General bounds on KK masses

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based on 2104.12773 with G.B. De Luca,
2109.11560, 2212.02511, 2306.05456, & WiP
with De Luca (physics, Stanford),
N. De Ponti (math, Bicocca), A. Mondino (math, Oxford)

São Paulo, Holography@25, June 2023

Introduction

KK spectrum: one of the most important pieces of data associated to a compactification

In holography:

- Checks of operator/state correspondence

[Kim, Romans, Van Nieuwenhuizen '85;
Fabbri, Fré, Gualtieri, Termonia '99;
Ceresole, Dall'Agata, D'Auria, Ferrara '99...]

- Relevant for scale separation

Exactly known vacua: $m_{\mathrm{KK}} \sim \frac{1}{\mathrm{diam}} \sim \frac{1}{r_{\mathrm{AdS}}} \sim \sqrt{|\Lambda|}$ 'no scale separation'

But in CFT: generically, no susy protection ⇨ many heavy operators?

[Polchinski, Silverstein '09]

This talk: several mathematical bounds on the masses of spin-two fields…

- valid for many gravity compactifications (not just string/M-theory)
- often also in presence of singularities (D-branes, O-planes…)
- in terms of both ‘size’ and ‘shape’

We will not solve the problem of scale separation, but constrain it in several ways.

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For example:

- $m_k^2 < 600 k^2 \mathbf{max}\left\{ m_1^2,\ |\Lambda| + \sigma^2 \right\}$ $\quad \sigma = \sqrt{D-2}\,\mathrm{sup}|\mathrm{d}A|$
- $m_k^2 \leqslant \left(|\Lambda| + (D-1)\sigma^2 \right) + \gamma \frac{k^2}{\mathrm{diam}^2}$ max. distance among any two points
- $m_k^2 \geqslant \frac{C}{k^6} h_k^2$ Cheeger constants: quantify ‘small necks’

Plan

- Mathematical background

  Curvature, warping, and the weighted Raychaudhuri equation

- Overview of bounds

  in terms of Planck mass; Cheeger constant; diameter

- Examples and applications

  scale separation; gravity localization

Mathematical background

- Spin-two masses: eigenvalues of weighted Laplacian

$$\Delta_f \psi \equiv -\mathrm{e}^{-f} \nabla^m (\mathrm{e}^f \nabla_m \psi)$$

[Csaki, Erlich, Hollowood, Shirman'00; Bachas, Estes '11]

total dimension

$$f = (D-2)A$$

$$\mathrm{d}s_D^2 = \mathrm{e}^{2A}(\mathrm{d}s_d^2 + \mathrm{d}s_n^2)$$

warping

Mass operators for other fields:
not universal, only known in some cases.

e.g. Freund–Rubin: [Duff, Nilsson, Pope '86]

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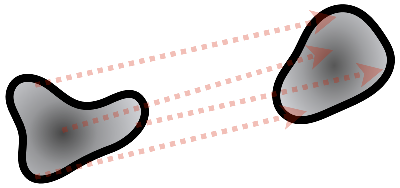
- Presence of warping invalidates old theorems on 'usual' Laplace–Beltrami

long history: [Lichnerowicz '58, Cheeger '70, Cheng '75, Li, Yau '80, Buser '82...]

- Natural in the setup of metric measure spaces:

integration measure = not just $\sqrt{g}$ but $e^f \sqrt{g}$

- Many ideas from recent progress in field of 'Optimal transport': best way to transport mass distributions



[Monge 1781, Kantorovich 1940...]
review: [Villani '08]

- Consider a distribution of particles moving geodesically $\rho(x)$ such that $\int_M \sqrt{g}\rho = 1$

Entropy: $S = -\int_M \sqrt{g}\rho \log \rho$

velocity field

using Raychaudhuri eq.: $\partial_t^2 S = -\int_M \sqrt{g}\rho(\nabla_m U_n \nabla^m U^n + R_{mn} U^m U^n)$

$$R_{mn} \geqslant 0 \;\Rightarrow\; \partial_t^2 S \leqslant 0$$

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- Weighted 'Tsallis entropy': homogeneous (rather than extensive) [$\sim$ log Rényi entropy]

[Havrda, Charvat '67; Patil, Taillie '82; Tsallis '88]

$$S_{N,f} \equiv N\left(1 - \int_M \sqrt{g} e^f \rho^{\frac{N-1}{N}}\right)$$

$$\partial_t^2 S_{N,f} \leqslant -\int_M \sqrt{g} e^f \rho^{\frac{N-1}{N}} \underbrace{\left(R_{mn} - \nabla_m\nabla_n f + \tfrac{1}{n-N}\nabla_m f \nabla_n f\right)}_{\substack{|||\\ R^{N,f}_{mn}}} U^m U^n$$

[McCann '19; Mondino, Suhr '19; De Luca, De Ponti, Mondino, AT '22]

"Bakry–Émery curvature"

[Bakry, Émery '85]

N 'effective dimension':
played $\sim$ role of rank of $\nabla_m U_n$

- Consider a higher-dimensional gravity $m_D^{D-2} \int \mathrm{d}^D x \sqrt{-g_D} R_D + \text{matter}$

  and a compactification $\mathrm{d}s_D^2 = \mathrm{e}^{2A}(\underset{\substack{\uparrow \\ \text{max.} \\ \text{symmetric}}}{\mathrm{d}s_d^2} + \mathrm{d}s_n^2)$

[De Luca, AT '20]

similar ideas in
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EoM: $R_{MN} = \frac{1}{2}m_D^{2-D}\left(T_{MN} - \frac{1}{D-2}g_{MN}T\right) \equiv \hat{T}_{MN}$

internal:

$$R_{mn} + (D-2)(-\nabla_m\nabla_n A + \partial_m A\partial_n A) = ((D-2)|\mathrm{d}A|^2 + \nabla^2 A)g_{mn} + \hat{T}_{mn}$$

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$$= \Lambda g_{mn} + (\hat{T}_{mn} - \tfrac{1}{d}g_{mn}\hat{T}_{(d)})$$

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[“Reduced Energy Condition”]

- for all bulk fields in type II and $d = 11$ sugra
- potentials
- for brane sources

[De Luca, AT ’20]

similar ideas in [Gautason, Schillo, Van Riet, Williams ’15]

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BE curvature

$f = (D-2)A$

$N = 2 - d < 0$

but still OK

- for all bulk fields in type II and $d = 11$ sugra
- potentials
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[“Reduced Energy Condition”]

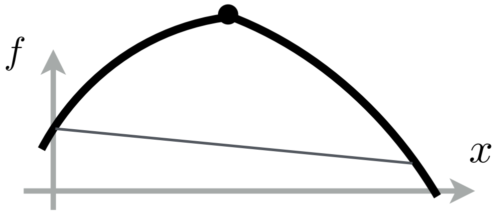
Sources create singularities. It would be best to avoid derivatives...

- Inspiration: functions of one variable

$$f'' \leqslant 0$$

generalize to
non-smooth functions:

concavity



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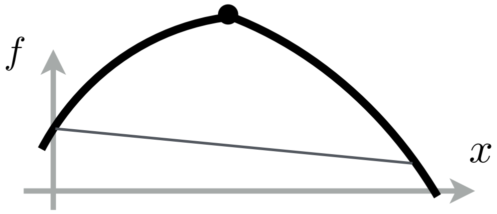

$$R^{N,f}_{mn} \geqslant 0 \Rightarrow \partial_t^2 S_{N,f} \leqslant 0$$

generalize to
non-smooth spaces:

concavity of $S_{N,f}$

this leads to the ‘Riemann-Curvature-Dimension’ [RCD] condition

[Sturm ’06; Lott, Villani ’07;
Ambrosio, Gigli, Savaré 14]

[One can also reformulate the Einstein equations in this language]

[McCann ’19; Mondino, Suhr ’19;
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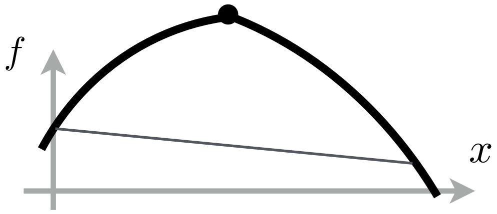

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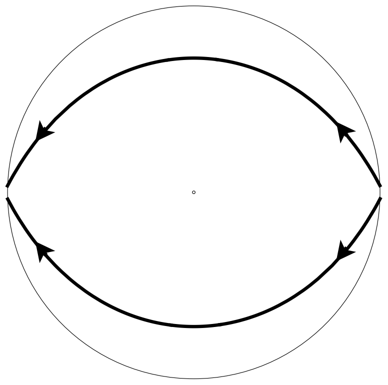
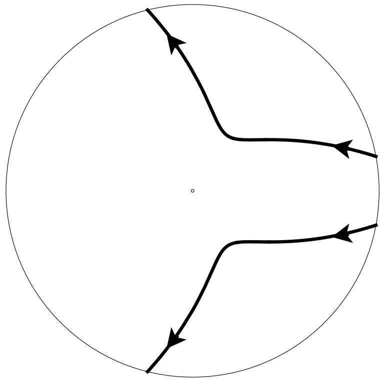
- Theorem: spaces with D-branes are RCD.

[De Luca, De Ponti, Mondino, AT '22]



intuitively, they attract particles.



… while this fails for O-planes, which repel

Overview of bounds

[De Luca, AT '20;
De Luca, De Ponti,
Mondino, AT '21,'22]

Rather than showing all bounds in detail, a summary. Definitions:

- 4d Planck mass $M_4^2 \sim M_D^{D-2} \int_M \sqrt{g} \mathrm{e}^{(D-2)A}$ [if unwarped: int. volume]

- diameter: max. distance between any two points in M

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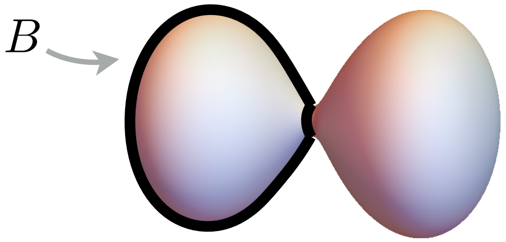
- Cheeger constant h_1: small when space has small 'neck'

$$h_1 = \min_B \frac{\mathrm{vol}_A(\partial B)}{\mathrm{vol}_A(B)}$$

'min. of $\frac{\text{perimeter}}{\text{area}}$'



$$\mathrm{vol}_A(B) \equiv \int_B \sqrt{g} e^{(D-2)A}$$

	4d Planck mass	Cheeger	diameter
upper bound	m_k [smooth; warp.]	m_1 [D-branes; warp.] $\frac{m_k}{m_1^2}$ [O-planes]	m_k [smooth; warp.]
lower bound		m_k [O-planes]	m_1 [D-branes]

- [smooth] $\subset$ [D-branes] $\subset$ [O-planes]

  spaces with D-brane sing. — spaces with also O-plane sing.

- [warp.]: bound contains $\sigma \equiv \sup_M \sqrt{D-2}|\mathrm{d}A|$

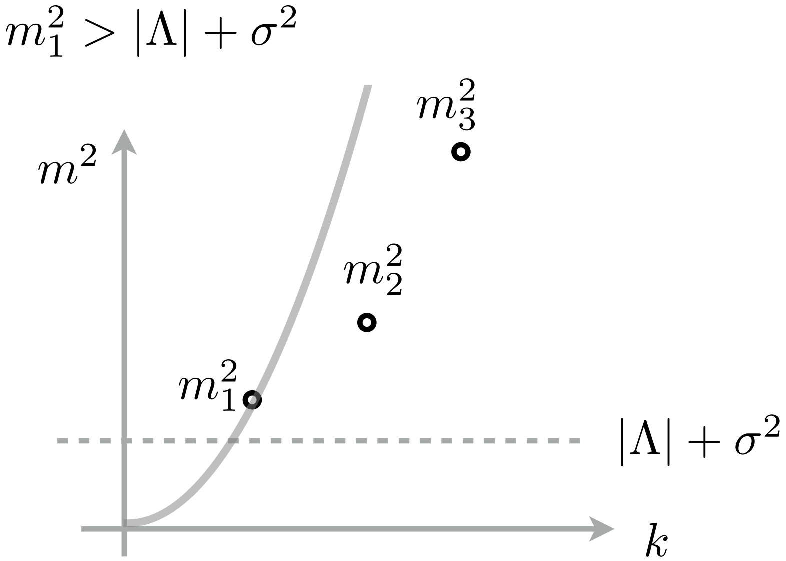
Scale separation

total dimension

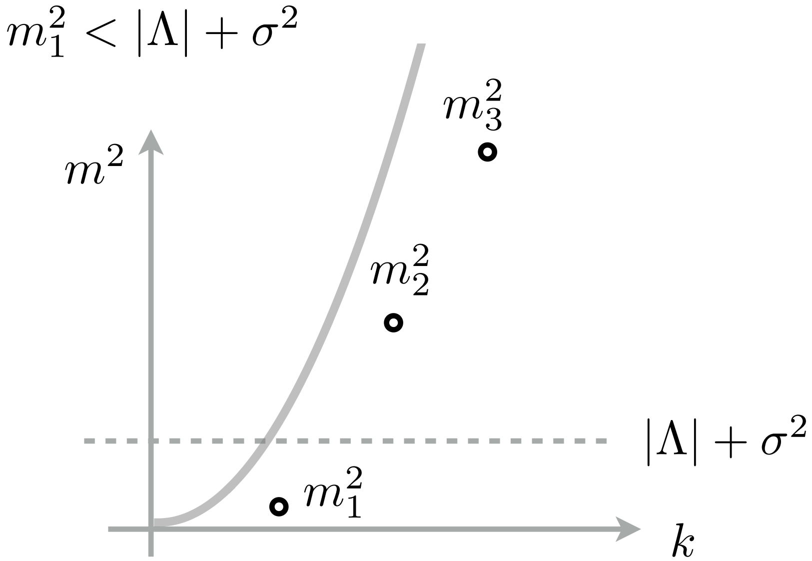
$$D = d + n$$

$$\sigma = \sqrt{D-2}\,\mathrm{sup}|\mathrm{d}A|$$

- combining several bounds: $m_k^2 < 600 k^2 \mathrm{max}\Big\{m_1^2,\ |\Lambda| + \sigma^2\Big\}$



higher masses constrained by m_1^2



higher masses constrained by $|\Lambda| + \sigma^2$

- a version of scale separation: second, third mass etc. can’t be arbitrarily heavy

- first case in agreement with the Spin-2 conjecture;
  in second case, counterexamples

[Klaewer, Lüst, Palti ’18]
[de Rham, Heisenberg, Tolley ’18]
[Bachas ’19]

- how about the first mass?

[M_n smooth]

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$$m_k^2 \leqslant \max\Big\{(D-2)\sigma^2,\ \tfrac{1}{n-1}(|\Lambda| + \sigma^2)\Big\} + \beta(k\, m_D^{D-2} m_d^{2-d})^{2/n}$$

[Planck masses]

[$\alpha, \beta, \gamma \sim 10^4$]

[De Luca, AT '21]
using [Hassannezhad '12]

but this doesn't exclude scale separation:

e.g. $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_p$ → large second term

$p \to \infty$

[Gautason, Schillo, Van Riet, Williams '15]
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- this issue is eliminated working with the diameter:

[De Luca, AT '21] using [Setti '98]

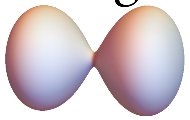
$$m_k^2 \leqslant \left(|\Lambda| + (D-1)\sigma^2\right) + \gamma\frac{k^2}{\mathrm{diam}^2}$$

but now problem is 'nonlocal': how large is diam?

for sphere quotients:
[Greenwald '00,
Gorodski, Lange, Lytchak, Mendes '19,
Collins, Jafferis, Vafa, Xu, Yau '22]

- lower bounds can be useful for establishing scale separation in a given solution.

Cheeger

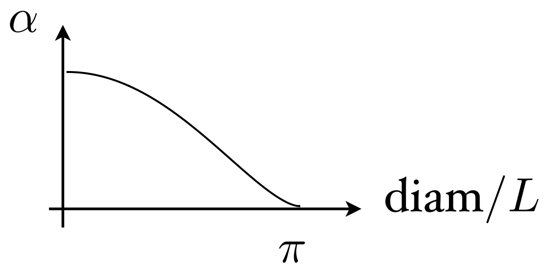


$$\frac{h_1^2}{4} \leqslant m_1^2$$

[even with O-planes]

$$\alpha\left(\frac{\mathrm{diam}}{L_{\mathrm{AdS}}}\right)\frac{1}{\mathrm{diam}^2} \leqslant m_1^2$$

[even with D-branes]



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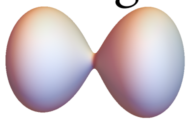

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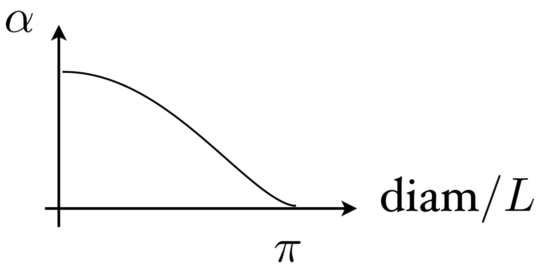

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- AdS$_4\times$ CY: estimate of Cheeger on 10d solution confirms

[$\int F_4 \sim N$]

$$m_1 \geqslant \frac{h_1}{2} \sim N^{-1/4} \gg \sqrt{|\Lambda|} \sim N^{-3/4}$$

[DeWolfe, Giryavets, Kachru, Taylor '05]
[Acharya, Benini, Valandro '06;
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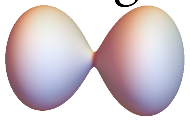

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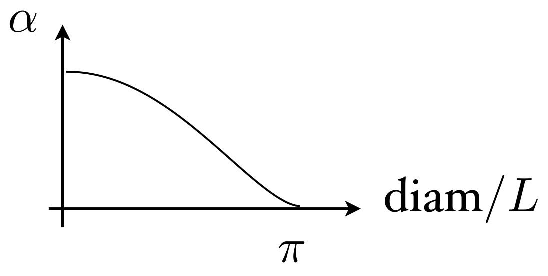

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- A potentially simpler example: M-theory lift smooth AdS$_4 \times$ (weakG_2)$_7$? $F_4 \sim N\text{vol}_{\text{AdS}_4}$

$$m_1 \geqslant \frac{c}{\text{diam}} \sim N^{-11/48} \gg \sqrt{|\Lambda|} \sim N^{-7/24}$$

[Cribiori, Junghans, Van Hemelryck,
Van Riet, Wrase '19]

- Another way to obtain scale separation: Casimir energy

[De Luca, De Ponti, Mondino, AT '22]

inspired by
[De Luca, Silverstein, Torroba '21]
[Arkani-Hamed, Dubovsky, Nicolis, Villadoro '07]

$$\langle T^{\text{Cas}}_{\mu\nu}\rangle = \tfrac{\ell_P}{R_7^{11}} g_{\mu\nu} \qquad \langle T^{\text{Cas}}_{mn}\rangle = -\tfrac{4}{7}\tfrac{\ell_P}{R_7^{11}} g_{mn}$$

'Freund–Rubin' $\text{AdS}_4 \times T^7$: $F_4 \sim N \text{vol}_{\text{AdS}_4}$

$$m_1 \geqslant \frac{c}{\text{diam}} \sim N^{-2/3} \gg \sqrt{|\Lambda|} \sim N^{-11/3}$$

$m_1 \sim |\Lambda|^{1/11}$ much slower than conjectural bound $|\Lambda|^{1/4}$ [Rudelius '21, Castellano, Herráez, Ibáñez '21]

Gravity localization

- Similar bounds also apply when $\int_M \sqrt{g} \mathrm{e}^{(D-2)A} = \infty$

⇨ no massless graviton: $m_0 \neq 0$

(could be evaded if $\exists\ L^2$ harmonic function: but this never happens)

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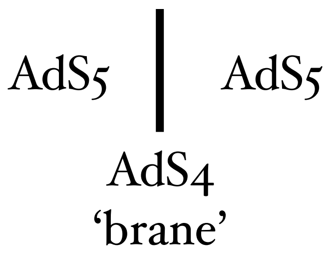
- maybe one can still obtain 4d gravitational potential in some regime?

5d model:

[Karch, Randall '00]



$$V \sim G M_1 M_2 \left(\frac{1}{R} + \frac{L_5^2}{L_4 R^2} + \dots \right)$$

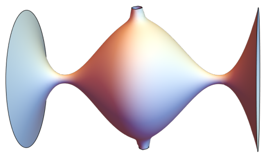
⇨ 4d behavior for $R \gg L_5^2 / L_4$

relies on:

- $m_1 \ll \sqrt{|\Lambda|} \ll m_2$
- wave-functions for $m_{k>1}$ small near brane

- string theory realizations?
  - hol. duals of defects in $\mathcal{N} = 4$ super-YM

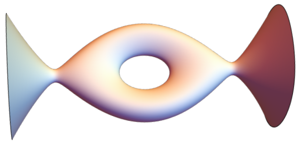
    $S^2 \times S^2$ fibred on

    [Bachas, Lavdas '17, '18] based on
    [D'Hoker, Estes, Gutperle '07]
    [Assel, Bachas, Estes, Gomis '11]

  - MN-type solutions over ∞-volume Riemann surfaces

    top. spheres fibred on

    [Maldacena, Nuñez '00...]

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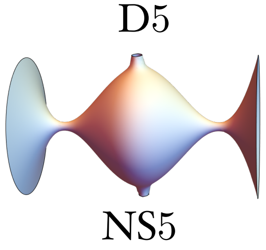

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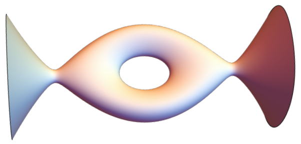

[Maldacena, Nuñez '00...]

- a version of the scale separation problem:

$$m_0 \ll \sqrt{|\Lambda|} \ll m_1$$

easy

problematic, because:

$$m_k^2 < 150k^2 \text{max}\Big\{m_0^2,\, |\Lambda| + \sigma^2\Big\}$$

⇨ 'localization' only up to cosmological scale?

- wave-function suppression might help, but hard to assess with our methods.

Conclusions

- No math evidence against scale separation; not even without sources

- Nontrivial relations among different masses; ‘bootstrap’ flavor

- We should improve by removing warping.

  requires advance in theory of optimal transport with ‘negative’ dimensions

- Bounds in terms of diameter, Cheeger constant:
  useful to estimate masses without computations

- Optimal transport approach to curvature in terms of entropy:
  might be of deeper significance