Problem 1

The Majorana-conjugate spinor is defined by

$$\bar{\Psi} = \Psi^{\mathrm{T}} C,$$

where C is the charge-conjugation matrix

$$C \Gamma_M C^{-1} = -\Gamma_M{}^{\mathrm{T}}.$$

Show that

1. The Dirac action
$$S = i \int d^D x \, \bar{\Psi} \Gamma^M \partial_M \Psi$$
is well-defined provided C is anti-symmetric:
$$C^{\mathrm{T}} = -C.$$

2. The Dirac Lagrangian is Lorentz-invariant.

3. The chirality constraint
$$\Gamma \Psi = \Psi$$
is consistent with Majorana conjugation only if
$$C \Gamma C^{-1} = -\Gamma^{\mathrm{T}}. \tag{1}$$

Hard part:

4. Show that
   (a) The charge conjugation matrix exists in $D = 2, 3, 4, 6, 7, 8 \mod 8$
   (b) C is anti-symmetric in $D = 2, 3, 4 \mod 8$.
   (c) Eqn. (1) holds in $D = 2, 6 \mod 8$.