

## I. DERIVATION OF MASTER EQUATION FOR CHIRAL WAVEGUIDE

Here the goal is to derive the master equation for the chiral waveguide within the Born-Markov approximation, which was presented in the lectures but left as an exercise. In case you are interested in learning more, this discussion closely follows Section 3 of Ref. [1]. This Born-Markov calculation to integrate out some “bath” degrees of freedom (in our case, the photons) to arrive at an open system description of a system is one of the paradigmatic calculations of quantum optics theory.

First, we write the Hamiltonian as

$$H = \underbrace{\int_{-\infty}^{\infty} dk ck a_k^{\dagger} a_k + \sum_i \omega_0 \sigma_{ee}^i}_{H_0} - \underbrace{\sqrt{\frac{\Gamma_{1DC}}{2\pi}} \sum_i \int dk (\sigma_{eg}^i a_k e^{ikz_i} + h.c.)}_{V}. \quad (1)$$

The terms in  $H_0$  represent free photons with a dispersion relation  $\omega_k = ck$  and atoms with a transition frequency of  $\omega_0$  between the ground and excited states.  $V$  represents the interaction between photons and each atom at position  $z_i$ .

We first transform from the Schrodinger picture, where the total atom-photon density matrix evolves as  $\frac{d\rho}{dt} = -i[H, \rho]$ , to the interaction picture, in which the density matrix becomes stationary under evolution of  $H_0$  alone. Concretely,  $\rho_I$  and  $V_I$  in the interaction picture are related to their Schrodinger counterparts by

$$\rho_I(t) = e^{iH_0 t} \rho(t) e^{-iH_0 t} \quad (2)$$

and similarly for  $V_I$ . Evaluating  $V_I$  explicitly, one finds

$$V_I(t) = -\sqrt{\frac{\Gamma_{1DC}}{2\pi}} \sum_i \int dk (\sigma_{eg}^i a_k e^{ikz_i} e^{i(\omega_0 - \omega_k)t} + h.c.). \quad (3)$$

The physical effect of moving to an interaction picture, when we integrate out the photons, is to assume that the emitted photons are centered around the frequency  $\omega_0$ .

In the interaction picture,

$$\frac{d\rho_I}{dt} = -i [V_I(t), \rho_I(t)]. \quad (4)$$

**Problem 1.** Show that formal integration of the above equation starting from initial time  $t = 0$  gives rise to the integro-differential equation

$$\frac{d\rho_I}{dt} = -i[V_I(t), \rho_I(0)] - \int_0^t d\tau [V_I(t), [V_I(t - \tau), \rho_I(t - \tau)]]. \quad (5)$$

end

We assume the initial state is factorizable, with the field starting in vacuum,  $\rho_I(t = 0) = \rho_a \otimes |0\rangle\langle 0|$ . We also want to perform a partial trace over the field (which we label “f”), to give an equation for the atomic density matrix alone,  $\rho_a$ .

$$\frac{d\rho_a}{dt} = \text{Tr}_f \frac{d\rho_I(t)}{dt} \quad (6)$$

$$= -\text{Tr}_f \int_0^t d\tau [V_I(t), [V_I(t - \tau), \rho_I(t - \tau)]], \quad (7)$$

using the fact that  $\text{Tr}_f [V_I, \rho_I(0)]$  vanishes. Note that the evolution of the density matrix at time  $t$  depends on its value at previous times  $t - \tau$ , which is an exact result, but inconvenient to evaluate.

The Born-Markov approximation assumes, that from the atomic point of view, the photons always appear to be in vacuum, and that the photons have a short "memory" time. That is, the integrand is dominated by values of the density matrix around  $\tau = 0$ , such that we can replace

$$\rho_I(t - \tau) \approx \rho_a(t) \otimes |0\rangle\langle 0|, \quad (8)$$

giving a time-local evolution. In a consistent manner, we can also take the upper bound of integration to  $t \rightarrow \infty$ . In summary, the **Born-Markov approximation** amounts to:

$$\frac{d\rho_a}{dt} \approx -\text{Tr}_f \int_0^\infty d\tau [V_I(t), [V_I(t - \tau), \rho_a(t) \otimes |0\rangle\langle 0|]]. \quad (9)$$

This is in fact a general result, independent of the system that we want to apply it to.

Now in the following, we will want to re-write the equation above in the "quantum jump" form

$$\frac{d\rho_a}{dt} = -i(H_{\text{eff}}\rho_a - \rho H_{\text{eff}}^\dagger) + \sum_i c_i \rho_a c_i^\dagger \quad (10)$$

as discussed in lectures.

**Problem 2.** Comparing Eqs. (9) and (10), we anticipate that the effective non-Hermitian Hamiltonian is given by

$$-iH_{\text{eff}}\rho_a = -\text{Tr}_f \int_0^\infty d\tau V_I(t)V_I(t - \tau)\rho_a(t) \otimes |0\rangle\langle 0|. \quad (11)$$

Evaluate this to show that the effective Hamiltonian is given by

$$H_{\text{eff}} = -\frac{i\Gamma_{1D}}{2} \sum_i \sigma_{ee}^i - i\Gamma_{1D} \sum_{z_i > z_j} e^{ik_0(z_i - z_j)} \sigma_{eg}^i \sigma_{ge}^j, \quad (12)$$

where  $k_0 = \omega_0/c$ . This reproduces the effective Hamiltonian for chiral waveguide QED presented in the lecture.

Hints: use the identities

$$\int_0^\infty d\tau e^{i(\omega_0 - \omega)\tau} = \pi\delta(\omega_0 - \omega) + i\mathcal{P} \frac{1}{\omega_0 - \omega}, \quad (13)$$

where  $\mathcal{P}$  denotes the Cauchy principal value, and

$$\mathcal{P} \int_{-\infty}^\infty dk \frac{e^{ikx}}{k} = i\pi \text{sign}(x), \quad (14)$$

where  $\text{sign}(x) = +1$  for  $x > 0$ ,  $\text{sign}(x) = -1$  for  $x < 0$ , and  $\text{sign}(x) = 0$  for  $x = 0$ .

*end*

**Problem 3.** Similarly, we can calculate the jump term, which comes from

$$\left( \frac{d\rho_a}{dt} \right)_{\text{jump}} = \text{Tr}_f \int_0^\infty d\tau V_I(t)\rho_a \otimes |0\rangle\langle 0|V_I(t - \tau) + V_I(t - \tau)\rho_a \otimes |0\rangle\langle 0|V_I(t), \quad (15)$$

and show that

$$\hat{c} = \sqrt{\Gamma_{1D}} \sum_i \hat{\sigma}_{ge}^i e^{-ik_0 z_i} \quad (16)$$

is the single jump operator for the system.

## II. SUPER- AND SUBRADIANCE FOR $N = 2$ ATOMS

In lecture, we numerically diagonalized the effective Hamiltonian for a 1D chain of atoms in the single-excitation sector for a large but finite atom number  $N$ , and we also discussed how the single-excitation sector for  $N \rightarrow \infty$  is diagonalized by Bloch's theorem. The goal of this exercise is to exactly diagonalize  $N = 2$  atoms, which are separated by a distance much smaller than the resonant wavelength of their atomic transition,  $r \ll \lambda_{eg}$ , in the entire Hilbert space (zero, one, and two excitations).

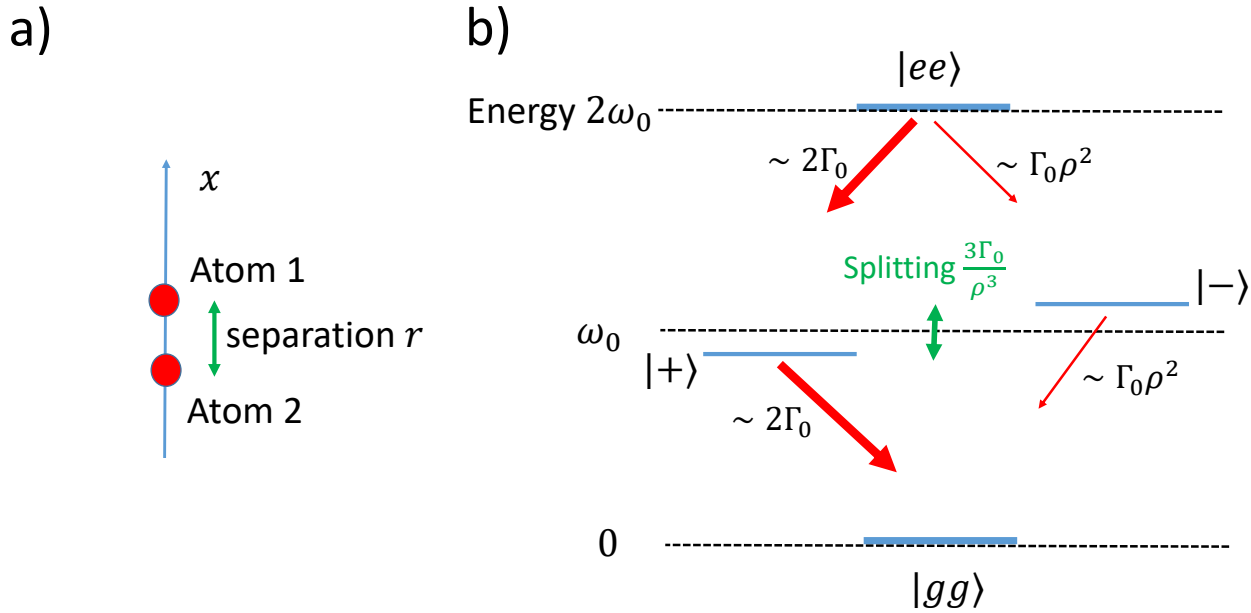


FIG. 1.

First, we have to choose an orientation of the atomic transitions, and the atomic positions. Let's choose the atoms to be separated by a distance  $r$  along the  $x$ -axis, i.e., the vector connecting them is  $\mathbf{r} = r\hat{x}$ . Let's also choose the orientation of the atomic transition to also be along  $x$ ,  $\hat{\phi} = \hat{x}$ , as illustrated in Fig. 1a. As discussed in lecture, the Green's function projected along  $\hat{\phi}$  is then

$$\mathbf{G}_{xx}(\mathbf{r}, 0, \omega_0) = \frac{3}{4}e^{i\rho} \left[ \left( \frac{1}{\rho} + \frac{i}{\rho^2} - \frac{1}{\rho^3} \right) + \left( -\frac{1}{\rho} - \frac{3i}{\rho^2} + \frac{3}{\rho^3} \right) \frac{(\hat{x} \cdot \mathbf{r})^2}{r^2} \right], \quad (17)$$

where  $\rho = \omega_0 r/c$  is a dimensionless distance. Taking  $\mathbf{r} = r\hat{x}$ , one has

$$\mathbf{G}_{xx}(r\hat{x}, 0, \omega_0) = \frac{3e^{i\rho}(1 - i\rho)}{2\rho^3}. \quad (18)$$

Let's now expand around  $\rho = 0$ . It turns out that the order of  $\rho$  that we should keep is different depending on whether we look at the real or imaginary part of  $G_{xx}$ . In particular, let's write

$$\mathbf{G}_{xx}(r\hat{x}, 0, \omega_0) = \left[ \frac{G_{xx} + G_{xx}^*}{2} \right] + i \left[ \frac{G_{xx} - G_{xx}^*}{2i} \right]. \quad (19)$$

Separately expanding the real and imaginary parts, the lowest non-trivial orders that we want to keep are

$$\mathbf{G}_{xx}(r\hat{x}, 0, \omega_0) \approx \frac{3}{2\rho^3} + i \left( \frac{1}{2} - \frac{\rho^2}{20} \right). \quad (20)$$

(You can check this if you want.)

The non-Hermitian, effective Hamiltonian is

$$H_{\text{eff}} = -\Gamma_0 \sum_{i,j=1}^2 G_{xx}(\mathbf{r}_i, \mathbf{r}_j, \omega_0) \sigma_{eg}^i \sigma_{ge}^j \quad (21)$$

Note that this Hamiltonian only has non-zero matrix elements between states having the same total number of excitations. Thus, we can separately diagonalize  $H_{\text{eff}}$  in different excitation number manifolds.

**Problem 4.** The easiest manifolds to diagonalize are the manifold with zero excitations (both atoms in their ground states  $|gg\rangle$ ), and two excitations (both atoms in excited states  $|ee\rangle$ ). There is only one state in each manifold, so each one must be the eigenstate of  $H_{\text{eff}}$  in that manifold. Show that the corresponding eigenenergies are  $\epsilon_{ee} = -i\Gamma_0$  and  $\epsilon_{gg} = 0$ . (Recall that exactly at  $\rho = 0$ , we should ignore the diverging term  $1/\rho^3$  in  $G_{xx}$  and set it to zero.)

**Problem 5.** Next, we diagonalize the manifold of one excitation, spanned by the states  $|ge\rangle$  and  $|eg\rangle$ . Intuitively, because the atoms are identical, the eigenstates should be given by  $|\pm\rangle = \frac{|ge\rangle \pm |eg\rangle}{\sqrt{2}}$ . Verify that these are eigenstates, with eigenenergies  $\epsilon_+ \approx -\frac{3\Gamma_0}{2\rho^3} - i\Gamma_0$ , and  $\epsilon_- \approx +\frac{3\Gamma_0}{2\rho^3} - \frac{i\Gamma_0\rho^2}{20}$ .

Recall that a complex eigenenergy  $\epsilon_\alpha = \Delta\omega_\alpha - i\Gamma_\alpha/2$  can be related to the energy shift  $\Delta\omega_\alpha$  of that state relative to the bare atomic transition frequency  $\omega_0$ , and a modified decay rate  $\Gamma_\alpha$  due to interference in emission. Note that the doubly excited state  $|ee\rangle$  has a decay rate  $2\Gamma_0$  that is twice that of a single excited atom, which makes sense because we have double the number of excitations. Interestingly, though, we see that the state  $|+\rangle$ , which only has a single excitation, *also* has a decay rate of  $2\Gamma_0$ , physically due to constructive interference! Conversely, the state  $|-\rangle$  has a vanishing decay rate as the distance of separation between the two atoms goes to zero,  $\rho \rightarrow 0$ . This result has a classical interpretation in terms of two oscillating dipoles that radiate in phase (constructive interference of emission), and out of phase (the net dipole moment of the system is zero). The two states also have energy shifts of  $\sim \mp \frac{1}{\rho^3}$ , which reflects the interaction of the two atoms via their optical near-fields.

The energies and decay rates are visually depicted in Fig. 1b. Note that although diagonalization of  $H_{\text{eff}}$  gives the total decay rate of  $|ee\rangle$ , it cannot determine whether the decay is primarily into the state  $|+\rangle$  or  $|-\rangle$ . It turns out that the decay is mostly into  $|+\rangle$ , which must be calculated by looking at the jump term in the Lindblad master equation.

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[1] Gross and Haroche, Superradiance: an essay on the theory of collective spontaneous emission, Phys. Rep. 93, 301 (1982).