle$$

$$\equiv \frac{1}{V_D} \int d^D\mathbf{k}\, \varepsilon_{i_0 ... i_D} n^{i_0} \partial_1 n^{i_1} ... \partial_D n^{i_D} e^{i\mathbf{k}\cdot\mathbf{R}}$$

Applications to 1D classes

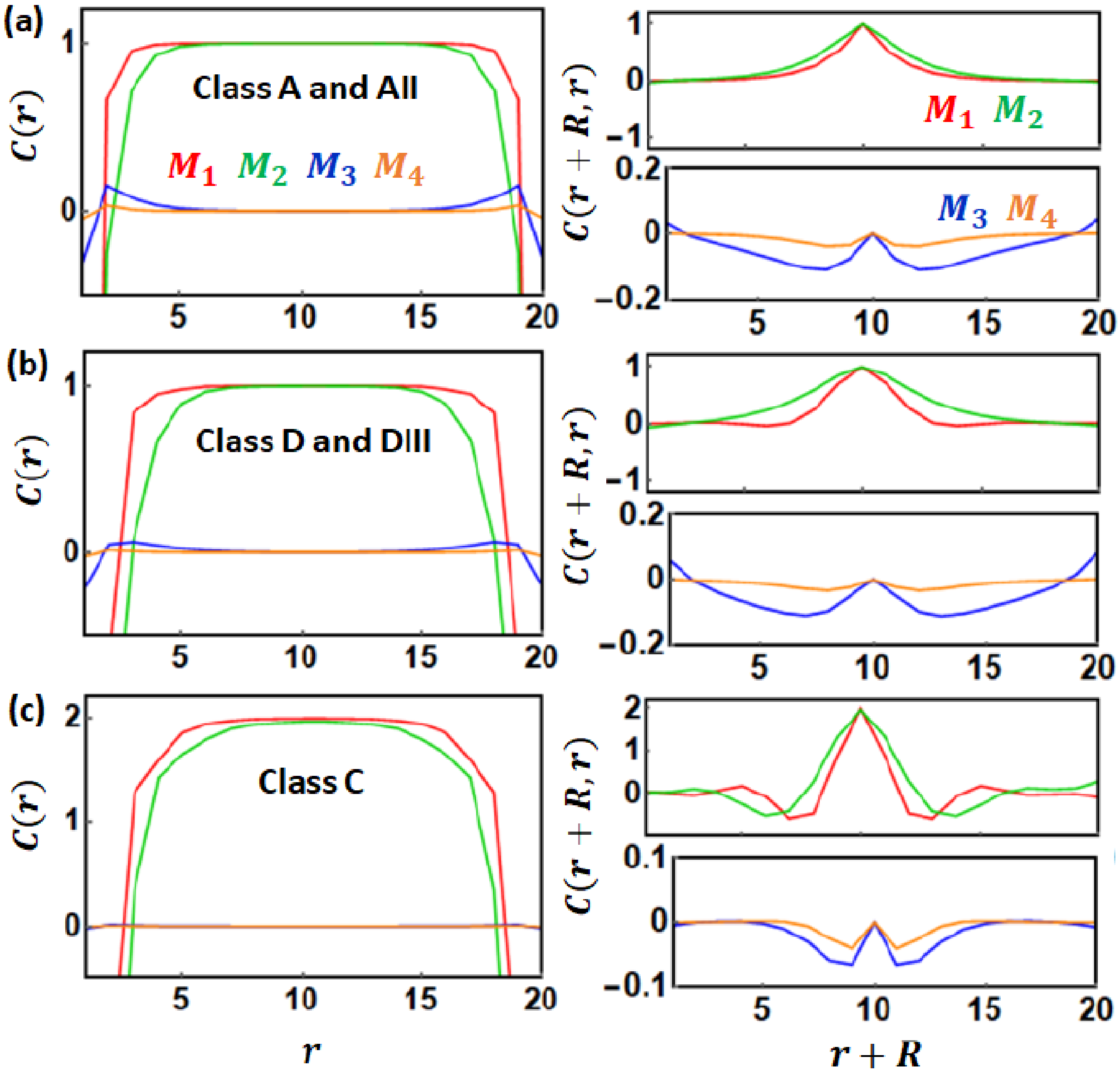


Topological operator in 1D

$$\hat{\mathcal{C}}_{1D} = N_D W\left[Q\hat{x}P + P\hat{x}Q\right]$$

Applications to 2D classes

Local marker | Nonlocal marker



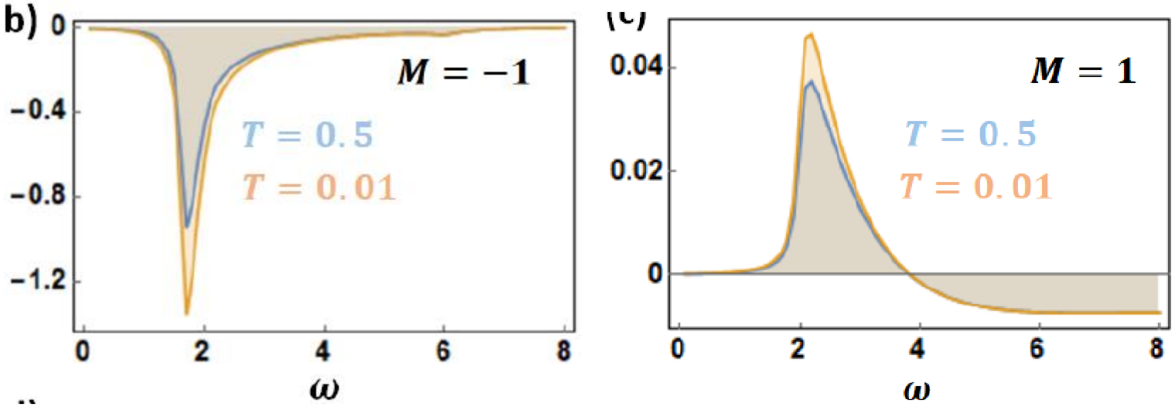
Topological operator in 2D

$$\hat{C}_{2D} = N_D W \left[Q\hat{x}P\hat{y}Q - P\hat{x}Q\hat{y}P \right]$$

Chern insulator, BHZ model

Chiral and Helical p-wave SC

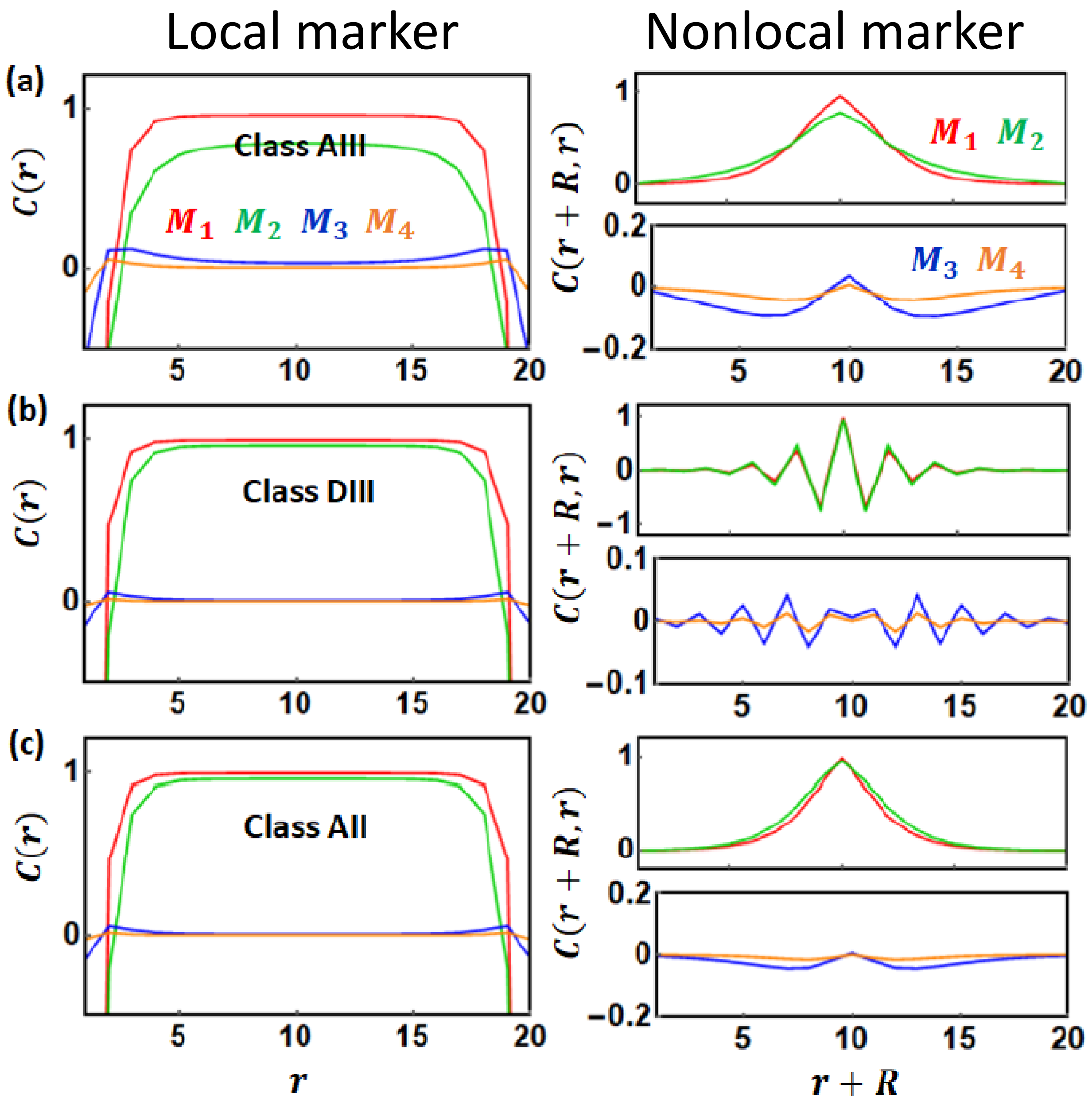
Nontrivial phase | Trivial phase



Circular dichroism = Chern marker Spec. Fn.

Molignini, Lapierre, Chitra, and WC, arXiv:2207.00016

Applications to 3D classes



Topological operator in 3D

$$\hat{C}_{3D} = N_D W \left[Q\hat{x}P\hat{y}Q\hat{z}P + P\hat{x}Q\hat{y}P\hat{z}Q \right]$$

B-phase of ^3He

3DTI like Bi_2Se_3, Bi_2Te_3

Summary

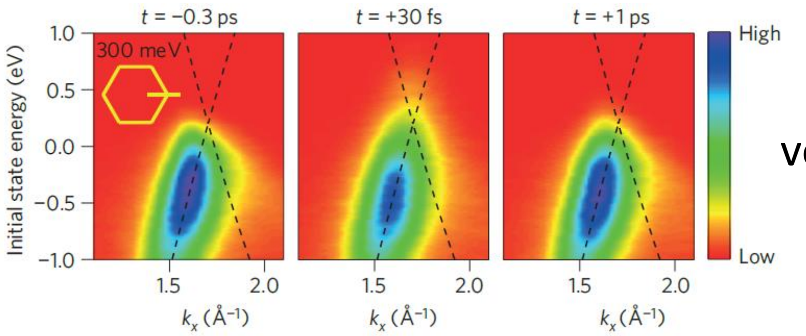
- Unification of topological invariants in Dirac models

$$\deg[\boldsymbol{n}] = \frac{1}{V_D}\int d^D k \;\; \epsilon_{i_0 \ldots i_D} n^{i_0} \partial_1 n^{i_1} \ldots \partial_D n^{i_D}$$

von Gersdorff, Panahiyan, WC, PRB 103, 245146 (2021)

- Metric-curvature correspondence and measurement of topological order

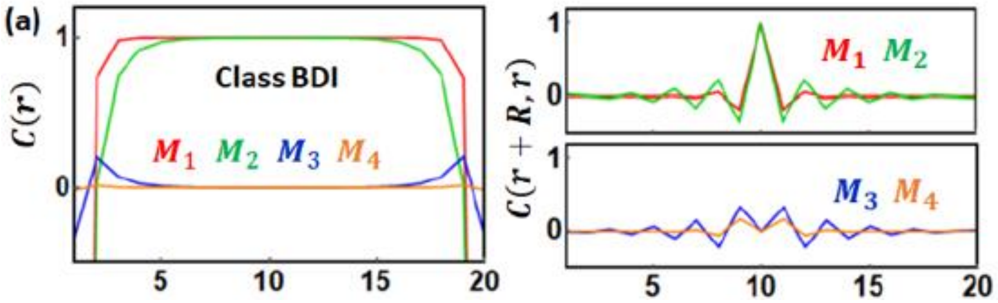
$$|J_{\boldsymbol{n}}| = \left(\frac{8}{N}\right)^{\frac{D}{2}} \sqrt{\det g_{\mu\nu}}$$



von Gersdorff and WC, PRB 104, 095113 (2021)

- Universal topological marker

$$\hat{\mathcal{C}} = N_D W \left[Q\,\hat{i}_1 P\,\hat{i}_2 \ldots \hat{i}_D \mathcal{O} + (-1)^{D+1} P\,\hat{i}_1 Q\,\hat{i}_2 \ldots \hat{i}_D \overline{\mathcal{O}} \right]$$



WC, PRB 107, 045111 (2023)