Applied Nonrelativistic Conformal Field Theory

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Plan

- Nonrelativistic conformal symmetry
- Nonrelativistic CFTs
- Fermions at unitarity
- Neutrons and "UnNuclear Physics"

Refs.: Y. Nishida, DTS 2007
H.-W. Hammer, DTS 2103.12610
S.D. Chowdhury, R. Mishra, DTS to appear

Schrödinger group

- group of symmetries of the free Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

- Phase rotation M $\psi \to e^{i\alpha}\psi$
- space and time translations $\mathbf{P}, H$; rotations J_{ij}
- Galilean boosts $\mathbf{K}$ $\psi(t,\mathbf{x}) \to e^{im\mathbf{v}\cdot\mathbf{x}-\frac{i}{2}mv^2t}\,\psi(t,\mathbf{x}-\mathbf{v}t)$
- Dilatation D $\psi(t,\mathbf{x}) \to \lambda^{3/2}\psi(\lambda^2 t, \lambda\mathbf{x})$

“Proper conformal transformation”

$$C: \ \psi(t, \mathbf{x}) \to \frac{1}{(1+\alpha t)^{3/2}} \exp\left(\frac{i}{2}\frac{m\alpha x^2}{1+\alpha t}\right)\psi\left(\frac{t}{1+\alpha t}, \frac{\mathbf{x}}{1+\alpha t}\right)$$

Schrödinger algebra

$X \setminus Y$	P_j	K_j	D	C	H
P_i	0	$-i\delta_{ij}M$	$-iP_i$	$-iK_i$	0
K_i	$i\delta_{ij}M$	0	iK_i	0	iP_i
D	iP_j	$-iK_j$	0	$-2iC$	$2iH$
C	iK_j	0	$2iC$	0	iD
H	0	$-iP_j$	$-2iH$	$-iD$	0

$SO(2,1)$

Schrödinger symmetry from light-cone reduction

- Start from a relativistic conformal algebra in 1 higher dimension
- Select operators that commute with the light-cone momentum $P^+ = (P^0 + P^{d+1})/\sqrt{2}$
- These operators form a Schrödinger algebra
- $P^- \to H \qquad D + M^{+-} \to D \qquad \frac{K^+}{2} \to C$ ...

Nonrelativistic CFTs

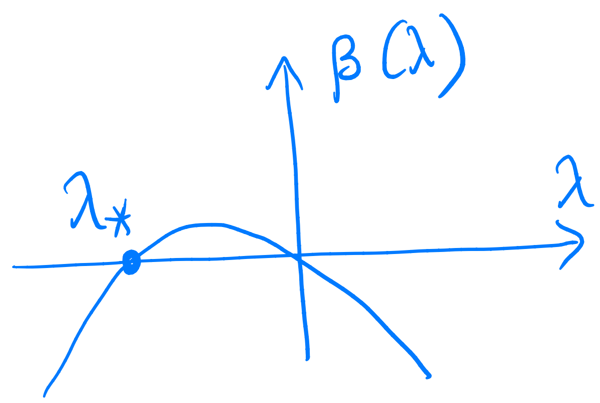
Y. Nishida, DTS, 2007

- are QFTs with Schrödinger symmetry
- primary operators: $[K_i, O(0)] = [C, O(0)] = 0$
- operator dimensions

$$\langle O(t,\mathbf{x}) O^\dagger(0,\mathbf{0}) \rangle = \frac{C}{t^\Delta} \exp\left(\frac{imx^2}{2t} \right)$$

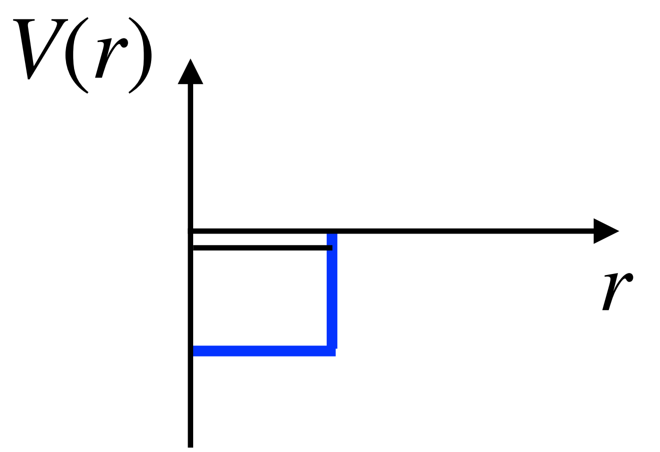
Example of NRCFT

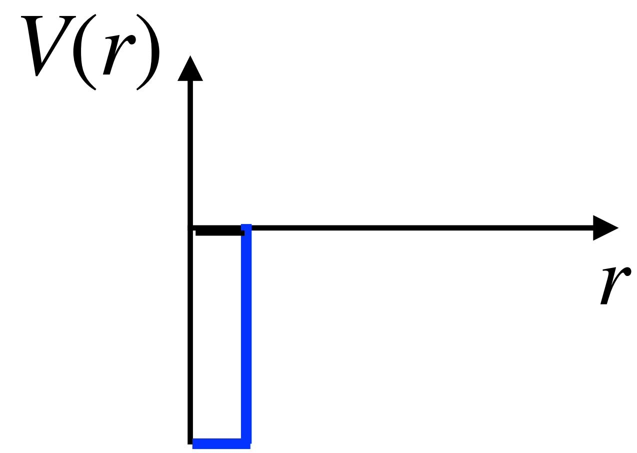
- $S = \int dt\, d^d\mathbf{x} \left(i\psi^\dagger \partial_t \psi - \frac{1}{2} |\nabla \psi|^2 - \lambda \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right)$
- NR power counting: $[E] = 2, [p] = 1, [\psi] = \frac{d}{2}$
- $\beta(\lambda) = \epsilon\lambda + \frac{1}{2\pi}\lambda^2$



- interacting CFT at $\lambda_* = -2\pi\epsilon$
- Does the fixed point survive at $d = 3$?
  - yes, this is the so-called "unitarity fermion"

“Unitarity regime”





- Take a potential of a certain shape, e.g.,
  $V(r) = -V_0$ for $r < r_0$, 0 for $r > r_0$

- fine-tune the depth so that there is one “bound state” at zero energy

$$V_0 = \frac{\pi^2 \hbar^2}{8m} \frac{1}{r_0^2}$$

- Then let $r_0 \to 0$: “unitarity regime” s-wave scattering saturates unitarity

Scattering length and unitarity regime

- This situation corresponds to low-energy resonant scattering in quantum mechanics
- s-wave scattering amplitude given by scattering length a and effective range r_0:

$$f(k) = \frac{1}{-ik + \frac{1}{a} + \frac{1}{2}kr_0^2}$$

- Unitarity regime: $a \to \infty$
  $r_0 \to 0$
  - no dimensionful length scale

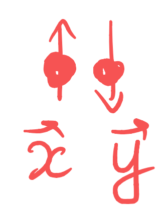
- The unitarity regime can be understood directly, without taking the limit $r_0 \to 0$

Unitarity fermions: QM

- Wave function of m spin-up and n spin-down fermions $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_m; \mathbf{y}_1, \ldots, \mathbf{y}_n)$

- ψ antisymmetric under exchanging two $\mathbf{x}$'s or $\mathbf{y}$'s

- When one spin-up and one spin-down fermions approach each other:



$$\psi(\mathbf{x}, \mathbf{y}) = \frac{C}{|\mathbf{x} - \mathbf{y}|} + b(\mathbf{x}, \mathbf{y}), \quad b(\mathbf{x}, \mathbf{x}) = 0$$

- $H = -\frac{1}{2} \sum_a \frac{\partial^2}{\partial \mathbf{x}_a^2} - \frac{1}{2} \sum_a \frac{\partial^2}{\partial \mathbf{y}_a^2}$

Charge-2 local operator

- Second-quantized formulation of QM:

$$\langle 0 | \psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y}) | \Psi \rangle = \Psi(\mathbf{x}, \mathbf{y})$$

- Limit $\mathbf{y} \to \mathbf{x}$ does not exist:

$$\langle 0 | \psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{x}) | \Psi \rangle = \Psi(\mathbf{x}, \mathbf{x}) = \infty$$

- but one can define

$$O_2(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{x}} |\mathbf{x} - \mathbf{y}| \psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})$$

- then

$$\langle 0 | O_2(\mathbf{x}) | \Psi \rangle = \lim_{\mathbf{y}\to\mathbf{x}} |\mathbf{x} - \mathbf{y}| \Psi(\mathbf{x}, \mathbf{y}) = \text{finite}$$

Dimension of O_2

- $O_2(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{x}} |\,\mathbf{x}-\mathbf{y}\,|\,\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})$
- $\Delta[O_2] = 2\Delta[\psi] - 1 = 2$
- In free theory: $\Delta[\psi\psi] = 3$
- Situation is reminiscent of holography: two boundary CFT with different dimensions of a scalar operators: $\Delta_+ + \Delta_- = d$

  here effectively # of spacetime dim is 5

Charge-3 operator

- Need to know hort distance behavior of $\Psi(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y})$
- 3-body problem solved by Efimov ~ 1970

$$\Psi(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}) \sim R^{-0.2273}$$

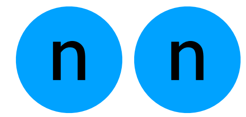
$$R^2 = |\mathbf{x}_1 - \mathbf{x}_2|^2 + |\mathbf{x}_1 - \mathbf{y}|^2 + |\mathbf{x}_2 - \mathbf{y}|^2$$

- Charge-3 operator

$$O_3(\mathbf{x}) = \lim_{\mathbf{x}_2 \to \mathbf{x}} \lim_{\mathbf{y} \to \mathbf{x}} R^{0.2273} \psi_\uparrow(\mathbf{x}) \psi_\uparrow(\mathbf{x}_2) \psi_\downarrow(\mathbf{y})$$

- $\Delta[O_3] = 4.2727$

NRCFT in real world



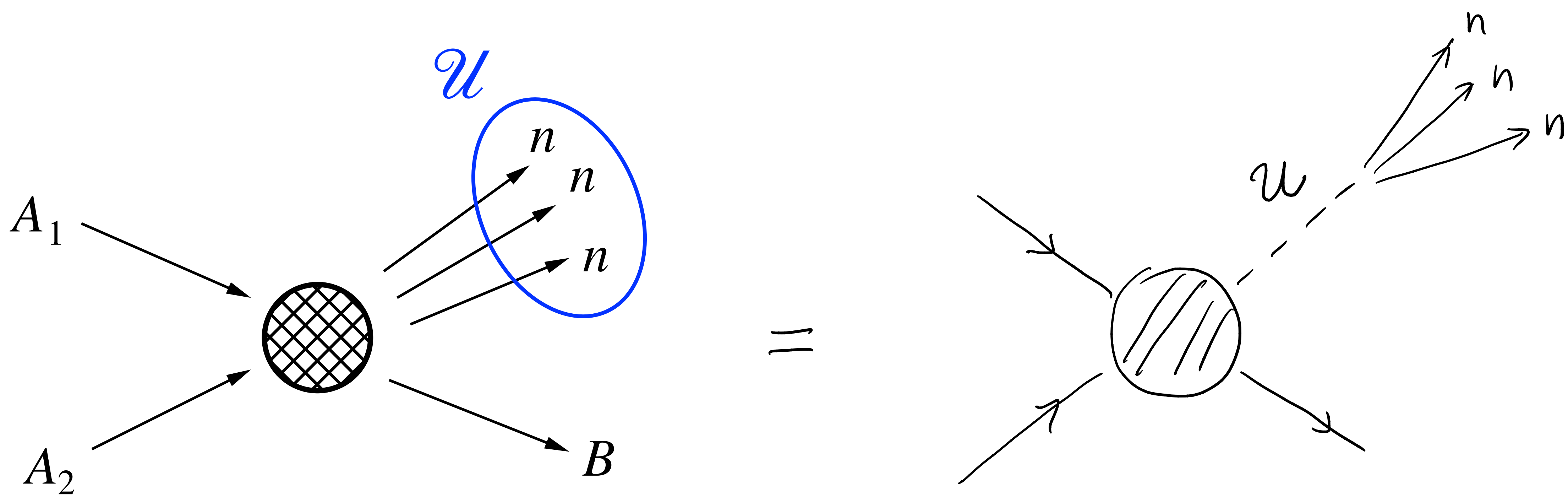
- 2 neutrons almost form a bound state
- Scattering length between 2 neutrons anomalously large: $a \approx -19$ fm, $r_0 \approx 2.75$ fm
- Neutrons are approximately described by a NRCFT in the energy range between $\hbar^2/ma^2 \sim 0.1$ MeV and $\hbar^2/mr_0^2 \sim 5$ MeV
- With a consequence for nuclear reactions

Nuclear reactions

H.-W. Hammer and DTS, 2103.12610

- Many nuclear reactions with emissions of neutrons:
  - $^{3}H + {}^{3}H \rightarrow {}^{4}He + 2n$
  - $^{7}Li + {}^{7}Li \rightarrow {}^{11}C + 3n$
  - $^{4}He + {}^{8}He \rightarrow {}^{8}Be + 4n$
- In some kinematic regime these reactions occur in two steps

"UnNuclear physics"

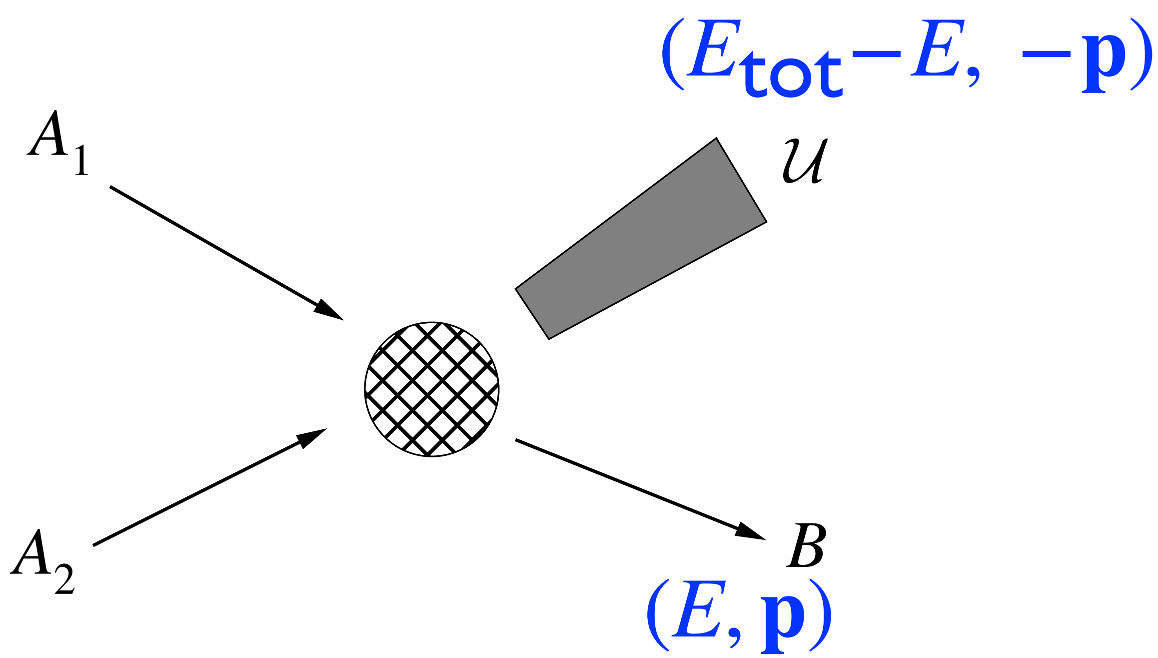


$$P(A_1 + A_2 \to B + 3n) = P(A_1 + A_2 \to B + \mathcal{U})P(\mathcal{U} \to 3n)$$

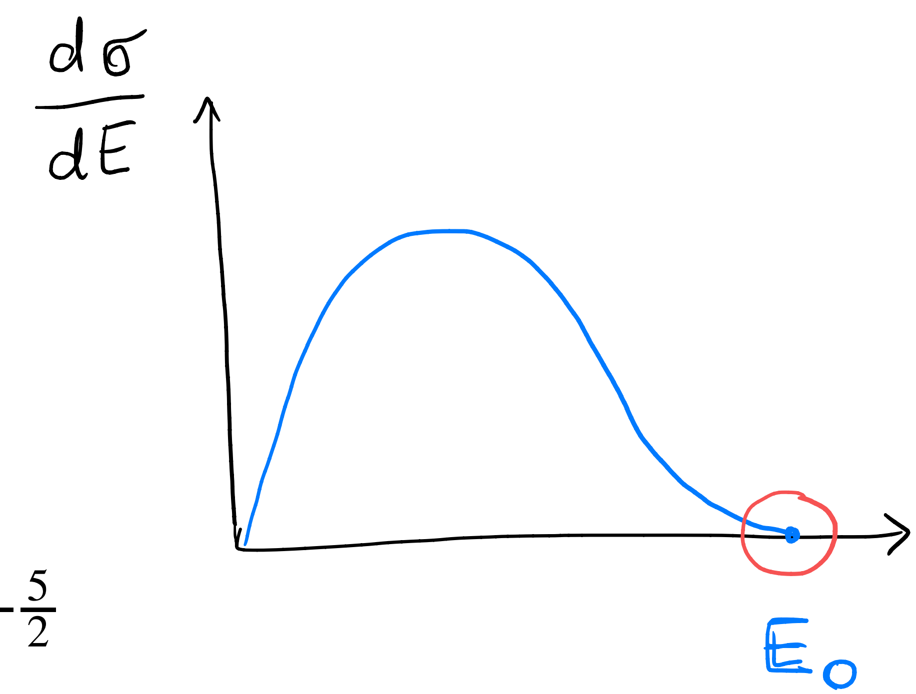
When energy scale of primary reaction is larger than $\mathcal{U} \to 3n$

$\mathcal{U}$ = unnucleus (nonrelativistic version of Georgi's unparticle)

Rates of unnuclear processes



$$E_{\text{tot}} = E + E_{\mathcal{U}}$$



- $$\frac{d\sigma}{dE} \sim |\mathcal{M}|^2 \sqrt{E} \times \underbrace{\operatorname{Im} G_{\mathcal{U}}(E_{\text{tot}} - E, \mathbf{p})}$$

  $\rightarrow (E_0 - E)^{\Delta - \frac{5}{2}}$

- Near end point: $\frac{d\sigma}{dE} \sim (E_0 - E)^{\Delta - \frac{5}{2}}$

$\Delta = \Delta[\mathcal{U}]$

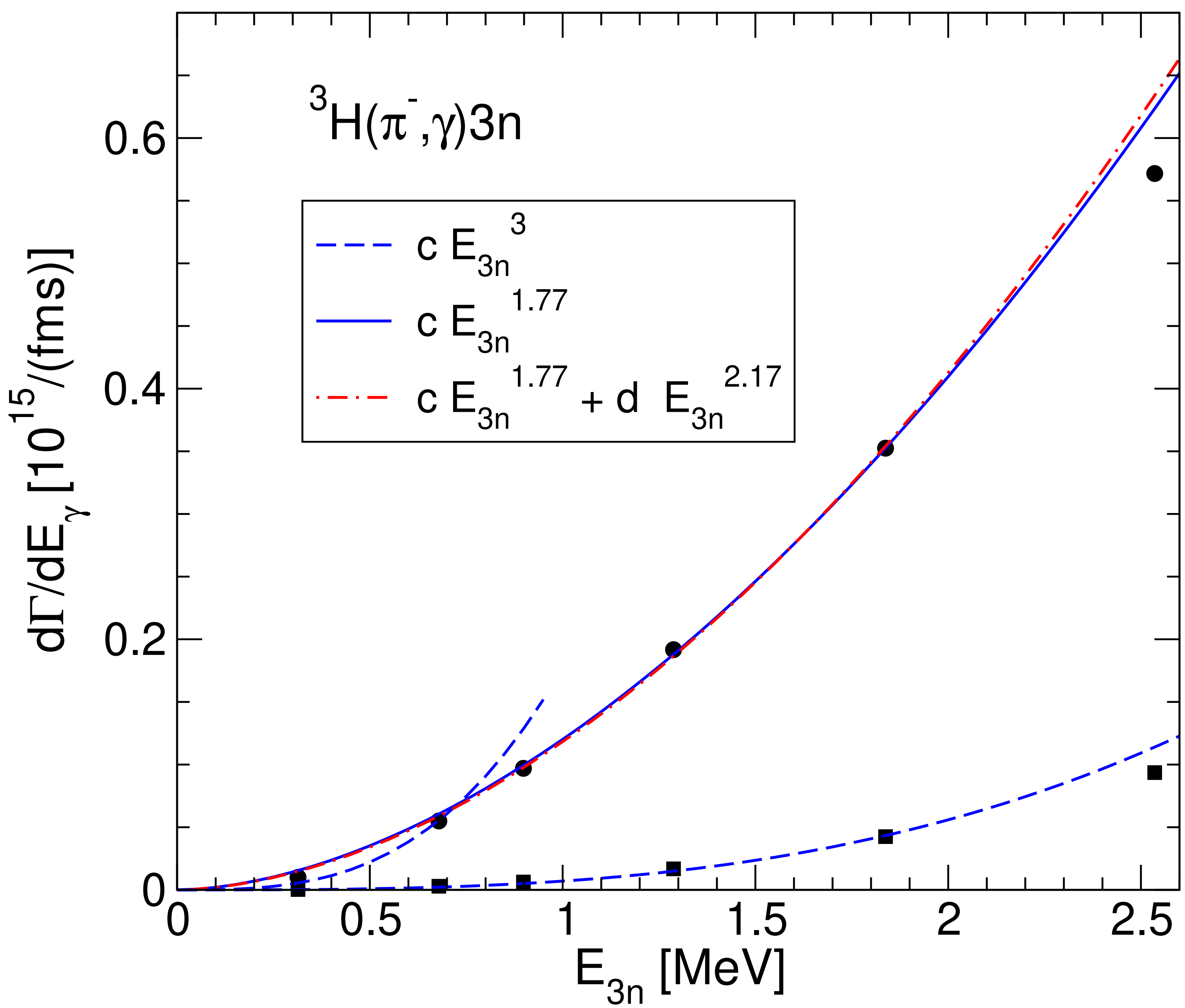
Nuclear reactions

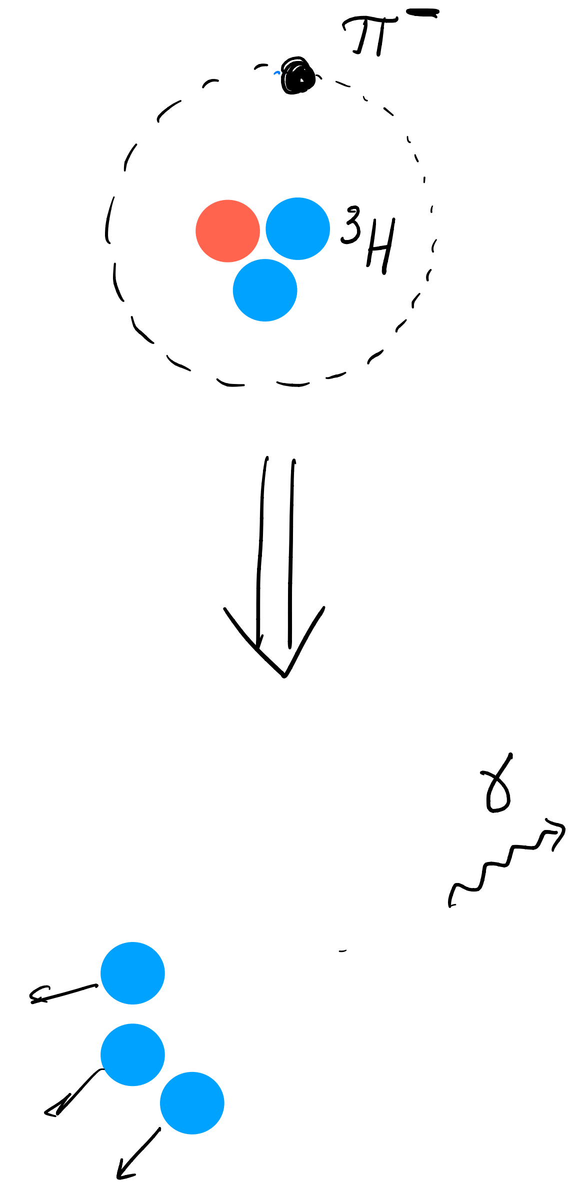
- $^3H + {}^3H \to {}^4He + 2n$ — $\alpha = -0.5$
- $^7Li + {}^7Li \to {}^{11}C + 3n$ — $\alpha = 1.77$
- $^4He + {}^8He \to {}^8Be + 4n$ — $\alpha = 2.5 - 2.6$

Watson-Migdal 1950's

- recall our prediction:
  - $\dfrac{d\sigma}{dE} \sim (E_0 - E)^{\alpha}$
- and regime of validity: kinetic energy of neutrons in their c.o.m. frame between $\hbar^2/ma^2 \sim 0.1$ MeV $\hbar^2/mr_0^2 \sim 5$ MeV

Comparison with “experiment”





$$\pi^- + {}^3\text{H} \rightarrow \gamma + 3n$$

Golak et al. PRC 98, 054001 (2018)

Away from conformality

- Finite scattering length and effective range can be included as perturbation from fixed point

$$L = L_{\text{CFT}} + \frac{1}{a} O_2^\dagger O_2 - r_0 O_2^\dagger \left(i\partial_t + \tfrac{1}{4}\nabla^2 \right) O_2$$

- Contribution to $\langle O_3 O_3^\dagger \rangle$ can be computed using conformal perturbation theory
  S.D. Chowdhury, R. Mishra, DTS to be published

- Gives the corrections to the conformal behavior as one approaches the two ends of the energy conformal window

$$\frac{d\sigma}{dE} \sim \omega^\alpha \left(1 + \frac{c_1}{a_0\sqrt{m\omega}} + c_2 r_0 \sqrt{m\omega} \right) \qquad c_2 = 0$$

Conclusion

- NR conformal field theories have Schrödinger symmetry
- Example: fermions at unitarity
- Approximately realized by neutrons in nuclear physics
- Leads to a power-law behavior of differential cross sections of certain processes near threshold
- Nonrelativistic conformal perturbation theory can be developed and used

Thank you