Non-relativistic corners in String Theory and AdS/CFT

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based on work:

to appear **230y.xxxxx** (Bidussi,Harmark,Hartong,NO,Oling)

2107.006542 (JHEP) (Bidussi,Harmark,Hartong,NO,Oling)

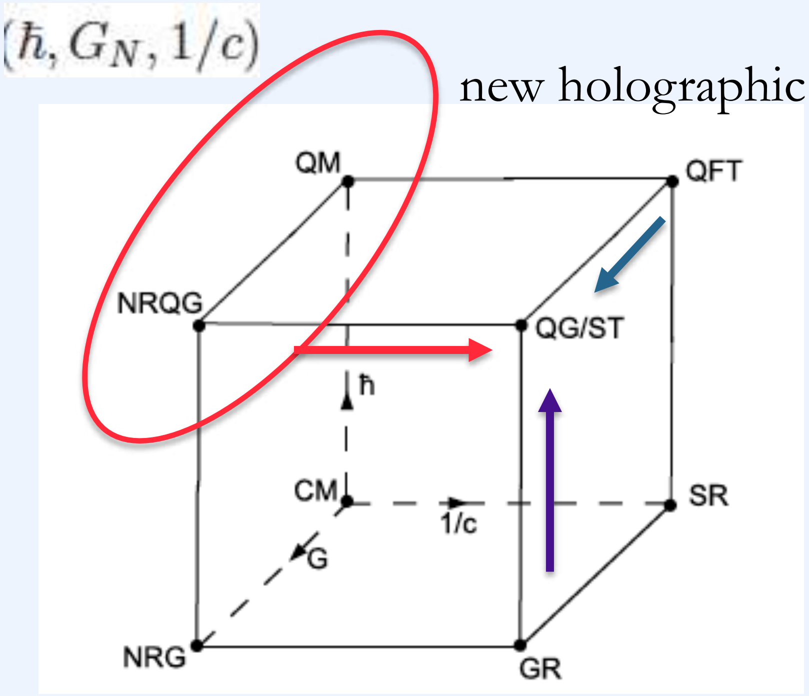
2011.02539 (JHEP) (Harmark,Hartong,NO,Oling)

1907.01663 (JHEP) (Harmark,Hartong,Menculini,NO,Oling)

& 1810.05560 (JHEP) (Harmark,Hartong,Menculini,NO,Yan)

1705.03535 (PRD) (Harmark,Hartong,NO)

Non-relativistic physics and cube of physical theories



new holographic dualities ?

a third route towards
(relativistic) quantum gravity

how does this fit with
string theory/holography ?

already (classical) non-relativistic gravity (NRG)
is more than just Newtonian gravity

Non-Lorentzian geometries

recent progress in understanding non-relativistic corners of:
gravity, quantum field theory and string theory:

→ builds on improved understanding of non-Lorentzian geometries
= spacetimes with local symmetries other than Lorentz

NL geometries appear in:

- bdry geometries in non-AdS holography (e.g. Lifshitz flat space)
- covariant formulations of PN approximation in GR
- covariant formulations of non-Lorentzian fluids and CMT systems (FQHE, fractons, ..)
- Horava–Lifshitz gravity, non-relativistic versions of CS, JT
- cosmology, black hole horizons, ultra-local GR (Carroll)
- double field theory
- **non-relativistic corners of String Theory**
- **near-BPS limits of string theory on AdS5 × S5**

Why non-relativistic (NR) string theory ?

- can we learn more about ordinary string theories from NR techniques

- what is landscape of UV complete non-Lorentzian theories (certain NR strings contained in double field theory)

- D-branes/open strings in non-Lorentzian string theory?

- can we build explicit examples of holographic dualities ?

- is NRQG a well-defined corner and does it have a ST description ?

Main messages

- non-relativistic strings naturally appear in the AdS/CFT corr.

- NR strings from near-BPS limits in N=4 SYM
- resulting sigma models are part of a larger landscape of NR strings:

NR string theory from near-critical B-field (Gomis,Ooguri/Danielsson et al)

curved spacetime generalization (still 2D CFTs):
strings with Lorentzian worldsheet/probing non-Lorentzian target space time

Further world-sheet limit gives:
Galilean string (with non-relativistic worldsheet)
→ novel class of sigma models that includes NR AdS/CFT corners

(Note: dual limit gives Carrollian string - relevant for AdS/CFT ?)

NR strings from N=4 SYM

- Spin Matrix theory (SMT) limits of AdS/CFT obtained by zooming in on unitarity bounds of N=4 SYM on RxS3: Harmark/Orselli (2014)

$$\lambda \to 0 \quad , \quad \frac{E-Q}{\lambda} = \text{fixed}$$

Q = linear sum of Cartan charges of PSU(2,2|4)

→ N=4 SYM simplifies and becomes QM theory

- reduces to nearest-neighbor spin chains in planar N limit

- low energy excitations of spin chains = magnons

$$E - Q = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} - 1$$

[Beisert(0511)]

becomes in SMT limt:

$$H - Q = \frac{g}{2\pi^2}\sin^2\frac{p}{2}$$

non-relativistic

- semi-classical limits of spin chains become sigma models: Kruczenski (0311)

e.g. Landau-Lifshitz model for SU(2) sector

$$\mathcal{L}_{\rm LL} = \frac{J}{4\pi}\left[\sin\theta\dot{\phi} - \frac{1}{4}\Big((\theta')^2 + \cos^2\theta(\phi')^2\Big)\right]$$

Stringy side of SMT gives NR sigma models

- using AdS/CFT dictionary: SMT (near-BPS) limit can be formulated as limit of type IIB string theory on AdS5xS5

  - correspond to non-relativistic world-sheet strings !

→ LL model (and generalizations for other near-BPS sectors) is example of a novel class of non-relativistic worldsheet strings with a non-Lorentzian target spacetime

- one of target space dimensions = position along the spin chain (zero momentum because of cyclicity of trace)

- strongly suggests: bulk description of SMT is a type of NR gravity

  - new class of flat-fluxed backgrounds obtained recently: analogue of flat Minkowski space using Penrose type limits
  → natural starting point to quantize the theory

NR strings

NR strings on flat spacetime = Gomis-Ooguri string

Gomis,Ooguri(2000); Danielsson et al.(2000);

→ Non-Lorentzian (stringy Newton-Cartan) geometries
when spacetime is curved

Andringa et al (2012), Harmark,Hartong,NO(2017); Bergshoeff,Gomis,Yan(2018);
Harmark,Hartong,Menculini,NO,Yan(2018); Gomis,Oh,Yan(2019);
Gallegos,Gursoy,Zinnato(2019), Harmark,Hartong,Menculini,NO,Oling(2019);
Bergshoeff,Gomis,Rosseel,Simsek,Yan(2019); Kluson (2018/19), …….
Yan (2021), Bergshoeff,,Lahnsteiner,,Romano,Rosseel,Simsek(2021);
Bidussi, Harmark,Hartong,NO,Oling(2021); …. (& many more refs since then)

also:

- tensionless strings — e.g.Lindstrom,Sundborg,Theodoridis(1991) Bagchi,Gopakumar(2009) Bagchi,Banerjee,Parekh(2019)
- Galilean strings — Battle,Gomis,Not(2016))
- relation to double field theory — Ko,Melby-Thompson,Meyer,Park(2015) Morand,Park(2017);Berman,Blair,Otsuki(2019);Blair(2019)

NR strings (on flat spacetime)

Gomis,Ooguri(2000); Danielsson et al.(2000)

zero Regge slope limit of relativistic string theory in near-critical B-field

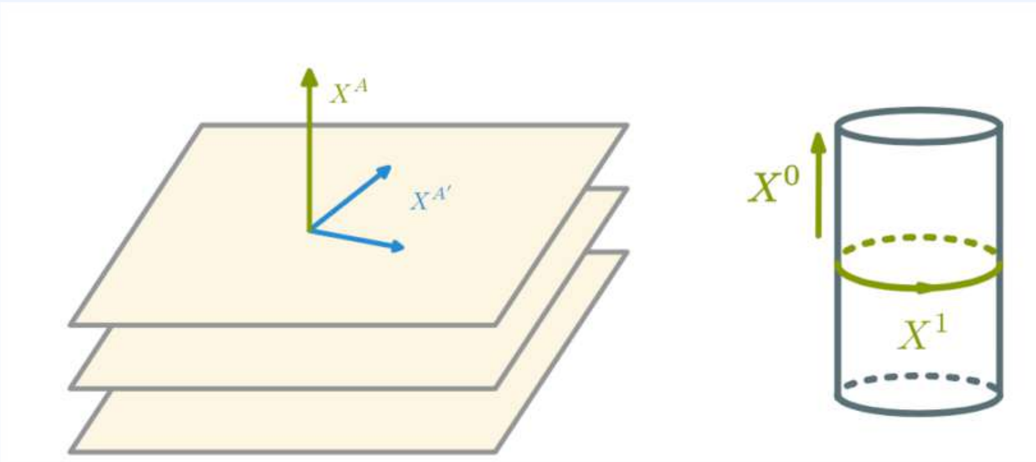
$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{h} \left(h^{\alpha\beta}\, \partial_\alpha X^{A'}\, \partial_\beta X_{A'} + \lambda \bar{\mathcal{D}} X + \bar{\lambda} \mathcal{D} \overline{X} \right),$$

in conformal gauge:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(\partial_\alpha X^{A'}\, \partial^\alpha X_{A'} + \lambda \bar{\partial} X + \bar{\lambda} \partial \overline{X} \right),$$

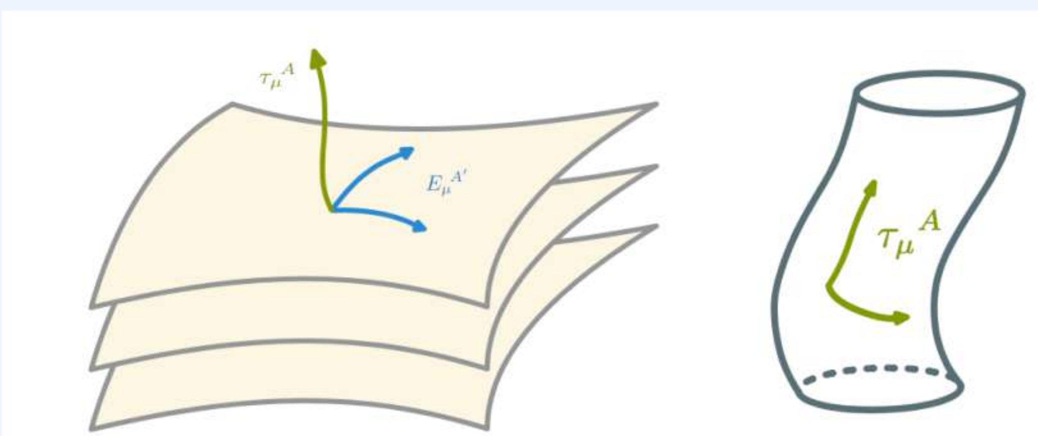
- Galilean invariant dispersion relation
- no massless physical states
- low-energy effective theory described by Newton-like gravity
- all asymptotic states carry non-zero winding along (compact) X1
- space-time S-matrix with NR symmetry

Gomis/Ooguri NR string lives in flat space



figures from
review on NRST
Gerben Oling & Ziqi Yan
(2202.12698)

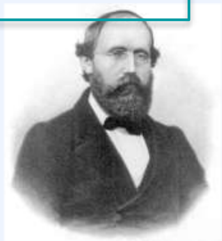
Q: what is the general target space probed by NR strings ?



Space-Time symmetries and Geometry

local symmetries of space and time ←→ geometry of space and time

Einstein: Poincare ←→ (pseudo-)Riemannian geometry



Cartan: Galilean/Bargmann ←→ Newton-Cartan geometry

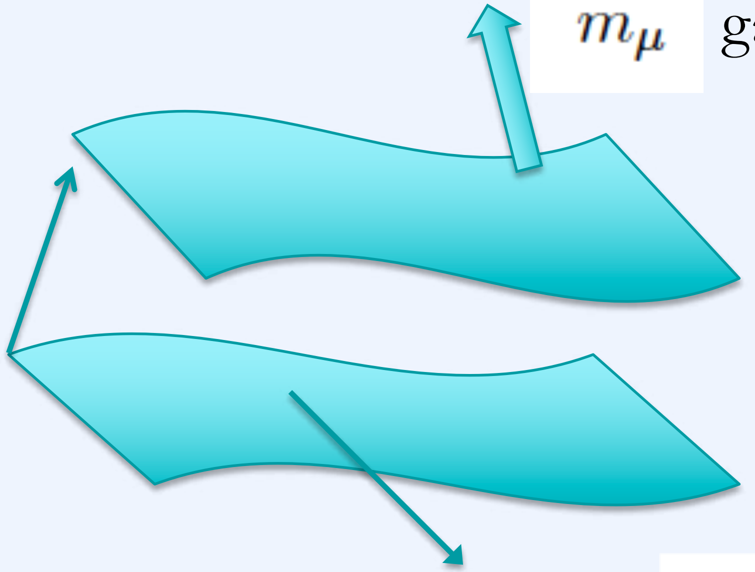
[Eisenhart,Trautman,Dautcourt,Kuenzle,Duval,Burdet,Perrin,Gibbons,Horvathy,Julia,Nicolai,…] ..

- geometrize Poisson equation of Newtonian gravity
  falling observers see Galilean laws of physics

torsional Newton-Cartan geometry (NR particles)

m_μ gauge field (of mass generator)

time τ_μ



space $h_{\mu\nu}$

$\nabla_\mu \tau_\nu = 0$

gives connection with intrinsic torsion $\sim d\tau$

NC = no torsion → $d\tau = 0$ absolute time

TTNC = twistless torsion → $\tau \wedge d\tau = 0$ preferred foliation equal time slices

TNC no condition on τ_μ

Christensen,Hartong,NO,Rollier(2013)

Coupling of non-relativistic particle to TNC geometry

two routes:

- null-reduction of relativistic particle
- c → infinity of extremal charged particle

$$q = mc^2.$$

$$S = \frac{m}{2}\int \frac{h_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}{\tau_\rho \dot{X}^\rho} d\lambda - m\int m_\mu \dot{X}^\mu d\lambda$$

[Kuchar], [Bergshoeff et al]

kinetic term

potential term: coupling to m_μ

m_0 ~ Newtonian potential

$$T^\mu = m\int d\tau\, \partial_\tau X^\mu \delta(x - X(\tau))$$

mass current

- action has TNC local target space symmetries

Non-relativistic string on curved spacetime

two routes:

- null reduction of relativistic string action
  (`light-like T-duality' along null isometry)
- c → infinity limit in near-critical B-field of relativistic string action
  (F-strings extremally charged under B-field: tension = charge)

→ NR strings move in torsional string Newton-Cartan geometry:

- Riemannian geometry (transverse) fibered over 2-dimensional Lorentzian base (longitudinal)

- pullback of longitudinal base on the worldvolume
  = Lorentzian metric on worldsheet

NRST action on TSNC target space

- Nambu-Goto form:

kinetic potential

$$S_{\rm NR} = -\frac{T}{2}\int d^2\sigma\left[\sqrt{-\tau}\,\eta^{AB}\tau_A^\alpha\tau_B^\beta h_{\alpha\beta} + \epsilon^{\alpha\beta}m_{\alpha\beta}\right],$$

A = 0,1 : longitudinal directions
a= 2,…D-1: transverse directions
alpha = sigma,tau: world-sheet

$$h_{\mu\nu} = e^a_\alpha e^b_\beta \delta_{ab}$$

torsional string Newton–Cartan geometry : τ_μ^A, $h_{\mu\nu}$, $m_{\mu\nu}$.

$m_{\mu\nu}$. couples to
worldsheet tension current

$$J_{\rm T}^{\mu\nu} = T\int d^2\sigma\,\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\delta(x - X(\sigma^\alpha)),$$

TSNC Polyakov action:

$$S = -\frac{T}{2}\int d^2\sigma\Big[\sqrt{-\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N h_{MN} + \epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N m_{MN}$$
$$+ \lambda\epsilon^{\alpha\beta}e^+{}_\alpha\tau^+{}_M\partial_\beta X^M + \bar\lambda\epsilon^{\alpha\beta}e^-{}_\alpha\tau^-{}_M\partial_\beta X^M\Big],$$

Symmetries of the TSNC action

- 2-form gauge syms: $\bar{\delta} m_{\mu\nu} = 2\partial_{[\mu}\lambda_{\nu]}.$

- transverse string Galilean boosts:

$$\bar{\delta} h_{\mu\nu} = -\lambda_{Ab}\left(\tau_\mu^A e_\nu^b + \tau_\nu^A e_\mu^b\right) \quad , \quad \bar{\delta} m_{\mu\nu} = -2\epsilon_{AB}\lambda^B{}_c\tau_{[\mu}^A e_{\nu]}^c.$$

- longitudinal Lorentz boosts

→ string analogue of the symmetries of NR particle coupling to Newton-Cartan

Remarks

- beta functions/effective spacetime actions for the NR string obtained in various different formulations/using different methods

Gomis,Oh,Yan(2019); Bergshoeff,Gomis,Rosseel,Simsek,Yan(2019); Yan,Yu(2019) Bergshoeff et al (2021); Yan (2021); Gallegos,Gursoy,Zinnato(2019); Gallegos,Gursoy,Verma,Zinnato(2020);….

→ describe the dynamics of (versions of) non-relativistic (super-) gravity

- limits vs. expansions

- limit geometry: type I (cancellation of divergent term)
- geometry from expansion: type II (each term in the action generates more gauge fields)

van den Bleeken (2018), Hansen,Hartong,NO (2019, 2020)

Hartong,Have (2021,2022)

Galilean limit of TSNC string

start with NR string action

$$\mathcal{L} = \frac{T_{\rm NR}}{2}\left[\tau\left(\tau_0^\alpha\tau_0^\beta - \tilde{c}^2\tau_1^\alpha\tau_1^\beta\right)h_{\alpha\beta} - \varepsilon^{\alpha\beta}m_{\alpha\beta}\right]$$

worldsheet scaling limit: $\tilde{c}\to\infty$ (also rescale tension and m-field)

$T_{\rm NR} = T_G\tilde{c}^{-2}$

gives Nambu-Goto action of Galilean string

$$S = -\frac{T}{2}\int d^2\sigma\left[\sqrt{-\tau}\,\tau_1{}^\alpha\tau_1{}^\beta h_{\alpha\beta} + \epsilon^{\alpha\beta}m_{\alpha\beta}\right],$$

- symmetries:

- longitudinal Galilei & transverse string Galilei boosts
- 2-form gauge trafos

- will see: reduces for subclass of spacetimes to actions found in near-BPS SMT decoupling limits of AdS/CFT

Galilean string (cont'd)

partial gauge fixing $\tau^0_\sigma = \tau^0_M X'^M = 0.$

residual gauge symmetries $\tau \to \tilde{\tau}(\tau)$ and $\sigma \to \tilde{\sigma}(\tau, \sigma)$

static gauge

$$X^t = a\tau\,, \qquad X^v = wR_v\sigma + f(\tau)$$

$X^v \sim X^v + 2\pi R_v$ and $w > 0$ is the winding number.

- EOM of f: level matching constraint

take: $\tau^0_M dx^M = dt$ and $\tau^1_M dx^M = dv.$ $\qquad X^M = (X^t, X^v, X^i)$

Lagrangian of Galilei string takes form:

$$\begin{aligned}\mathcal{L} \;=\; T\Big[& m_{iv}\left(\dot{X}^v X'^i - \dot{X}^i X'^v\right) - \frac{\dot{X}^t}{2X'^v} h_{ij} X'^i X'^j \\ & -\frac{1}{2}(h_{tt} + 2m_{tv})\dot{X}^t X'^v - (h_{vi} + m_{ti})\dot{X}^t X'^i - m_{ij}\dot{X}^i X'^j \Big]\end{aligned}$$

Galilean string (cont'd)

- theories of form: $$\mathcal{L} = T\left[m_{iv}\left(\dot{X}^v X'^i - \dot{X}^i X'^v\right) - \frac{\dot{X}^t}{2X'^v} h_{ij} X'^i X'^j\right]$$

  - can be obtained by starting with relativistic strings on AdS5xS5 and taking SMT limit

- AdS5xS5 admits null Killing vector

can null reduce and then take limit $\tilde{c} \to \infty$ $\tilde{c} = (4\pi g_s N)^{-1/2}$

→ gives particular SMT (most general one) and others as special subcases

Remarks

- AdS5xS5 null reduction vs large c limit (need `dual' background when taking large c limit)
- Q: what is dual field theory description after null reduction but before taking SMT limit ?

SU(2) case and LL model as non-rel 2D CFT

start with AdS5xS5 in appropriate coordinates

consider BPS bound $E \geq Q = J = J_1 + J_2$. $\quad g_s \to 0 \ , \ \frac{E-Q}{g_s} = \text{fixed}$

$E = i\partial_t$ and $J = -i\partial_\gamma$. $\quad t = x^0 - \frac{1}{2}u \ , \ \gamma = x^0 + \frac{1}{2}u$

after limit $c = \frac{1}{\sqrt{4\pi g_s N}}$ to infinity

$$\tau = d\tilde{x}^0 \ , \ m = -\frac{\cos\theta}{2}d\phi \ , \ h_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2)$$

NR background R x S2 and non-zero “magnetic” flux

gives LL model

$$S = \frac{Q}{4\pi}\int d^2\sigma[\dot{\phi}\cos\theta - \frac{1}{4}(\theta'^2 + \sin^2\theta\phi'^2)]$$

free magnon limit: S2 -> R2
(corresponds to pp-wave limit)

$$\tau = d\tilde{x}_0 \ , \ m = \frac{x}{2}dy \ , \ h = \frac{1}{4}[dx^2 + dy^2]$$

action

$$S \sim \int d^2\sigma[x\dot{y} - \frac{1}{4}(x'^2 + y'^2)]$$

Polyakov action for Galilean strings

$$S=-\frac{T}{2}\int d^2\sigma\Big[\left(e\,e_1{}^{\alpha}e_1{}^{\beta}h_{MN}+\epsilon^{\alpha\beta}m_{MN}\right)\partial_\alpha X^M\partial_\beta X^N+\omega\epsilon^{\alpha\beta}e^0{}_\alpha\tau^0{}_\beta+\psi\epsilon^{\alpha\beta}(e^0{}_\alpha\tau^1{}_\beta+e^1{}_\alpha\tau^0{}_\beta)\Big]$$

- exhibits Galilean world-sheet structure
- Weyl and local (2D) Galilean boost act on the zweibeine:

$$e^0{}_\alpha\to fe^0{}_\alpha\,,\quad e^1{}_\alpha\to fe^1{}_\alpha+\hat{f}e^0{}_\alpha\,,\quad \omega\to\frac{1}{f}\omega-\frac{\hat{f}}{f^2}\psi\,,\quad \psi\to\frac{1}{f}\psi$$

- analysis of kinetic terms suggests setting $e^0{}_\sigma=0$ (otherwise wrong sign)
- natural from pov of Galilean ws. structure:
  → spatial foliation in terms of ws. coordinate time: $e^0{}_\alpha=f\,d\tau,$
- expected full reparametrization freedom is:

$$\tau\to g(\tau),\qquad \sigma\to h(\tau,\sigma).$$

→ Hamiltonian analysis: R x Vir symmetries

Carrollian limit of TSNC string

- scaling limit and $\tilde{c} \to 0$

Nambu-Goto action

$$S = -\frac{T}{2}\int d^2\sigma\Big[-\sqrt{-\tau}\,{\tau_0}^{\alpha}{\tau_0}^{\beta}h_{\alpha\beta} + \epsilon^{\alpha\beta}m_{\alpha\beta}\Big]$$

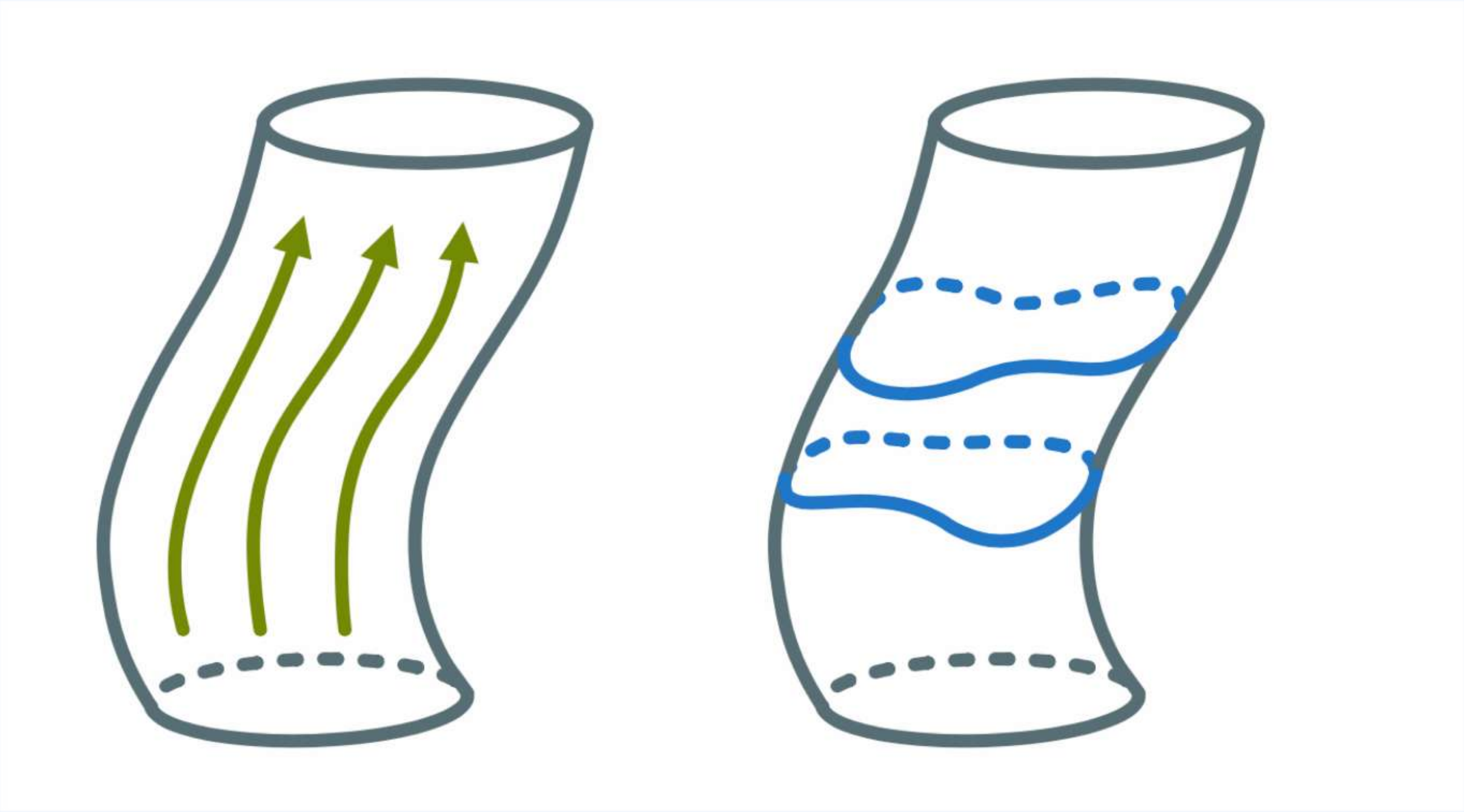
- related to Galilean case by 0 interchanged with 1

- Polyakov formulation:
worldsheet structure with Carrollian metric structure that is Weyl invariant

- gauge-fixed action has residual symmetry: BMS_3
(contraction of Vir x Vir)

- corresponds to a limit in AdS/CFT?

Visualisation of world-sheet



(drawing: courtesy of Gerben Oling)

Further developments (NRST)

- open strings and branes:

  - non-relativistic open string sector and DBI actions
    Gomis,Yan,Yu (2020)
  - connection to NR D/M-branes
    Kluson/Blair,Gallegos,Zinnato (2021)/Ebert,Sun,Yan(2021)
  - strings/branes as background solutions
    Bergshoeff,Lahnsteiner,Romano,Rosseel(2022)
  - generalize procedure to non-relativistic limit of extremal p-branes
    TSNC analogue for p-branes (incl. D/M)
    Bidussi,Harmark,Hartong,NO,Oling (in progress

- SUSY generalization of (include RR fields)
  NR limit & relations to DFT/exceptional FT
  - non-perturbative dualities in NR string theory

- connection to integrable models
  Gomis,Gomis,Kamimura(2005)/Roychowdhury(2019/
  Fontanella,NietoGarcia,Torielli(2021),Fontanello,van Tongeren(2022)

Further developments (NR worldsheets0

- Hamiltonian analysis

  Kluson (2021), Bidussi,Harmark,Hartong,NO,Oling (to appear)

- obtain beta functions for Galilean string

- connection with explicit construction of SMT using classical reduction of N=4 SYM & suitable quantization method

  Harmark,Wintergerst (2019),Baiguera,Harmark,Wintergerst(2020)
  Baiguera,Harmark,Lei,Wintergerst (2020)

- connections to Carrollian (small speed of light) gravity

Henneaux (1979), Bergshoeff,Gomis,Rollier,ter Veldhuis(2017),Hartong(2015)
Henneaux,Salgado-Rebolledo(2021),
de Boer,Hartong,NO,Sybesma,Vandoren(2021),Perez(2021),Hansen,NO,Oling,Soegaard(2021)

Outlook

- NR (a la GO) describes closed subsector of relativistic ST
  has covariant string Newton-Cartan formulation (still Lor. CFT2 on ws)

what can we add to 90s ST?
- covariant formulation of DLCQ of strings
- further connection with Matrix string theory/M-theory
- other expansions ?

- Spin Matrix limits give
  - tractable subsector of N=4 SYM
  - strings with Galilean structure on worldsheet
    → quantize

- similar Carrollian worldsheet models exist
  - from usual tensionless limit
  - BMS3 residual symmetries

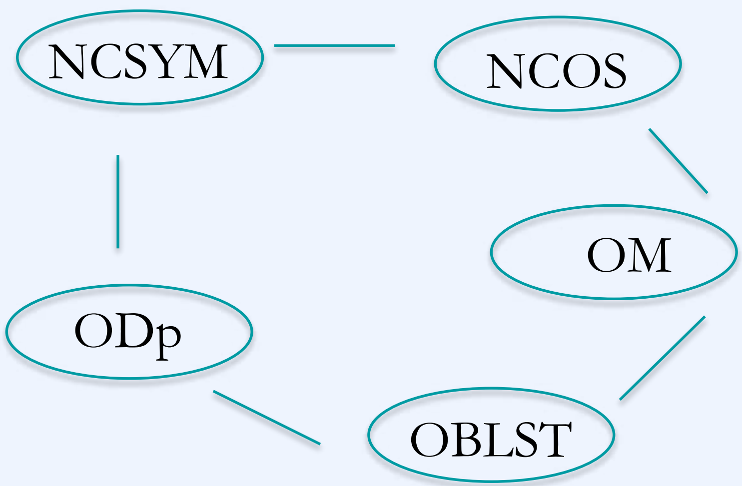
The end

extra stuff (varia)

Duality web of `non-Lorentzian' string theories?

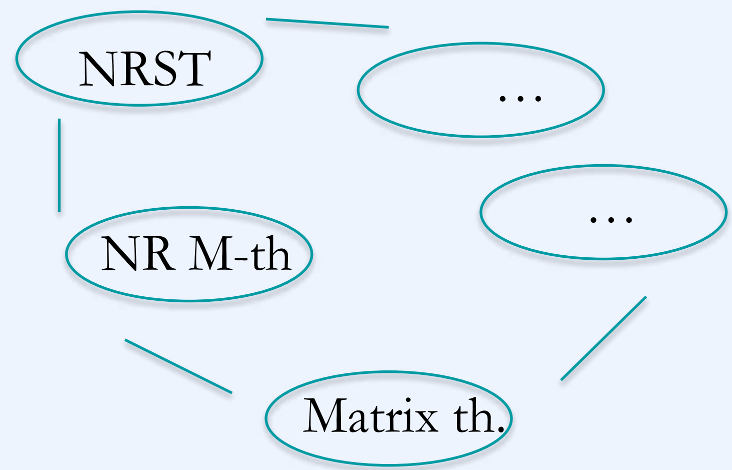
web of decoupled non-gravitational theories

(`open string sector')



web of non-Lorentzian gravitational string theories

(`closed string sector'_)



back to 2000s…

- self-contained corners of ST w. own geometry
- new window on non-perturbative effects ?

Non-AdS holography & NR holography

bulk: Lorentzian gravity (GR) — non-Lorentzian gravity

AdS

non-AdS (e.g. Lifshitz)

NC type gravities

boundary: CFT — NonLor-FT with scaling — NonLor-FT

non-Lorentzian holography

Routes towards non-Lorentzian holography

I. branes

- Dp-branes as probes of TSNC geometry
- Dp-branes as backgrounds solutions of NR (super) gravity actions

Can one find decoupling limits giving rise to avatars of AdS/CFT ?

II. limits of AdS/CFT

- limits of the NR world-sheet theories
- study quantization
- Hamiltonian analysis
- beta-functions

→ tractable limit of AdS/CFT, finite N,
simpler moduli space/genus expansion . ?

Galilean string (cont'd):

- Consider part that is quadratic in X^i. For constant m_{ij} that term is a total derivative and $\delta m_{ij} = 0$ implies $\delta m_{iv} = \partial_i \sigma$.
- We want a kinetic term for every X^i so m_{iv} must be such that its field strength is invertible: symplectic structure.
- Need $i = 1, \ldots, 2n$. Hence $m_{iv} = -\omega_{ij} X^j$ where ω_{ij} is a $2n \times 2n$ invertible antisymmetric matrix.
- WLOG the most general quadratic action that preserves the rotations leaving ω invariant is

$$\mathcal{L} = T\omega_{ij} X^j \left(\dot{X}^i - \dot{f}(\tau) X'^i \right) - \frac{T}{2} X'^i X'^i - \frac{\alpha^2}{2} X^i X^i$$

  where we used the strongest possible gauge fixing.
- n decoupled Schrödinger fields $\Phi^I = X^{2I-1} + iX^{2I}$ with $I = 1, \ldots, n$ and masses determined by eigenvalues of ω_{ij}.

Polyakov action for Carrollian strings

$$S=-\frac{T}{2}\int d^2\sigma\Big[\Big(-e\,e_0{}^{\alpha}e_0{}^{\beta}h_{MN}+\epsilon^{\alpha\beta}m_{MN}\Big)\partial_\alpha X^M\partial_\beta X^N+\omega\epsilon^{\alpha\beta}e^1{}_\alpha\tau^1{}_\beta+\psi\epsilon^{\alpha\beta}(e^0{}_\alpha\tau^1{}_\beta+e^1{}_\alpha\tau^0{}_\beta)\Big]$$

- exhibits Carrollian world-sheet structure
- Weyl and local (2D) Carrollian boost act on the zweibeine:

$$e^0{}_\alpha\to fe^0{}_\alpha+\hat{f}e^1{}_\alpha\,,\quad e^1{}_\alpha\to fe^1{}_\alpha\,,\quad \omega\to\frac{1}{f}\omega-\frac{\hat{f}}{f^2}\psi\,,\quad \psi\to\frac{1}{f}\psi$$

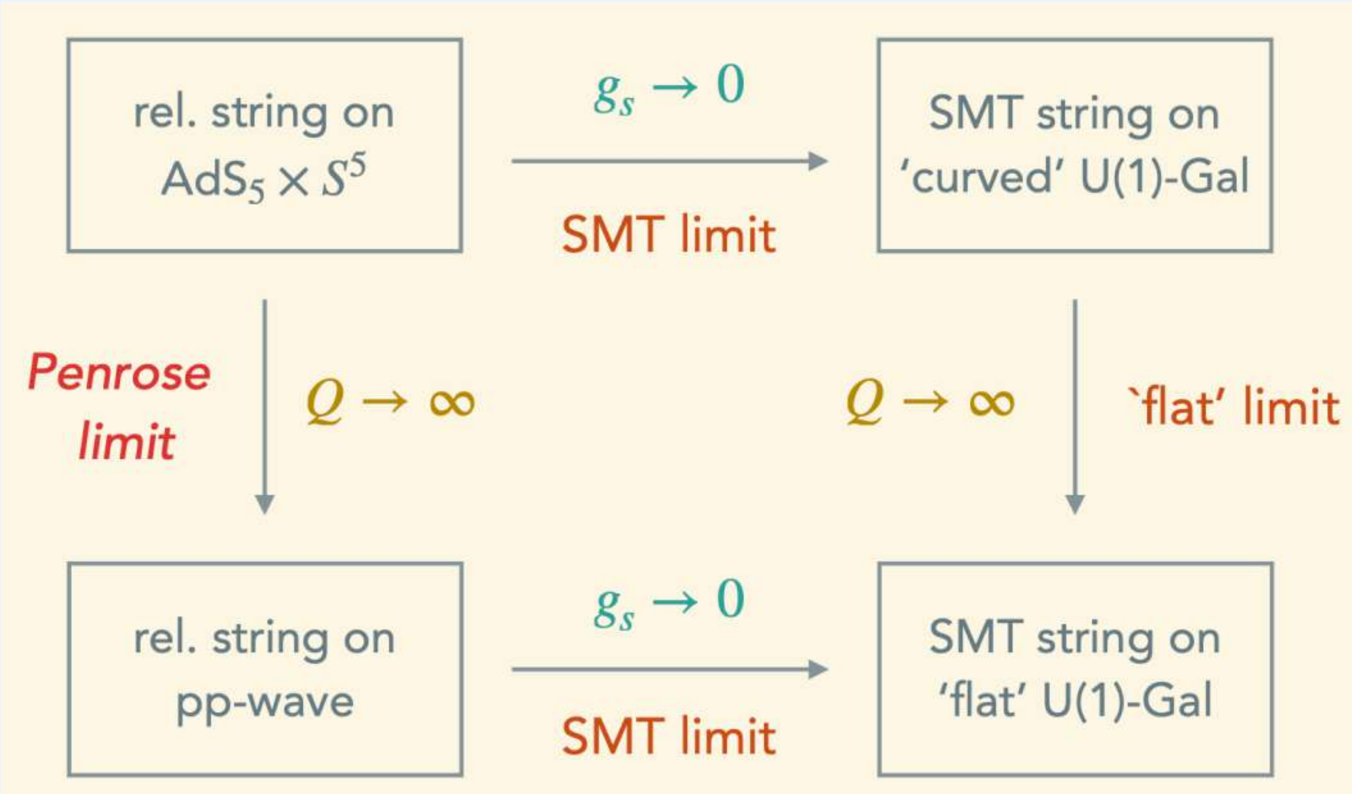
- no problem with (wrong-sign) kinetic terms
- Hamiltonian analysis for (subclass) of target spacetimes $x^M=(x^\mu,v)$,

v = compact isometry

$$\tau^0{}_M=\delta^t_M\,,\quad \tau^1{}_M=\delta^v_M\,,\quad m_{\mu\nu}=0\,,\quad m_{v\mu}=-m_\mu$$

gauge-fixed action has residual symmetry: BMS_3
(contraction of Vir x Vir)

Penrose limit and SMT limit commute



FF (flat-fluxed) backgrounds
→ natural starting point to quantize the theory

flat WS gauge
& "light-cone" gauge:

$$S = -\frac{Q}{2\pi}\int d^2\sigma\left[m_\mu \dot{X}^\mu + \frac{1}{2}h_{\mu\nu}X'^\mu X'^\nu\right]$$

TNC geometry from null reduction

Lorentzian metric with null isometry

$$ds^2 = g_{MN} dX^M dX^N = 2\tau_\mu dx^\mu (du - m_\nu dx^\nu) + h_{\mu\nu} dx^\mu dx^\nu ,$$

$$\tau_\mu h^{\mu\nu} = 0$$

torsional Newton–Cartan (TNC) geometry: τ_μ , $h_{\mu\nu}$, m_μ ,

local syms:

$$\delta\tau_\mu = \mathcal{L}_\xi \tau_\mu , \quad \delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} + \lambda_\mu \tau_\nu + \lambda_\nu \tau_\mu ,$$
$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \lambda_\mu + \partial_\mu \sigma ,$$

λ_μ Galilean (Milne) boosts

σ U(1) (mass) parameter

Non-perturbative string theory

complete understanding of non-perturbative regime is still lacking despite much progress made in last many decades:

- non-perturbative dualities

- Matrix theory:
  infinite boost limit of ST on spacelike circle = DLCQ of ST

  can be viewed as ST on light-like circle → non-relativistic behavior

• NRST as a novel way to study corners of relativistic string theory

TNC geometry from null reduction

Lorentzian metric with null isometry

$$ds^2 = g_{MN} dX^M dX^N = 2\tau_\mu dx^\mu (du - m_\nu dx^\nu) + h_{\mu\nu} dx^\mu dx^\nu \,,$$

$$\tau_\mu h^{\mu\nu} = 0$$

torsional Newton–Cartan (TNC) geometry: $\tau_\mu \quad , \quad h_{\mu\nu} \quad , \quad m_\mu \,,$

local syms:

$$\delta\tau_\mu = \mathcal{L}_\xi \tau_\mu \,, \quad \delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} + \lambda_\mu \tau_\nu + \lambda_\nu \tau_\mu \,,$$
$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \lambda_\mu + \partial_\mu \sigma \,,$$

λ_μ Galilean (Milne) boosts

σ U(1) (mass) parameter

TNC geometry as background geometry for NRFTs

putting relativistic field theory on a curved spacetime

$$\delta S_{\rm rel.matter} \sim \int d^4x \, T_{\mu\nu} \, \delta g^{\mu\nu}$$

- non-relativistic FT naturally couples to torsional Newton-Cartan:

$$\delta S_{\rm non-rel\,matter} \sim \int d^3x [E^\mu \delta\tau_\mu + S^{\mu\nu}\delta h_{\mu\nu} + T^\mu \delta m_\mu]$$

energy current — spatial stress & momentum current — mass current

see e.g. Son (2013),Hartong,Kiritsis,NO(2014),Jensen(2014)

Other properties

- geodesic equation on flat NC space with:

$$m_t = \Phi_{\text{Newt}}$$

→ Newton's law

- TNC geometry can also be obtained by gauging Bargmann algebra

Andringa,Bergshoeff,Gomis,de Roo (2012)

$$[G_a, P_b] = -\delta_{ab} N \quad , \qquad [G_a, H] = -P_a$$

& rotations

↑ mass generator

(just as pseudo-Riemannian geometry follows from gauging Poincare)

Warmup: Non-Relativistic particle from null reduction

null-reduction of relativistic particle $S = \int \frac{1}{2e} g_{MN} \dot{X}^M \dot{X}^N d\lambda =$

→ reduce on target space with null Killing vector :

probe mass is conserved momentum in null direction: $p_u = m$

$$S = \frac{m}{2} \int \frac{h_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}{\tau_\rho \dot{X}^\rho} d\lambda - m \int m_\mu \dot{X}^\mu d\lambda$$

[Kuchar], [Bergshoeff et al]

kinetic term

potential term: coupling to m_μ

m_0 ~ Newtonian potential

$$T^\mu = m \int d\tau \, \partial_\tau X^\mu \delta(x - X(\tau))$$

mass current

- action has TNC local target space symmetries

NR particle from limit of extremal particle

action of charged relativistic particle:

$$S = -mc\int \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\lambda + q\int A_\mu \dot{x}^\mu d\lambda$$

- time-space split in metric: $g_{\mu\nu} = -c^2 T_\mu T_\nu + h_{\mu\nu}$

expand for large c:

$$S = -mc^2\int \left[T_\mu - \frac{q}{mc^2}A_\mu\right]\dot{x}^\mu d\lambda + \frac{m}{2}\int \frac{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{T_\rho \dot{x}^\rho}d\lambda + \mathcal{O}(c^{-2})$$

extremal particle

$$q = mc^2.$$

divergent term. cancels with:

$$T_\mu = \tau_\mu + \frac{1}{2c^2}m_\mu\,,$$
$$A_\mu = \tau_\mu - \frac{1}{2c^2}m_\mu\,,$$

algebra level:
IW contraction

$$H = cP_0 + Q \quad , \qquad N = \frac{1}{2c^2}\left(cP_0 - Q\right)$$

energy mass

Kalb-Ramond B-field from string Poincare

metric and B-field symmetries

$$\bar{\delta} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \quad , \qquad \bar{\delta} B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} + 2\partial_{[\mu}\lambda_{\nu]} .$$

→ can be obtained from string extension of Poincare:
(with extra set of translational generators (cf. double field theory)

$$\begin{aligned}
[M_{\underline{ab}}, M_{\underline{cd}}] &= \eta_{\underline{ac}} M_{\underline{bd}} - \eta_{\underline{bc}} M_{\underline{ad}} + \eta_{\underline{bd}} M_{\underline{ac}} - \eta_{\underline{ad}} M_{\underline{bc}}, \\
[M_{\underline{ab}}, P_{\underline{c}}] &= \eta_{\underline{ac}} P_{\underline{b}} - \eta_{\underline{bc}} P_{\underline{a}}, \\
[M_{\underline{ab}}, Q_{\underline{c}}] &= \eta_{\underline{ac}} Q_{\underline{b}} - \eta_{\underline{bc}} Q_{\underline{a}},
\end{aligned}$$

Lie algebra valued connection:

$$\mathcal{A}_\mu = e_\mu^{\underline{a}} P_{\underline{a}} + \frac{1}{2}\omega_\mu{}^{\underline{ab}} M_{\underline{ab}} + \pi_\mu^{\underline{a}} Q_{\underline{a}};$$

$$g_{\mu\nu} = \eta_{\underline{ab}}\, e_\mu^{\underline{a}} e_\nu^{\underline{b}} . \qquad B_{\mu\nu} = \eta_{\underline{ab}}\, e_{[\mu}^{\underline{a}} \pi_{\nu]}^{\underline{b}} .$$

(see e.g. Ne'eman,Regge/D'Auria,Fre)

NR string action from c → infinity limit

$$S = S_{\text{NG}} + S_{\text{WZ}},$$

$$S_{\text{NG}} = -cT_{\text{F}} \int d^2\sigma \sqrt{-\det g_{\alpha\beta}}, \qquad S_{\text{WZ}} = -c\frac{T_{\text{F}}}{2} \int d^2\sigma B_{\alpha\beta}\epsilon^{\alpha\beta}.$$

- use vielbein decomposition of the NSNS target space:

$$g_{\mu\nu} = \eta_{\underline{ab}} e_\mu^{\underline{a}} e_\nu^{\underline{b}} \quad , \qquad B_{\mu\nu} = \eta_{\underline{ab}} e_{[\mu}^{\underline{a}} \pi_{\nu]}^{\underline{b}},$$

- split tangent space into
A=0,1: longitudinal
a=2,..d-1: transverse

$$e_\mu^{\underline{a}} = (cE_\mu^A, e_\mu^a) \quad , \qquad \pi_\mu^{\underline{a}} = (c\Pi_\mu^A, \pi_\mu^a),$$

- reparametrize longitudinal vielbeins:

$$E_\mu^A = \tau_\mu^A + \frac{1}{2c^2}\pi_\mu^B \epsilon_B{}^A,$$

$$\Pi_\mu^A = \epsilon^A{}_B \tau_\mu^B + \frac{1}{2c^2}\pi_\mu^A,$$

divergent term in action cancels since F-string is extremal

NRST action on TSNC target space

after limit:

kinetic potential

$$S_{\rm NR} = -\frac{T}{2}\int d^2\sigma\left[\sqrt{-\tau}\,\eta^{AB}\tau^\alpha_A\tau^\beta_B h_{\alpha\beta} + \epsilon^{\alpha\beta}m_{\alpha\beta}\right],$$

$$h_{\mu\nu} = e^a_\alpha e^b_\beta \delta_{ab} \qquad m_{\mu\nu} = \eta_{AB}\tau^A_{[\mu}\pi^B_{\nu]} + \delta_{ab}e^a_{[\mu}\pi^b_{\nu]}.$$

torsional string Newton–Cartan geometry : $\tau_\mu^A, \quad h_{\mu\nu}, \quad m_{\mu\nu}.$

$m_{\mu\nu}$. couples to worldsheet tension current $J_{\rm T}^{\mu\nu} = T\int d^2\sigma\,\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\delta(x - X(\sigma^\alpha)),$

conserved charges: $M^\mu = \int d\sigma J^{0\mu}.$

gravitational mass: $M^v = 2\pi TR.$ (for compact longitudinal direction v)

m_{0v} = gravitational potential

- weak equivalence principle for NR string

F-string Galilean (FSG) algebra

- decompose string Poincare algebra:

$$\underline{a} = (A, a)$$

longitudinal, transverse

$$P_A, \quad Q_A, \quad P_a, \quad Q_a, \quad J_{AB} = \epsilon_{AB} J = M_{AB}, \quad J_{ab} = M_{ab}, \quad c\,G_{Ab} = M_{Ab}.$$

$$H_A = c(P_A + Q_B \epsilon^B{}_A) \quad , \quad N_A = \frac{1}{2c}(\epsilon_A{}^B P_B + Q_A)$$

(basis transformation)

after IW contraction (c → infinity):

$$[G_{Ab}, H_C] = \eta_{AC} P_b + \epsilon_{AC} Q_b,$$
$$[G_{Ab}, P_c] = -\delta_{bc} \epsilon_A{}^B N_B,$$
$$[G_{Ab}, Q_c] = -\delta_{bc} N_A,$$

& further commutators involving SO(1,1) x SO(d-2) rotations

- symmetry trafos follow from FSG-valued connection:

$$\mathcal{A}_\mu = \tau_\mu^A H_A + e_\mu^a P_a + \omega_\mu J + \frac{1}{2}\omega_\mu{}^{ab} J_{ab} + \omega_\mu{}^{Ab} G_{Ab} + \pi_\mu^A Z_A + \pi_\mu^a Q_a,$$

Non-relativistic strings from null reduction

- start from Polyakov action (including NSNS) and reduce along null isometry

- implement conservation of string momentum along null isometry using Lagrange multipliers

- go to dual formulation that exchanges the (fixed) momentum along null direction for fixed winding of string along compact dual direction

→ action of non-relativistic strings moving in torsional string Newton-Cartan target space

& FSG symmetries can also be derived

Limit vs. 1/c expansion

- limit geometry: type I (cancellation of divergent term)

- geometry from expansion: type II
(each term in the action generates more gauge fields)

type II studied for:

- NR particle (and coupling to non-relativistic gravity from expanding GR) van den Bleeken (2018), Hansen,Hartong,NO (2019, 2020)
- NR string Hartong,Have (2021)

spectrum:
(compact long. direction)
center of mass velocity << c

$$E = \frac{c^2 w R_{\text{eff}}}{\alpha'_{\text{eff}}} + \frac{N_{(0)} + \tilde{N}_{(0)}}{w R_{\text{eff}}} + \frac{\alpha'_{\text{eff}}}{2 w R_{\text{eff}}} p^2_{(0)} + \mathcal{O}(c^{-2})$$

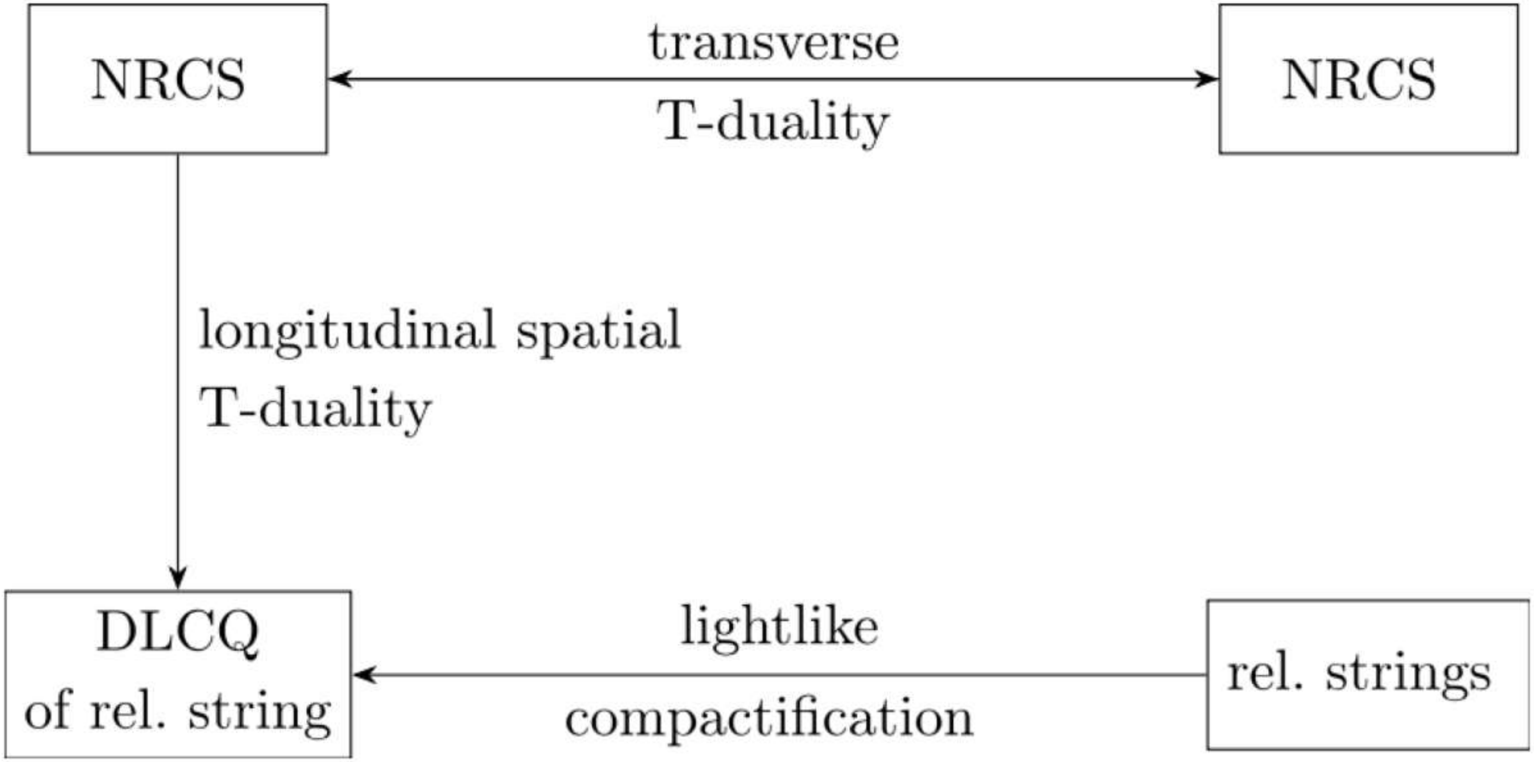


FIGURE 2 | T-duality transformations of NR closed string theory (NRCS).

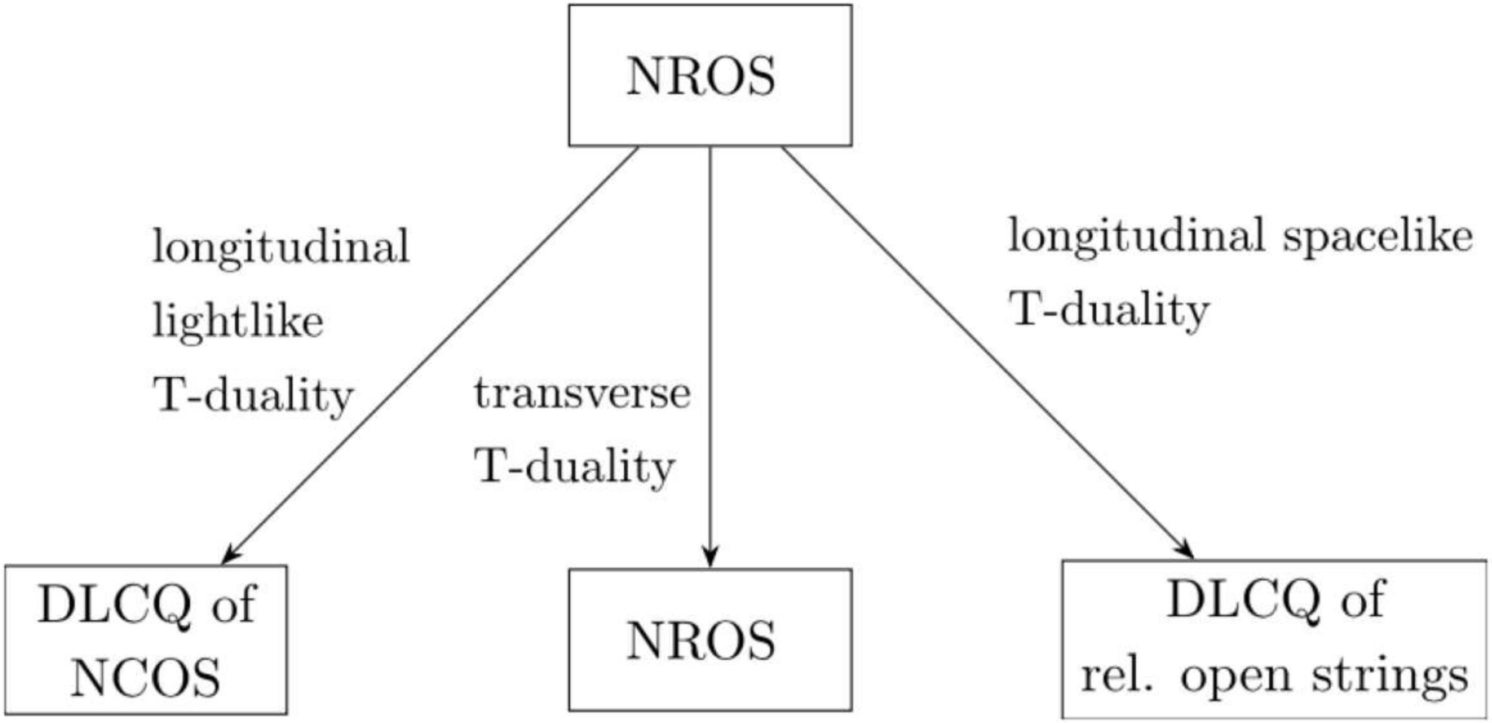


FIGURE 3 | T-duality relations for NR open string theory (NROS). The transverse T-duality swaps Dirichlet and Neumann boundary conditions in the usual way.