Surface operators and holographic M2-branes

Nadav Drukker

King's College London

Based on:
`arXiv:2003.12372` with M. Probst and M. Trépanier
`arXiv:2004.04562` with S. Giombi, A. Tseytlin and X. Zhou
`arXiv:2009.10732` with M. Probst and M. Trépanier
`arXiv:2012.11087` with M. Trépanier
`arXiv:2111.09385` with M. Trépanier
`arXiv:2204.12627` with M. Trépanier
`arXiv:2210.07251` with M. Trépanier
`arXiv:23??.?????` with O. Shahpo and M. Trépanier

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Holography at 25 — ICTP SAIFR

Motivation

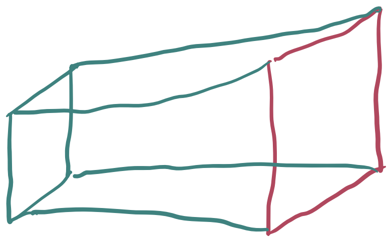
- Approach the 6d $\mathcal{N} = (2,0)$ theory from its surface operator observables.
- Throw at it all tools that worked for Wilson loops in $\mathcal{N} = 4$ SYM.
- Learn more nonperturbative methods in CFTs.

Outline

- Old observables:
  - String-antistring potential.
  - Anomaly coefficients.
- New BPS surface operators.
- New observables:
  - VEVs of anomaless surfaces (torus).
  - BPS and non-BPS creases: “Generalised string-antistring potential.”

Old observables

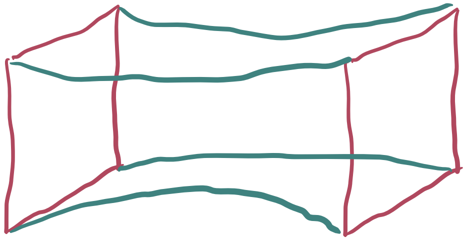
- An M2-brane ends on a 2-surface on the boundary of $AdS_7 \times S^4$
- Simplest configuration: Flat $\mathbb{R}^2$.



- Next simplest: A sphere.
  [Berenstein, Corrado / Fischler, Maldacena]

- Next simplest: A pair of planes.
  [Maldacena]
  equal to:

$$\log \langle V_{\parallel} \rangle = \frac{8N\sqrt{\pi}\,\Gamma(2/3)}{\Gamma(1/6)^3} \frac{\text{Area}}{R^2}\,.$$



Anomalies [Deser Schwimmer]

- Calculating correlation functions of local operators gives rise to logarithmic divergences, which account for anomalous dimensions.
- Smooth line operators have finite expectation values.
- Cusped Wilson loops have anomalies and the anomaly of the almost straight cusp is related to the Bremsstrahlung function.
- Surface operators again have logarithmic divergences, signalling anomalies.

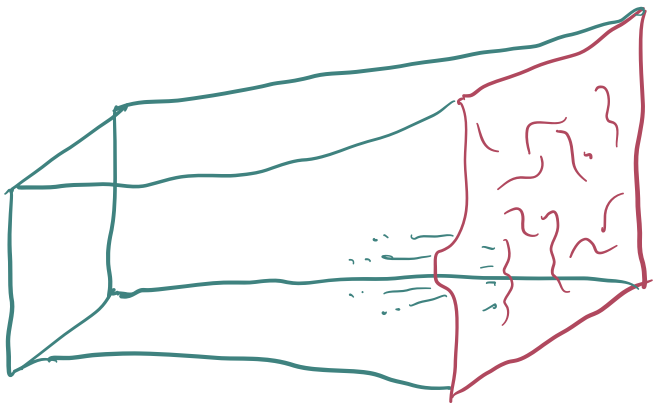
$$\log \langle V_\Sigma \rangle \sim \frac{1}{4\pi} \log \epsilon \int_\Sigma \mathrm{vol}_\Sigma \left[a_1 \mathcal{R}^\Sigma + a_2 \left(H^2 + 4 \operatorname{tr} P \right) + b \operatorname{tr} W + c(\partial n)^2 \right].$$

  - R^Σ is the Ricci scalar on Σ.
  - H is the mean curvature
  - P the pullback of the Schouten tensor.
  - W is the pullback of the Weyl tensor.
  - $(\partial n)^2 = \partial_m n^i \partial^m n^i$ are couplings to scalar fields.

Holographic anomalies [Graham Witten]

- Near boundary analysis of generic M2-brane in $AdS_7 \times S^4$
- The result is

$$a_1^{(N\to\infty)} = 0\,,\qquad a_2^{(N\to\infty)} = -N\,,\qquad b^{(N\to\infty)} = 0\,,\qquad c^{(N\to\infty)} = +N\,.$$



- Using defect CFT techniques we proved that $a_2 = -c$ and $b = 0$. [ND, Probst, Trépanier]
- Exact expressions for a_1 and a_2 exist for all N and any representation. For the fundamental representation they are [Wang][Chalabi, Estes, Jensen, Krym O'Bannon, Robinson, Rogers, Sisti][ND, Giombi Tseytlin, Zhou]

$$a_1^{(N)} = \frac{1}{2} - \frac{1}{2N}\,,\qquad a_2^{(N)} = -N + \frac{1}{2} + \frac{1}{2N}\,.$$

- For a spherical surface or radius R

$$\langle V_{S^2}\rangle \propto \frac{1}{R^{2a_1+4a_2}} = R^{4N-3-1/N}\,.$$

Is this all?

BPS surface operators [ND, Trépanier]

- The theory has 32 supercharges.
- “Locally BPS” operators have a half-rank projector at every point along the surface.
  - Generally more than one half-rank projector equation will only have a trivial solution.
- Want compatible equations at all points along the surface.
  - The plane has the same equation at all points, hence globally 1/2 BPS.
- Found 4 classes (and several subclasses) of geometries that allow for BPS observables, with appropriate choice of n^i:
  - Type-$\mathbb{R}$: Any curve in $\mathbb{R}^5$ times a line.
  - Type-$\mathbb{C}$: Any holomorphic curve in $\mathbb{R}^6 \sim \mathbb{C}^3$
  - Type-$\mathbb{H}$: Any surface in $\mathbb{R}^4 \subset \mathbb{R}^6$.
  - Type-S: Any surface in $S^3 \subset \mathbb{R}^6$.

Type-$\mathbb{R}$

A curve in $\mathbb{R}^5$ extended over x^6:

$$x^i(u) \subset \mathbb{R}^5 \; (i = 1, \ldots, 5) \quad \text{and} \quad x^6 = v\,,$$

$$n^i(u, v) = \frac{\partial_u x^i}{|\partial_u x|}\,.$$

- Restricting the curve to $\mathbb{R}$, we get a plane.
- With a curve in $\mathbb{R}^2$ can make a crease.
- They are uplifts of "Zarembo Wilson loops" in $\mathcal{N} = 4$ SYM.
- No anomaly.
- Finite quantity: VEV per unit length.

Type-$\mathbb{C}$

We choose a complex structure such that $\mathbb{R}^6 \cong \mathbb{C}^3$ and take any surface which is holomorphic, with fixed $n^i = \delta^{i5}$.

- Generally preserve two supercharges.
- In $\mathbb{C}^2$ preserves 4.
- In $\mathbb{C}$, it's just the plane, so 16.

- The anomaly is

$$\int_\Sigma \mathcal{A}_\Sigma^{\mathbb{C}} \, \mathrm{vol}_\Sigma = a_1 \chi(\Sigma) \,.$$

Type-$\mathbb{H}$

Arbitrary surface in $\mathbb{R}^4 \subset \mathbb{R}^6$. n^i chosen from the tangent space by projecting to the self-dual part, which is in S^2

$$n^i = \epsilon^{ab} \eta^i_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \, .$$

- Generally preserve a single supercharge.
- Restricting to $\mathbb{R}^3$ preserve two supercharges.
- Lagrangian manifolds: $n^i \in S^1$ preserve two supercharges.

- Anomaly related to degree of Gauss map.

Type-S

Take $x^\mu \in S^3$ and n^i according to

$$n^i = \frac{1}{2}\epsilon^{ab}\epsilon^{ijkl}\partial_a x^j \partial_b x^k x^l \,.$$

- Generic surface preserves two combination of Q and S supercharges.
- Restricting to S^2 gives the $1/2$ BPS sphere.
- Can construct spherical crease, which is $1/4$ BPS.
- Infinitesimal surfaces lead to the same as the $\mathbb{R}^3$ restriction of Type-$\mathbb{H}$.

- Anomaly includes topological part and part related to the area of the surface

$$\int \mathcal{A}^S_\Sigma \, \mathrm{vol}_\Sigma = (a_1 + a_2)\,\chi(\Sigma) + a_2 \frac{\mathrm{vol}(\Sigma)}{2\pi} \,.$$

Calibrated surfaces

- A BPS M2-brane satisfies a projection equation of the form

$$-\frac{i}{6}\varepsilon\Gamma_{MNP}\partial_m X^M \partial_n X^N \partial_p X^P \varepsilon^{mnp} = \varepsilon\,,$$

- For any preserved supercharge ε, can define a 3-form

$$\phi = -i\,\frac{\varepsilon\Gamma_{MNP}\varepsilon^\dagger}{\varepsilon\varepsilon^\dagger}\,dX^M \wedge dX^N \wedge dX^P\,.$$

- We could reduce the equations of motion to first order as

$$\partial_m X^M = \frac{1}{2} g_{ml}\varepsilon^{lnp} G^{ML}\phi_{LNP}\partial_n X^N \partial_p X^P\,.$$

- If ϕ is closed, the brane is a calibration: its action is equal to the integral of the form on any brane of the same homology class. [Harvey Lawson][Dymarsky, Gubser Guralnik, Maldacena][Joyce]

$$S = T_{M2}\int_V \phi\,.$$

- Using the metric

$$ds^2 = \frac{y}{L} dx_m dx_m + \frac{L^2}{y^2} dy_i dy_i \,,$$

- The forms for type-$\mathbb{R}$, $\mathbb{H}$ and $\mathbb{C}$ are

$$\phi^{\mathbb{R}} = -dx_6 \wedge \sum_{I=1}^{5} (dx_i \wedge dy_i) \;.$$

$$\phi^{\mathbb{H}} = \frac{1}{2} \eta_{I\mu\nu}\, dx_\mu \wedge dx_\nu \wedge dy_I - \left(\frac{L}{y}\right)^3 dy_1 \wedge dy_2 \wedge dy_3 \,.$$

$$\phi^{\mathbb{C}} = (dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6) \wedge dy_1 \,.$$

- All are exact, so we can integrate to the boundary and find that if there is no anomaly

$$S_{\text{ren}} = -\frac{N}{4\pi} \int_\Sigma \sqrt{h} H_\nu H_\nu d^2\sigma + \mathcal{O}(y^{-1}) \,.$$

Lagrangian surfaces

- Surfaces of type-$\mathbb{H}$ are restricted to $\mathbb{R}^4$. If the pullback of the symplectic from $\omega_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ to the surface

$$\omega_0|_\Sigma = 0\,,$$

  the surface is Lagrangian.
- The image of the Gauss map is now in S^1 and SUSY is doubled.
- Can define two calibration forms $\phi^{\mathbb{H}}$, $\phi^{\overline{\mathbb{H}}}$ satisfying

$$\phi^L|_V \equiv \frac{1}{2}\left[\phi^{\mathbb{H}} + \phi^{\overline{\mathbb{H}}}\right]_V = \text{vol}_V\,, \qquad \left(\phi^{\mathbb{H}} - \phi^{\overline{\mathbb{H}}}\right)|_V = 0\,.$$

- Eplicitly

$$\phi^L = (dx_1 \wedge dx_4 + dx_2 \wedge dx_3) \wedge dy_1 + (dx_3 \wedge dx_1 + dx_2 \wedge dx_4) \wedge dy_2\,,$$

$$\phi^{\mathbb{H}} - \phi^{\overline{\mathbb{H}}} = 2\left(dx_1 \wedge dx_2 + dx_3 \wedge dx_4 - \left(\frac{L}{y}\right)^3 dy_1 \wedge dy_2\right) \wedge dy_3\,.$$

- We can define a special Kähler structure on the 6d space with constant $y_{4,5,6}$, $x_{5,6}$ and the minimal M2 is special Lagrangian.

Summary

Type	**Geometry**	n^I **in**	**SUSYs**	**Also in**	**Anomaly**
$\mathbb{R}$	$\mathbb{R}\times\gamma,\ \gamma\subset\mathbb{R}^5$	S^4	1 Q		0
	$\mathbb{R}\times\gamma,\ \gamma\subset\mathbb{R}^4$	S^3	1 Q		
	$\mathbb{R}\times\gamma,\ \gamma\subset\mathbb{R}^3$	S^2	2 Q	$\mathbb{H}$	
	$\mathbb{R}\times\gamma,\ \gamma\subset\mathbb{R}^2$	S^1	4 Q	$\mathbb{H}$	
$\mathbb{C}$	$\Sigma\subset\mathbb{C}^3$ (holo.)	point	2 Q		yes
	$\Sigma\subset\mathbb{C}^2$ (holo.)	point	4 Q	$\mathbb{H}$, L	
$\mathbb{H}$	$\Sigma\subset\mathbb{R}^4$	S^2	1 Q		yes
subclass L	Lagrangian	S^1	2 Q		
subclass N	$\Sigma\subset\mathbb{R}^3$	S^2	2 Q	(S)	
S	$\Sigma\subset S^3$	S^3	$2\,(\mathrm{Q}+\mathrm{S})$		yes

Special examples with enhanced SUSY

Name	Type	Geometry	n^I in	SUSYs	see
cones	$\mathbb{H}$	over $\gamma \subset S^3$	S^2	Q, S	[Mezei Pufu Wang]
	$\mathbb{H}$, N	over $\gamma \subset S^2$	S^2	2Q, 2S	
crease	$\mathbb{R}$, $\mathbb{H}$, N	2 half-planes	S^0	$4Q, 4S \subset \mathfrak{osp}(4^*\|2)$	[Agmon Wang]
tori	S	$T^2 \subset S^3$	S^4	$2(Q+S)$	
	$\mathbb{H}$, L	$T^2 \subset \mathbb{R}^4$	S^1	2Q, 2S	
spheres	$\mathbb{H}$, N	$S^2 \subset \mathbb{R}^3$	S^2	2Q, 2S	[Berenstein Corrado Fischler Maldacena]
	S	latitude $S^2 \subset S^3$	S^3	$4(Q+S)$	
	S	large $S^2 \subset S^3$	point	$16(Q+S) \subset \mathfrak{osp}(4^*\|2)^2$	
plane	$\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$ L N	$\mathbb{R}^2$	point	$8Q, 8S \subset \mathfrak{osp}(4^*\|2)^2$	[Maldacena]

Anomaless surfaces

Having evaluate the anomaly coefficients, need new observables.

- If $\int \mathcal{A}_\Sigma = 0$, the surface has no anomaly and a finite VEV.
  - Type-$\mathbb{R}$.
  - Lagrangian surface of type-$\mathbb{H}$.
  - Other fine-tuned examples.
- The anomaly density is

$$\mathcal{A}_\Sigma = a_1 \mathcal{R}^\Sigma + c\left((\partial n)^2 - \left(H^2 + 4\,\mathrm{tr}\,P\right)\right).$$

Torus

A flat torus in $\mathbb{R}^4$ is Lagrangian and of type-$\mathbb{H}$, so anomaless

- Explicitly

$$x^1 = R_1 \cos\varphi_1\,, \qquad x^2 = R_1 \sin\varphi_1\,,$$
$$x^3 = R_2 \cos\varphi_2\,, \qquad x^4 = R_2 \sin\varphi_2\,.$$

with fixed R_1, R_2 and scalar couplings

$$n^1 = -\sin(\varphi_1+\varphi_2)\,, \qquad n^2 = \cos(\varphi_1+\varphi_2)\,.$$

- Preserves 4 supercharges.

Holographic torus

- With metric
$$ds^2 = \frac{y}{L}\,dx^\mu\,dx^\mu + \frac{L^2}{y^2}\,dy^I\,dy^I\,,$$
- Take ansatz
$$x^1 = r_1(\rho)\cos\varphi_1\,,\qquad x^2 = r_1(\rho)\sin\varphi_1\,,$$
$$x^3 = r_2(\rho)\cos\varphi_2\,,\qquad x^4 = r_2(\rho)\sin\varphi_2\,,$$
$$y_1 = -\rho\sin(\varphi_1+\varphi_2)\,,\qquad y_2 = \rho\cos(\varphi_1+\varphi_2)\,,\qquad y_3(\rho)\,.$$
- Solution:
$$y_3 = \text{const}\,,$$
$$r_1^2(\rho) = R_1^2 - \frac{2L^3}{\sqrt{\rho^2+y_3^2}} = R_1^2 - \frac{2L^3}{y}\,,$$
$$r_2^2(\rho) = R_2^2 - \frac{2L^3}{\sqrt{\rho^2+y_3^2}} = R_2^2 - \frac{2L^3}{y}\,.$$

Torus VEV

- action:

$$S = \frac{\pi N y}{L^3} R_1 R_2 - \pi N \left(\frac{R_1}{R_2} + \frac{R_2}{R_1} \right) .$$

  Agrees with results from calibration form.
- If we take $R_1 \to \infty$ we get a cylinder of length $D = 2\pi R_2$ and expect an extensive expression

$$\left\langle V_{\mathbb{R}\times S^1} \right\rangle = \exp \frac{s_1 D}{2\pi R_2}$$

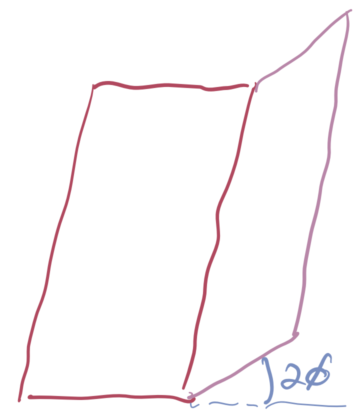
- Indeed we find $s_1 = \pi N$.
- For general N expect

$$\log \langle V_{T^2} \rangle = s_0 + s_1 \left(\frac{R_1}{R_2} + \frac{R_1}{R_2} \right) .$$

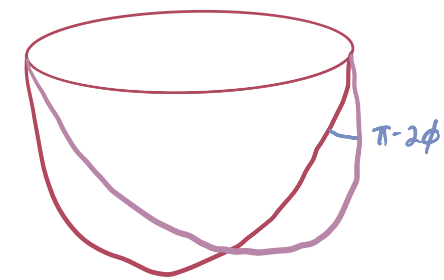
  What are $s_0(N)$ and $s_1(N)$ beyond the large N limit?
  $s_0(1) = s_1(1) = 0.$

Crease

- Surface extended in x^3 direction and a cusp of angle ϕ in the (x^1, x^2) plane.
- Allow to couple to two scalars with angle θ.
- $\mathfrak{so}(2,1) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ symmetry.
- For $\phi = \theta$, it is $1/4$ BPS and preserves $\mathfrak{osp}(4^*|2)$.
- Log of expectation value extensive in area of half plane with hyperbolic metric, so vanishes.



- Alternative picture: Spherical crease. Two hemispheres glued along the equator.



- Alternative: conformal transform to $\mathbb{H}_2 \times S^4$:

$$\begin{aligned} ds^2 &= du^2 + \cos^2 u \, dv^2 + \sin^2 u \, d\Omega_4^2 \\ &= \frac{1}{\cosh^2 \rho} \left(d\rho^2 + \sinh^2 \rho \, dv^2 + d\Omega_4^2 \right) . \end{aligned}$$

Generalised potential

- Finite observable is ratio

$$U(\phi,\theta)=\frac{1}{2\pi}\log\frac{V_{\phi,\theta}}{V_{S^2}}\,.$$

-2π is the regularized area of $\mathbb{H}_2$.

- Same anomaly for numerator and denominator $\langle V_{S^2}\rangle\sim 1/R^{2a_1+4a_2}$.
- In the abelian theory:

$$U^{(1)}(\phi,\theta)=\frac{1}{2\pi}\left(\log\cos\phi+\frac{\cos^2\phi-\cos^2\theta}{2\cos^2\phi}\right).$$

- In *AdS*:

$$U^{(N)}(\phi,\theta)=\frac{N}{\pi}\left(\log\cos\phi-(\phi-\theta)\tan\phi-\frac{(\phi-\theta)^2}{2}\left[\frac{1}{\cos^2\phi}+\frac{2\tan^3\phi}{3(\phi-\tan\phi)}\right]+\dots\right).$$

Defect CFT

- We can prove using dCFT techniques that

$$U(\phi,\theta) = U(\phi,\phi) + (\phi-\theta)\frac{dU(\phi,\phi)}{d\phi} + \ldots$$

- Analog of the "generalized bremsstrahlung function". [Correa, maldacena, Sever]
- Missing proof of

$$U(\phi,\phi) = \frac{c}{\pi}\log\cos\phi\,,$$

with c the anomaly coefficient.

Comparison to Wilson loops

Wilson loop	**Surface operator**
Circle: $\langle W_\circ \rangle = \frac{1}{N} L^1_{N-1}(-\lambda/4N) e^{\lambda/8N}$	Cylinder: $\langle V_{\mathbb{R}\times S^1} \rangle = \exp\left(\frac{NT}{2D} + \cdots\right)$
	Sphere: $\langle V_{S^2} \rangle \propto R^{4N-3-\frac{1}{N}}$
Two longitudes: $\langle W_\phi \rangle = \langle W_\circ((1-\frac{\phi^2}{\pi^2})\lambda) \rangle$	BPS crease: $\langle V_{\phi,\phi} \rangle / \langle V_{S^2} \rangle \stackrel{?}{=} \cos^{2N-1-\frac{1}{N}} \phi$
Bremsstrahlung function: $B = \frac{1}{2\pi^2} \lambda \partial_\lambda \log \langle W_\circ \rangle$	Near BPS: $\partial_\phi \log \frac{\langle V_{\phi,\phi} \rangle}{\langle V_{S^2} \rangle} \stackrel{?}{=} -(2N-1-\frac{1}{N}) \tan \phi$

Conclusion and outlook

- Surface operators are natural observables in the 6d theory.
- They are similar to Wilson loops.
- We can generalize some of the tools used for line operators to study the surface operators.
- We found relations among b, a_2 and c.
- Found very rich examples of BPS observables.
- Started calculating finite quantites
  - Torus and cylinder.
  - Crease: $U(\phi, \theta)$.
- First signs of exact results in $\log \cos \phi$ and its derivative.

The end