Boundary modes in fracton models

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My story with Eduardo  #TBT

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Introduction to Bosonization

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“The highly pedagogical notes of Eduardo Miranda, which are self-contained and assume just some basic knowledge of second quantization, are of particular relevance as an introduction to the use of modern bosonization techniques.” [BJP (2003)]
My story with Eduardo  #TBT

- First met in 1999 during my undergrad at Unicamp.
- Masters 2002-2004. Worked on 1D Kondo lattice, magnetic domain walls and impurities in Luttinger liquids.
- I keep going back to the lessons I learned from Eduardo. This talk is an example.
Outline

- Topological phases with boundaries
- Fractons
- Gapped boundaries of the Chamon model
- Effective field theory for boundary modes

In collaboration with Weslei Fontana (Natal)  
Gapped edges of topological phases

Topological-ordered phases without symmetries, e.g. fractional quantum Hall states and quantum spin liquids.

Long-range entanglement. Ground state degeneracy depends on topology of closed surface in real space. Fractionalization: elementary excitations are anyons.

Gapped edges classified by condensation of anyons with self/mutual bosonic statistics: Lagrangian subgroups of the set of quasiparticles.

[Kapustin & Saulina, NPB (2011); Levin, PRX (2013); Wang & Wen, PRB (2015)]
**Example: Z₂ topological order**

Present in Anderson’s resonating valence bond state.

[Anderson (1973); Kivelson, Rokhsar & Sethna (1987)]  

Realized in quantum simulation platforms: superconducting qubits, Rydberg atoms, trapped ions.

[Satzinger et al. (Google Quantum AI), Science (2021)]  
[Semeghini et al., Science (2021)]  
[Iqbal et al. (Quantinuum), arXiv:2302.01917]
Anyons in the $\mathbb{Z}_2$ spin liquid

**Toric code:** $e$ and $m$ particles have bosonic self-statistics, mutual semionic statistics; bound state of $e$ and $m$ is a fermion.


**Stabilizers:**

$$A_s = \prod_{j \in s} \sigma_j^x$$

$$B_p = \prod_{j \in p} \sigma_j^z$$

H = \sum_s A_s - \sum_p B_p

[Diagram of stabilizer operators and a lattice with anyons]
Two types of gapped edges


Boundary stabilizers:

\[ B'_p = \sigma_1^z \sigma_2^z \sigma_3^z \]

\[ A'_s = \sigma_1^x \sigma_2^x \sigma_3^x \]
Two types of gapped edges


Boundary stabilizers:

$$B'_p = \sigma_1 \bar{z} \sigma_2 \bar{z} \sigma_3$$

$$A'_s = \sigma_1^x \sigma_2^x \sigma_3^x$$
Two types of gapped edges

Open boundaries (surface codes): e particles condense on rough edges and m particles on smooth edges. Different boundary phases separated by quantum phase transition (described by Ising CFT).


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Boundary stabilizers:

For rough edges:

$$B'_p = \sigma_1^z \sigma_2^z \sigma_3^z$$

For smooth edges:

$$A'_s = \sigma_1^x \sigma_2^x \sigma_3^x$$
Fracton “topological” order

Restricted mobility: particles unable to move in isolation, move along certain directions by forming bound states (type-I, e.g. Chamon model, X-cube); or are completely immobile (type-II, e.g. Haah code).

[Chamon, PRL (2005); Bravyi, Leemhuis & Terhal, Ann. Phys. (2011); Haah, PRA (2011); Vijay, Haah & Fu, PRB (2015)]

Ground state degeneracy depends on geometric data; typically increases with system size.

Not described by topological quantum field theories; effective field theories depend on microscopic details (UV-IR mixing).

Lattice models involve multi-spin interactions. Proposals with Majorana zero modes and quantum circuits with time evolution and measurements.

[You & von Oppen, PRR (2019); Verresen et al., arXiv(2021); Lu et al., PRX Quantum (2022)]
Chamon model

Stabilizers defined on octahedra in the fcc lattice.

\[ O = \sigma_1^x \sigma_2^x \sigma_3^y \sigma_4^y \sigma_5^z \sigma_6^z \]

\[ H = -J \sum_I O_I \]

Sum over octahedra

[Chamon, PRL (2005)]
Chamon model

Stabilizers defined on octahedra in the fcc lattice. [Chamon, PRL (2005)]

\[ \mathcal{O} = \sigma_1^x \sigma_2^x \sigma_3^y \sigma_4^y \sigma_5^z \sigma_6^z \]

\[ H = -J \sum_I \mathcal{O}_I \]

\( \sum \text{over octahedra} \)

\( \mathbb{Z}_2 \) constraints on four sublattices:

\[ \prod_{I \in A} \mathcal{O}_I = \prod_{I \in B} \mathcal{O}_I = \prod_{I \in C} \mathcal{O}_I = \prod_{I \in D} \mathcal{O}_I = 1 \]
Bulk properties

Ground state degeneracy on 3-torus with dimensions $L_x \times L_y \times L_z$:

$$\mathcal{O}_I |\Psi_0^{(\alpha)}\rangle = |\Psi_0^{(\alpha)}\rangle \quad \forall I \quad \alpha = 1, \ldots, 2^{4n} \quad n = \gcd\left(\frac{L_x}{2}, \frac{L_y}{2}, \frac{L_z}{2}\right)$$


Defect: $\mathcal{O}_I = -1$

Spin operator creates four defects on a plane. Pair of defects (dipole) moves along rigid lines.
Single defects

Local operators cannot move single defect (fracton) without creating additional excitations.
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Local operators cannot move single defect (fracton) without creating additional excitations.
Chamon model with a boundary

Gapped boundaries of fracton models not classified by bulk braiding statistics.

Five-site stabilizers on (001) boundary: \( \mathcal{O}_I' = \sigma_1^x \sigma_2^x \sigma_3^y \sigma_4^y \sigma_5^z \)

Boundaries on AB and CD planes violate different constraints. Spin operator on the boundary creates three defects.

In both cases boundary defects cost energy. Boundary modes are gapped.

[Fontana & RP, arXiv (2022)]
**Boundary processes**

Dipoles can be converted into boundary fracton:  

\[(b, c) \rightarrow b \quad \checkmark \quad (b, c) \rightarrow c \quad \times\]
\[(a, d) \rightarrow a \quad \checkmark \quad (a, d) \rightarrow d \quad \times\]

Allowed processes depend on the parity of boundary plane coordinate \(z\).
Boundary processes

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Allowed processes depend on the parity of boundary plane coordinate \( z \).
Effective field theory in the bulk

Write spin operators in terms of two bosonic fields. Ground state subspace described by effective Chern-Simons-like action with higher derivatives:

\[ S = \sum_{l=1,2} \sum_{m,n=1,2} \int d^3r \, dt \left( \frac{1}{2\pi} K_{mn} A^{(l)}_m \partial_t A^{(l)}_n + \frac{1}{\pi} A^{(l)}_0 K_{mn} D_m A^{(l)}_n \right) \]

\[ K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad D_1 = \partial_x^2 - \partial_z^2 \quad D_2 = \partial_y^2 - \partial_z^2 \]

Add layer index \( l = 1, 2 \) for (001) planes with odd/even \( z \).

Gauge invariance: \( A^{(l)}_n \rightarrow A^{(l)}_n + \sqrt{a} D_n \zeta^{(l)} \) \( A^{(l)}_0 \rightarrow A^{(l)}_0 + \sqrt{a} \partial_t \zeta^{(l)} \)
Boundary theory

Analogy with QHE: gauge invariance requires physical degrees of freedom at the boundary. Write gauge fields as derivatives of two-component bosonic field.

\[ A_{IJ}^{(l)} = \sqrt{a} \, D_{IJ} \varphi_l \]

\[ [\varphi_l(r), D_{xy} \varphi_{l'}(r')] = i\pi (K^{-1})_{ll'} \delta(r - r') \]

\[ S_{bd} = \frac{1}{2\pi} \int d^2 r \, dt \left( K_{ll'} \varphi_l \partial_t D_{xy} \varphi_{l'} - M_{ll'} D_{xy} \varphi_l D_{xy} \varphi_{l'} \right) \]

Boundary modes are gapless. Dispersion has signature of UV-IR mixing.

\[ \omega = \mu |k_x^2 - k_y^2| \]

\[ \mu = \sqrt{\det(M)} \]
Symmetries and charged operators

Conserved charges: \[ Q_l = \frac{1}{\pi} \int dxdy K_{l\nu} D_{xy} \varphi_\nu \]
\[ D_{xy} = \partial_x^2 - \partial_y^2 \]

Charged operators: \[ \Psi_q(r) \sim e^{iq_l \varphi_l(r)} \]
\[ q = (q_1, q_2) \]
\[ [Q_l, \Psi_q(r)] = -q_l \Psi_q(r) \]

Dipoles created by exponentials of derivatives:
\[ \Psi_q^\dagger(r + \delta) \Psi_q(r) \sim e^{-i q_l \cdot \nabla \varphi_l(r)} \]

Boundary action has U(1) subsystem symmetries:
\[ \varphi_l(x, y) \rightarrow \varphi_l(x, y) + f_l(x + y) + g_l(x - y) \]

Generalized neutrality condition:
\[ \left\langle \prod_\alpha e^{iq_\alpha \varphi_l(r_\alpha)} \right\rangle \neq 0 \quad \text{only if } r_\alpha \text{ vertices of rectangular membrane.} \]
Boundary condition

To match number of d.o.f. in the boundary theory, we impose that normal derivative must be linear combination of bosonic fields:

\[ a \partial_z \varphi_l \rightarrow c_{ll'} \varphi_{l'} \]  
(coefficients depend on the type of boundary)

Ribbon operators that extend to the boundary create boundary fracton at the end point.

\[
W_l = e^{\frac{i}{\sqrt{a}} \int d\xi_1 \int_{\xi_1}^{\xi_1+w} d\bar{\xi}_1 A^{(l)}_2}
\]

\[
W_l = e^{i\sqrt{2}w[\partial_z \varphi_l(\mathbf{r}_0) + \partial_y \varphi_l(\mathbf{r}_0)]} \sim e^{i\sqrt{2}w[c_{ll'} \varphi_{l'}(\mathbf{r}_0) + \cdots]}
\]
How to gap out the boundary modes?

Perturbations are similar to cosine terms in Luttinger liquid theory.

\[ \delta H_q = -g_q \int dxdy \cos(q \varphi_l) \]

\[ \sim \frac{g_q}{2} \int dxdy (q \varphi_l)^2 \]

which leaves (353) unchanged. In this case, the Umklapp term becomes

\[ H_{\text{Umklapp}} \rightarrow \frac{\Delta}{2\pi^2 \alpha^2} \int dx \cos [4\sqrt{\pi}g\phi(x)]. \]  

(356)

It is known from renormalization group arguments that for \( g > g_{\text{crit}} = 1/2 \), the Umklapp term is irrelevant: its effect on the low-energy sector is simply to renormalize the effective parameters \( u \) and \( g \) but, other than this, it can be ignored. However, if \( g < 1/2 \), then the Umklapp term is relevant: it is responsible for the opening of a gap in the spectrum. In this case, a Luttinger liquid description is no longer valid. The case \( g = g_{\text{crit}} = 1/2 \) (which coincides with the isotropic Heisenberg model as we saw) is a marginal case. The renormalization factor of (356) is finite in this case.

[Edwardo’s “Introduction to bosonization”]
Correlations at weak coupling fixed point

Gapless boundary theory is scale invariant with dynamical exponent $z = 2$. But two-point correlation is (ultra) short-range:

$$\langle e^{iq_1 \varphi_1(r)} e^{-iq'_1 \varphi_1(0)} \rangle = 0 \quad (r \neq 0) \quad \text{can be small}$$

Dipole correlations can be long range:

$$\langle e^{i\omega q_1 \partial_\xi \varphi_1(\xi)} e^{-i\omega q'_1 \partial_\xi \varphi_1(0)} \rangle = \left( \frac{1}{\alpha^2 + \xi^2} \right)^{\eta_q} \quad \text{can be relevant}$$

Gapless phase is stable. BKT-type transition to non-fractonic phase.

Same theory appears in 2D model: XY plaquette model for Bose metal.

[Paramekanti, Balents & Fisher, PRB (2002); Seiberg & Shao, SciPost (2021); You & Moessner, PRB (2022); Grosvenor et al., PRB (2023)]
Strong coupling regime

Solvable lattice model is at strong coupling: vanishing dispersion \( \mu \to 0 \)

Transition between gapped boundaries governed by competition between two cosine terms:

\[ \delta H = - \int dxdy \left[ g_1 \cos(q_0 \varphi_1) + g_2 \cos(q_0 \varphi_2) \right] \]

First-order boundary phase transition; mapping to 2D compass model/Xu-Moore model.

Weak coupling regime requires perturbations that enhance quantum fluctuations and layer mixing at the boundary.

Outlook: microscopic model for gapless phase? Boundary phase transitions?

[in collaboration with C. Chamon]
Summary

- Fracton phases represent a new type of “topological” order in which quasiparticles have restricted mobility.

- The Chamon model has two types of gapped (001) boundaries distinguished by broken parity constraints and processes that turn bulk dipoles into boundary fractons.

- Effective field theory predicts more boundary phases. Need better understanding of boundary phase transitions.
Conservation laws

Electric and magnetic fields vanish in the ground state manifold.

\[ E_{n}^{(l)} = \partial_{t} A_{n}^{(l)} - D_{n} A_{0}^{(l)} = 0 \quad \text{and} \quad B^{(l)} = D_{1} A_{2}^{(l)} - D_{2} A_{1}^{(l)} = 0 \]

Subsystem symmetries: conservation of line/ribbon operators.

\[ W_{l} = \exp \left[ \frac{i q}{\sqrt{a}} \int d\xi_{1} \int_{\bar{\xi}_{1}+w}^{\xi_{1}} d\bar{\xi}_{1} A_{2}^{(l)} \right] \]