A spindle story:
from AdS to equivariant localization and back

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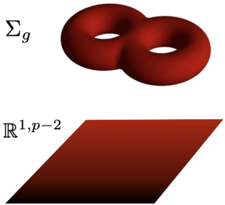
Outline

1. Holography and wrapped branes
2. Branes wrapped on a spindle
3. Extremal problems and factorization in gravitational blocks
4. Branes wrapped on higher dimensional orbifolds
5. Equivariant localization in field theory and in supergravity
6. Outlook

AdS/CFT dualities from wrapped branes

- [Maldacena,Nuñez] in 2000 enlarged hugely the list of examples of AdS/CFT dualities by considering p-branes wrapped on a Riemann surface Σ_g with genus g

- At low energies the world-volume theory is a $(p-1)$-dimensional QFT



- If this is a CFT $\rightarrow$ AdS$_p \times \Sigma_g$ solution in some gauged supergravity $\hookrightarrow$ AdS$_p \times M$ solution in $D = 10, 11$ supergravity

- Supersymmetry preserved on $\Sigma_g \rightarrow$ SCFT$_{p-1}$ with susy AdS$_p$ dual

Supersymmetry with the topological twist

- Couple the theory to a background R-symmetry gauge field A_μ

$$\nabla_\mu \epsilon = (\partial_\mu + \omega_\mu)\epsilon = 0 \quad \rightarrow \quad (\partial_\mu + \omega_\mu - A_\mu)\epsilon = 0$$

where $\omega_\mu \equiv {\omega_\mu}^{12}$ is the spin connection on Σ_g

- Choosing $A_\mu = \omega_\mu$ supersymmetry is preserved by $\partial_\mu \epsilon = 0$
- ϵ becomes effectively a scalar $\longrightarrow$ topologically twisted theory
- Metrics on $\Sigma_g = H_2/\Gamma$ with $\Gamma \subset SL(2,\mathbb{R})$ have constant curvature
- Geometrically: A is the connection on a line bundle L, that gets identified with the tangent bundle of Σ_g:

$$\int_{\Sigma_g} c_1(L) = \int_{\Sigma_g} \frac{dA}{2\pi} = \int_{\Sigma_g} \frac{d\omega}{2\pi} = \int_{\Sigma_g} c_1(T\Sigma_g) = 2(g-1)$$

Supersymmetry with no twist

- For genus $g = 0$, namely S^2, supersymmetry can also be preserved differently, by the standard Killing spinors that exist on all spheres

$$\nabla_\mu \epsilon = \gamma_\mu \epsilon$$

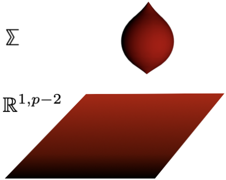
- In this case there is no background R-symmetry gauge field A
- More generally, we can couple the theories to a number of "flavour" background fields A_I, with the Killing spinors charged under a linear combination, e.g. the diagonal $A_R = \sum_I A_I$
- The supersymmetry constraints on the fluxes $n_I \equiv \int_{\Sigma_g} \frac{dA_I}{2\pi}$ become

topological twist $\sum_I n_I = 2(1 - \mathrm{g})$

no twist $\sum_I n_I = 0$

A spindle story

- [Ferrero,Gauntlett,DM,Perez-Ipiña,Sparks] in 2020 expanded further the list of examples of AdS/CFT dualities proposing that p-branes can be wrapped on a "spindle" Σ
- This is a nickname for the weighted projective line $\mathbb{WP}^1_{[n_+,n_-]}$, that is a two-sphere with two different orbifold singularities $\mathbb{C}/\mathbb{Z}_{n_+}$, $\mathbb{C}/\mathbb{Z}_{n_-}$ at its poles, making it look a bit like a spindle



- If the $(p-1)$-dimensional theory is a CFT $\rightarrow$ AdS$_p \times \Sigma$ solution in some gauged supergravity $\hookrightarrow$ AdS$_p \times M$ solution in $D = 10, 11$

The spindle's new features

Besides the obvious difference with respect to the smooth case due to the orbifold singularities, the spindle yields additional surprises

1. it does not admit metrics with constant curvature
2. it has a $U(1)_\Sigma$ symmetry
3. supersymmetry preserved in only two ways [Ferrero,Gauntlett,Sparks]

$$\sum_I n_I = \frac{\sigma_1}{n_+} + \frac{\sigma_2}{n_-} \qquad \sigma_1\sigma_2 = \sigma = \pm 1 : \quad \text{twist/anti-twist}$$

$\sigma = +1$ is topologically a twist: $\pm \int_\Sigma \frac{dA_R}{2\pi} = \int_\Sigma c_1(T\Sigma) = \frac{1}{n_+} + \frac{1}{n_-}$

$\sigma = -1$: for $n_+ = n_- = 1$ reduces to S^2 with no twist

Branes wrapped on the spindle: explicit examples

- After the first example involving D3 branes, several additional examples have been constructed [many authors]
- M2-branes: $\mathrm{AdS}_2 \times M_9$ in $D = 11$ dual to SCQM $\rightarrow$ BPS black hole entropy $\rightarrow$ I-extremization
- M5-branes: $\mathrm{AdS}_5 \times M_6$ in $D = 11$ dual to SCFT_4 $\rightarrow$ a central charge $\rightarrow$ a-maximization
- D4-branes: $\mathrm{AdS}_4 \times M_6$ in massive Type IIA dual to SCFT_3 $\rightarrow$ F_{S^3} free energy $\rightarrow$ F-extremization
- D2-branes: $\mathrm{AdS}_2 \times M_8$ in massive Type IIA dual to SCQM $\rightarrow$ BPS black hole entropy $\rightarrow$ I-extremization
- Variants of these "basic" cases...

Branes wrapped on the spindle: general lessons

- In all cases in the SCFT an observable† "F" is extremized over a suitable parameter space, associated with the selection of the exact R-symmetry of the SCFT_{p-1}
- R-symmetry of the SCFT_{p+1} can "mix" with the $U(1)_{\mathbb{\Sigma}}$ isometry

  $\rightarrow$ Parameters: chemical potentials $\Delta_I \sim U(1)^m$ global symmetries of the $\mathrm{SCFT}_{p+1} + \epsilon \sim U(1)_{\mathbb{\Sigma}}$
- The relevant extremal function $F(\Delta_I, \epsilon)$ can be decomposed in "blocks" $\mathcal{F}_m(\Delta_I)$, encoding universal properties of the SCFT_{p+1}
- In all cases there should be an extremal problem in supergravity

† F is for "off-shell Free energy" or "extremal Function"

Extremal functions and their factorization

Conjecture [Faedo,DM]: the extremal functions for M2, D3, D4, M5 branes wrapped on spindles can be written in terms of gravitational blocks as

$$\boxed{F(\varphi_I, \epsilon) = \frac{1}{\epsilon}\mathcal{F}_m(\varphi_I + \frac{n_I}{2}\epsilon) \pm \frac{1}{\epsilon}\mathcal{F}_m(\varphi_I - \frac{n_I}{2}\epsilon)}$$

where the parameters φ_I, ϵ and the fluxes n_I obey the constraints

$$\sum_I \varphi_I - \frac{1}{2}\left(\frac{\sigma^1}{n_+} - \frac{\sigma^2}{n_-}\right)\epsilon = 2 \qquad \sum_I n_I = \frac{\sigma^1}{n_+} + \frac{\sigma^2}{n_-}$$

On-shell the $F(\varphi_I^*, \epsilon^*)$ coincide with the relevant observables:

for D3, M5 agree with anomaly polynomial computations in the SCFT$_{p+1}$

for M2, D4 predict the large N limit of entropy/free energy

Gravitational blocks

[Hosseini,Hristov,Zaffaroni]

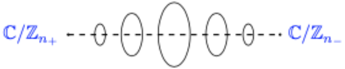
- The building blocks $\mathcal{F}_m(\Delta_I)$ for dimensions $p+1=3,4,5,6$:

brane	sugra solution	$\mathcal{F}_m(\Delta_I)$	QFT interpretation
M2	$\mathrm{AdS}_4 \times S^7$	$N^{3/2}(\Delta_1\Delta_2\Delta_3\Delta_4)^{1/2}$	S^3 free energy
D3	$\mathrm{AdS}_5 \times S^5$	$N^2\Delta_1\Delta_2\Delta_3$	4d anomaly polynomial
D4	$\mathrm{AdS}_6 \times S^4$	$N^{5/2}(\Delta_1\Delta_2)^{3/2}$	S^5 free energy
M5	$\mathrm{AdS}_7 \times S^4$	$N^3(\Delta_1\Delta_2)^2$	6d anomaly polynomial

- p-branes compactified on Σ give rise to $\mathrm{AdS}_p \times M$ solutions

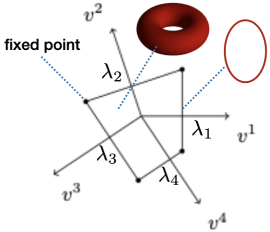
From the spindle to toric orbifolds

- The spindle is the only toric orbifold in two real dimensions



$\mathbb{T}^1 = U(1)$ action with two fixed points at the north and south poles

- Toric orbifolds $\mathbb{M}_{2n}$: $\mathbb{T}^n$ fibration over a convex rational polytope $\mathcal{P} = \{y \in \mathbb{R}^n \; : \; y_i v_i^a - \lambda_a \geq 0\}$, with $a = 1, \dots, d$



symplectic form: $\omega = dy_i \wedge d\phi_i$

coordinates on $\mathbb{T}^n$: $\phi_i \to \xi = \epsilon_i \frac{\partial}{\partial \phi_i}$

moment maps: $y_i \to H = \epsilon_i y_i$

facets/divisors: $D_a = \{y_i v_i^a = \lambda_a\}$

line bundles: L_a with $c_1(L_a)$

Branes wrapped on higher dimensional orbifolds

Conjecture [Faedo,Fontanarossa,DM]: the extremal functions for D4, M5 branes wrapped on $\mathbb{M}_4$ can be written in terms of gravitational blocks as

$$F(\Delta_I, \epsilon_i) = \sum_a \frac{1}{d_{a,a+1}\epsilon_1^a \epsilon_2^a} \mathcal{F}_m(\Delta_I - p_I^a \epsilon_1^a - p_I^{a+1} \epsilon_2^a)$$

where, denoting with $\epsilon = (\epsilon_1, \epsilon_2)$ the two equivariant parameters:

$$d_{a,a+1} = \langle v^a, v^{a+1} \rangle \equiv \epsilon^{ij} v_i^a v_j^{a+1} \qquad \epsilon_1^a = -\frac{\langle v^{a+1}, \epsilon \rangle}{d_{a,a+1}} \qquad \epsilon_2^a = \frac{\langle v^a, \epsilon \rangle}{d_{a,a+1}}$$

The parameters Δ_I and the fluxes p_I^a obey the constraints

$$\Delta_1 + \Delta_2 = 2 \qquad p_1^a + p_2^a = \sigma^a \qquad \sigma^a = \pm 1$$

with the p_I^a parameterizing the "physical fluxes" $q_I^a \equiv \int_{D_a} \frac{dA_I}{2\pi} = D_{ab} p_I^b$

- Some explicit supergravity solutions corroborated this conjecture

Geometrization of the extremal problems

- For M2 and D3 $\rightarrow$ Sasakian volume extremization [DM,Sparks,Yau]
- For M2 and D3 wrapped on $\mathbb{\Sigma}$ (and Σ_g) $\rightarrow$ extremization related to the master volume $\mathcal{V}(\lambda_a, \epsilon_i)$ [(Couzens),Gauntlett,DM,Sparks]
- GK geometry [Gauntlett,Kim]: $X_{2n-1} \rightarrow Y_{2n+1} \rightarrow \mathbb{\Sigma}$

$$F = S_{SUSY}(\lambda_a, \epsilon_i) \qquad\qquad \mathcal{F}_m = \mathcal{V}(\lambda_a, \epsilon_i)$$

- Gravitational block form of the extremal function S_{SUSY} proved in [Boido,Gauntlett,DM,Sparks]. For details see poster by A. Boido
- For each brane there should be a geometric extremal problem in supergravity dual to the extremal problems in SCFT

Branes wrapped on toric orbifolds: general lessons

Let's recap the lessons from examples/special cases in gravity and QFT

1. In all the SCFT examples one observes that there exist an extremal function that determines the R-symmetry of the SCFT and this takes the form of blocks summed over the fixed points of an orbifold
2. In all the supergravity examples one observes that there exist an extremal function that governs the geometry and this takes the form of gravitational blocks summed over the fixed points of an orbifold
3. For M2 and D3 branes we have a good understanding of the extremal problems in (GK) geometry. What about other branes?
4. In all these problems the data are always topological or have to do with a choice of vector field (i.e. equivariant)

→ equivariant localization should be useful in holography! [DM,Zaffaroni]

Equivariant localization

- Consider a manifold (or orbifold) $\mathbb{M}_{2n}$ with the action of a torus $\mathbb{T}^k$
- For a vector field $\xi = \epsilon_i \frac{\partial}{\partial \phi_i} \in \text{Lie}(\mathbb{T}^k)$ one can define the equivariant exterior derivative as $d_\xi \equiv d + 2\pi i_\xi$
- Notice that d_ξ maps p-forms in $(p+1)$-forms $\oplus$ $(p-1)$-forms, so it acts naturally on formal sums of forms of different degree
- Such a polyform $\alpha^{\mathbb{T}}$ is said to be equivariantly closed if it satisfies

$$d_\xi \alpha^{\mathbb{T}} = 0$$

- It is natural to work in the framewok of toric geometry: $k = n$. In this context the Berline-Vergne fixed point theorem states

$$d_\xi \alpha^{\mathbb{T}} = 0 \quad \Rightarrow \quad \int_{\mathbb{M}_{2n}} \alpha^{\mathbb{T}} = \sum_{A=1} \frac{\alpha^{\mathbb{T}}|_{y_A}}{d_A \, e^{\mathbb{T}}|_{y_A}}$$

Trial central charges from equivariant localization

- Consider M5 branes wrapped on a toric orbifold $\mathbb{M}_4$ [DM,Zaffaroni]
- The anomaly polynomial (at leading order in N) reads

$$\mathcal{A}_{6\mathrm{d}} = \frac{N^3}{24} c_1(F_1)^2 c_1(F_2)^2 \qquad c_1(F_I) = \Delta_I c_1(F_R^{2\mathrm{d}}) - p_I^a c_1^{\mathbb{T}}(L_a)$$

where we have introduced the equivariant first Chern classes

$$c_1^{\mathbb{T}}(L_a) = c_1(L_a) + 2\pi \mu_a^i \epsilon_i \qquad d_\xi c_1^{\mathbb{T}}(L_a) = 0$$

- Integrating $\mathcal{A}_{6\mathrm{d}}$ on $\mathbb{M}_4$ yields the equivariant integral

$$F(\Delta_I, \epsilon_i) = \frac{N^3}{24} \int_{\mathbb{M}_4} (\Delta_1 - p_1^a c_1^{\mathbb{T}}(L_a))^2 (\Delta_2 - p_2^a c_1^{\mathbb{T}}(L_a))^2$$

- The Berline-Vergne fixed point theorem reproduces the M5 brane extremal function conjectured in [Faedo,Fontanarossa,DM]!

The equivariant volume

In [DM,Zaffaroni] we have proposed that a key object to consider is the equivariant volume of a (toric) orbifold $\mathbb{M}_{2n}$, defined as

$$\mathbb{V}(\lambda_a, \epsilon_i) = \frac{1}{(2\pi)^n} \int_{\mathbb{M}_{2n}} \mathrm{e}^{-H} \frac{\omega^n}{n!} = (-1)^n \int_{\mathbb{M}_{2n}} \mathrm{e}^{-\frac{\omega}{2\pi} - H}$$

1. It depends (manifestly) on the equivariant parameters $\epsilon_i \in \mathbb{R}^n$ and (through the shape of $\mathcal{P}$) on the Kähler parameters $\lambda_a \in \mathbb{R}^d$
2. $\omega^{\mathbb{T}} \equiv \omega + 2\pi H\,,\ d_\xi \omega^{\mathbb{T}} = 0 \ \Rightarrow$ evaluated with the Berline-Vergne fixed point theorem (below, the expression for toric $\mathbb{M}_4$)

$$\boxed{\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{a=1}^{d} \frac{1}{d_{a,a+1}\epsilon_1^a \epsilon_2^a} \mathrm{e}^{-\lambda_a \epsilon_1^a - \lambda_{a+1}\epsilon_2^a}}$$

Geometric interpretation in the compact case

1. Using the relation (true in co-homology) [Guillemin]

$$\frac{\omega^{\mathbb{T}}}{2\pi} = -\sum_a \lambda_a c_1^{\mathbb{T}}(L_a)$$

the equivariant volume can be interpreted as the generating functional for the "equivariant intersection numbers" $D_{a_1 \dots a_p}$ (polynomial in ϵ_i):

$$\mathbb{V}(\lambda_a, \epsilon_i) = (-1)^n \sum_p \frac{1}{p!} \sum_{a_1,\dots,a_p=1}^{d} \lambda_{a_1} \cdots \lambda_{a_p} D_{a_1 \dots a_p}$$

where

$$D_{a_1 \dots a_p} = \int_{\mathbb{M}_{2n}} c_1^{\mathbb{T}}(L_{a_1}) \dots c_1^{\mathbb{T}}(L_{a_p}) = (-1)^n \frac{\partial^p \mathbb{V}(\lambda_a, \epsilon_i)}{\partial \lambda_{a_1} \dots \partial \lambda_{a_p}}\bigg|_{\lambda_a = 0}$$

2. Also interpreted as the volume of the polytope $\mathcal{P}$ with measure e^{-H}:

$$\mathbb{V}(\lambda_a, \epsilon_i) = \int_{\mathcal{P}} \mathrm{e}^{-\epsilon_i y_i} \mathrm{d}y_1 \dots \mathrm{d}y_n$$

which can be evaluated using Stokes' theorem!

Non-compact equivariant volume

- Thanks to the convergence factor $\mathrm{e}^{-\epsilon_i y_i}$ we can also define the equivariant volume for asymptotically conical toric orbifolds

- For partial resolutions of Calabi-Yau singularities $X_{2n} = C(Y_{2n-1})$ it is interesting to consider the formal expansion

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{k=0}^{\infty} \mathbb{V}^{(k)}(\lambda_a, \epsilon_i) \qquad \text{where} \qquad \mathbb{V}^{(k)}(\lambda_a, \epsilon_i)$$

  - is a polynomial in λ_a, homogeneous of degree k
  - is a rational function in ϵ_i, homogeneous of degree $k - n$

Sasakian volume

$$\mathrm{Vol}[Y](\epsilon_i) = \frac{2\pi^n}{(n-1)!}\mathbb{V}(0, \epsilon_i)$$

master volume & S_{SUSY}

$$\mathcal{V}(\lambda_a, \epsilon_i) = (2\pi)^n \mathbb{V}^{(n-1)}(\lambda_a, \epsilon_i)$$

$$S_{SUSY} = \epsilon_1 (2\pi)^n \mathbb{V}^{(n-2)}(\lambda_a, \epsilon_i)$$

- The equivariant volume of a CY singularity captures and generalizes the Sasakan volume [MSY] and the master volume [GMS] at once!

Equivariant localization in supegravity

- The local geometry of M2 ($n = 4$) and D3 ($n = 3$) branes wrapped on Σ can be modelled by (toric) fibrations

$$\mathrm{CY}_n \hookrightarrow \mathrm{CY}_{n+1} \to \Sigma$$

where the CY_n encode information on the higher-dimensional SCFT

- Using the fixed point theorem one can prove

$$\boxed{\mathbb{V}_{CY_{n+1}} = \frac{1}{\epsilon_0}\mathbb{V}_{CY_n}(\lambda_a^+, \epsilon_i^+) - \frac{1}{\epsilon_0}\mathbb{V}_{CY_n}(\lambda_a^-, \epsilon_i^-)}$$

- The order $\mathcal{O}(\lambda_a^{n-1})$ of the above identity gives as a corollary an equivariant localization proof of the gravitational block formula

$$\boxed{S_{SUSY}|_{CY_{n+1}} = 2\pi\frac{\epsilon_1}{\epsilon_0}\left(\mathcal{V}_{CY_n}(\lambda_a^+, \epsilon_i^+) - \mathcal{V}_{CY_n}(\lambda_a^-, \epsilon_i^-)\right)}$$

in GK geometry, previously proved in [Boido,Gauntlett,DM,Sparks]

All branes wrapped on the spindle

- What is the geometric/physical interpretation of $\mathbb{V}^{(k)}(\lambda_a, \epsilon_i)$?
- We proposed that for other branes (M5, D4, D2) wrapped on the spindle, the geometry can still be modelled as $\mathrm{CY}_n \hookrightarrow \mathrm{CY}_{n+1} \rightarrow \Sigma$
- In all cases, for some suitable integers k_1, k_2 we can write

$$F = \mathbb{V}^{(k_1)}_{CY_{n+1}} \qquad M_a = -\frac{\partial \mathbb{V}^{(k_2)}_{CY_{n+1}}}{\partial \lambda_a}$$

where M_a is a set of integer fluxes defined by $\sum_a M_a v_i^a = 0$, reproducing the expected gravitational block decomposition

$$\boxed{F(\Delta_I, \epsilon) = \frac{1}{\epsilon}\left(\mathcal{F}_m(\Delta_I^+) \pm \mathcal{F}_m(\Delta_I^-)\right)}$$

See also [Benetti-Genolini,Gauntlett,Sparks] + talk by J. Sparks

Conclusions

- Equivariant localization plays a key role in holography, both in field theory as well as in characterizing supergravity solutions
- Anomaly polynomial computations for theories compactified on orbifolds can be reformulated in terms of equivariant localization
- Equivariant orbifold indices are the building blocks of supersymmetric partition functions/indices on orbifolds [Inglese,DM,Pittelli]
- Equivariant localization in supergravity explains the factorization in gravitational blocks of all extremal/entropy functions
- We expect that the equivariant volume is a central object encoding several extremal problems for supersymmetric geometries

Thank you!