School on Emergent Phenomena in Non-Equilibrium Many-Body Physics
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Driven Open Quantum Systems: From Micro- to Macrophysics

Sebastian Diehl

Institute for Theoretical Physics, University of Cologne

Outline

Keldysh theory general: A. Kamenev, *Field theory or non-equilibrium systems*, Cambridge University Press

Review: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016)

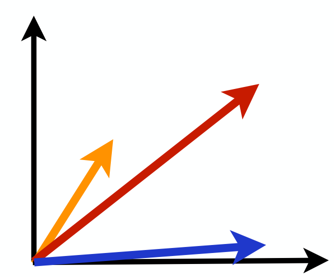
Lecture I & II: Theoretical background & non-equilibrium phases

$$\partial_t \rho = -i[H, \rho] + \mathcal{L}[\rho]$$

$$\mathrm{e}^{\mathrm{i}\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi \mathrm{e}^{\mathrm{i}S_M[\Phi+\delta\Phi]}$$

- From the quantum master equation to the Keldysh functional integral
  - construction
  - semiclassical limit, connection to exciton-polariton systems
  - "what is non-equilibrium about it?"

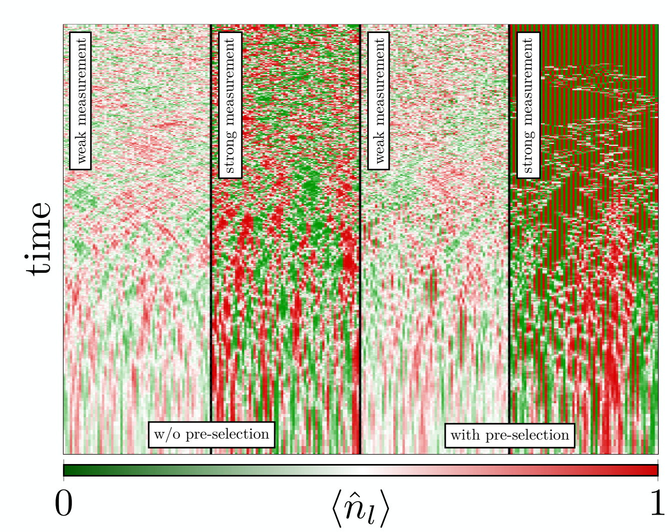
- Applications: stationary states of driven open quantum systems
  - fate of BKT physics out of equilibrium
  - phase transition driven by non-equilibrium drive

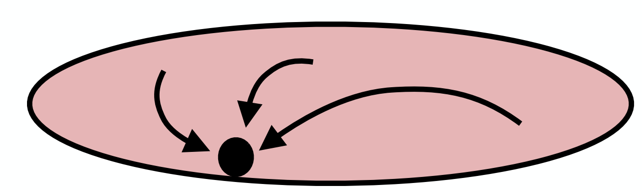


Lecture III: More quantum

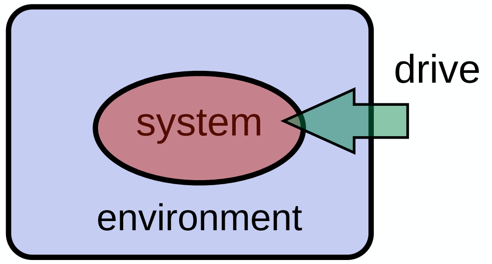
- dark state dynamics and topology?

- measurements?





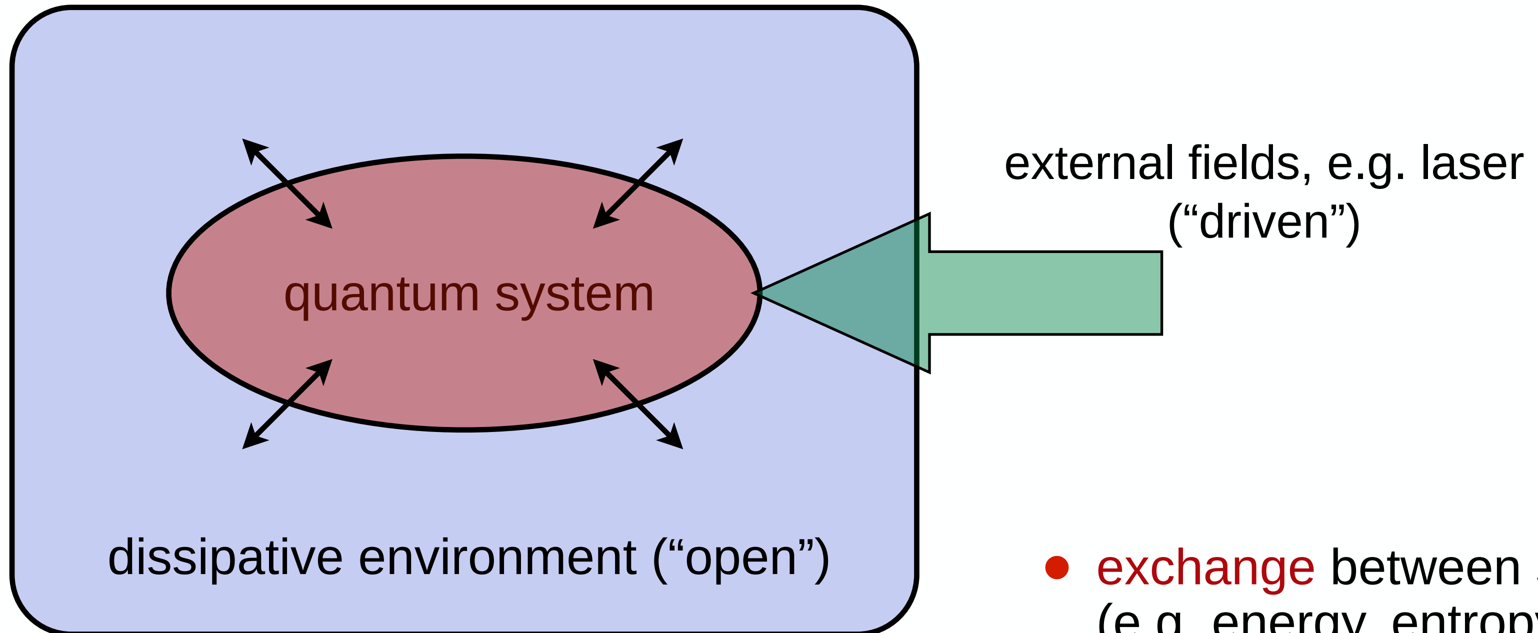
Lindblad quantum master equation:
From few to many degrees of freedom



$$\partial_t \rho = -i[H,\rho] + \kappa \sum_i L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\}$$

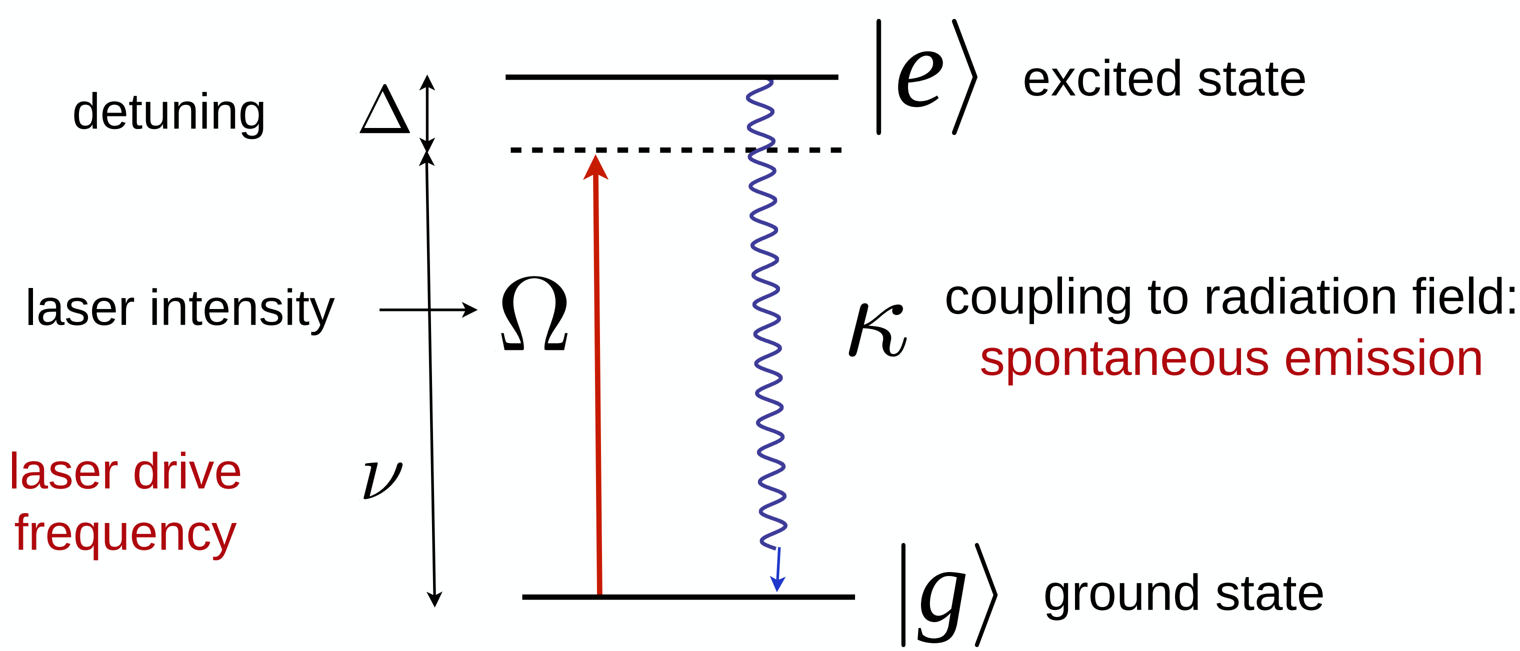
What is a driven open quantum system?

- quantum optics:



- exchange between system and bath (e.g. energy, entropy, particle number)

- example: laser driven atom coupled to the radiation field (two-level system)



- simple fact: drive essential to access upper level

- implications:

  - no minimisation of energy
  - no guarantee for detailed balance
  - no obedience of the second law of thermodynamics (state purification)

Driven open quantum systems: microscopic description

- quantum master equation

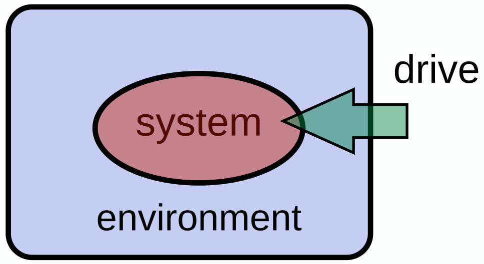
$$\partial_t \hat\rho = \underbrace{-i[\hat H, \hat\rho]}_{\text{coherent evolution}} + \underbrace{\sum_i \gamma_i [2\hat L_i \hat\rho \hat L_i^\dagger - \hat L_i^\dagger \hat L_i \hat\rho - \hat\rho \hat L_i^\dagger \hat L_i]}_{\text{driven-dissipative evolution}}$$

Lindblad operators

lattice site, spin …

$$\equiv \hat{\mathcal{L}}[\hat\rho]$$

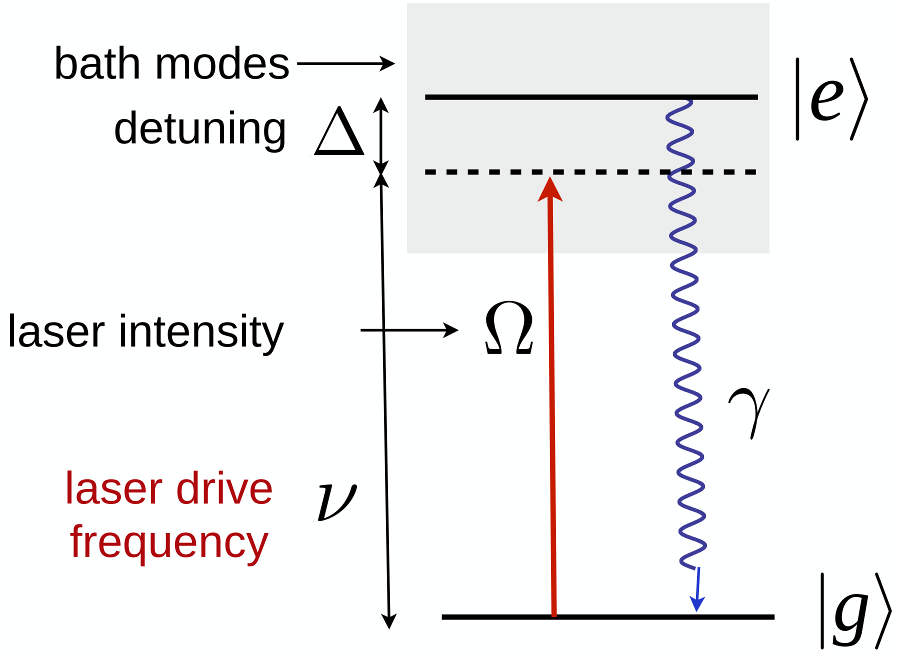
Lindbladian; also: Liouvillian



- derivation from system-bath setting: second order time dependent perturbation theory

see Darrick's lecture

- example: two-level system



$$H = (|e\rangle, |g\rangle) \begin{pmatrix} \Delta & \Omega \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} \langle e| \\ \langle g| \end{pmatrix}$$

$$\hat L = |g\rangle\langle e| = \sigma^-$$

$$g_\mu \sim \Omega$$

- starting point: system-bath setting

$$\hat H_{\mathrm{t}} = \hat H + \hat H_{\mathrm{b}} + \hat H_{\text{s-b}}$$

$$\hat H_{\mathrm{b}} = \sum_\mu \epsilon_\mu \hat b_\mu^\dagger \hat b_\mu$$

$$\hat H_{\text{s-b}} = \sum_\mu g_\mu e^{-i\nu t}\, \hat L\, \hat b_\mu^\dagger + \mathrm{h.\ c.}$$

(for single L)

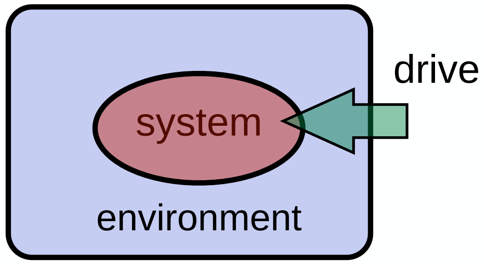
- 3 approximations:
  - Born: weak system-bath coupling -> bath unaffected by system (2nd order pert. th.)
  - Markov: system evolution slow wrt bath -> time-local evolution
  - rotating wave: drive ν selects relevant energy regimes

Driven open quantum systems: microscopic description

- quantum master equation

Lindblad operators

$$\partial_t \hat{\rho} = \underbrace{-i[\hat{H},\hat{\rho}]}_{\text{coherent evolution}} + \sum_i \underbrace{\gamma_i[2\hat{L}_i\hat{\rho}\hat{L}_i^\dagger - \hat{L}_i^\dagger\hat{L}_i\hat{\rho} - \hat{\rho}\hat{L}_i^\dagger\hat{L}_i]}_{\text{driven-dissipative evolution}}$$



$$\equiv \hat{\mathcal{L}}[\hat{\rho}]$$ Lindbladian; also: Liouvillian

- derivation from ‘symmetry’ (i.e. implementing key physical requirements)
  - Lindbladian defines a dynamical map $\hat{\rho}(t+\Delta t) = \hat{\rho}(t) + \Delta t \cdot \hat{\mathcal{L}}[\hat{\rho}]$
  - with properties
    - Hermiticity: $\hat{\rho}(t)^\dagger = \hat{\rho}(t) \implies \hat{\rho}^\dagger(t+\Delta t) = \hat{\rho}(t+\Delta t)$ since $\hat{\mathcal{L}}[\hat{\rho}]^\dagger = \hat{\mathcal{L}}[\hat{\rho}]$
    - complete positivity: $\hat{\rho}(t) \geq 0 \implies \hat{\rho}(t+\Delta t) \geq 0$
    - trace preservation / probability conservation $\partial_t \mathrm{tr}\hat{\rho}(t) = 0$ since $\mathrm{tr}\hat{\mathcal{L}}[\hat{\rho}] = 0$

Göran Lindblad

➡ up to a unitary transformation (above: diagonal form in index i), $\hat{\mathcal{L}}[\hat{\rho}]$ is the most general time-local generator with these properties

G. Lindblad, Commun. Math. Phys. (1976)

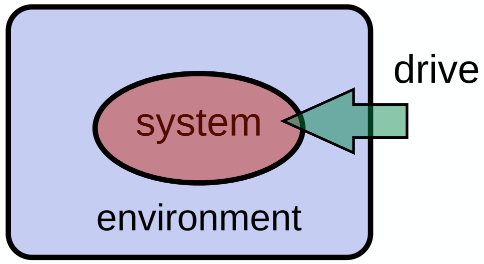
Nielsen & Chuang, Chap. 8

Driven open quantum systems: microscopic description

- quantum master equation

Lindblad operators

$$\partial_t \hat\rho = \underbrace{-i\left[\hat H, \hat\rho\right]}_{\text{coherent evolution}} + \sum_i \underbrace{\gamma_i [2\hat L_i \hat\rho \hat L_i^\dagger - \hat L_i^\dagger \hat L_i \hat\rho - \hat\rho \hat L_i^\dagger \hat L_i]}_{\text{driven-dissipative evolution}}$$



$$\equiv \hat{\mathcal{L}}[\hat\rho]$$ Lindbladian; also: Liouvillian

- interpretation:

$$\partial_t \hat\rho = -i(\hat H - i \sum_i \gamma_i \hat L_i^\dagger \hat L_i)\hat\rho + \mathrm{h.c.} + 2\sum_i \gamma_i \hat L_i \hat\rho \hat L_i^\dagger$$

energy

decay (dissipation)

ensures probability conservation (fluctuation)

$$\text{“}E - i\Gamma\text{”}$$

$$\partial_t \mathrm{tr}\hat\rho(t) = 0$$

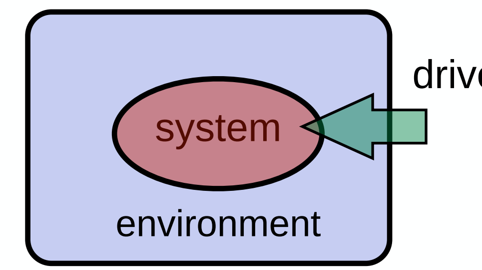
Driven open quantum systems: microscopic description

- quantum master equation

Lindblad operators

$$\partial_t \hat{\rho} = \underbrace{-i[\hat{H}, \hat{\rho}]}_{\text{coherent evolution}} + \sum_i \underbrace{\gamma_i [2\hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \hat{L}_i^\dagger \hat{L}_i \hat{\rho} - \hat{\rho} \hat{L}_i^\dagger \hat{L}_i]}_{\text{driven-dissipative evolution}}$$

$$\equiv \hat{\mathcal{L}}[\hat{\rho}]$$ Lindbladian; also: Liouvillian



- So far: few degrees of freedom in the "system"
- Question: What if we replace few by many degrees of freedom?

➡ The interface of quantum optics and many-body physics

➡ Quantum Optics:
coherent and driven-dissipative dynamics on equal footing

➡ Many-Body Physics:
continuum of spatial degrees of freedom

➡ Statistical Mechanics:
physics at the largest distances

microphysics ⟶ macrophysics

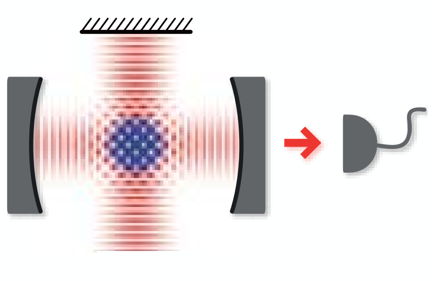
The interface of quantum optics and many-body physics

➡ Quantum Optics ➡ Many-Body Physics ➡ Statistical Mechanics

microphysics ⟶ macrophysics

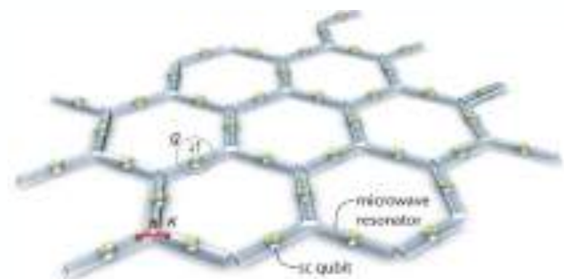
- The experimental platforms: light-matter systems realize driven open quantum matter

Atoms



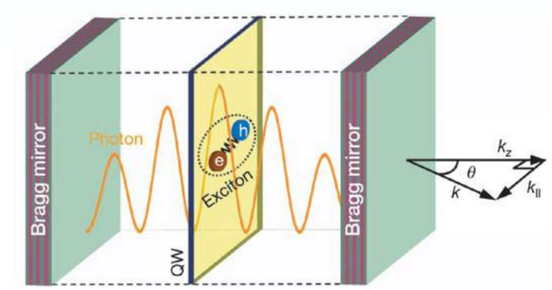
Bose-Einstein condensate in a cavity
Baumann et al., Nature (2010)

Light



Microcavity arrays
Houck, Türeci, Koch, Nat. Phys. (2012)

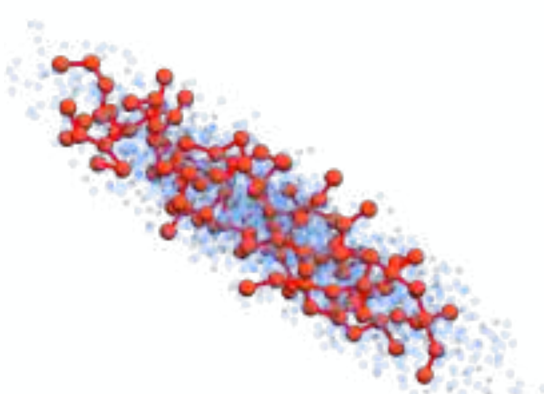
Solids



Exciton-polariton condensates
Kasprzak et al., Nature (2006)

and more:

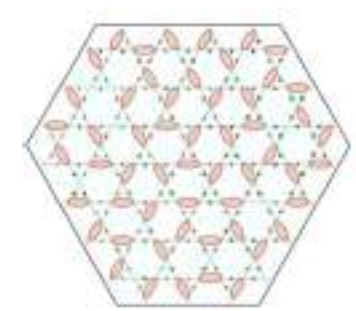
- polar molecules
- nano-mechanics
- photon BECs



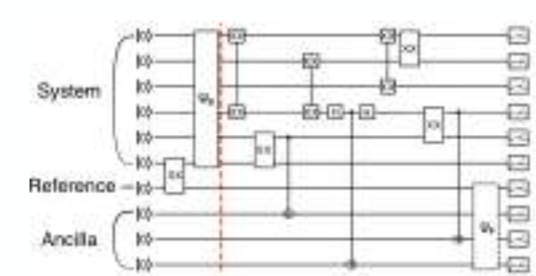
driven-dissipative Rydberg gases
S. Helmrich, A. Arias, G. Lochhead, M. Buchhold, SD, S. Whitlock, Nature (2020); T. Wintermantel, … SD, S. Whitlock, Nat. Comm. (2021)

Quantum devices / NISQ Platforms

superconducting circuits
K. Satzinger et al. Science (2021)



Rydberg tweezers
G. Semeghini et al. Science (2021)



trapped ions
C. Noel et al. Nat. Phys. (2022)

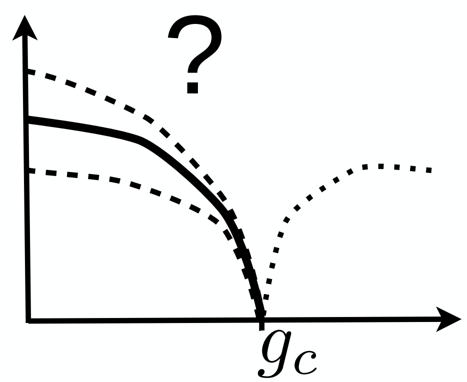
The interface of quantum optics and many-body physics

➡ Quantum Optics ➡ Many-Body Physics ➡ Statistical Mechanics

microphysics	→	macrophysics
Microscopic	“Thermodynamic”	Long wavelength

- Questions and challenges to theory: physics at various length scales

Novel universal phenomena ?



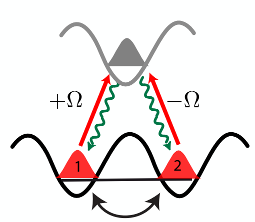
Efficient theoretical tools ?

$$Z[J] = \int \mathcal{D}\varphi \, e^{i(S[\varphi] + \int J\varphi)}$$

perform the transition form micro-to macrophysics:

quantum field theory out of equilibrium

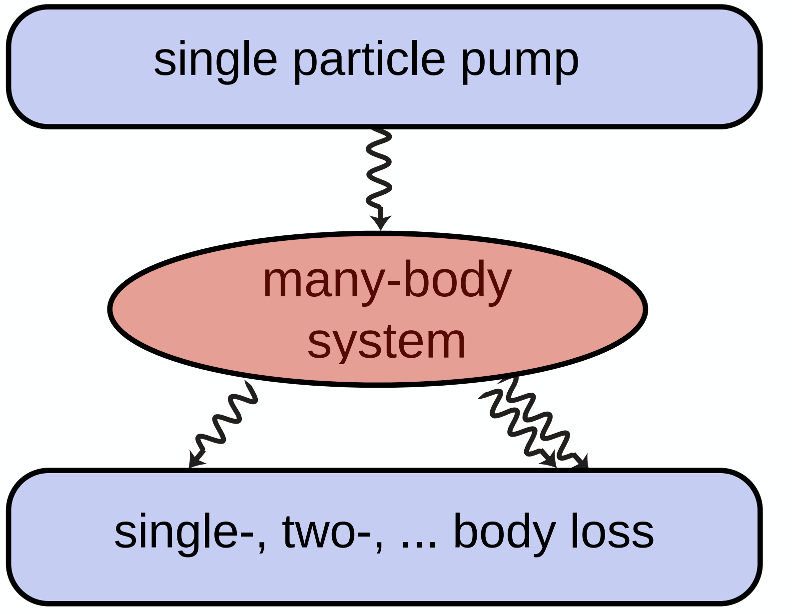
Experimental platforms ?



cold atoms, light-driven semiconductors, microcavity arrays, trapped ions, NISQ …

A workhorse model: Lindbladian formulation

exercise: derive equation for field expectation value!



- generic microscopic many-body model: 'Lindblad ϕ^4 theory'

$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \hat\phi^\dagger_{\mathbf{x}} \left(\tfrac{\triangle}{2M} - \mu\right) \hat\phi_{\mathbf{x}} + \tfrac{\lambda}{2}(\hat\phi^\dagger_{\mathbf{x}}\hat\phi_{\mathbf{x}})^2$$

$$\mathcal{D}[\rho] = \underbrace{\gamma_p \int_{\mathbf{x}} [\hat\phi^\dagger_{\mathbf{x}}\,\rho\,\hat\phi_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi_{\mathbf{x}}\hat\phi^\dagger_{\mathbf{x}},\rho\}]}_{\text{single particle pump}} \;+\; \underbrace{\gamma_l \int_{\mathbf{x}} [\hat\phi_{\mathbf{x}}\,\rho\,\hat\phi^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi^\dagger_{\mathbf{x}}\hat\phi_{\mathbf{x}},\rho\}]}_{\text{single particle loss}} \;+$$

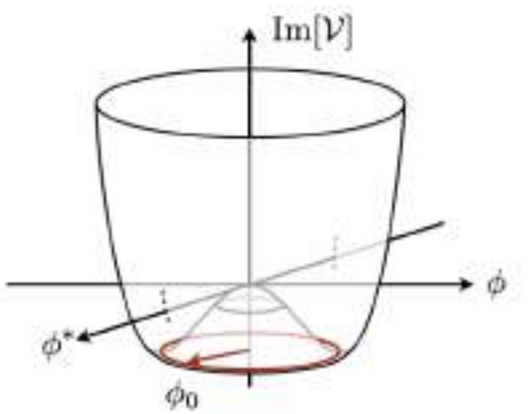
$$\underbrace{\kappa \int_{\mathbf{x}} [\hat\phi^2_{\mathbf{x}}\,\rho\,\hat\phi^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi^{\dagger\,2}_{\mathbf{x}}\hat\phi^2_{\mathbf{x}},\rho\}]}_{\text{two particle loss}}$$

- basic physics: mean field theory
  - study evolution of $\phi(\mathbf{x},t) \equiv \langle\hat\phi(\mathbf{x})\rangle(t) = \mathrm{tr}[\hat\phi(\mathbf{x})\hat\rho(t)]$ $\quad \hat\rho = \prod_{\mathbf{x}} \hat\rho(\mathbf{x}), \quad \underbrace{\hat\rho(\mathbf{x}) = |\phi(\mathbf{x})\rangle\langle\phi(\mathbf{x})|}_{\text{coherent state}}$
  - homogenous limit $\phi(\mathbf{x},t) = \phi(t)$

$$i\partial_t \phi(t) = [-\mu - i(\gamma_l - \gamma_p) + (\lambda - i\kappa)|\phi(t)|^2]\phi(t)$$

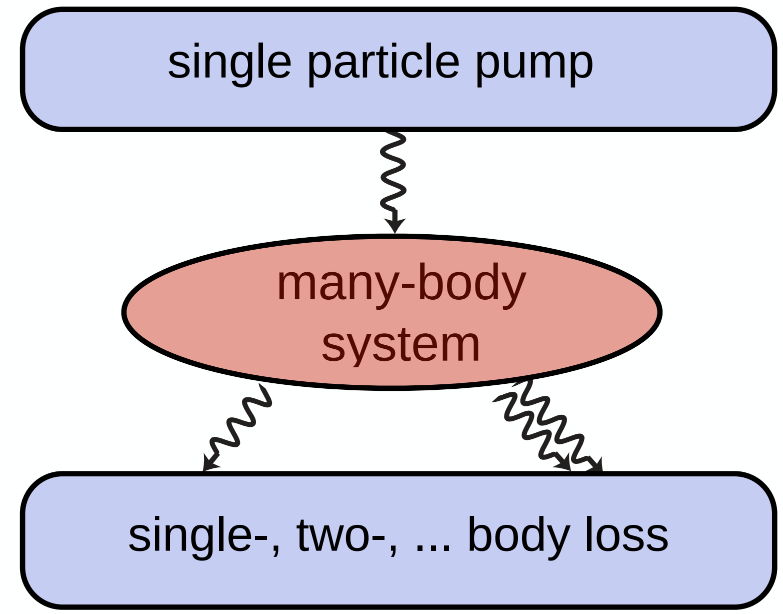
➡ overdamped motion in potential landscape

➡ condensation / spontaneous U(1) symmetry breaking for $\gamma_l - \gamma_p < 0$



A workhorse model: Lindbladian formulation

- generic microscopic many-body model: ‘Lindblad ϕ^4 theory’

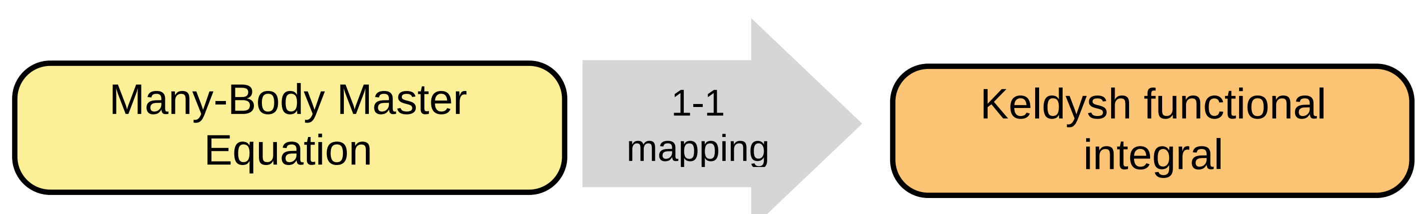


$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu\right) \hat{\phi}_{\mathbf{x}} + \frac{\lambda}{2} (\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}})^2$$

$$\mathcal{D}[\rho] = \underbrace{\gamma_p \int_{\mathbf{x}} [\hat{\phi}^\dagger_{\mathbf{x}} \, \rho \, \hat{\phi}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}}, \rho\}]}_{\text{single particle pump}} + \underbrace{\gamma_l \int_{\mathbf{x}} [\hat{\phi}_{\mathbf{x}} \, \rho \, \hat{\phi}^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}}, \rho\}]}_{\text{single particle loss}} + \underbrace{\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}} \, \rho \, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^2_{\mathbf{x}}, \rho\}]}_{\text{two particle loss}}$$

- plan:
  - translate to Lindblad-Keldysh functional integral



  - how does this model relate e.g. to exciton-polariton systems? (semiclassical limit)
  - ‘what is non-equilibrium about it’?
  - how to extract the phase structure?

Leonid W. Keldysh

Keldysh functional integral for stationary states of driven open quantum systems

- Construction from quantum master equation
- Semiclassical limit
- “What is non-equilibrium about it?”

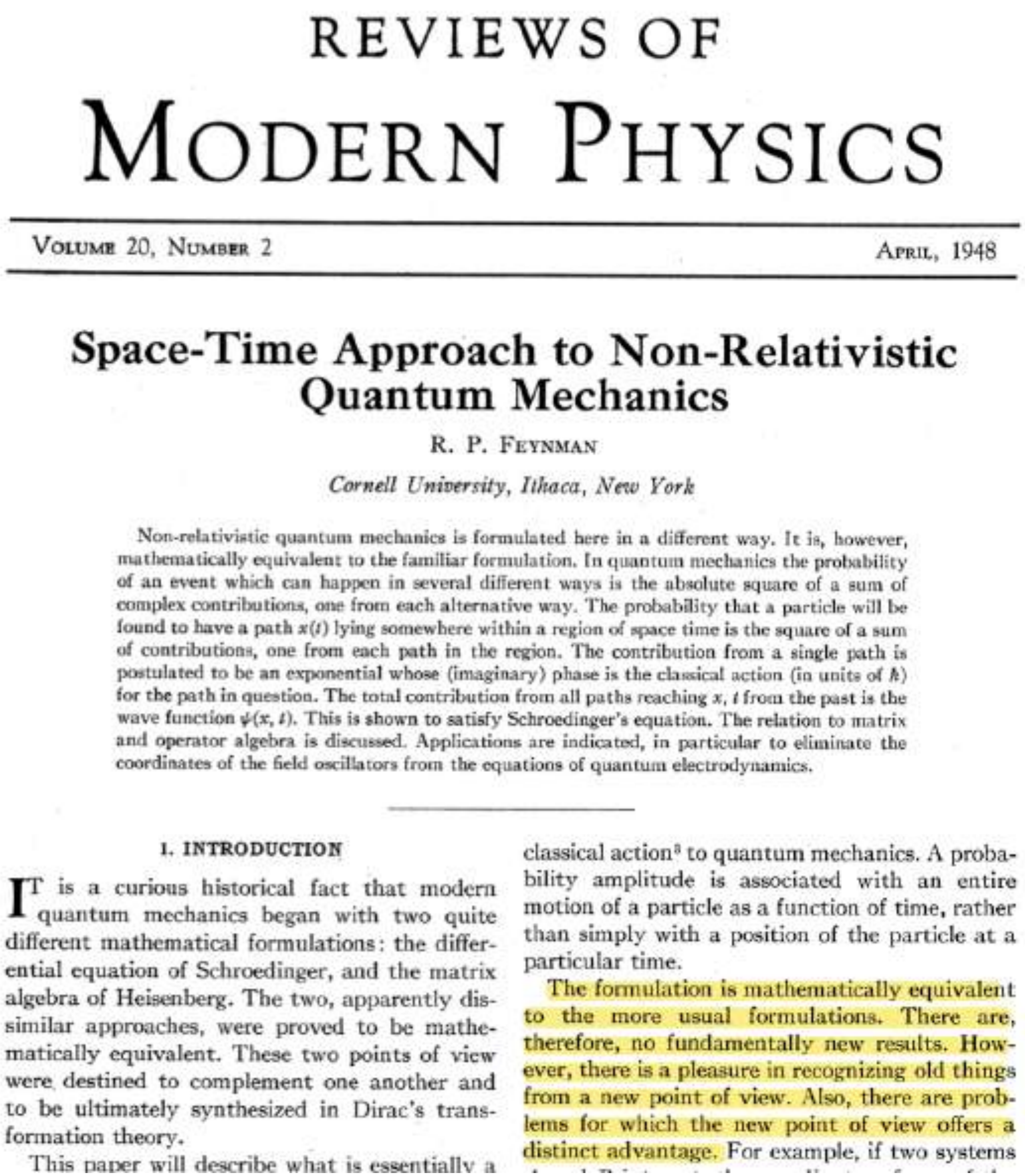
Many-Body Master Equation → 1-1 mapping → Keldysh functional integral

$$\partial_t \hat\rho = -i(\hat H - \sum_i \gamma_i \hat L_i^\dagger \hat L_i)\hat\rho + \text{h.c.} + 2\sum_i \gamma_i \hat L_i \hat\rho \hat L_i^\dagger$$

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

Keldysh functional integrals: Why?

- Feynman's formulation of quantum mechanics



REVIEWS OF

MODERN PHYSICS

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

It is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems

- Useful language for systems with many degrees of freedom

- general: powerful techniques
  - diagrammatic perturbation theory;
  - collective variables;
  - renormalization group

- non-equilibrium Keldysh
  - closer to the real-time formulations of quantum mechanics
  - gives unified view on and principles (e.g. symmetries) for equilibrium and non-equilibrium systems
  - indispensable for many systems:
    - disorder
    - dissipation

    infinite harmonic baths!
  - open the powerful toolbox of quantum field theory for many-body non-equilibrium situations

Keldysh functional integral

more details: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016)

- The basic idea in three steps:

$$\hbar = 1$$

$$U(t,t_0) = e^{-iH(t-t_0)}$$

1. Schrödinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

2. Heisenberg-von Neumann equation: evolving a state (density) matrix

$$\partial_t\rho(t) = -i[H,\rho(t)] \quad \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

- identical for pure (separable) states $\rho = |\psi\rangle\langle\psi|$

3. The same is true for the Lindblad equation:

$$\partial_t\rho = -i[H,\rho] + \kappa\sum_i L_i\rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i,\rho\} \equiv \mathcal{L}[\rho]$$

$$\Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

linear superoperator (acts from both sides on density matrix)

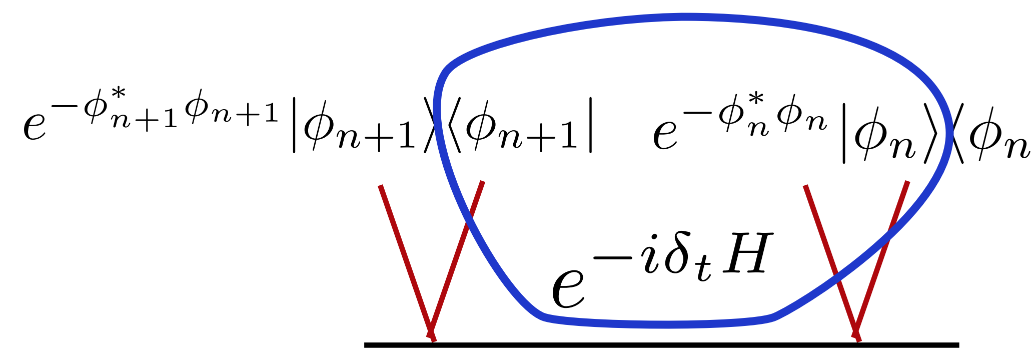
Keldysh functional integral (bosons)

fermions: see appendix!

1. Functional integral idea:

➡ “Trotterization” of time interval and insertion of coherent states: $e^{iH(t-t_0)} = \lim_{N\to\infty} (1+i\delta_t H)^N$

- one time step



$$e^{-\phi^*_n\phi_n}\langle\phi_{n+1}|e^{-i\delta_t H[a^\dagger,a]}|\phi_n\rangle$$

$$\approx e^{-\phi^*_n\phi_n}\langle\phi_{n+1}|1-i\delta_t H[a^\dagger,a]|\phi_n\rangle$$

H normally ordered

$$= e^{-\phi^*_n\phi_n}e^{+\phi^*_{n+1}\phi_n}(1-i\delta_t H[\phi^*_{n+1},\phi_n])$$

$$\approx e^{i\delta_t[-i\frac{(\phi^*_{n+1}-\phi^*_n)}{\delta_t}\phi_n - H[\phi^*_{n+1},\phi_n]]}$$

continuum limit

$$dt \qquad -i\partial_t\phi^*(t)\cdot\phi(t) \qquad H[\phi^*(t),\phi(t)]$$

coherent states (bosons):

$$a|\phi\rangle = \phi|\phi\rangle$$

$$\langle\phi'|\phi\rangle = e^{\phi'^*\phi}$$

$$\mathbf{1} = \int\frac{d\phi^* d\phi}{\pi}e^{-\phi^*\phi}|\phi\rangle\langle\phi|$$

Keldysh functional integral

1. Functional integral idea:

➡ "Trotterization" of time interval and insertion of coherent states: $e^{iH(t-t_0)} = \lim_{N\to\infty}(1+i\delta_t H)^N$

- many time steps

$$\underbrace{\int \prod_t \frac{d\phi^*(t)d\phi(t)}{\pi}}_{=: \int \mathcal{D}(\phi^*,\phi)} e^{i\int_{t_0}^{t_f} dt[-i\partial_t\phi^*(t)\cdot\phi(t)-H[\phi^*(t),\phi(t)]]}$$

functional integral measure

- Discussion

➡ operator H -> complex, time dependent functional H

➡ time evolution from overlap of neighbouring states

➡ no reference to single particle or many-body Hamiltonian, lattice or continuum!

➡ single set of degrees of freedom for vector evolution

Keldysh functional integral

2. Schrödinger vs. Heisenberg-von Neumann

$$U(t,t_0) = e^{-iH(t-t_0)}$$

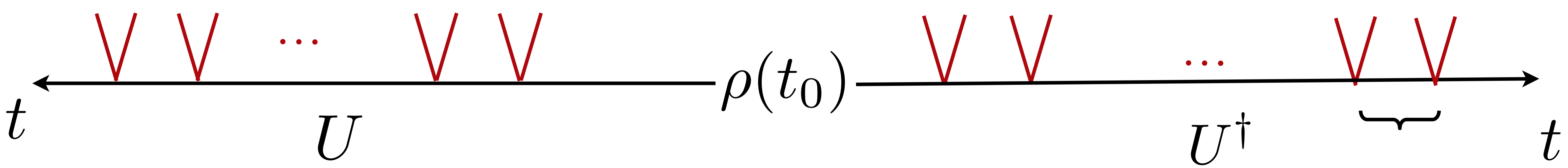
- Schrödinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

- Heisenberg-von Neumann equation: evolving a state (density) matrix

$$\partial_t\rho(t) = -i[H,\rho(t)] \quad \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

- Second case: “Trotterization” on both sides:



$$e^{iH(t-t_0)} = \lim_{N\to\infty}(1+i\delta_t H)^N \qquad \delta_t = \frac{t-t_0}{N}$$

➡ two sets of degrees of freedom for matrix evolution

Keldysh functional integral

3. Schrödinger vs. Lindblad

$$U(t,t_0) = e^{-iH(t-t_0)}$$

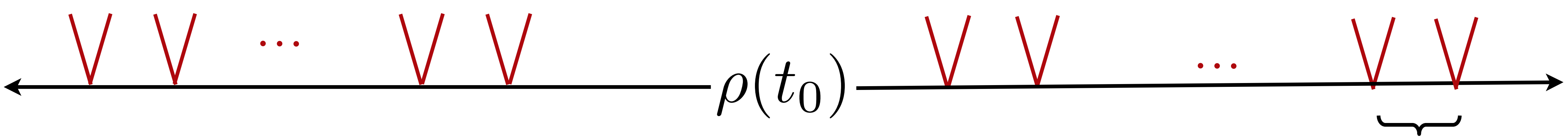
- Schrödinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

- Lindblad equation: evolving a state (density) matrix

$$\partial_t\rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho] \qquad \Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

- Identical program for Liouville generator of dynamics (left and right action on density matrix)



$$\rho(t) = e^{(t-t_0)\mathcal{L}}\,\rho_0 = \lim_{N\to\infty}\left(1+\delta_t\mathcal{L}\right)^N \rho_0 \qquad \delta_t = \frac{t-t_0}{N}$$

➡ two sets of degrees of freedom for matrix evolution

Keldysh functional integral

3. Schrödinger vs. Lindblad

$$U(t,t_0) = e^{-iH(t-t_0)}$$

- Schrödinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

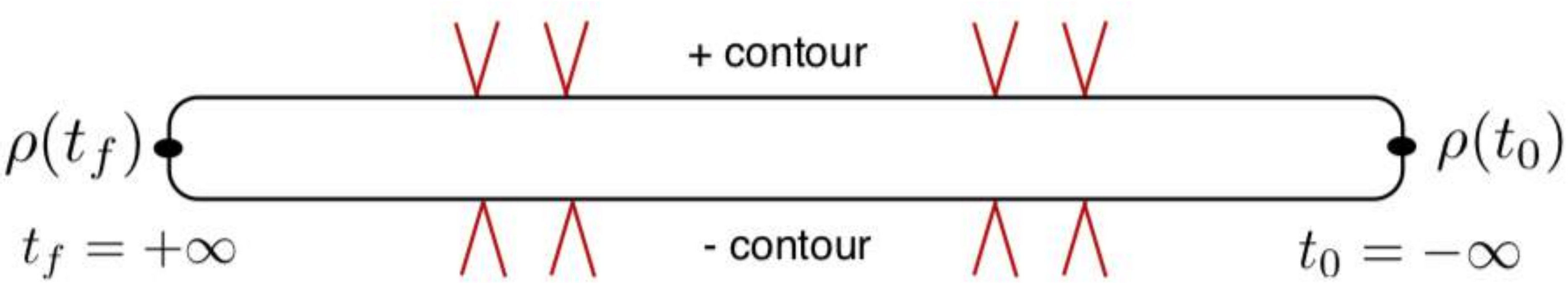
- Lindblad equation: evolving a state (density) matrix

$$\partial_t\rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho] \qquad \Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

- final step: Keldysh “partition function”

$$Z = \mathrm{tr}\rho(t) = \mathrm{tr}\rho(t_0) = 1$$

$t_0 \to -\infty, t_f \to +\infty$



information on all stages;
stationarity reached
(boundary conditions
irrelevant)

Keldysh functional integral: Final result

- Lindblad equation: $$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho]$$

$$= -i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2} \rho L_i^\dagger L_i)$$

- equivalent Keldysh functional integral:

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) e^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

$$\Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi_\pm^* \end{pmatrix}$$

$$S_M[\Phi_+,\Phi_-] = \int dt (\phi_+^* i\partial_t \phi_+ - \phi_-^* i\partial_t \phi_- - i\mathcal{L}[\Phi_+,\Phi_-])$$

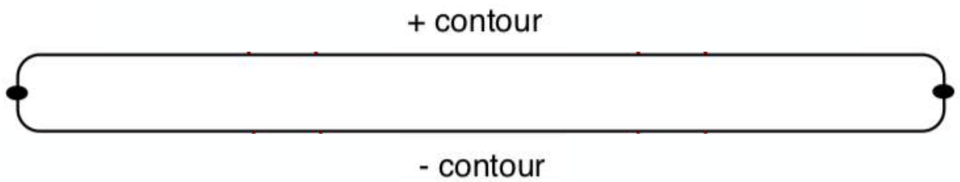
$$\boxed{\mathcal{L}[\Phi_+,\Phi_-] = -i\,(H_+ - H_-) - \kappa \sum_i \left(L_{i,+} L_{i,-}^\dagger - \tfrac{1}{2} L_{i,+}^\dagger L_{i,+} - \tfrac{1}{2} L_{i,-}^\dagger L_{i,-}\right)}$$

$$H_\pm = H(\Phi_\pm) \text{ etc.}$$

➡ recognize Lindblad structure

➡ simple translation table (for contour normal ordered Lindbladian)

- operator right of density matrix -> - contour
- operator left of density matrix -> + contour



➡ caveat: contour diagonal Lindblad terms need temporal regularisation to track operator ordering

Keldysh functional integral: structural properties

- Keldysh functional integral:

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) e^{i(S_M[\Phi_+,\Phi_-]}$$

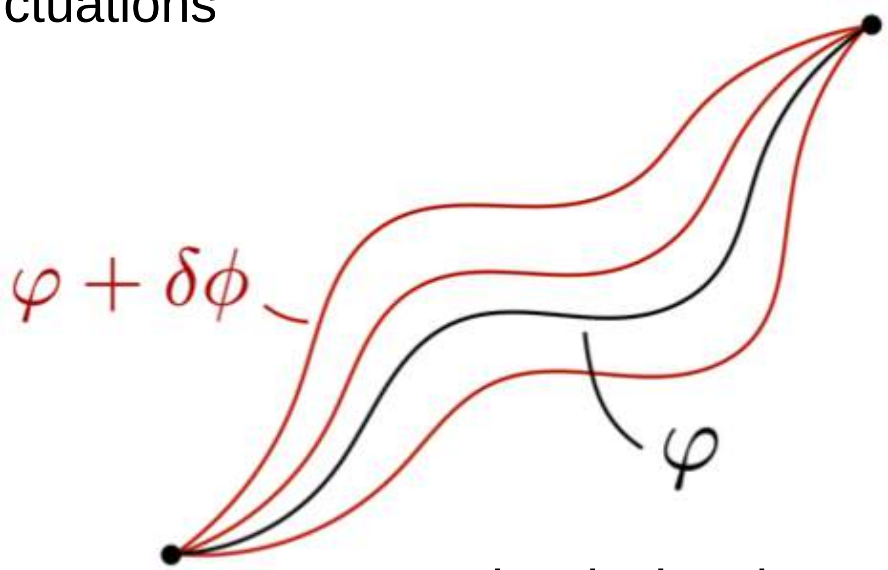
$$\Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi^*_\pm \end{pmatrix}$$

$$S_M[\Phi_+, \Phi_-] = \int dt(\phi^*_+ i\partial_t \phi_+ - \phi^*_- i\partial_t \phi_- - i\mathcal{L}[\Phi_+, \Phi_-])$$

$$\boxed{\mathcal{L}[\Phi_+, \Phi_-] = -i\,(H_+ - H_-) - \kappa \sum_i \left(L_{i,+} L^\dagger_{i,-} - \tfrac{1}{2} L^\dagger_{i,+} L_{i,+} - \tfrac{1}{2} L^\dagger_{i,-} L_{i,-} \right)}$$

$H_\pm = H(\Phi_\pm)$ etc.

- focus on three key aspects, reflecting different levels of inclusion of fluctuations
    - probability conservation (zero order)
    - deterministic limit (first order)
    - fluctuations (quantum and class. statistical) fully included



- next lecture, compromise: semiclassical limit, classical statistical fluctuations included

classical path (minimizes S)

Probability conservation / "causality"

- trace / probability conservation:
  - Lindblad: $$\partial_t \mathrm{tr}\rho = \mathrm{tr}\Big(-i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2}\rho L_i^\dagger L_i)\Big) = 0$$
    cyclicity
  - Keldysh: $$Z = \mathrm{tr}\rho(t) = 1 \qquad Z = \int \mathcal{D}(\Phi_+, \Phi_-) e^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

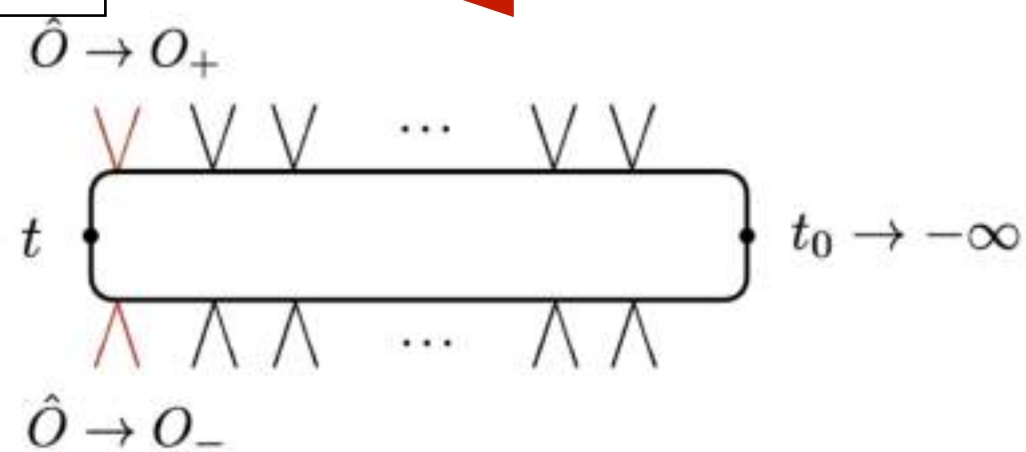
- will argue: reflected on the action as

$$\Phi_+ = \Phi_- \Rightarrow \boxed{S_M[\Phi_+, \Phi_-] = 0}$$

  - redundancy

$$\langle \hat{O} \rangle(t) = \mathrm{tr}[\hat{O}\hat{\rho}(t)] = \mathrm{tr}[\hat{\rho}(t)\hat{O}]$$

$$\langle \hat{O} \rangle(t) = \langle O_+(t) \rangle = \langle O_-(t) \rangle$$



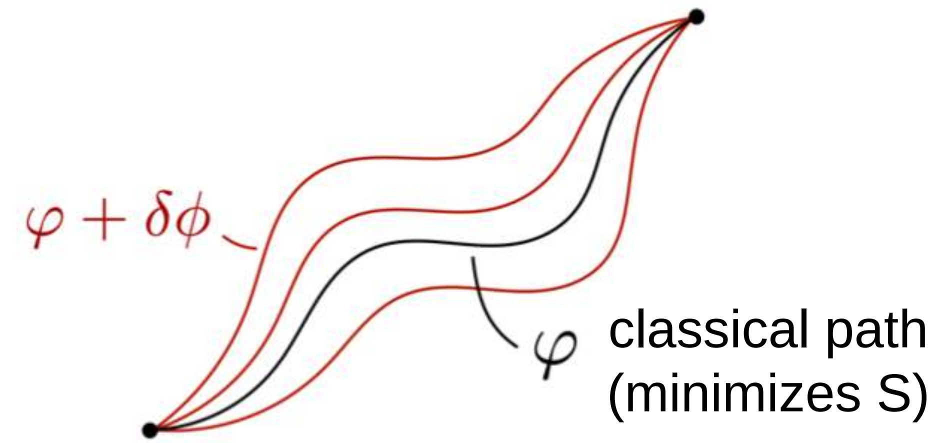
$$\Longrightarrow \; 0 = \partial_t Z = i\langle s(\Phi_+(t), \Phi_-(t))\rangle = i\langle s(\Phi_+(t), \Phi_+(t))\rangle \quad \forall t \qquad S = \int_{t_0}^{t} dt'\, s(\Phi_+(t'), \Phi_-(t'))$$

- mnemonic: taking trace = ignoring contour order
- motivates Keldysh rotation

Probability conservation and Keldysh rotation

- trace / probability conservation:
  - Lindblad: $$\partial_t \mathrm{tr}\rho = \mathrm{tr}\big(-i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2}\rho L_i^\dagger L_i)\big) = 0$$ cyclicity
  - Keldysh: $$Z = \mathrm{tr}\rho(t) = 1 \qquad Z = \int \mathcal{D}(\Phi_+, \Phi_-) e^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$
- make probability conservation more handy:
  - starting point: contour basis: $S[\Phi_+, \Phi_-]$ with $S[\Phi_+, \Phi_- = \Phi_+] = 0$
  - Keldysh rotation $$\begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_+ + \phi_- \\ \phi_+ - \phi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$$ center-of-mass / relative
  - action in Keldysh/RAK basis $S[\Phi_c, \Phi_q]$
  - probability conservation $$\boxed{S[\Phi_c, \Phi_q = 0] = 0}$$
- interpretation of the fields: use $\langle \hat{\phi} \rangle = \langle \phi_+ \rangle = \langle \phi_- \rangle$
  - "classical" field can acquire expectation value (<—> condensation, spontaneous symmetry breaking) $$\langle \hat{\phi} \rangle = \langle \phi_c \rangle / \sqrt{2}, \quad \langle \phi_q \rangle = 0$$
  - "quantum" field cannot

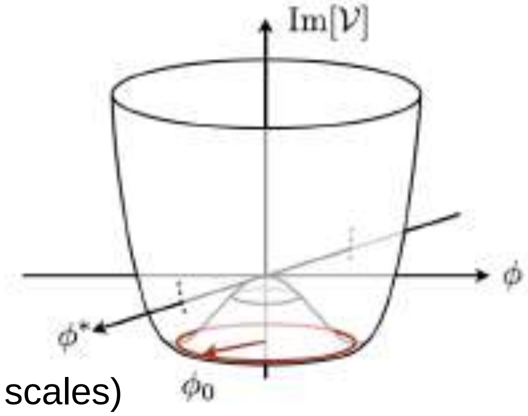
Deterministic limit



- Keldysh functional integral

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$

- probability conservation: zero order in the quantum field $S[\Phi_c, \Phi_q = 0] = 0$



- first order in quantum field: ordering principle due to condensation
  - classical / occupation field: macroscopic occupation $N \to \infty$ (more precisely: only q=0 mode scales)

$$\phi_c(q) \sim N^{1/2} \implies \phi_c(x) \sim \frac{N^{1/2}}{V^{1/2}} \sim N^0 \qquad \phi_c(x) = V^{-1/2} \sum_q e^{-iqx} \phi_c(q)$$

  - quantum field:

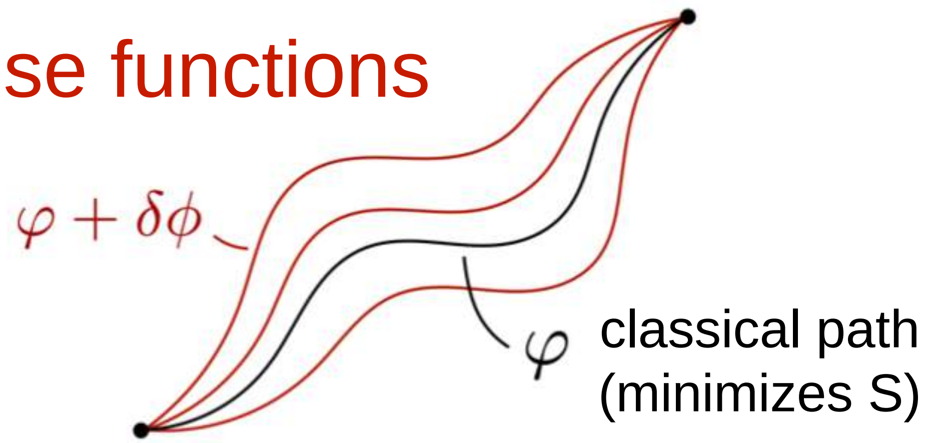
$$\phi_q(q) \sim N^0 \implies \phi_q(x) \sim \frac{1}{V^{1/2}} \sim N^{-1/2}$$

- expand the action to leading order $S[\Phi_+ = (\Phi_c + \Phi_q)/\sqrt{2}, \Phi_- = (\Phi_c - \Phi_q)/\sqrt{2}] \approx \int_{\mathbf{x},t} [\phi_q \frac{\delta S}{\delta \phi_c} + \phi_q^* \frac{\delta S}{\delta \phi_c^*}]$
- can do integral over quantum field $Z = \int D[\phi_c, \phi_c] \delta \left[\frac{\delta S}{\delta \phi_c} \right] \delta \left[\frac{\delta S}{\delta \phi_c^*} \right]$
- only deterministic configuration contributes with $\frac{\delta S}{\delta \phi_c} = 0 = \frac{\delta S}{\delta \phi_c^*}$

➡ deterministic limit Lindblad phi^4: dissipative Gross-Pitaevskii mean field theory

exercise: make connection to operator mean field theory

Including fluctuations: Correlation and response functions



- Keldysh functional integral

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$

- probability conservation: zero order in the quantum field $S[\Phi_c, \Phi_q = 0] = 0$

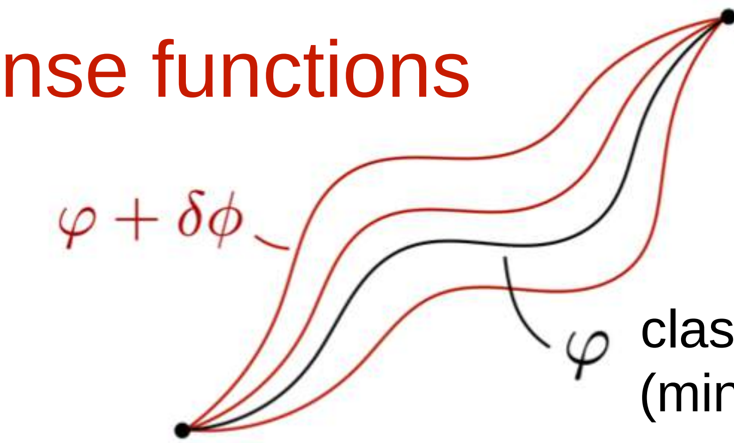
- deterministic limit: first order in quantum field (good if there is condensation = dominant field configuration)

- how to quantify deviations from the deterministic limit?

➡ correlation functions: fluctuations around deterministic configuration $\sim \langle \phi_c^*(\vec{x}, t) \phi_c(\vec{x}', t') \rangle$

➡ response functions: impact of an external perturbation $\sim \langle \phi_q^*(\vec{x}, t) \phi_c(\vec{x}', t') \rangle$

Including fluctuations: Correlation and response functions



classical path (minimizes S)

- Keldysh functional integral

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$

- probability conservation: zero order in the quantum field $S[\Phi_c, \Phi_q = 0] = 0$

- correlation functions
  - origin: - quantum fluctuation (non-commuting operators)
    - statistical fluctuation (many quantum states contribute due to coupling to environment)
  - definition:
    - field fluctuation $\delta\phi_c(\vec{x}, t) \equiv \phi_c(\vec{x}, t) - \langle \phi_c(\vec{x}, t) \rangle$

$$\Longrightarrow \langle \delta\phi_c(\vec{x}, t) \rangle = 0 \qquad \text{(always)},$$

$$\langle \delta\phi_c^*(\vec{x}, t) \delta\phi_c(\vec{x}', t') \rangle = \underbrace{\langle \phi_c^*(\vec{x}, t)\phi_c(\vec{x}', t') \rangle - \langle \phi_c^*(\vec{x}, t) \rangle \langle \phi_c(\vec{x}', t') \rangle}_{\text{covariance of the field}} \neq 0$$

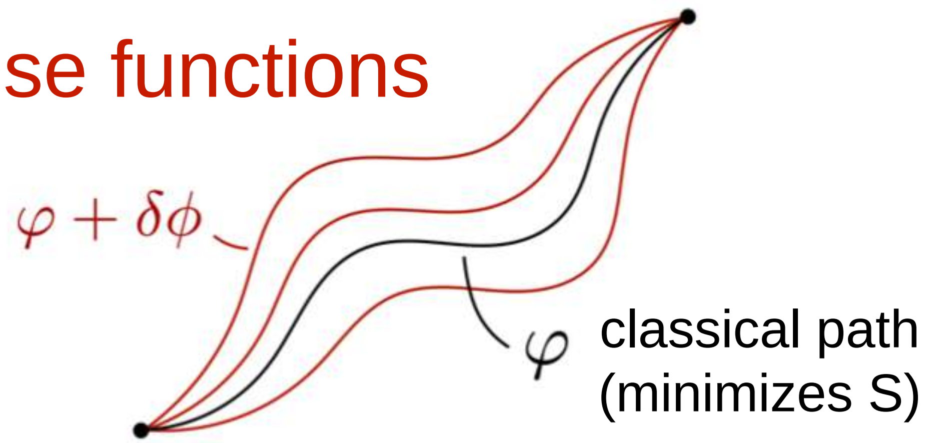
= 0 in det. limit

- Keldysh Green function (correlation function)

$$\boxed{G^K(x, t, x', t') \equiv -i \langle \phi_c(x, t) \phi_c^*(x', t') \rangle}$$

- remark: higher order correlators don't generally factorize, coupled hierarchy (Dyson-Schwinger equations)

Including fluctuations: Correlation and response functions



- Keldysh functional integral

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

- probability conservation: zero order in the quantum field $\quad S[\Phi_c, \Phi_q = 0] = 0$

- response functions
  - perturb the system: $\quad \hat{H} \to \hat{H} + \int_{\vec{x}} h(\vec{x},t)[\hat{\phi}^\dagger(\vec{x}) + \hat{\phi}(\vec{x})]$

    weak real valued external field

  - modification of Keldysh action

$$S \to S + \int_{\vec{x},t} h(\vec{x},t)[\phi_+^*(\vec{x},t) - \phi_-^*(\vec{x},t) + \text{c.c.}] = S + \int_{\vec{x},t} \sqrt{2} h(\vec{x},t)[\phi_q^*(\vec{x},t) + \text{c.c.}]$$

external field couples to quantum field

  - e.g. linear response of the field expectation

$$\langle \phi_c(\vec{x},t)\rangle_\Delta \equiv \int \mathcal{D}[\Phi,\Phi^*]\phi_c(\vec{x},t) e^{-i(S+\Delta S)} = \langle \phi_c(x,t) e^{i\int h(\phi_q^* + \phi_q)}\rangle$$

perturbed

$$\approx \langle \phi_c(x,t)\rangle + i \int_{\vec{x},t} h(\vec{x},t)\langle \phi_c(\vec{x},t)\phi_q^*(\vec{x},t)\rangle$$

  - retarded Green function (response function)

$$\boxed{G^R(\vec{x},t,\vec{x}',t') = -i\langle \phi_c(\vec{x},t)\phi_q^*(\vec{x}',t')\rangle}$$

exercise: verify relation to operator formalism!

Correlation vs. response functions: Calculation

more details: L. Sieberer, M. Buchhold, SD, Reports on Progress in Physics (2016)

- introduce complex contour dependent sources (cf. StatMech)

$$Z[j] = \langle e^{\mathrm{i}\int (j_+\phi_+^* - j_-\phi_-^* + c.c.)}\rangle = \langle e^{\mathrm{i}\int (j_c\phi_q^* + j_q\phi_c^* + c.c.)}\rangle \qquad Z[j=0] = Z = 1$$

normalization

- order parameter / occupation field:

$$\langle \phi_c(t,\mathbf{x})\rangle = -i\frac{\delta Z[j]}{\delta j_q^*(t,\mathbf{x})}\Big|_{j=0}$$

- single particle response: how does the field react to external perturbations?

relation to operator formalism (once and for all)

response to coherent field

t = t', x = x'

$$G^R(t-t',\mathbf{x}-\mathbf{x}') = i\frac{\delta^2 Z}{\delta j_q^*(t,\mathbf{x})\delta j_c(t',\mathbf{x}')}\Big|_{j=0} = -i\langle \phi_c(t,\mathbf{x})\phi_q^*(t',\mathbf{x}')\rangle = -i\theta(t-t')\langle[\hat\phi(t,\mathbf{x}),\hat\phi^\dagger(t',\mathbf{x}')]\rangle = 1$$

- single particle correlations: how are states occupied?

t = t', x=x'

$$G^K(t-t',\mathbf{x}-\mathbf{x}') = i\frac{\delta^2 Z}{\delta j_q^*(t,\mathbf{x})\delta j_q(t',\mathbf{x}')}\Big|_{j=0} = -i\langle \phi_c(t,\mathbf{x})\phi_c^*(t',\mathbf{x}')\rangle = -i\langle\{\hat\phi(t,\mathbf{x}),\hat\phi^\dagger(t',\mathbf{x}')\}\rangle = 2\langle \hat n(\mathbf{x})\rangle + 1$$

time and space translation invariance assumed

- total Green's function

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} \qquad G^A = (G^R)^\dagger, \qquad (G^K)^\dagger = -G^K$$

prob. conservation

Hermitian conjugates

anti-Hermitian

Correlation vs. response: relation to Keldysh action

exercise: prove formulas on this page!

more details: L. Sieberer, M. Buchhold, SD, Reports on Progress in Physics (2016)

- by example: master equation for decaying cavity

$$\partial_t \rho = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger \hat{a}, \rho\})$$

- action:

time domain $a_\nu(t)$

$$S = \int dt (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & \overbrace{i\partial_t - \omega_0 - i\kappa}^{} \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

$$\equiv P^A(\omega)$$

frequency domain $a_\nu(\omega)$

$$= \int \frac{d\omega}{2\pi} (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \omega - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

$$\underbrace{\equiv P^R(\omega) \qquad \equiv P^K}_{\text{claim: } G^{-1}(\omega)}$$

Correlation vs. response: relation to Keldysh action

exercise: prove formulas on this page!

more details: L. Sieberer, M. Buchhold, SD, Reports on Progress in Physics (2016)

- by example: master equation for decaying cavity

$$\partial_t \rho = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger \hat{a}, \rho\})$$

- action:

time domain $a_\nu(t)$

$$S = \int dt (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & \underset{\equiv P^A(\omega)}{i\partial_t - \omega_0 - i\kappa} \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

frequency domain $a_\nu(\omega)$

$$= \int \frac{d\omega}{2\pi} (a_{cl}^*, a_q^*) \underbrace{\begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \underset{\equiv P^R(\omega)}{\omega - \omega_0 + i\kappa} & \underset{\equiv P^K}{2i\kappa} \end{pmatrix}}_{G^{-1}(\omega)} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

- partition function: completion of the square

$$Z[j_c, j_q] = \langle e^{i\int \frac{d\omega}{2\pi}(j_c^* a_q + j_q^* a_c + \text{h.c.})} \rangle = e^{i\int \frac{d\omega}{2\pi}(j_q^*, j_c^*) G(\omega) \begin{pmatrix} j_q \\ j_c \end{pmatrix}}$$

- single particle Green's function

$$\begin{pmatrix} \langle \phi_c(\omega)\phi_c^*(\omega')\rangle & \langle \phi_c(\omega)\phi_q^*(\omega')\rangle \\ \langle \phi_q(\omega)\phi_c^*(\omega')\rangle & \langle \phi_q(\omega)\phi_q^*(\omega')\rangle \end{pmatrix} = -\begin{pmatrix} \frac{\delta^2 Z}{\delta j_q^*(\omega)\delta j_q(\omega')} & \frac{\delta^2 Z}{\delta j_q^*(\omega)\delta j_c(\omega')} \\ \frac{\delta^2 Z}{\delta j_c^*(\omega)\delta j_q(\omega')} & \frac{\delta^2 Z}{\delta j_c^*(\omega)\delta j_c(\omega')} \end{pmatrix}\Bigg|_{j=0} = iG(\omega)\delta(\omega - \omega')$$

- summary in matrix components (valid beyond example):

$$\boxed{G^{-1} = \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \implies G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} \qquad G^{R/A} = [P^{R/A}]^{-1} \qquad G^K = -G^R P^K G^A}$$

action matrix kernel

single particle Green's function

exercise: verify calculations on this page!

Correlation vs. response: relation to Keldysh action

more details: L. Sieberer, M. Buchhold, SD, Reports on Progress in Physics (2016)

- by example: master equation for decaying cavity

$$\partial_t \rho = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a}, \rho\})$$

- action:

$$S = \int dt (a^*_{cl}, a^*_q) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i\kappa \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

time domain $a_\nu(t)$

$$= \int \tfrac{d\omega}{2\pi} (a^*_{cl}, a^*_q) \begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \omega - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

frequency domain $a_\nu(\omega)$

- observables from the Green's functions:

response

- decay of single-particle response: $G^R(t-t') = \int_\omega e^{i\omega(t-t')} G^R(\omega) = \theta(t-t') e^{i\omega_0(t-t')} e^{-\kappa(t-t')}$
- Lorentzian spectral density: $A(\omega) = \mathrm{Im} G^R(\omega) = \dfrac{2\kappa}{(\omega-\omega_0)^2 + \kappa^2}$

correlation

- cavity mode occupation in stationary state $2\langle \hat{n}(t)\rangle + 1 = \langle \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}(t)\hat{a}^\dagger(t)\rangle = iG^K(t-t) = i\int_\omega e^{i\omega(t-t)} G^K(\omega) = 1$

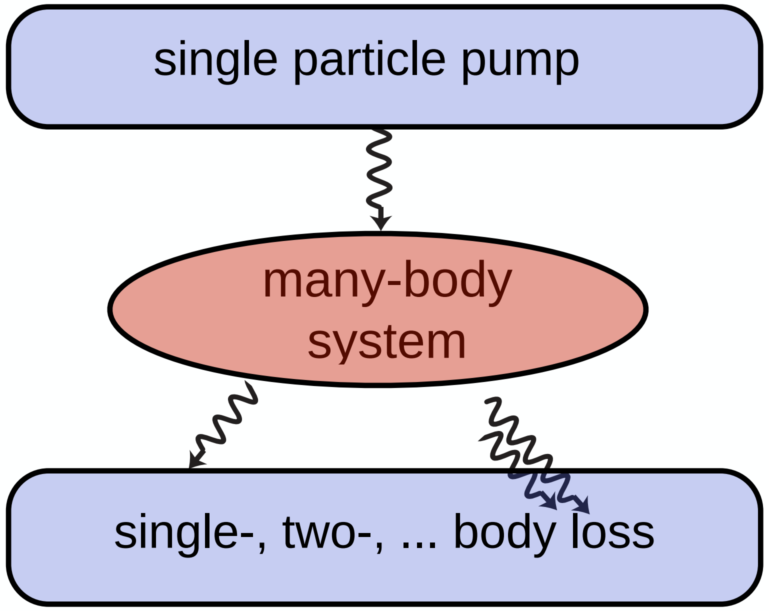
$\langle \hat{n}(t\to\infty)\rangle = 0$ $\qquad (t \to \infty)$

➡ correlation / statistical properties: G^K

➡ response / spectral properties: G^R

Back to many-body model: Workhorse Lindbladian

- generic microscopic many-body model:

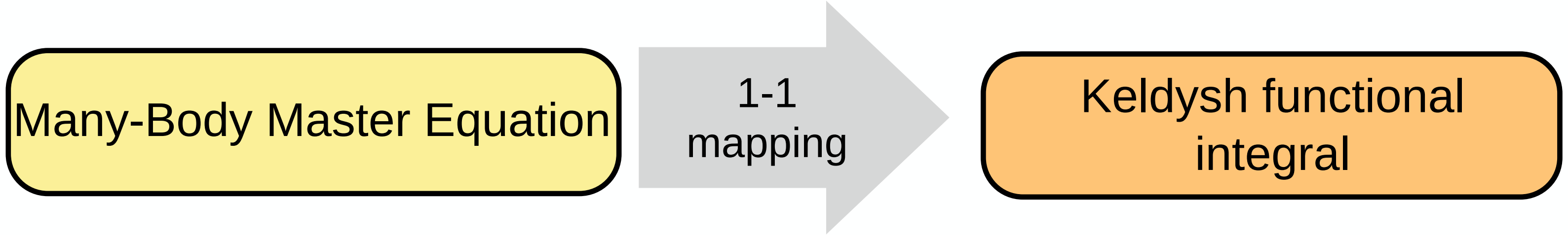


$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \underbrace{\hat\phi^\dagger_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu\right) \hat\phi_{\mathbf{x}}}_{\text{kinetic energy}} + \underbrace{\frac{\lambda}{2}(\hat\phi^\dagger_{\mathbf{x}}\hat\phi_{\mathbf{x}})^2}_{\text{two-particle interaction}}$$

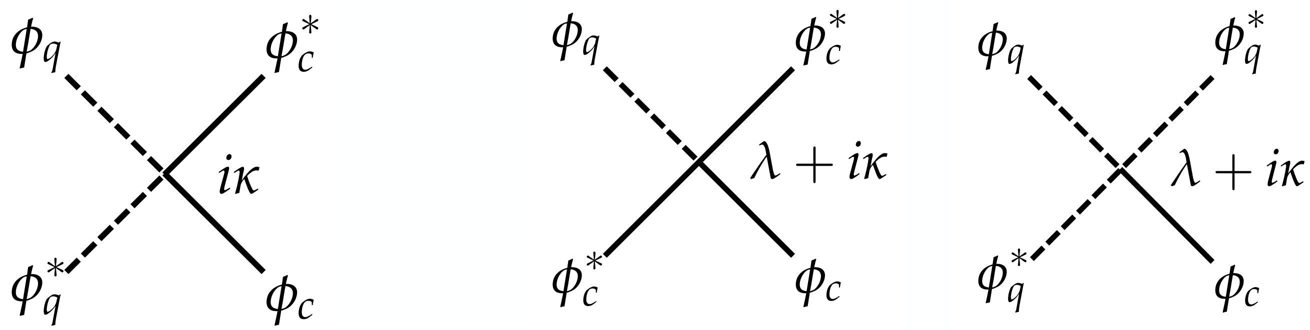
$$\mathcal{D}[\rho] = \underbrace{\gamma_p \int_{\mathbf{x}} [\hat\phi^\dagger_{\mathbf{x}}\, \rho\, \hat\phi_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi_{\mathbf{x}}\hat\phi^\dagger_{\mathbf{x}}, \rho\}]}_{\text{single particle pump}} + \underbrace{\gamma_l \int_{\mathbf{x}} [\hat\phi_{\mathbf{x}}\, \rho\, \hat\phi^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi^\dagger_{\mathbf{x}}\hat\phi_{\mathbf{x}}, \rho\}]}_{\text{single particle loss}} +$$

$$\underbrace{\kappa \int_{\mathbf{x}} [\hat\phi^2_{\mathbf{x}}\, \rho\, \hat\phi^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi^{\dagger\,2}_{\mathbf{x}}\hat\phi^2_{\mathbf{x}}, \rho\}]}_{\text{two-particle loss}}$$



Many-body model: Workhorse Lindblad-Keldysh action

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa\phi_c^*\phi_c\phi_q^*\phi_q - \tfrac{1}{2}\left[(\lambda + i\kappa)\left(\phi_c^{*2}\phi_c\phi_q + \phi_q^{*2}\phi_c\phi_q\right) + c.c. \right] \right\}$$



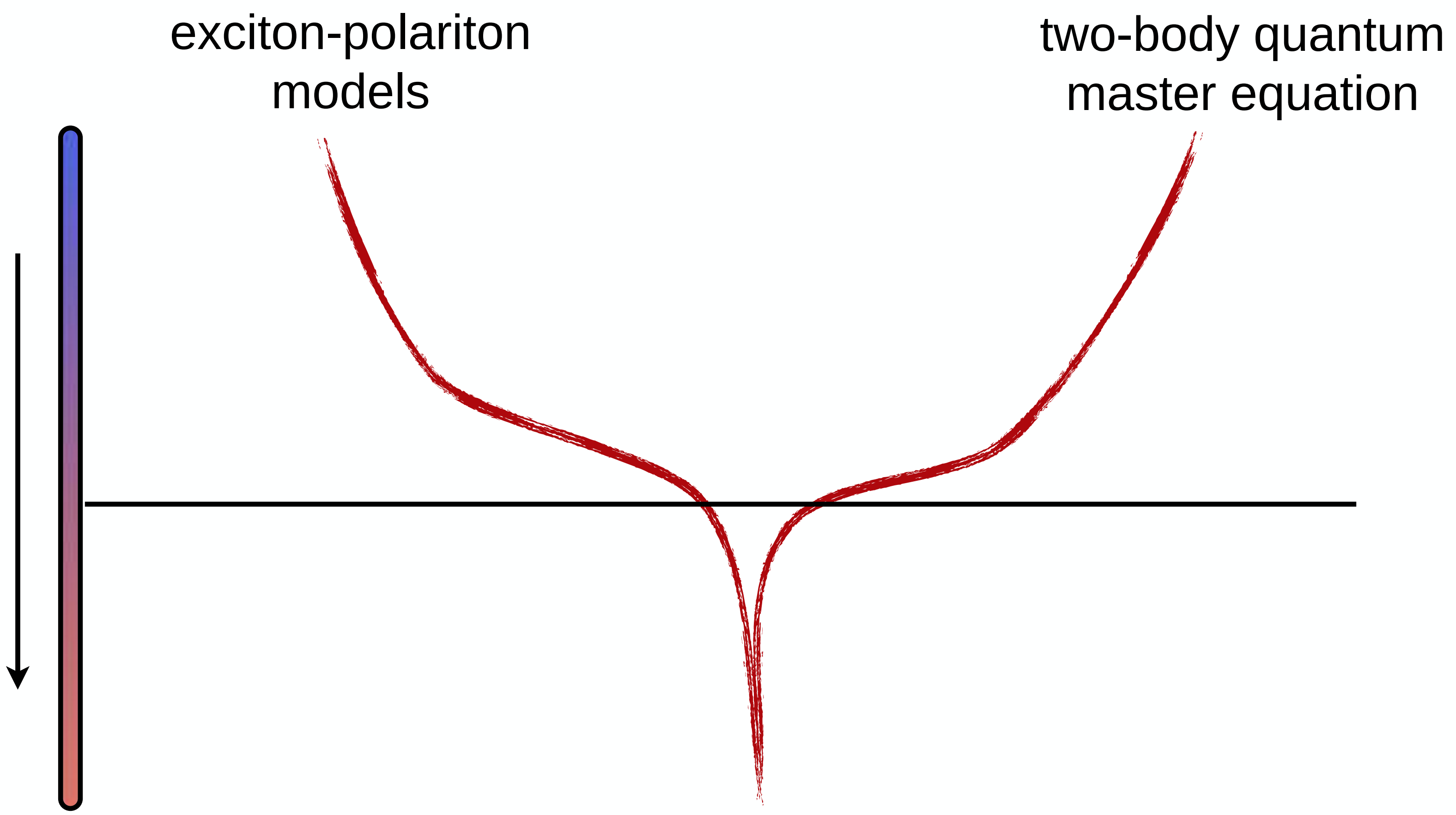
- Gaussian sector: inverse Green's function
  - retarded/advanced $$P^R(\omega, \mathbf{q}) = \omega - (\mathbf{q}^2 - \mu) + i\,(\gamma_l - \gamma_p)\,/2$$
  - Keldysh component $$P^K = i\,(\gamma_l + \gamma_p)$$

difference: distance from a phase transition

sum: noise of loss and pumping add up

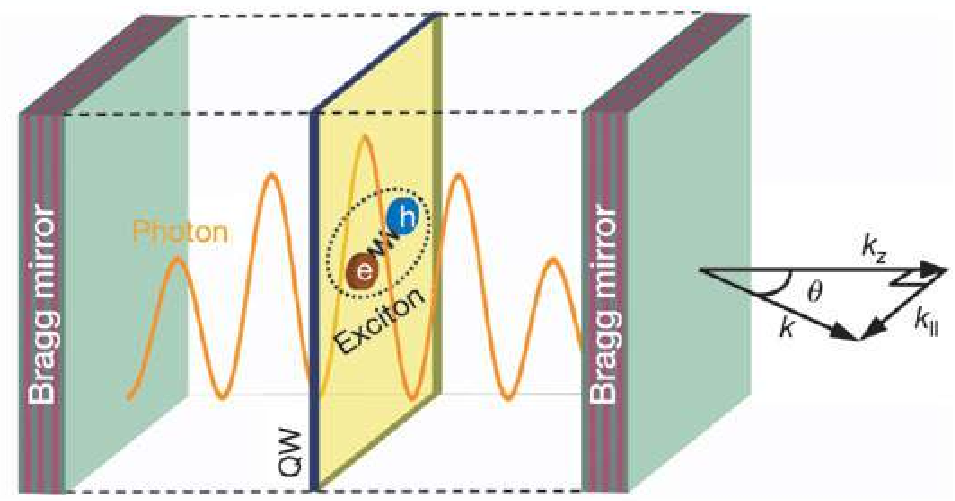
- now: simplifications in the semiclassical limit:
  - sharp argument close to a critical point
  - provides intuition for a frequency regime $\omega \ll \gamma = \gamma_l + \gamma_p$
- now: “what is non-equilibrium about it?”

Semi-classical limit and Langevin equations

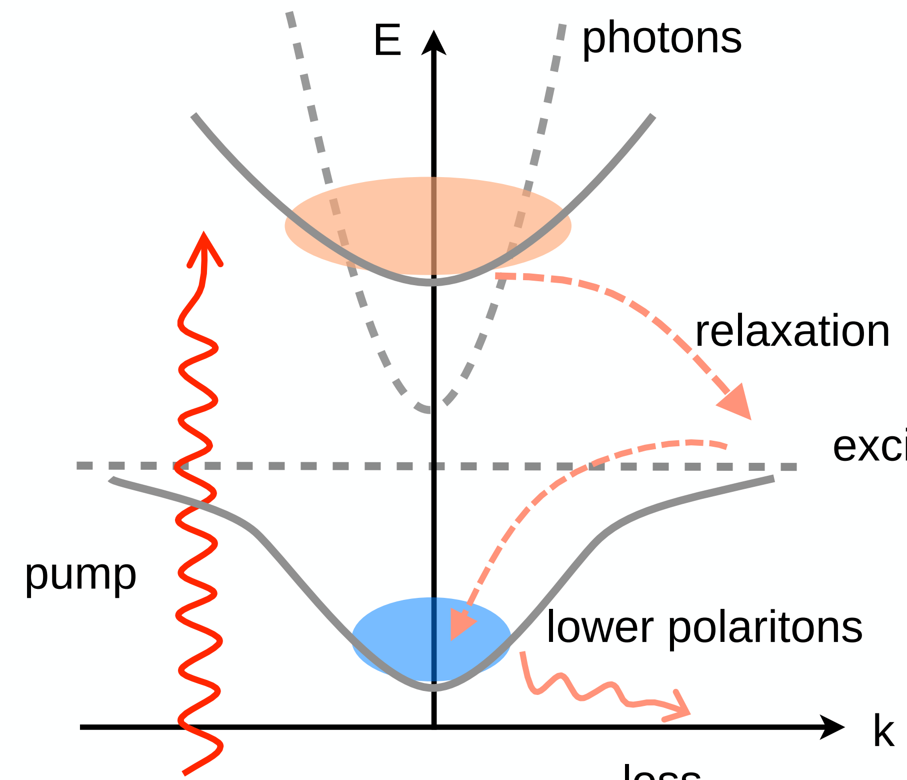


Intermezzo: Exciton-polariton systems

Kasprzak et al., Nature 2006



Imamoglu et al., PRA 1996



- phenomenological description: stochastic driven-dissipative Gross-Pitaevskii-Eq

$$i\partial_t\phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\,|\phi|^2\right]\phi + \zeta$$

propagation — pump & loss rates — elastic collisions — two-body loss

Szymanska, Keeling, Littlewood PRL (04,06); PRB (07);
Wouters, Carusotto PRL (07,10)

$$\langle\zeta^*(t,\mathbf{x})\zeta(t',\mathbf{x}')\rangle = \gamma\delta(t-t')\delta(\mathbf{x}-\mathbf{x}')$$

- structure: $$i\partial_t\phi = \frac{\delta\mathcal{H}_c}{\delta\phi^*} + i\frac{\delta\mathcal{H}_d}{\delta\phi^*} + \zeta$$

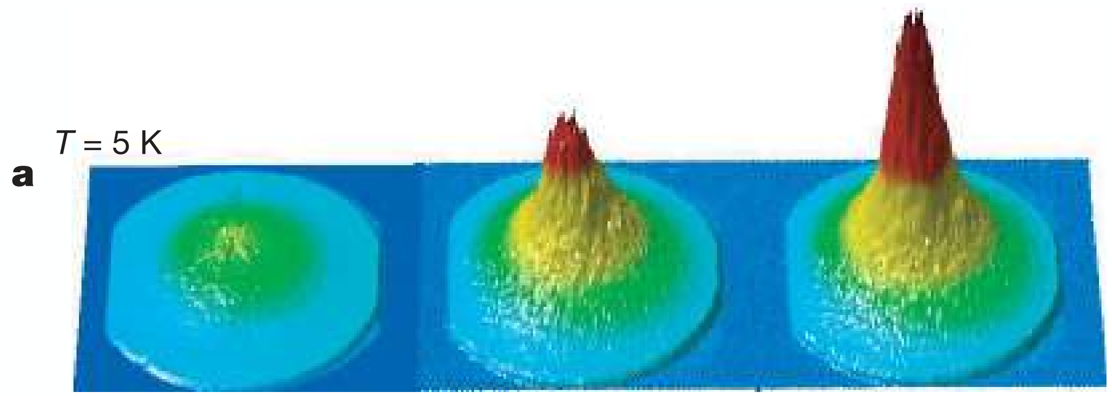
coherent (reversible) — dissipative (irreversible) — noise

$$\mathcal{H}_\alpha = \int d^3x[r_\alpha|\phi|^2 + K_\alpha|\nabla\phi|^2 + \lambda_\alpha|\phi|^4]$$

$$\alpha = c, d$$

Intermezzo: Exciton-polariton systems

- Bose condensation seen despite non-equilibrium conditions



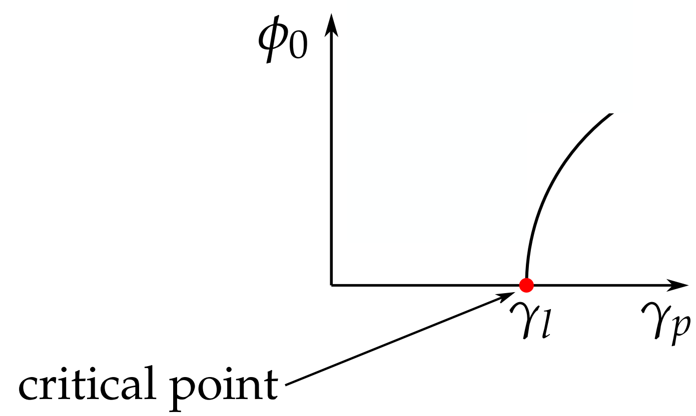
Kasprzak et al., Nature 2006

- stochastic driven-dissipative Gross-Pitaevskii-Eq

$$\cancel{i\partial_t}\phi = \left[\cancel{-\frac{\nabla^2}{2m}} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\left|\phi\right|^2\right]\phi + \cancel{\zeta}$$

Szymanska, Keeling, Littlewood PRL (04, 06); PRB (07)); Wouters, Carusotto PRL (07,10)

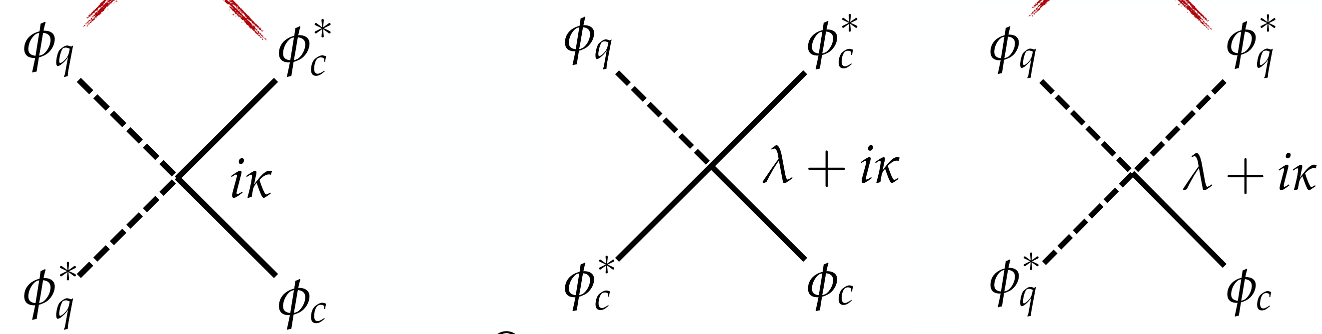
- mean field
  - neglect noise
  - homogeneous solution $\phi(\mathbf{x},t) = \phi_0$



- naively, just as Bose condensation in equilibrium!
- Q1: How does this model relate to the Lindbladian and Lindblad-Keldysh field theory?
- Q2: What is “non-equilibrium” about it?

Semiclassical limit of Lindblad-Keldysh action: power counting

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \cancel{\phi_c^* \phi_c \phi_q^* \phi_q} - \tfrac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \cancel{\phi_q^{*2} \phi_c \phi_q} \right) + c.c. \right] \right\}$$



- Gaussian sector close to a critical point:
  - retarded/advanced $\quad P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 \overbrace{- \mu + i(\gamma_l - \gamma_p)/2}^{\to 0} \sim q^2$
  - Keldysh component $\quad P^K = i(\gamma_l + \gamma_p) \sim q^0$
- canonical field dimensions: $\boxed{[\phi_c] = \dfrac{d-2}{2} < [\phi_q] = \dfrac{d+2}{2}}$
  - action is dimensionless: phase e^{iS} in the functional integral
  - d > 2: couplings with more than two quantum fields irrelevant in the RG sense
- massive simplification: semi-classical Martin-Siggia-Rose-Janssen-de Dominicis action

$$S = \int_{t,\mathbf{x}} \left\{ \underset{\text{linear}}{\phi_q^* \frac{\delta \bar{S}[\phi_c]}{\delta \phi_c^*}} + c.c. + \underset{\text{quadratic}}{i2\gamma \phi_q^* \phi_q} \right\} \qquad \bar{S} = \int_{t,\mathbf{x}} \{ \phi_c^* i\partial_t \phi_c - \underset{\text{Hermitian}}{\mathcal{H}_c} + \underset{\text{anti-Hermitian}}{i\mathcal{H}_d} \}$$

Relation to driven-dissipative Gross-Pitaevskii-equaiton

- Equivalence of semiclassical theory and Langevin equations
  - structure of action in semiclassical limit (Martin-Siggia-Rose-Janssen-de Dominicis functional integral)

$$Z = \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*]\, e^{iS[\phi_c, \phi_c^*, \phi_q, \phi_q^*]}$$

$$S = \int_{t,\mathbf{x}} \left\{ \phi_q^* \frac{\delta \bar{S}[\phi_c]}{\delta \phi_c^*} + c.c. + i2\gamma \phi_q^* \phi_q \right\} \qquad \bar{S} = \int_{t,\mathbf{x}} \{\phi_c^* i\partial_t \phi_c - \mathcal{H}_c + i\mathcal{H}_d\}$$

- Decouple the quantum field (complete the square via Hubbard-Stratonovich trick)

$$Z = \int \mathcal{D}[\xi, \xi^*]\, e^{-\frac{1}{2\gamma}\int_{t,\mathbf{x}} \xi^* \xi} \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*] e^{i\left[\phi_q^*\left(i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c{}^*} + i\frac{\delta \mathcal{H}_d}{\delta \phi_c{}^*} - \xi\right) + c.c.\right]}$$

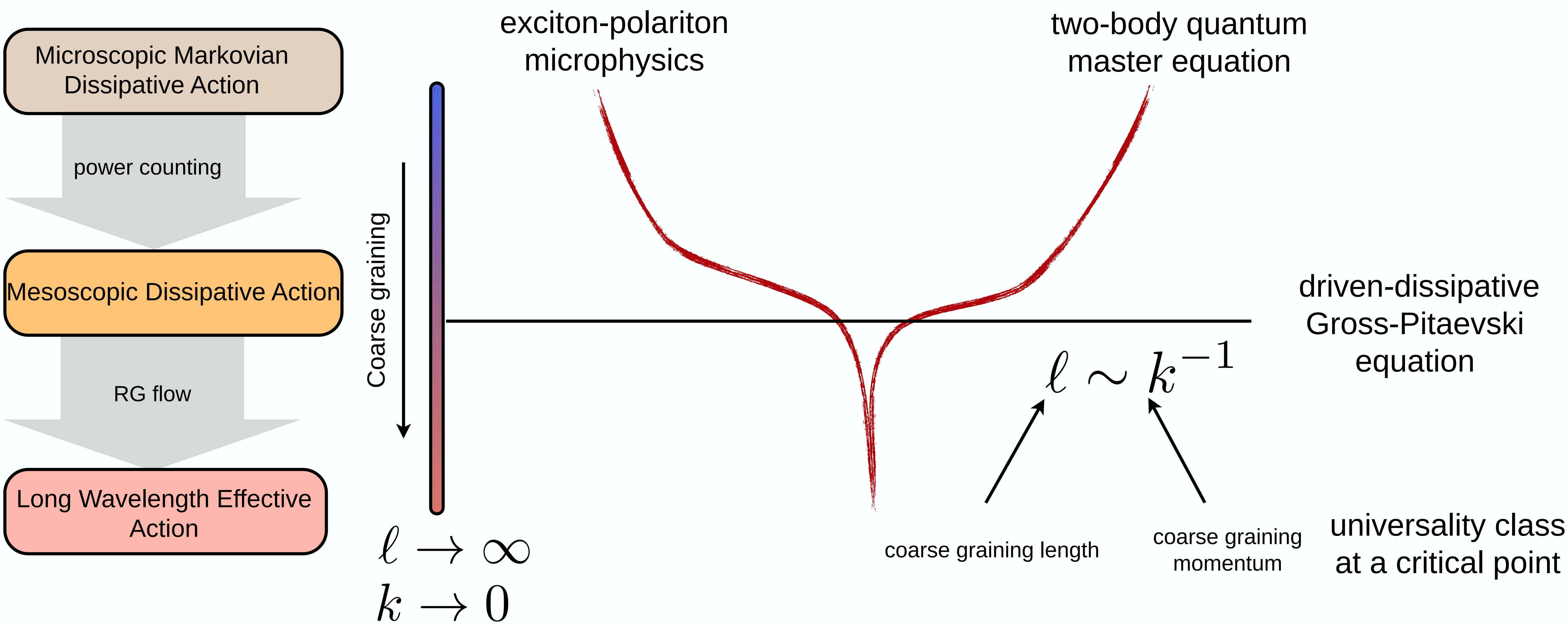
- Recognize Fourier representation of delta-constraint: Langevin equation

$$Z = \int \mathcal{D}[\xi, \xi^*]\, e^{-\frac{1}{2\gamma}\int_{t,\mathbf{x}} \xi^* \xi} \int \mathcal{D}[\phi_c, \phi_c^*] \delta\left(\boxed{i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c^*} + i\frac{\delta \mathcal{H}_d}{\delta \phi_c^*} - \xi}\right) \delta\left(c.c.\right)$$

➡ noise averaging ➡ at each instant of time: ➡ driven-dissipative Gross-Pitaevski equation

Semiclassical limit and exciton-polariton model

- example of “weak” universality

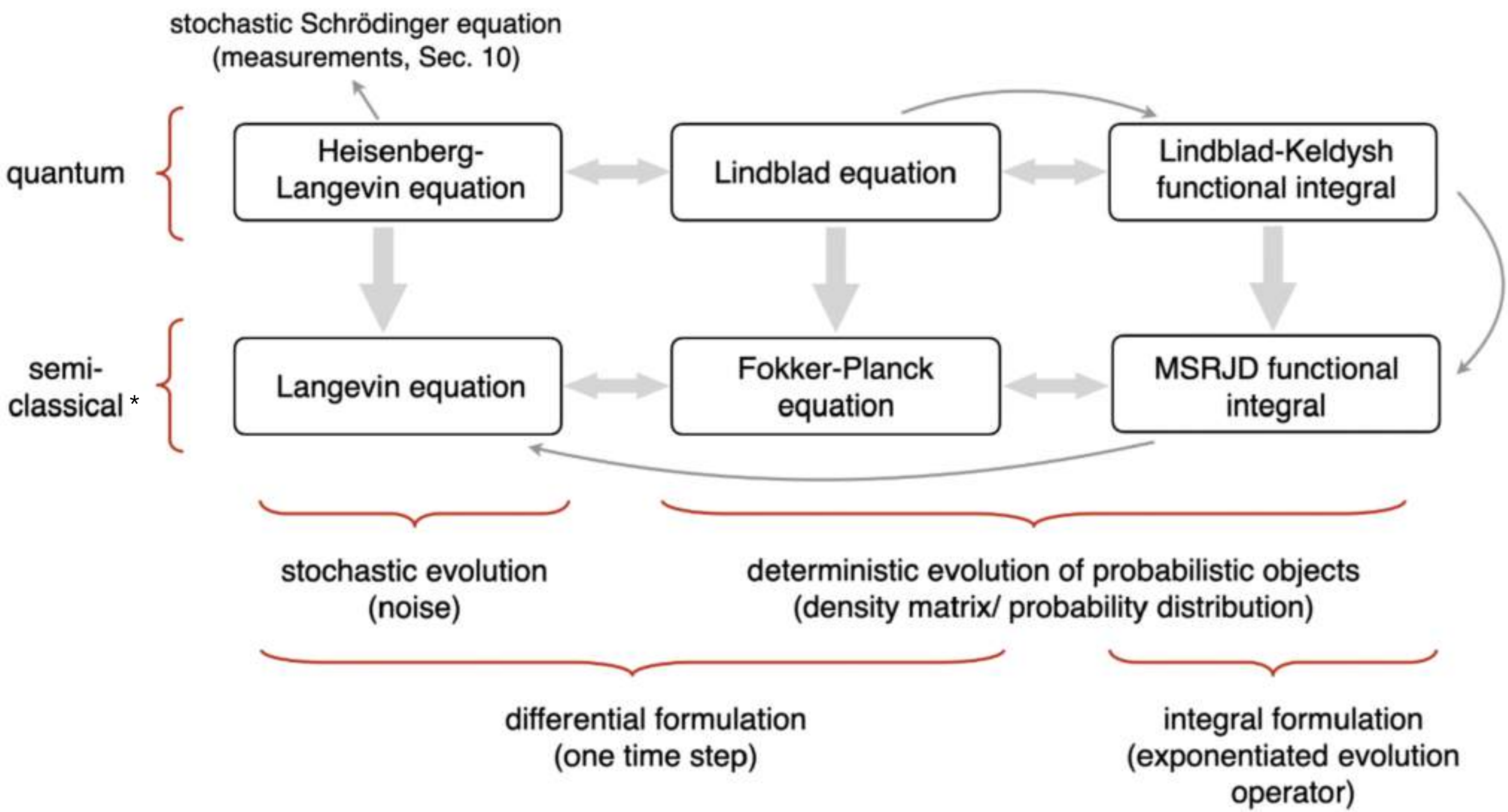


➡ many microscopic models collapse to an effective low energy model

➡ form dictated by microscopic symmetries

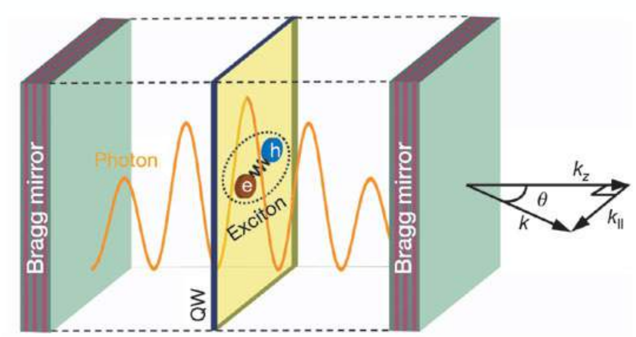
➡ longer wavelength behavior to be determined by calculation

Overview: Langevin equations, master equation, Keldysh integral

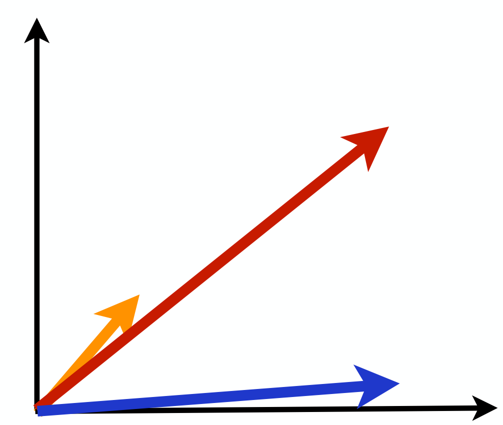


* effects of phase coherence still present (cf. BEC as classical wave)

“What is non-equilibrium about it?”



$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$



"What is non-equilibrium about it?"

- different notions of 'non-equilibrium'

Time evolution

➡ time translation invariance broken (e.g. thermalization, Floquet..)

Stationary states (considered here)

➡ flux equilibria

- not in static observables:

$$\rho = e^{-\beta H} / \mathrm{tr} e^{-\beta H}$$

➡ any positive semidefinite Hermitian operator can be written like this

- dynamical observables, e.g.:

$$\langle \psi^\dagger(t)\psi(0)\rangle \qquad\qquad \psi(t) = e^{iHt}\psi e^{-iHt}$$

➡ non-equilibrium conditions are encoded in the generator of dynamics

➡ thermal equilibrium realized if generator of dynamics coincides with statistical weight

➡ otherwise must expect non-equilibrium conditions (Lindbladian)

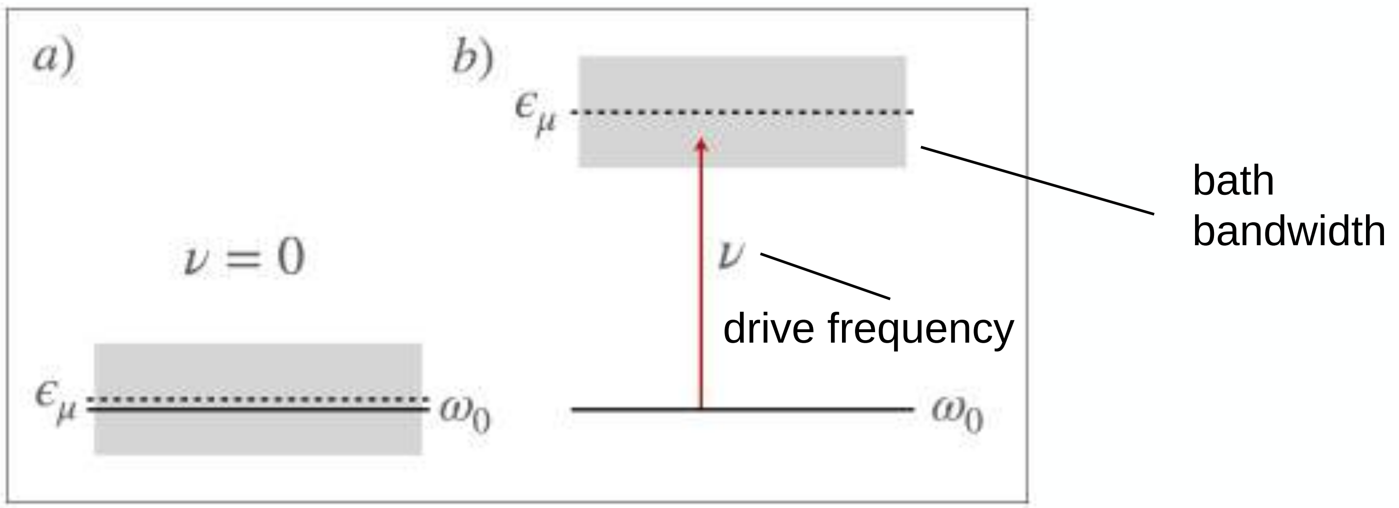
“What is non-equilibrium about it?”

more details: see appendix!

- non-equilibrium stationary states:
  - open system: is it the coupling to a bath —> irreversibility?
    - ➡ no, can be compatible with thermal equilibrium (Caldeira-Leggett Models)
  - driven & open system: it is in the way how we couple to a bath:

$$\hat{H}_{\mathrm{t}} = \hat{H} + \hat{H}_{\mathrm{int}} + \hat{H}_{\mathrm{b}} \qquad \hat{H}_{\mathrm{b}} = \sum_\mu \epsilon_\mu b_\mu^\dagger b_\mu \qquad \hat{H}_{\mathrm{int}} = \sum_\mu g_\mu e^{-i\nu t} L b_\mu^\dagger + \mathrm{h.~c.}$$

system — bath — system-bath



equilibrium: $\nu = 0 \ll g_\mu \ll \omega_0 \approx \epsilon_\mu$

non-equilibrium: $g_\mu \ll \omega_0 \ll \nu \approx \epsilon_\mu$

- the state of the bath is fixed, distribution function $n^{\mathrm{b}}_{B/F}(\omega)$
- singles out a frame of reference —> drive scale cannot be removed
  - ➡ driven open nature incompatible with thermal equilibrium

“What is non-equilibrium about it?”

- more formally: quantum master equation $$\underbrace{\partial_t \rho = -i[H,\rho]}_{\Longrightarrow S_H} + \underbrace{\mathcal{D}[\rho]}_{\Longrightarrow S_{\mathcal{D}}}$$

- equivalent Keldysh functional integral: $Z = \int \mathcal{D}\phi_\pm e^{i(S_H[\phi_\pm]+S_{\mathcal{D}}[\phi_\pm])}$

- equilibrium dynamics microscopically generated by a time-independent (undriven) Hamiltonian alone

$$S_{\mathcal{D}} = 0$$

➡ symmetry of Keldysh action under discrete transformation

L. Sieberer, A Chiocchetta, U. Täuber, A. Gambassi, SD PRB (2015); F. Haehl, R. Loganayagam, M. Rangamani, JHEP (2016); M. Crossley, P. Glorioso, H. Liu, JHEP (2016)

$$\boxed{\mathcal{T}_\beta : \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t+i\beta/2,\mathbf{x}), \quad i \to -i} \qquad \mathcal{T}_\beta^2 = 1 \qquad \beta = 1/T$$

- associated “Ward identities” are equilibrium quantum fluctuation-dissipation relations of arbitrary order

e.g. single-particle sector

$$\boxed{G^K(\omega,\mathbf{q}) = (2n_B(\omega/T)+1)[G^R(\omega,\mathbf{q}) - G^A(\omega,\mathbf{q})]}$$

correlations | Bose distribution | responses

any order <=> detailed balance <=> global thermal equilibrium

➡ the Lindbladian ($S_{\mathcal{D}}$) violates this symmetry and therefore detailed balance explicitly

➡ intuition: underlying is a (rapidly) driven system with no energy conservation

Equilibrium symmetry: some details

L. Sieberer, A Chiocchetta, U. Täuber, A. Gambassi, SD PRB (2015)

- Undriven system: equilibrium dynamics generated by a time-independent Hamiltonian

➡ symmetry of Schwinger-Keldysh action under discrete transformation

$$\mathcal{T}_\beta: \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t+i\beta/2,\mathbf{x}), \quad i \to -i \qquad \mathcal{T}_\beta^2 = 1 \qquad \beta = 1/T$$

- symmetry: invariance of $Z = \int \mathcal{D}\phi_\pm e^{i(S_H[\phi_\pm]+S_\mathcal{D}[\phi_\pm])}$

- implies for correlation functions

$$\langle \mathcal{O}[\phi_\pm]\rangle = \langle \mathcal{T}_\beta(\mathcal{O}[\phi_\pm])\rangle \qquad \langle \mathcal{O}[\phi_\pm]\rangle = \int \mathcal{D}\phi_\pm \mathcal{O}[\phi_\pm] e^{iS[\phi_\pm]}$$

- physical consequence: Fluctuation-dissipation relations, of any order, e.g. single particle sector:

$$\boxed{G^K(\omega,\mathbf{q}) = (2n_B(\omega/T)+1)[G^R(\omega,\mathbf{q}) - G^A(\omega,\mathbf{q})]}$$

any order <=> detailed balance
<=> global thermal equilibrium

correlations Bose distribution responses

- connection to operator formalism: compact functional formulation of Kubo-Martin-Schwinger boundary condition: for any two operators A,B,

$$\langle A(t)B(t')\rangle = \langle B(t'-i\beta)A(t)\rangle. \qquad \langle \mathcal{O}\rangle = \mathrm{tr}(\mathcal{O}\rho)$$

- reason: $A(t) = e^{iHt}Ae^{-iHt}, \rho = e^{-\beta H}/\mathrm{tr}e^{-\beta H}$
  $\Rightarrow A(t)\rho = \rho A(t-i\beta)$ & cyclic invariance

Equilibrium symmetry: Semiclassical limit

- Undriven system: equilibrium dynamics generated by a time-independent Hamiltonian

➡ symmetry of Schwinger-Keldysh action under discrete transformation

$$\mathcal{T}_\beta : \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t + i\beta/2, \mathbf{x}), \quad i \to -i \qquad \mathcal{T}_\beta^2 = 1 \qquad \beta = 1/T$$

$$= e^{\pm i \frac{\beta}{2}\partial_t} \phi_\pm(-t,\mathbf{x})$$

- semiclassical limit: T large => $e^{\pm i \frac{\beta}{2}\partial_t} \approx 1 \pm i\frac{\beta}{2}\partial_t$

- action on the fields:

irrelevant by power counting

$$\mathcal{T}_\beta \phi_c(t,\mathbf{x}) = \phi_c^*(-t,\mathbf{x}) + \cancel{\frac{i}{2T}\partial_t \phi_q^*(-t,\mathbf{x})},$$

$$\mathcal{T}_\beta \phi_q(t,\mathbf{x}) = \phi_q^*(-t,\mathbf{x}) + \frac{i}{2T}\partial_t \phi_c^*(-t,\mathbf{x})$$

reproduces classical result

H. K. Janssen (1976); C. Aron et al, J Stat. Mech (2011)

➡ obtain geometric interpretation of the equilibrium symmetry

Geometric interpretation: equilibrium vs. non-equilibrium dynamics

- couplings spanning the Keldysh action lie in the complex plane

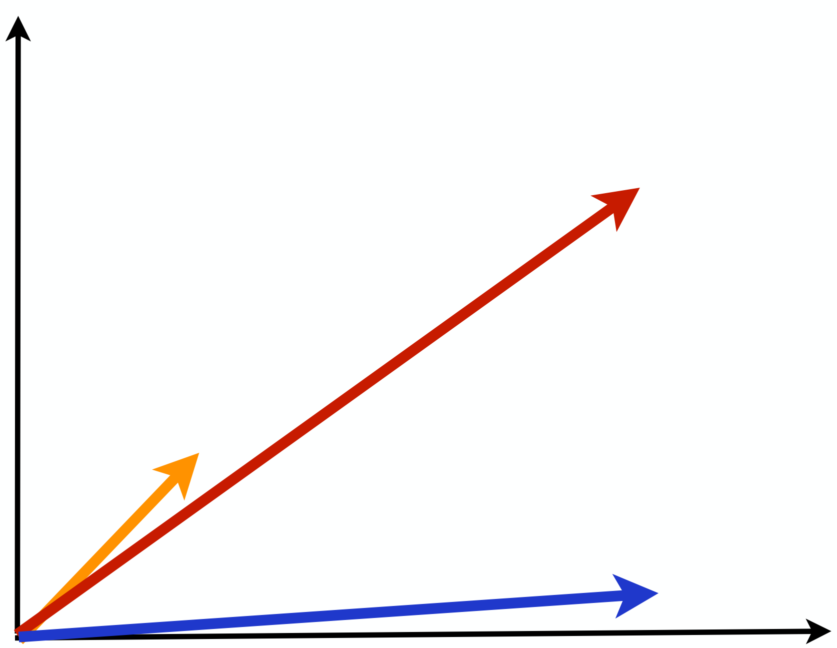
$$\underbrace{\partial_t \rho = -i[H,\rho]}_{\Leftrightarrow\, S_H} + \underbrace{\mathcal{D}[\rho]}_{\Leftrightarrow\, S_{\mathcal{D}}} \qquad \Leftrightarrow \qquad Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_H[\Phi_+,\Phi_-] + S_{\mathcal{D}}[\Phi_+,\Phi_-])}$$

& semiclassical limit: higher terms in ϕ_q irrelevant

Im

incoherent/ irrev. dynamics

$\Leftrightarrow S_{\mathcal{D}}$



Re

coherent/ reversible dynamics

$\Leftrightarrow S_H$

example: two-body processes λ

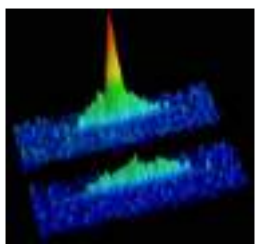
$\mathrm{Re}\lambda$ elastic two-body collisions

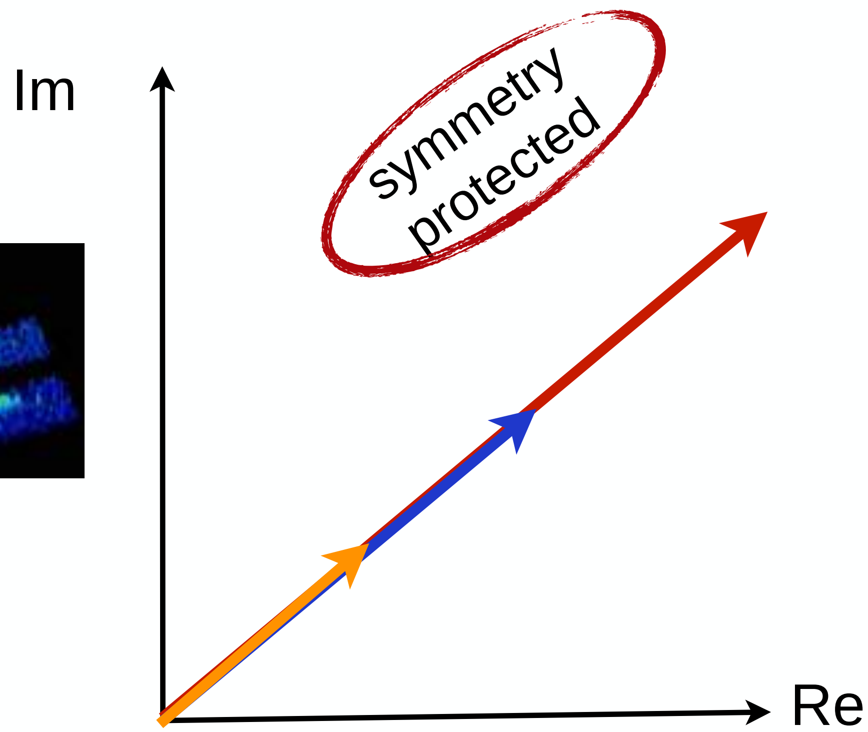
$\mathrm{Im}\lambda$ inelastic two-body losses

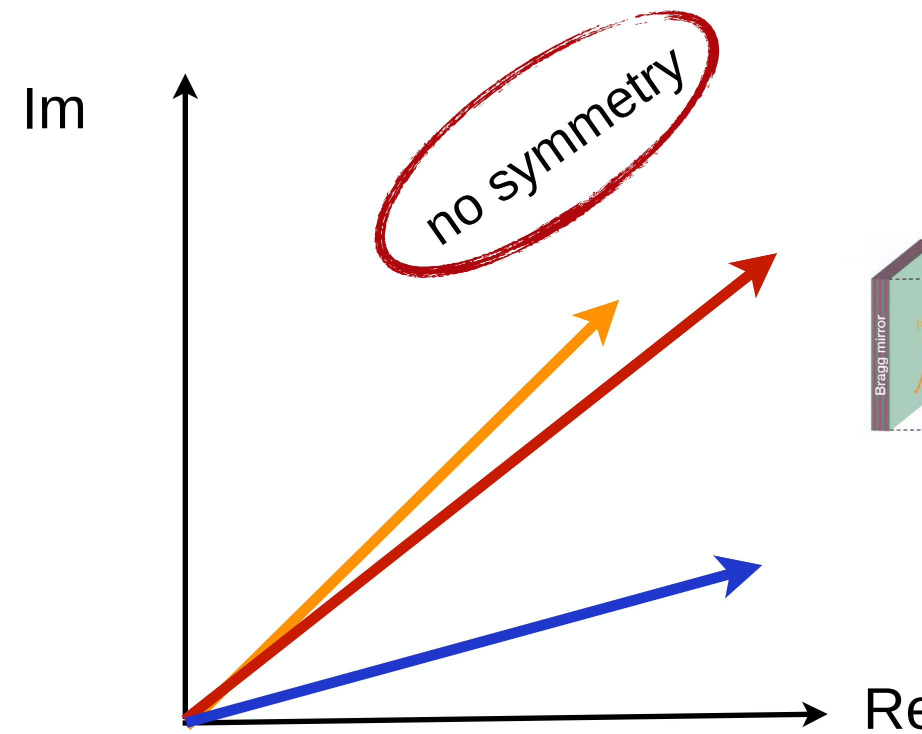
Geometric interpretation: equilibrium vs. non-equilibrium dynamics

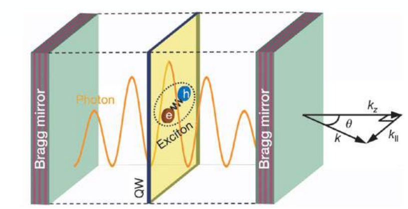
equilibrium dynamics

non-equilibrium dynamics







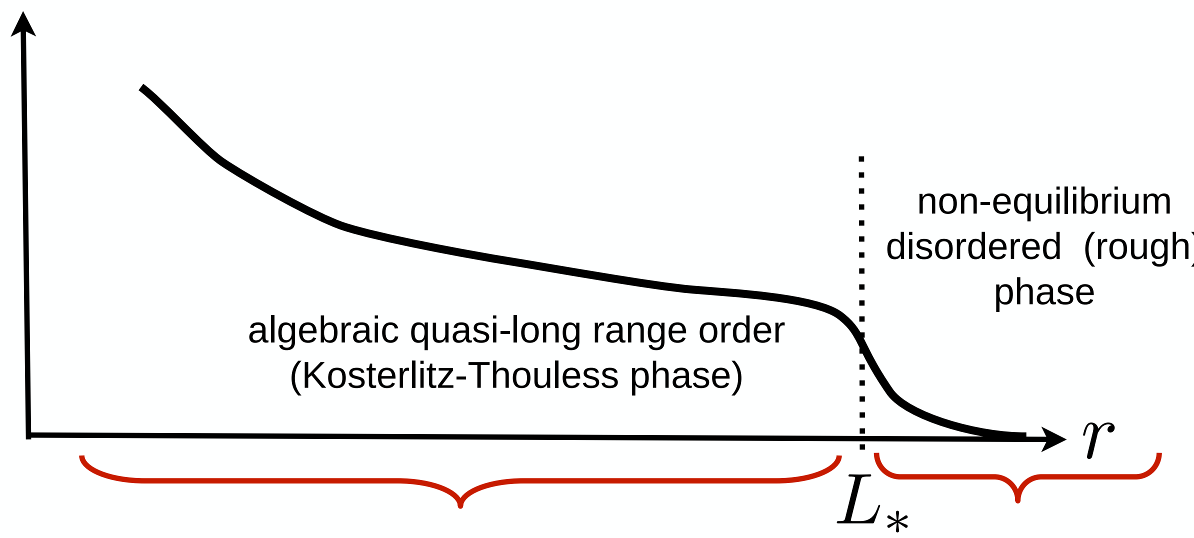


- coherent and dissipative dynamics may occur simultaneously
- but they are not independent

- coherent and dissipative dynamics do occur simultaneously
- they result from different dynamical resources

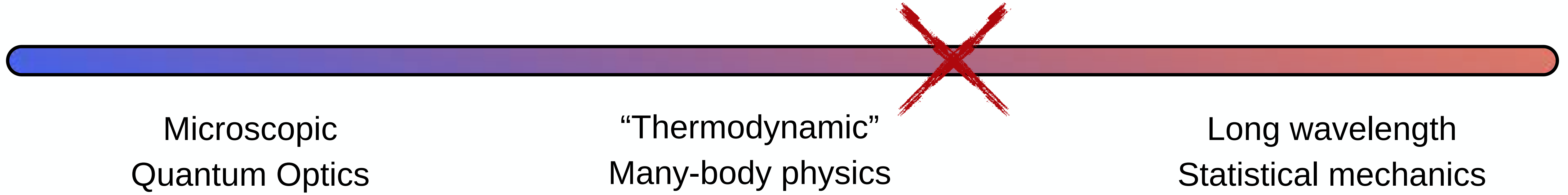
➡ what are the physical consequences of the spread in the complex plane?

Application:
Fate of BKT physics in driven open quantum systems



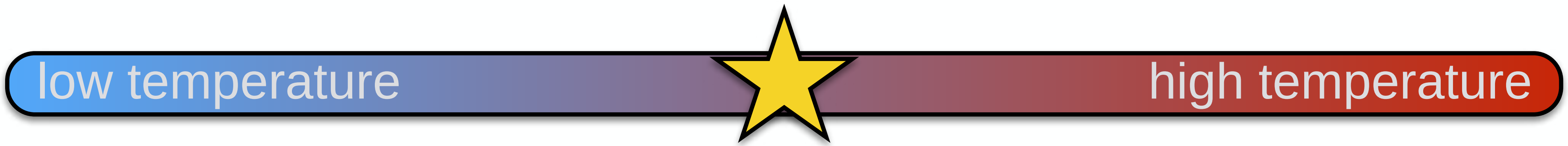
E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)
G. Wachtel, L. Sieberer, SD, E. Altman, PRB (2016)
L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)

L. He, L. Sieberer, E. Altman, SD, PRB (2015)
L. He, L. Sieberer, SD, PRL (2017)



Phase transitions in two dimensions

- continuous symmetry U(1): no spontaneous symmetry breaking, but a phase transition



- correlations

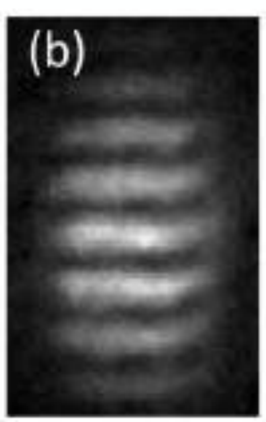
$$\langle \phi^*(\mathbf{x})\phi(0)\rangle \sim r^{-\frac{1}{2\pi K}} \qquad\qquad \sim e^{-r/\xi}$$

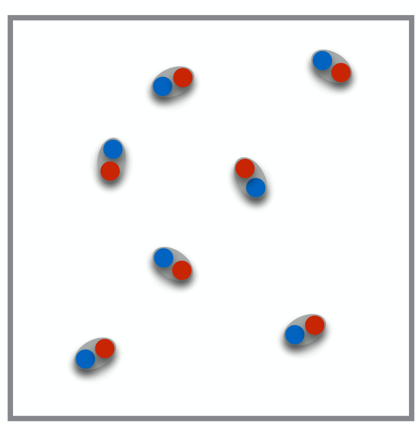
- responses: superfluidity

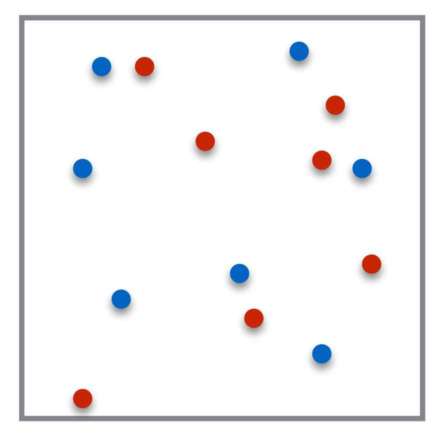
$$\rho_s \neq 0 \qquad\qquad \rho_s = 0$$

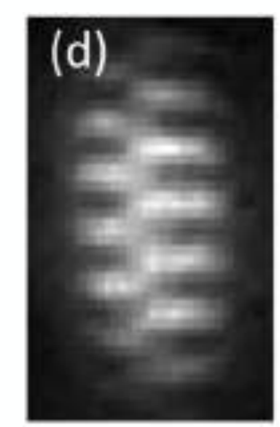
- BKT transition: unbinding of vortex-antivortex pairs

J. M. Kosterlitz, D. J. Thouless J. Phys. C (1973)





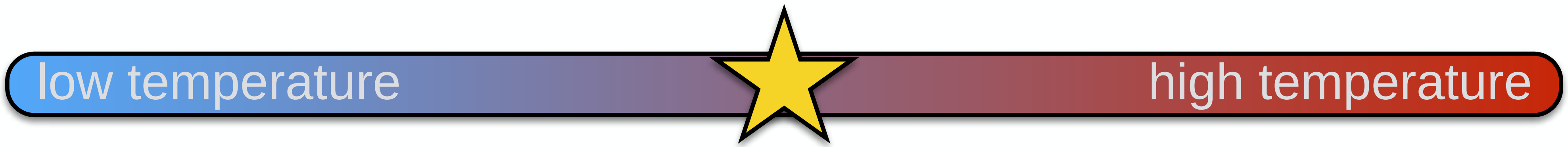




matter wave interferometry:
Z. Hadzibabic et al. Nature (2006)

… fate in driven open condensates?

Short reminder: Algebraic correlations



- correlations

$$\langle \phi^*(\mathbf{x})\phi(0)\rangle \sim r^{-\frac{1}{2\pi K}} \qquad\qquad \sim e^{-r/\xi}$$

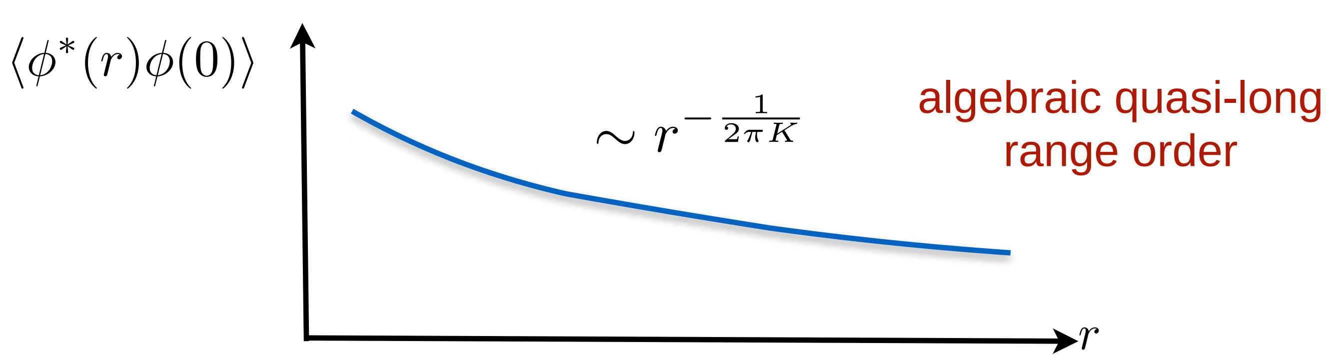
- physical reason: gapless spin wave fluctuations

  - spin wave action $S = \frac{K}{2}\int d^2x(\nabla\theta)^2 = \frac{K}{2}\int \frac{d^2q}{(2\pi)^2}\, q^2\, \theta(-\mathbf{q})\theta(\mathbf{q})$

  - phase-amplitude decomposition $\phi(\mathbf{x}) = \rho(\mathbf{x})^{1/2}e^{i\theta(\mathbf{x})} \approx \sqrt{n_0}e^{i\theta(\mathbf{x})}$

$$\langle \phi^*(\mathbf{x})\phi(0)\rangle \approx n_0\langle e^{i(\theta(\mathbf{x})-\theta(0))}\rangle = n_0 e^{-\frac{1}{2}\langle(\theta(\mathbf{x})-\theta(0))^2\rangle}$$

  - phase correlator $\frac{1}{2}\langle(\theta(\mathbf{x})-\theta(0))^2\rangle = \int^{1/a}\frac{d^2q}{(2\pi)^2}\frac{(e^{iqr}-1)}{Kq^2} = \frac{1}{2\pi K}\log(r/a)$



Description: Effective model

- mesoscopic starting point: driven-dissipative stochastic Gross-Pitaevski equation

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (g - i\kappa)\,|\phi|^2\right]\phi + \zeta$$

$$\phi = \rho e^{i\theta}$$

- effective low frequency dynamics

see also: G. Grinstein et al., PRL (1993)

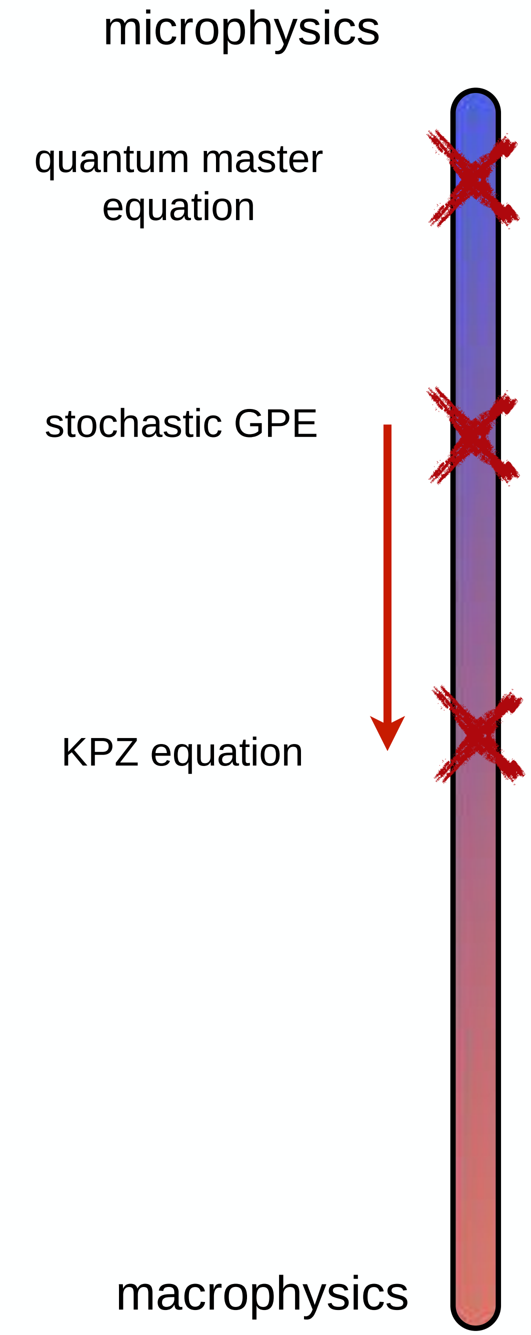
$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$

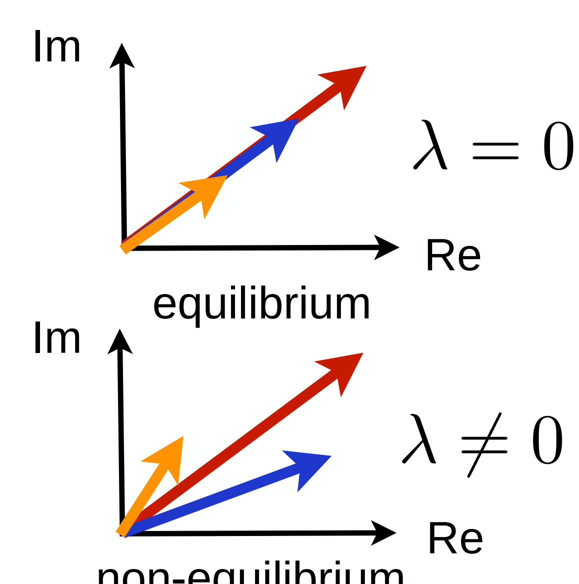
phase diffusion phase nonlinearity Markov noise

form of the KPZ equation

Kardar, Parisi, Zhang, PRL (1986)

- meaning: non-linear spin wave mode
- nonlinearity: single-parameter measure of non-equilibrium strength (ruled out in equilibrium by symmetry)





KPZ equation: A paradigm of non-equilibrium stat mech

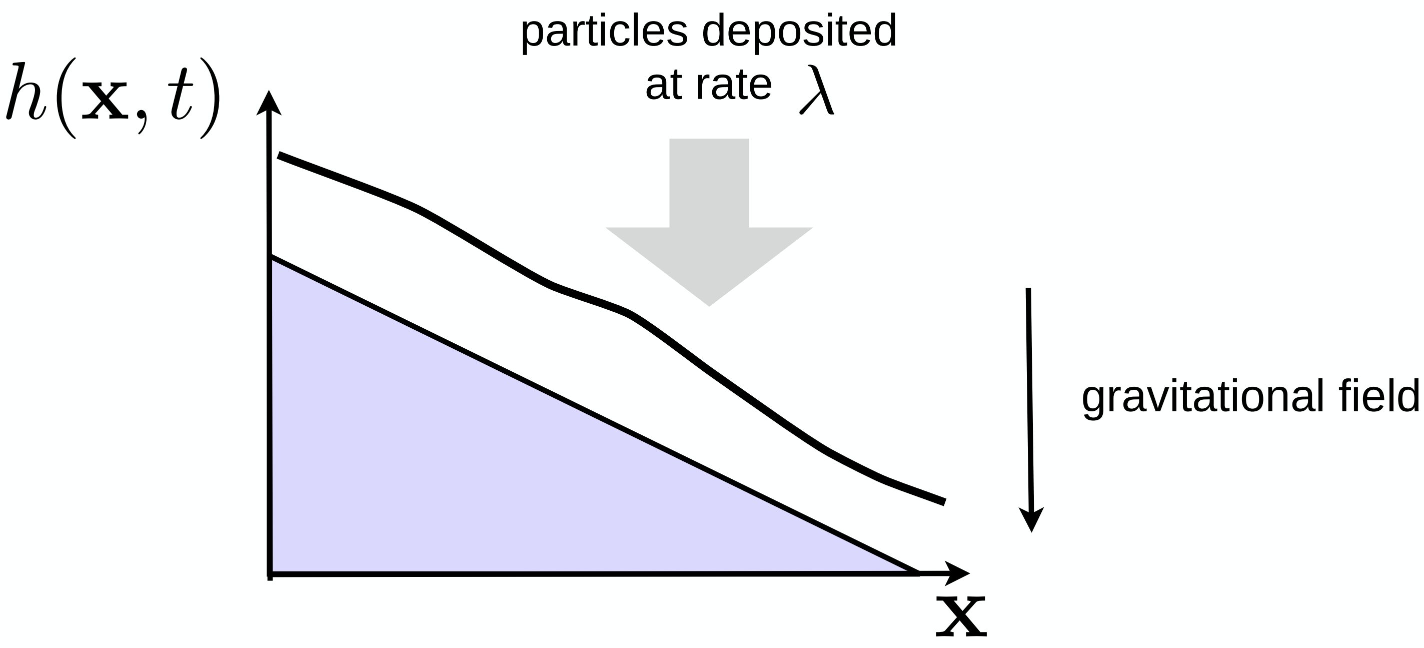
- originally: describes stochastic roughening of surface height $h(\mathbf{x},t)$

$$\partial_t h = D\nabla^2 h + \lambda(\nabla h)^2 + \xi$$

smoothens nonlinear growth noise

Kardar, Parisi, Zhang, PRL (1986)
Review: Krug, Adv. Phys. (1997)

- simplest physical scenario



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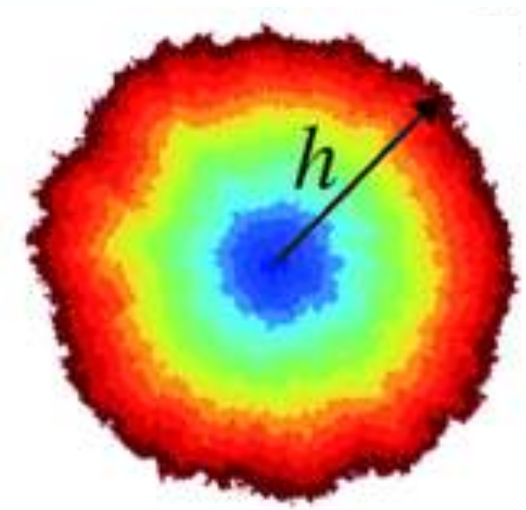
- multiple physical contexts



defect growth in liquid crystals

drive: electric field

from Takeuchi et al., Scientific Reports (2011)

bacterial colony growth

drive: sugar

Wakita et al., J. Phys. Jpn. Soc. (1997)

burning paper

drive: oxygen

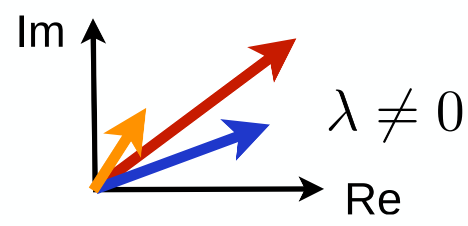
Maunuksela et al., PRL (1997)

E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)

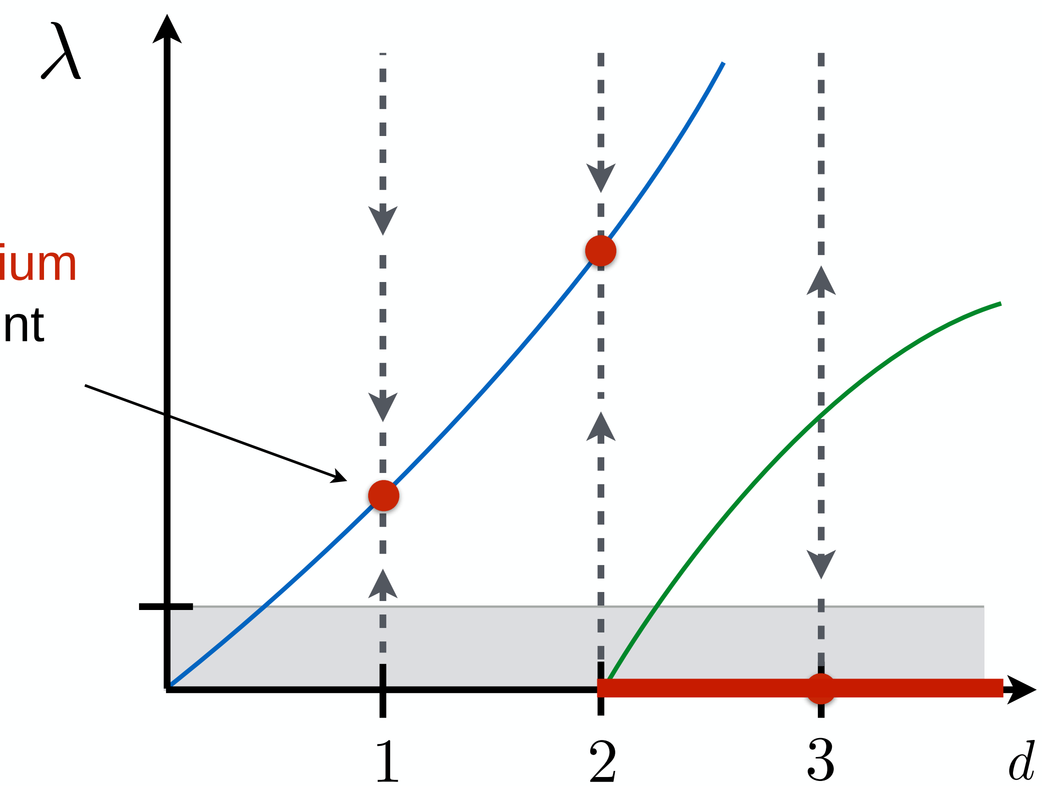
Physical implication I: Smooth KPZ fluctuations

- How important are non-equilibrium conditions at large distance?

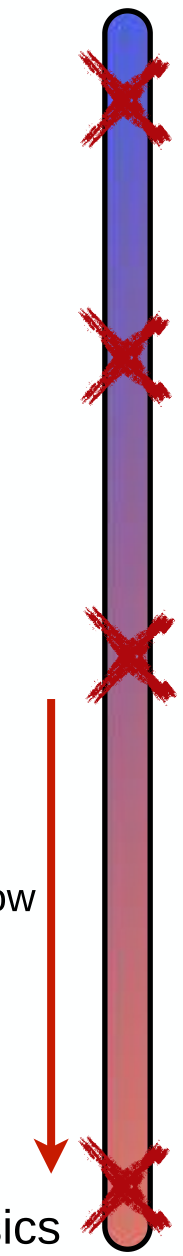
- behavior of non-equilibrium strength under coarse graining (RG flow)



strong non-equilibrium
<-> KPZ fixed point



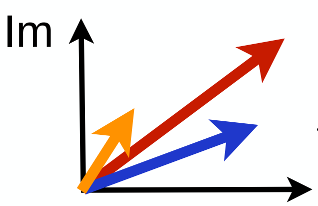
weak non-equilibrium
<-> equilibrium fixed point

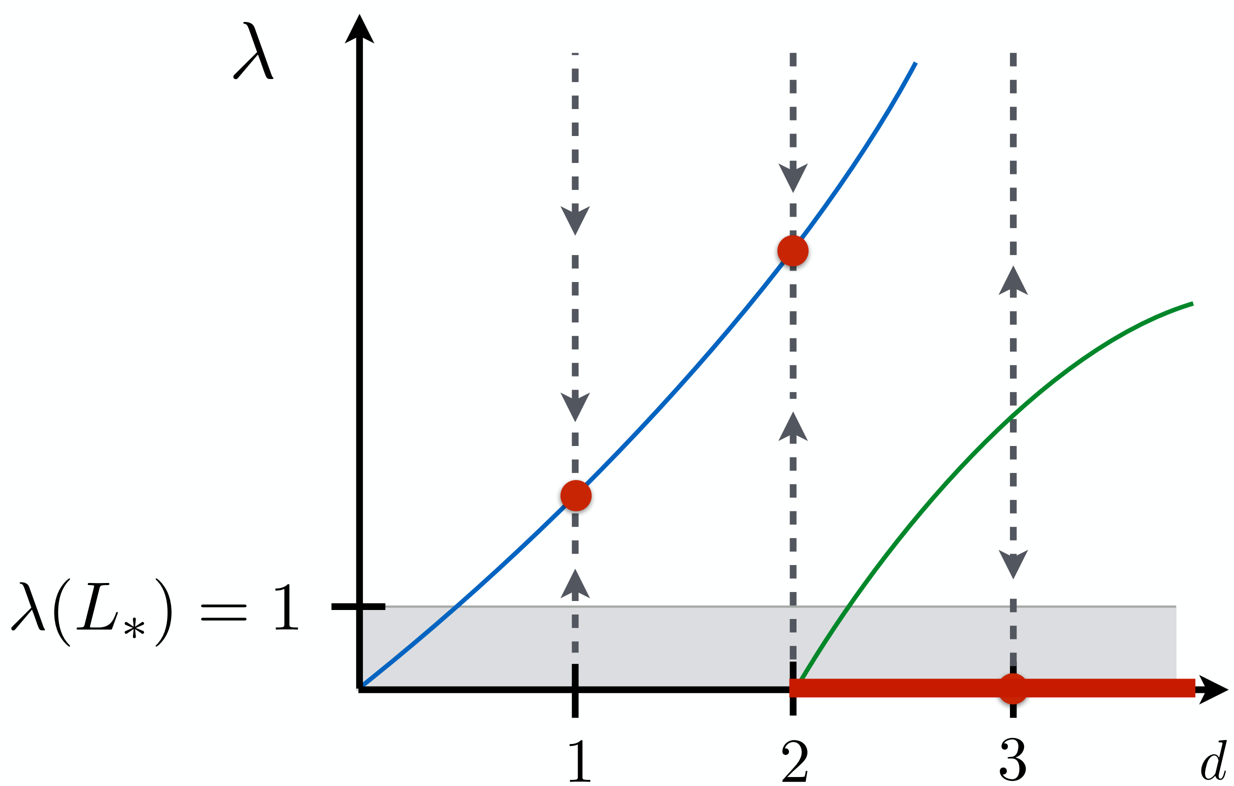


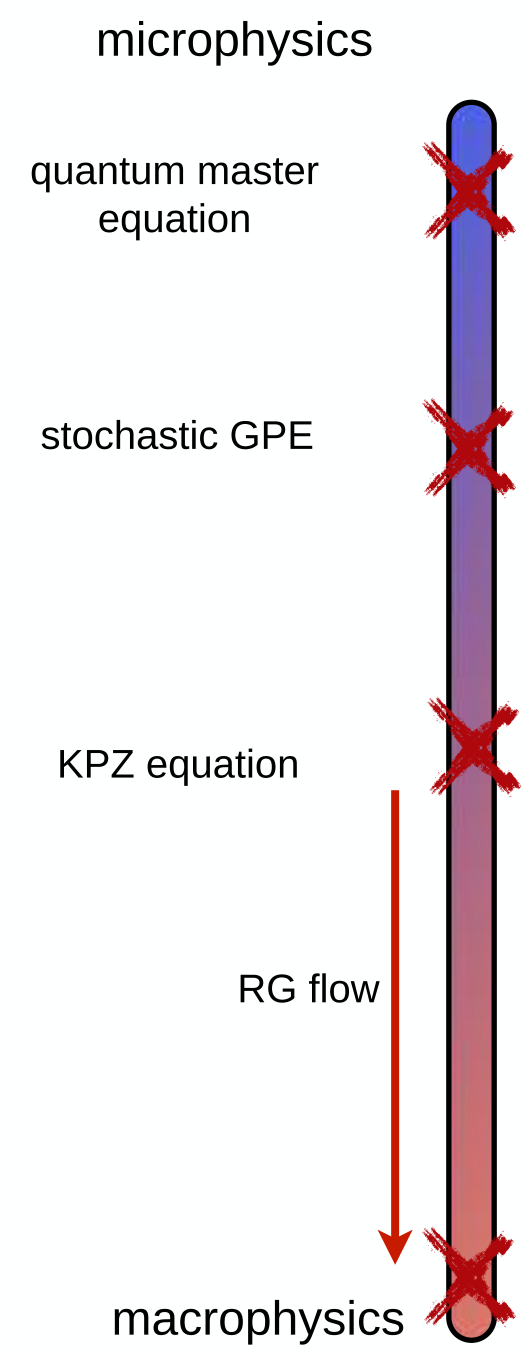
- general trend: non-equilibrium effects in systems with soft mode are
  - enhanced in d = 1,2
  - softened in d = 3 (below a threshold)

Physical implication I: Smooth KPZ fluctuations

- How important are non-equilibrium conditions at large distance?
- behavior of non-equilibrium strength under coarse graining (RG flow)







- 2D: implication: a length scale is generated

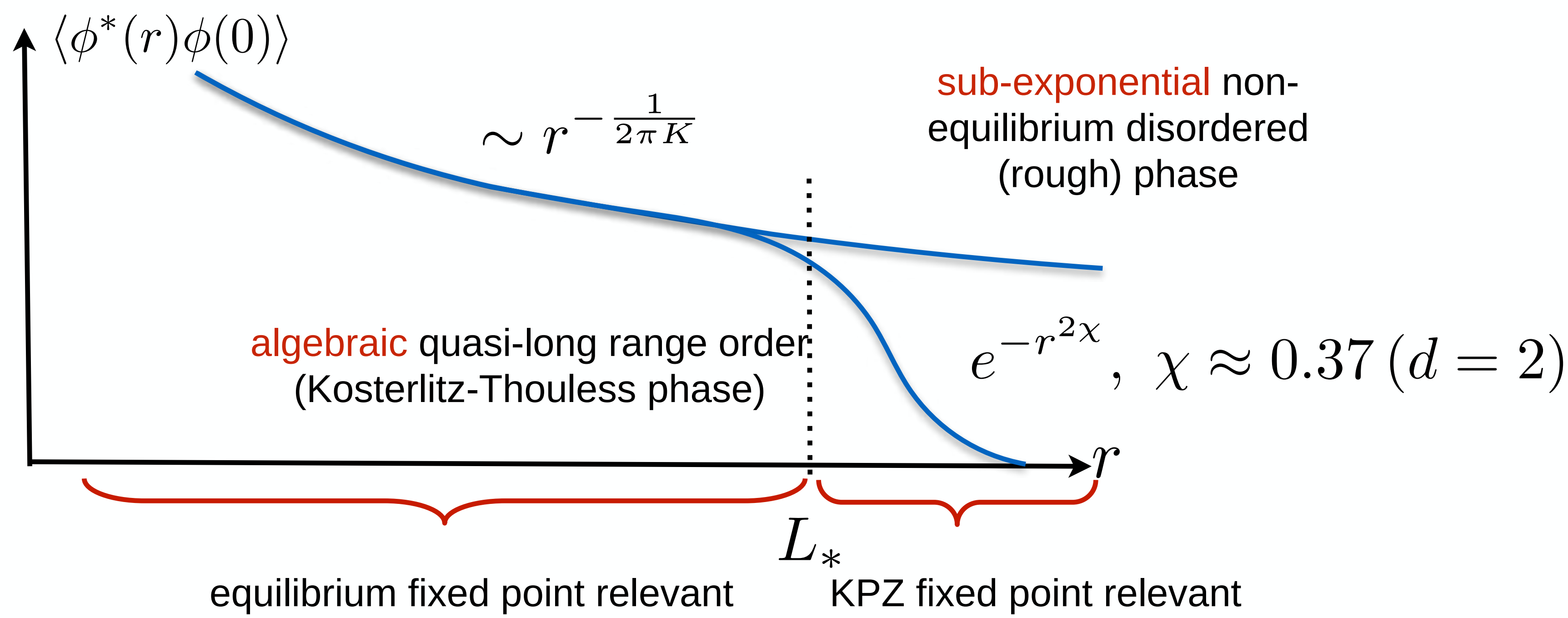
$$L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$$

microscopic (healing) length

E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)

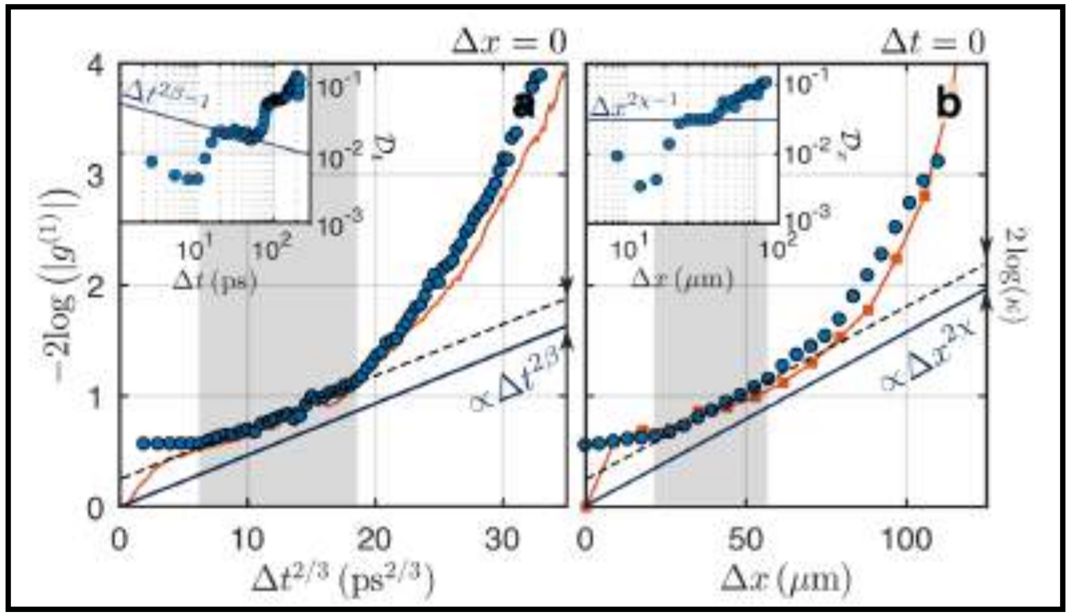
2D: Absence of algebraic order out of equilibrium

- generated length scale distinguishes two regimes: $L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$



- ➡ algebraic order absent in any two-dimensional driven open system at the largest distances
- ➡ but crossover scale exponentially large for small deviations from equilibrium
- ➡ observation in 1D systems (KPZ scaling exponents found)

$\beta = 1/3$ temporal scaling



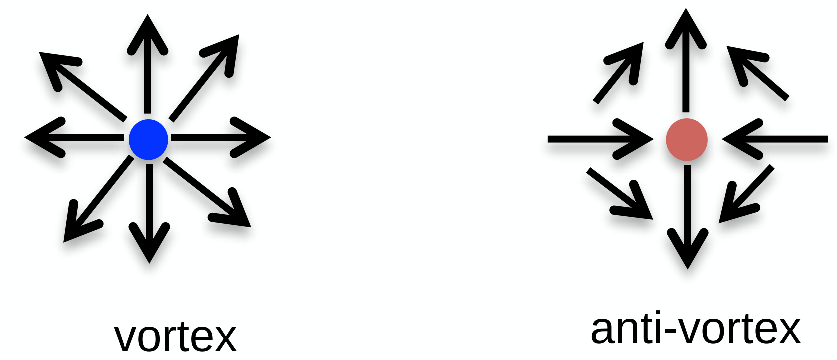
spatial scaling $\chi = 1/2$

Q. Fontaine et al., Nature (Aug 24 2022)

Physical implications II: Non-equilibrium Kosterlitz-Thouless

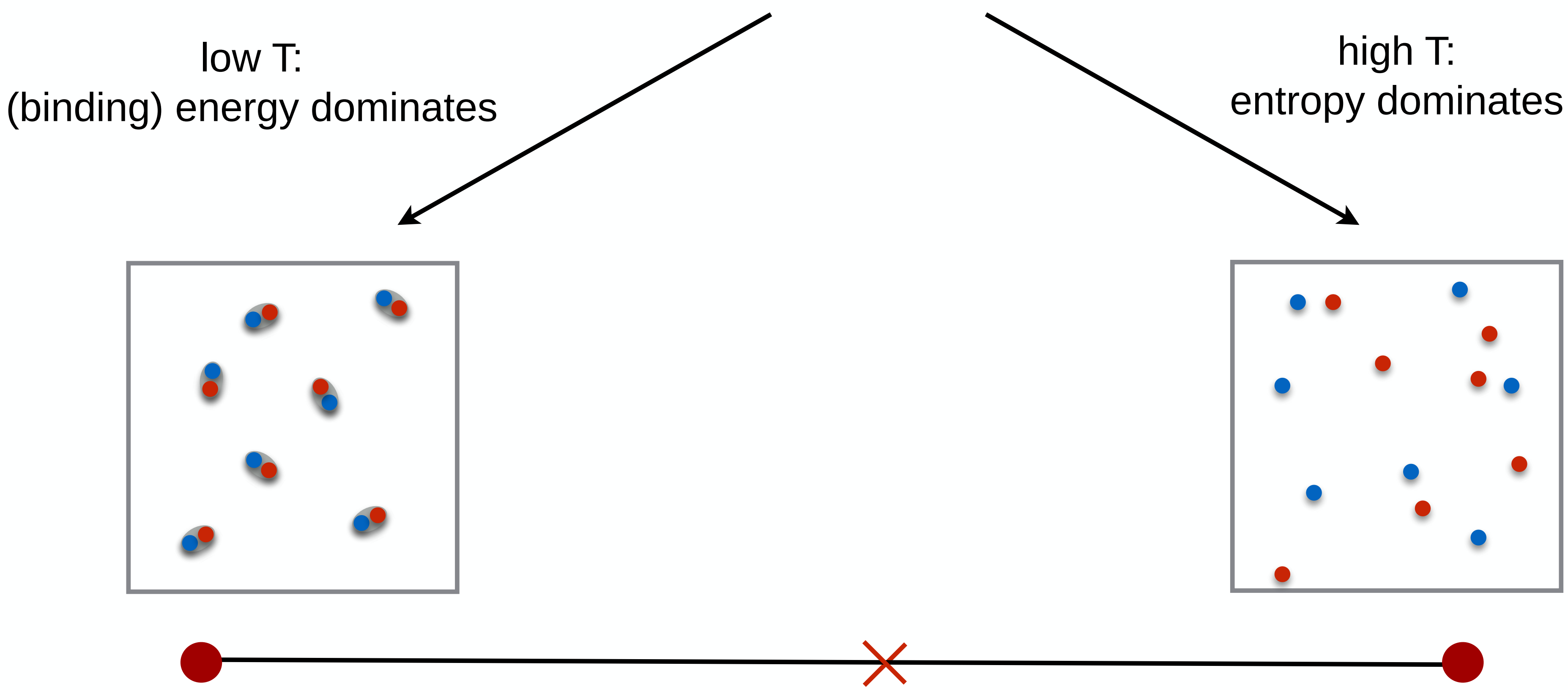
- KPZ equation for phase variable

$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$



- compact nature of phase allows for vortex defects in 2D!
- key ingredient of Kosterlitz-Thouless transition

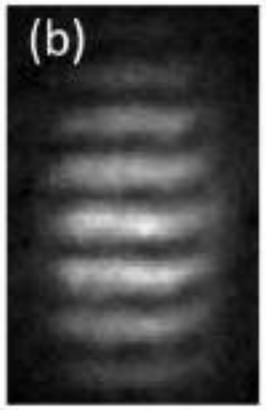
$$F = E - TS$$

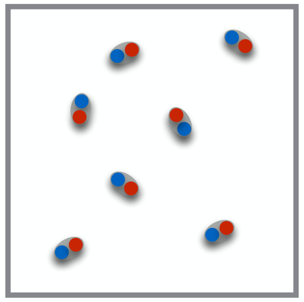


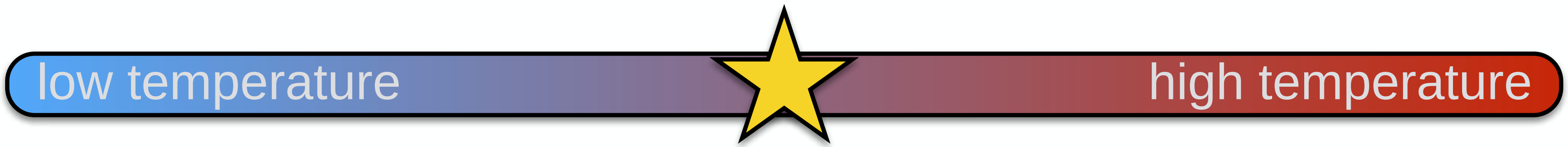
➡ so far, compactness of phase variable ignored

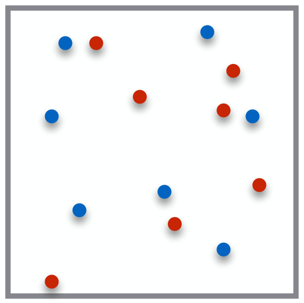
➡ how is this scenario modified in the driven system?

Mini-review: BKT transition









- BKT transition: unbinding of vortex-antivortex pairs
  J. M. Kosterlitz, D. J. Thouless J. Phys. C (1973)

- Single vortex picture: balance of energy (deterministic) and entropy (statistic)
  - Low T: vortices and antivortices bound in neutral pairs (irrelevant at long distance)
  - Q: when is it favorable (free energy minimum) to have unbound vortices?

- energy of single free vortex:

vortex configuration: mapping $(r,\varphi) \to \theta(r,\varphi) = \varphi \qquad \Longrightarrow \nabla\theta = \frac{1}{r}\hat{e}_\phi \partial_\varphi \theta(r,\varphi) = \frac{1}{r}\hat{e}_\phi$

$$\Longrightarrow \; E = \frac{K}{2}\int d^2x (\nabla\theta)^2 = \pi K \int_a^L dr r \frac{1}{r^2} = \pi K \log(L/a)$$

- entropy: sum all equally probable possibilities of placing vortices in 2D plane at minimal distance a:

$$S = -k_B \sum_i p_i \log p_i = k_B \log(L/a)^2$$

- free energy $\quad F = E - TS = (\pi K - 2k_B T)\log(L/a)$

- vortex proliferation above KT critical temperature $\quad T_{KT} = \dfrac{\pi K}{2k_B}$

➡ but out of equilibrium: no free energy at hand!

➡ field theory approach (analogous Kosterlitz' real space RG for vortices)

J. M. Kosterlitz, J. Phys. C (1974)

Field theoretical duality approach: Phase compactness

G. Wachtel, L. Sieberer, SD, E. Altman, PRB (2016)
L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)

- we ignored a fundamental symmetry of polaritons so far

$$\phi(t, \mathbf{x}) = \rho(t, \mathbf{x}) e^{i\theta(t,\mathbf{x})}$$

- phase compactness = local discrete gauge invariance under

$$\theta_{t,\mathbf{x}} \mapsto \theta_{t,\mathbf{x}} + 2\pi n_{t,\mathbf{x}}$$

➡ how to teach to the KPZ equation? lattice regularization and discrete stochastic update

$$\theta_{t+\epsilon,\mathbf{x}} = \theta_{t,\mathbf{x}} + \epsilon\left(\mathcal{L}[\theta]_{t,\mathbf{x}} + \eta_{t,\mathbf{x}}\right) + 2\pi n_{t,\mathbf{x}}$$

lattice regularized deterministic term

$$\boxed{Z = \sum_{\{\tilde{n}_{t,\mathbf{x}}\}} \int \mathcal{D}[\theta] e^{iS[\theta,\tilde{n}]}}$$

stochastic difference equation $\Leftrightarrow$ discrete noise MSR functional integral

non-equilibrium lattice gauge theory (manifestly gauge invariant)

Duality: Emergent non-linear electrodynamics

- non-equilibrium discrete gauge theory
- 2D: dual description in terms of noisy electrodynamics

$$Z = \sum_{\{\tilde{n}_{t,\mathbf{x}}\}} \int \mathcal{D}[\theta] e^{iS[\theta,\tilde{n}]} \quad \Leftrightarrow \quad Z \propto \underbrace{\sum_{\substack{\{n_{vX},\tilde{n}_{vX},\\ \mathbf{J}_{vX},\tilde{\mathbf{J}}_{vX}\}}}}_{\text{charges and currents}} \underbrace{\int \mathcal{D}[\phi,\tilde{\phi},\mathbf{A},\tilde{\mathbf{A}}]}_{\text{electromagnetic potentials}} e^{iS[\phi,\tilde{\phi},\mathbf{A},\tilde{\mathbf{A}},n_v,\tilde{n}_v,\mathbf{J}_v,\tilde{\mathbf{J}}_v]}$$

- Electrodynamic duality:

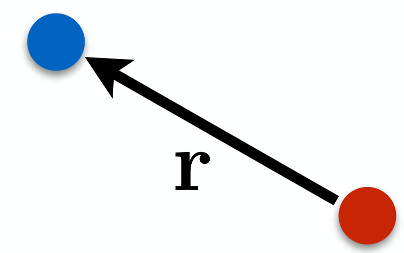
polaritons		"photons"
density fluctuations (gapped)	$\Leftrightarrow$	magnetic field
smooth phase fluctuations		electric field
vortices		charges

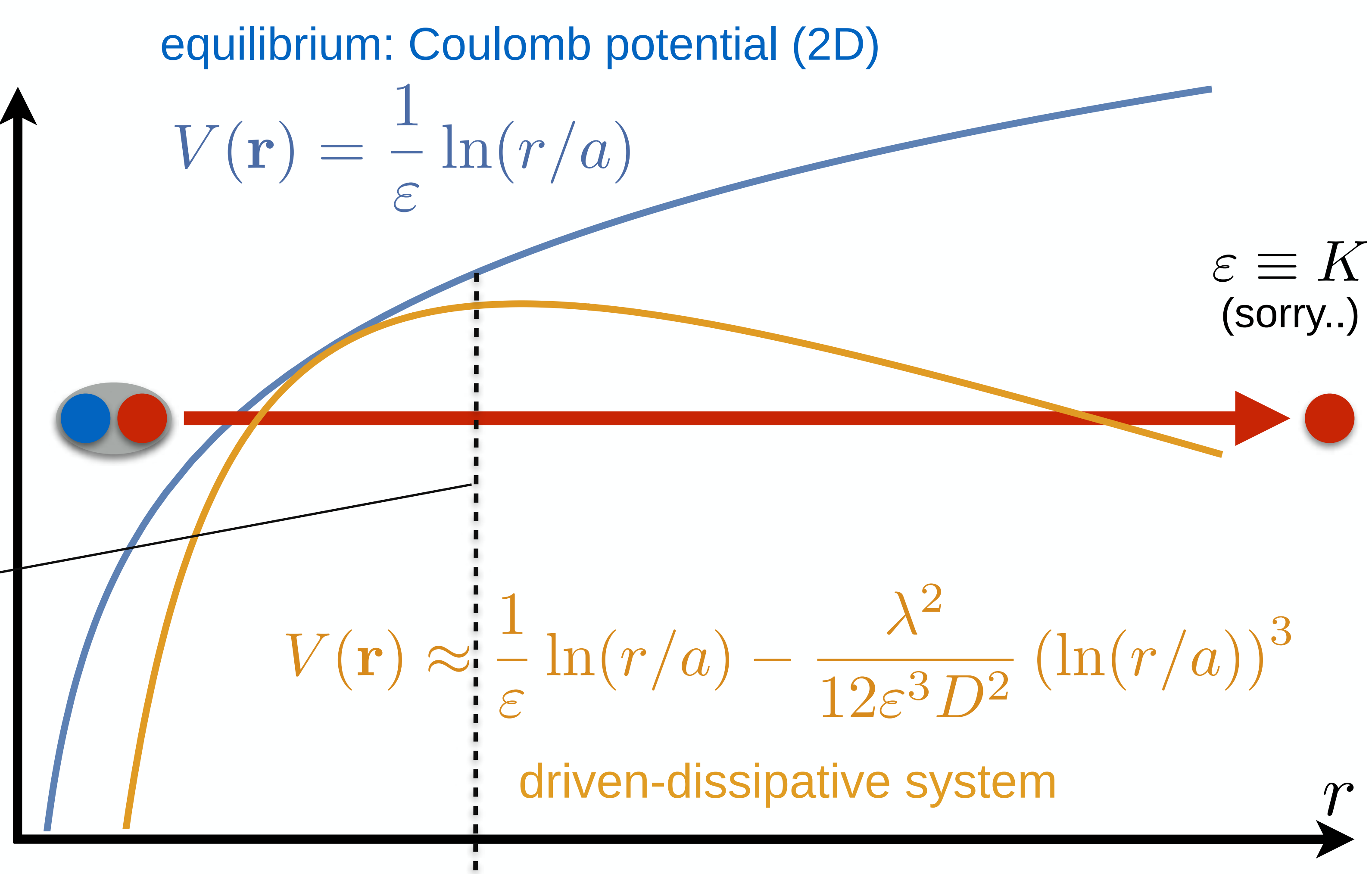
- key feature of non-equilibrium: effective Maxwell equation for electric field non-linear
  - proceed analogously to equilibrium: integrate out smooth fluctuations => effective (static) vortex theory (low mobility limit)

Effective theory for a single vortex-antivortex pair

- equation of motion for a single vortex-antivortex pair

$$\frac{d\mathbf{r}}{dt} = -\mu \nabla V(r) + \boldsymbol{\xi}$$





length scale:

$$L_v = a_0 e^{\frac{2D}{\lambda}}$$

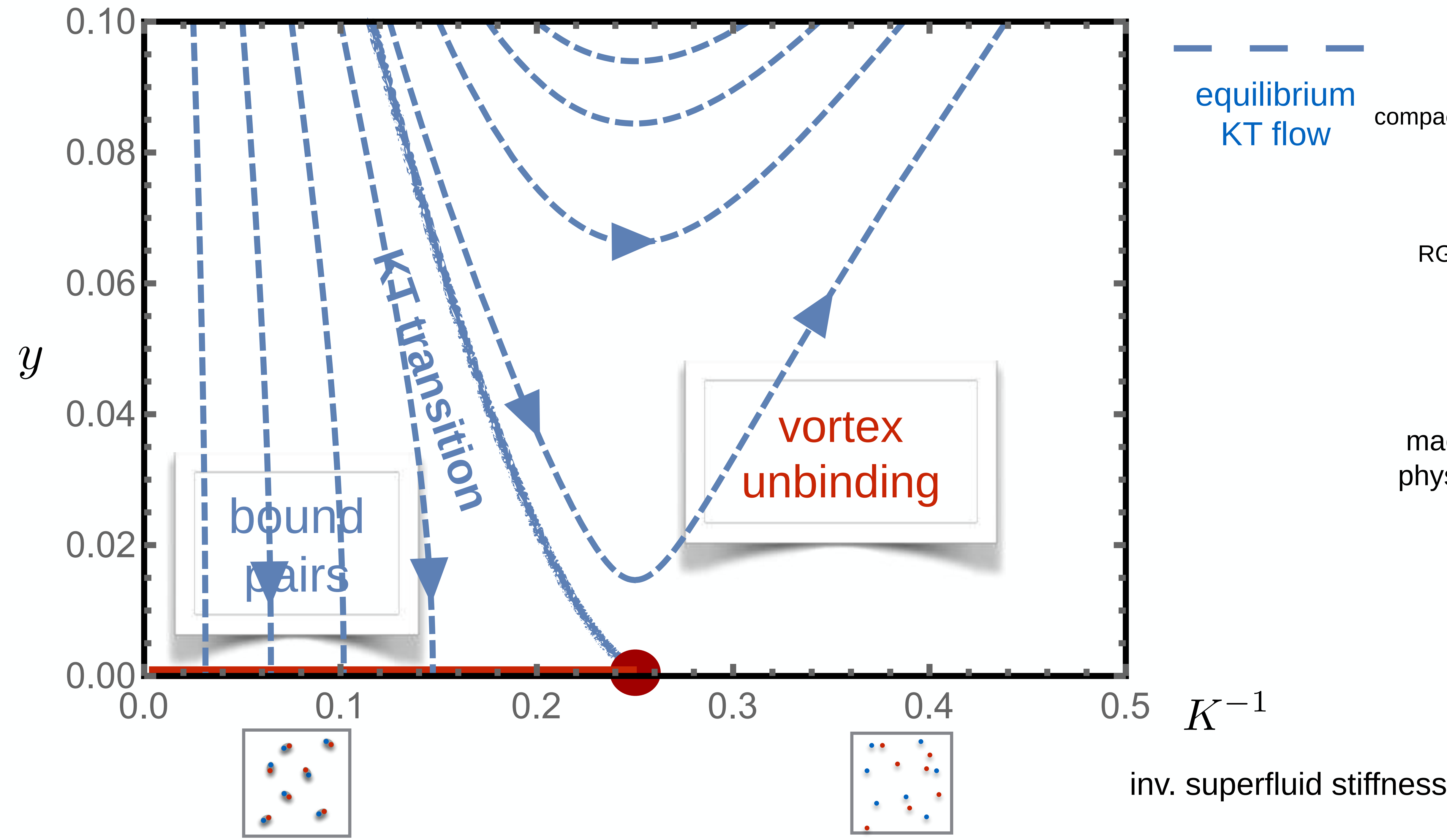
see also: I Aranson et al., PRB (1998) two-vortex problem

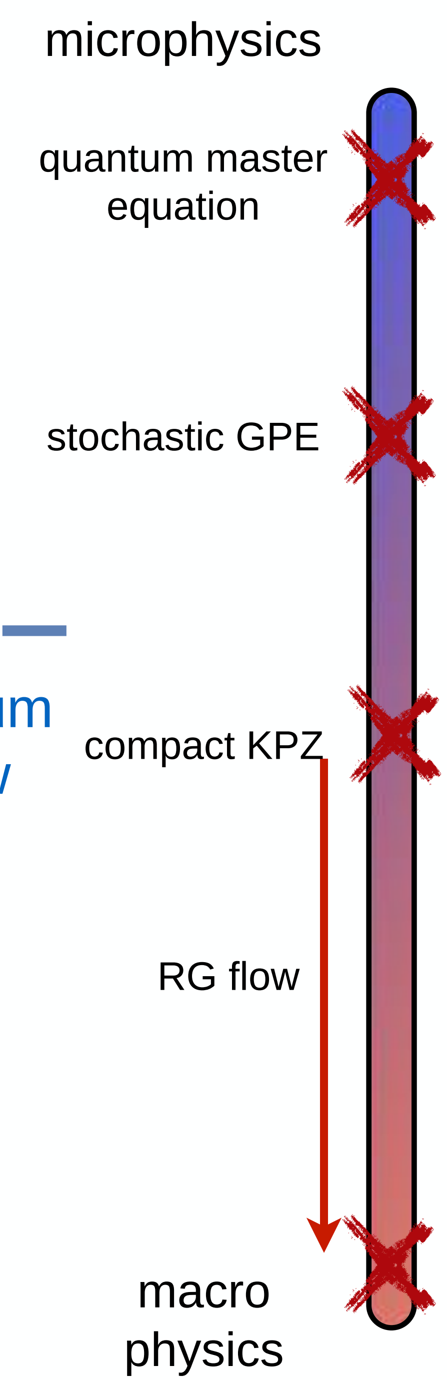
➡ noise-activated unbinding for a single pair (at exp small rate)

Many pairs: Corrections to Kosterlitz-Thouless flow

$$\frac{dK}{d\ell} = \frac{2\pi^2 y^2}{T} \qquad \frac{dy}{d\ell} = \left[2 - \frac{1}{2KT} + \cancel{\frac{\lambda^2}{4K^2 D^2}\left(\frac{1}{4} + \ell\right)}\right] y \qquad \frac{dT}{d\ell} = \cancel{\frac{\lambda^2 T}{2K^2 D^2}}\left(\frac{1}{4} + \ell\right)$$

- parameter y: ~ probability to create single vortex at distance l (flow parameter)





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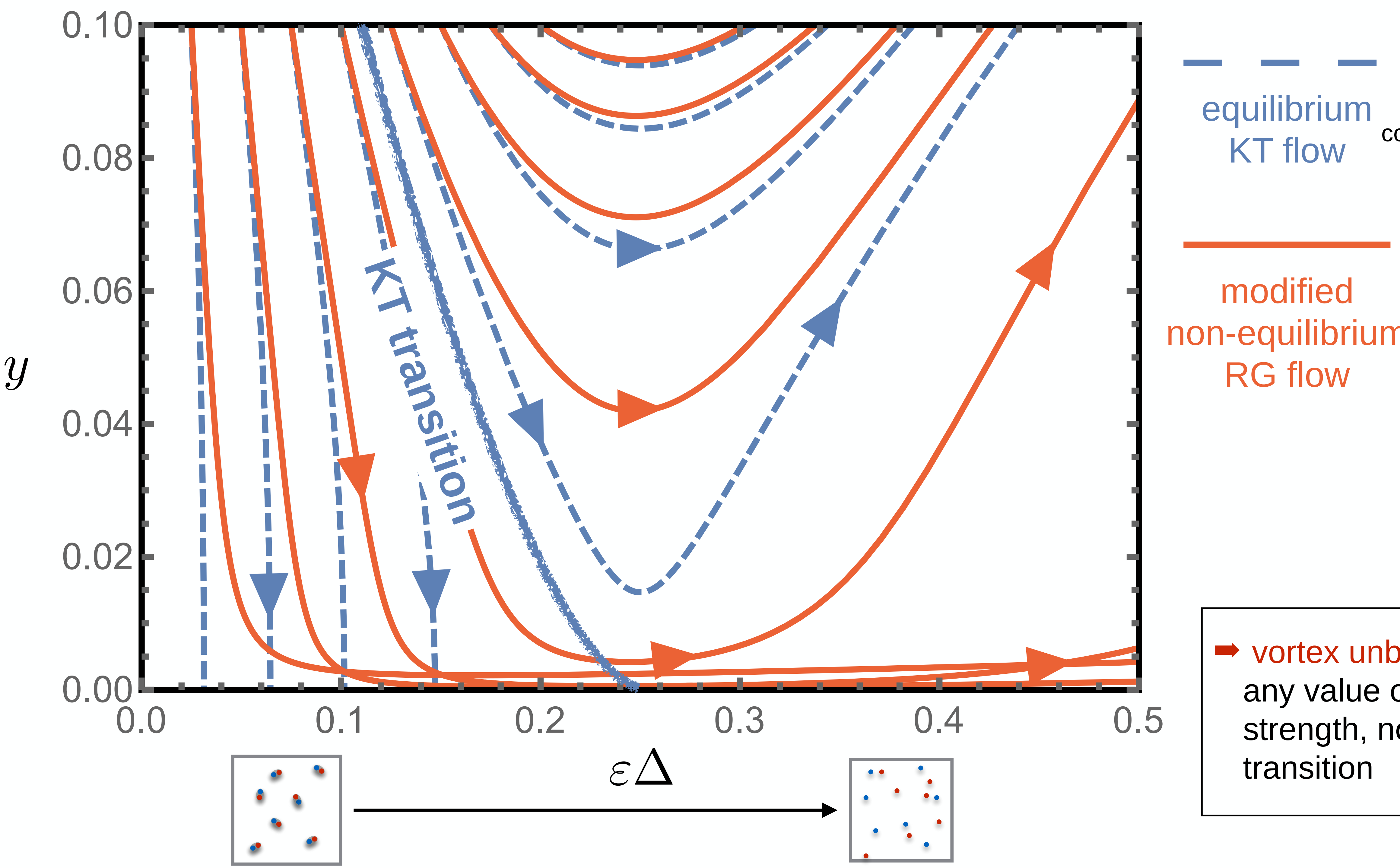
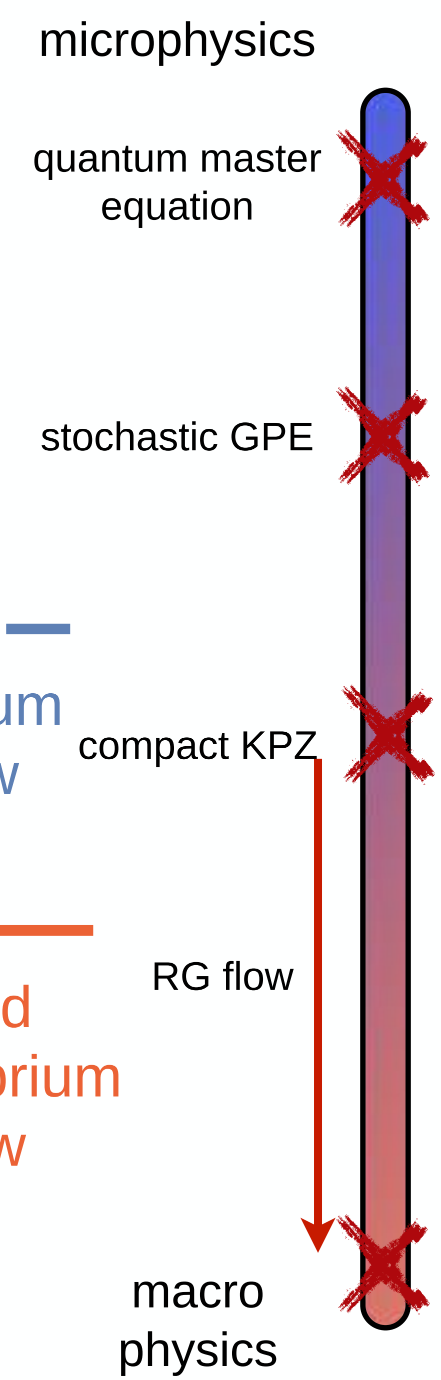




➡ vortex unbinding for any value of the noise strength, no phase transition

L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)
G. Wachtel, L. Sieberer, SD, E. Altman, PRB (2016)

Competing length scales and suppression of KT

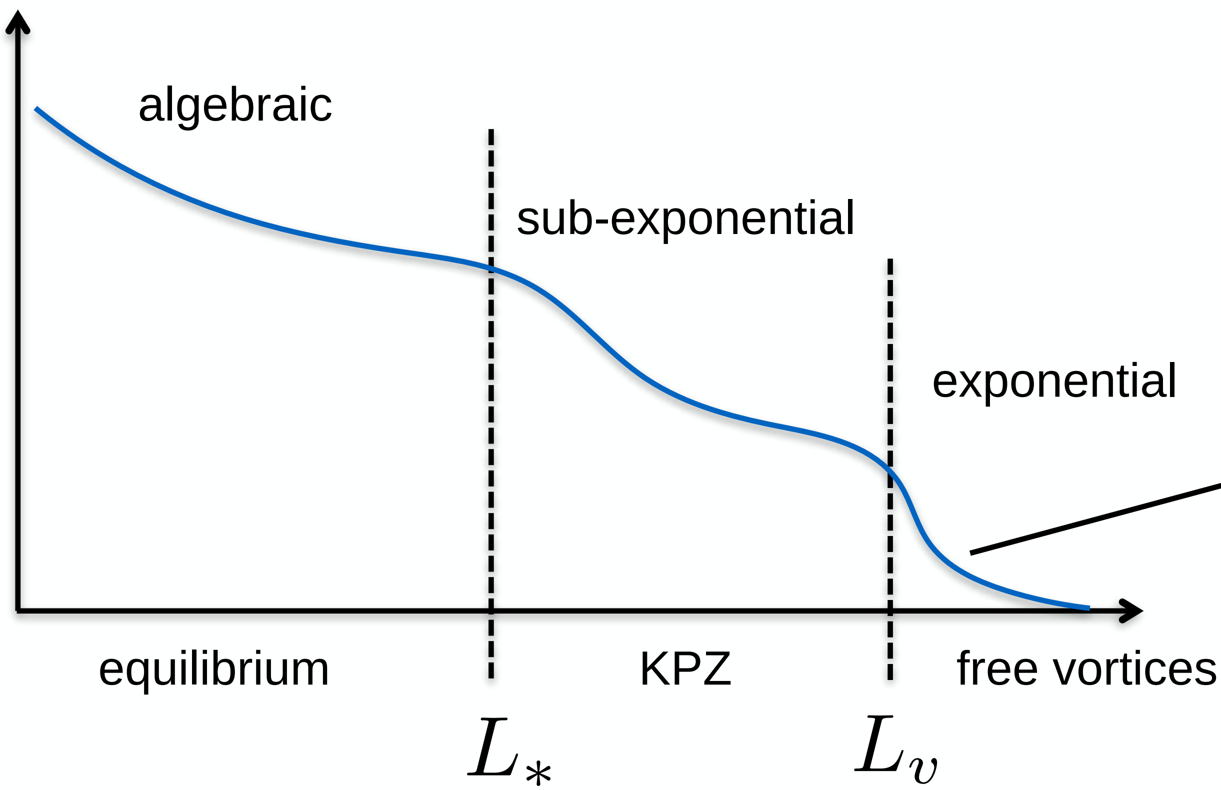
- two emergent length scales in complementary approaches:

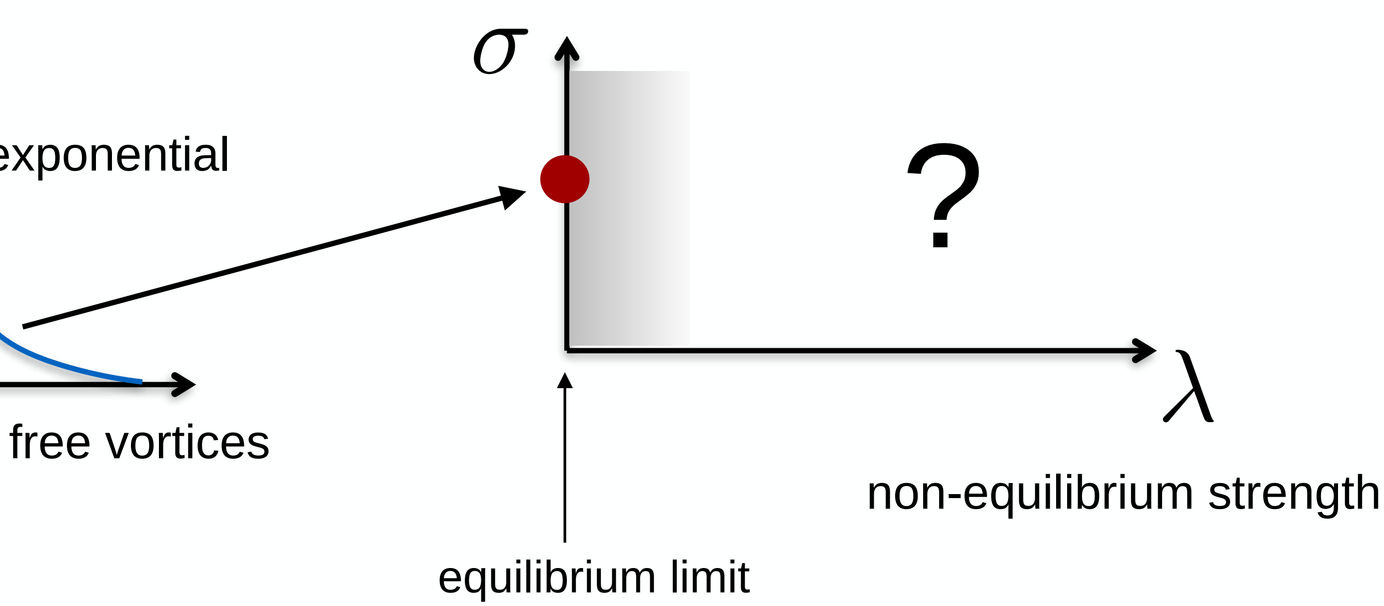
$$L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$$

KPZ length

$$L_v = a_0 e^{\frac{2D}{\lambda}}$$

vortex length

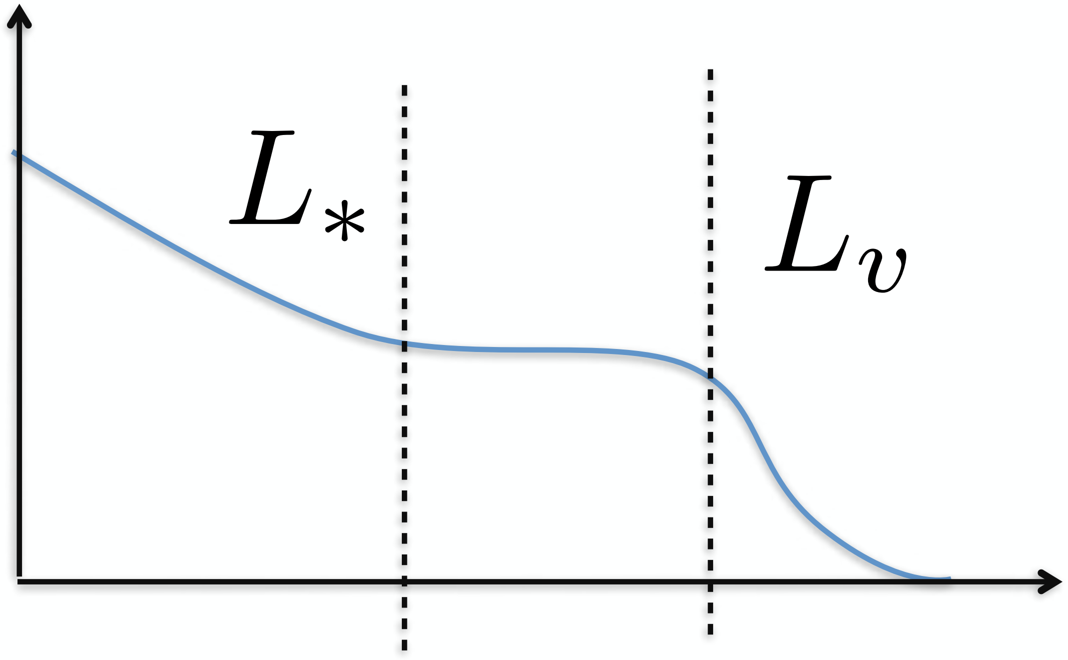




- numerical confirmation of two-scale scenario in 1D (defects: vortices in (1+1)D space-time)
  L. He, L. Sieberer, SD PRL (2017)
- 2D simulations demanding

Kosterlitz-Thouless physics fragile to non-equilibrium perturbation

Case study: 1 Dimension

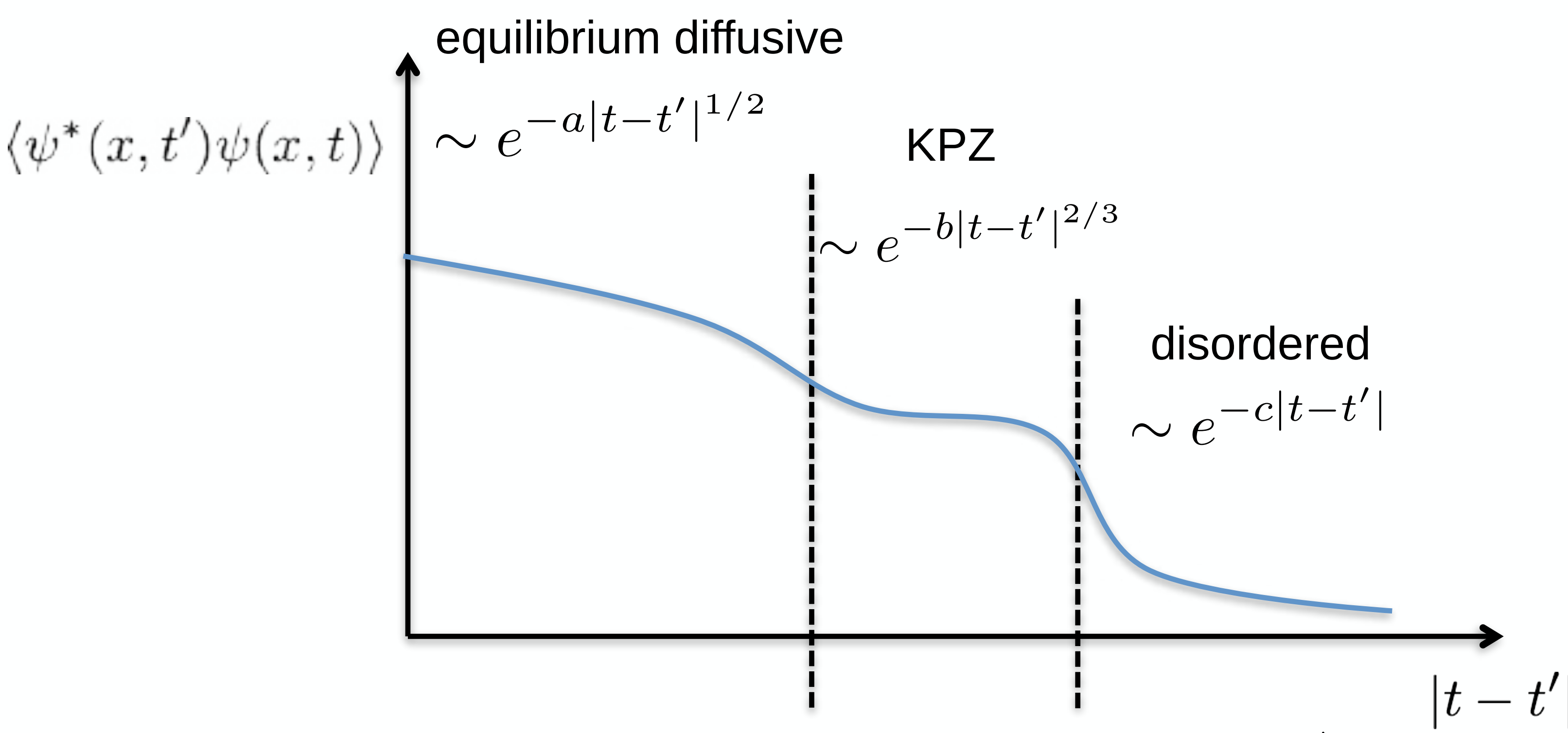


L. He, L. Sieberer, E. Altman, SD, PRB (2015)
L. He, L. Sieberer, SD, PRL (2017)

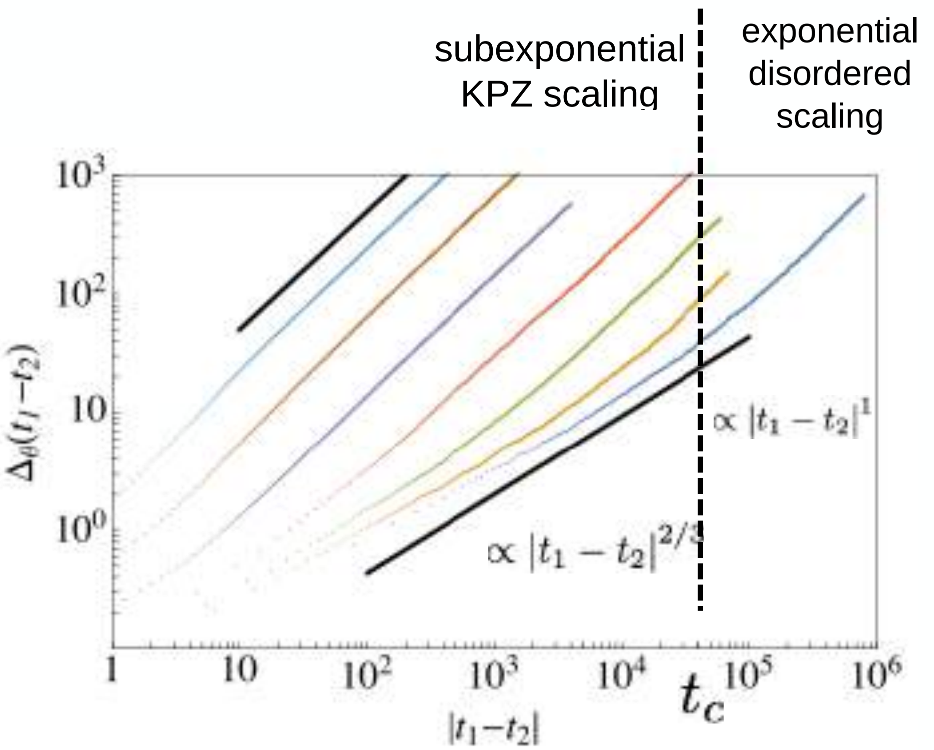
L. He, L. Sieberer, E. Altman, SD, PRB (2015)
see also: K. Yi, V. Gladilin, M. Wouters, PRB (2015)
L. He, L. Sieberer, SD, PRL (2017)

Sequence of Scales

- direct numerical solution of driven-dissipative GPE in one dimension
- Study temporal instead of spatial coherence function



numerical evidence



- Crossover scale $T_* \sim \sigma^{-2}$ $T_v \sim e^{E_c/\sigma}$

noise level

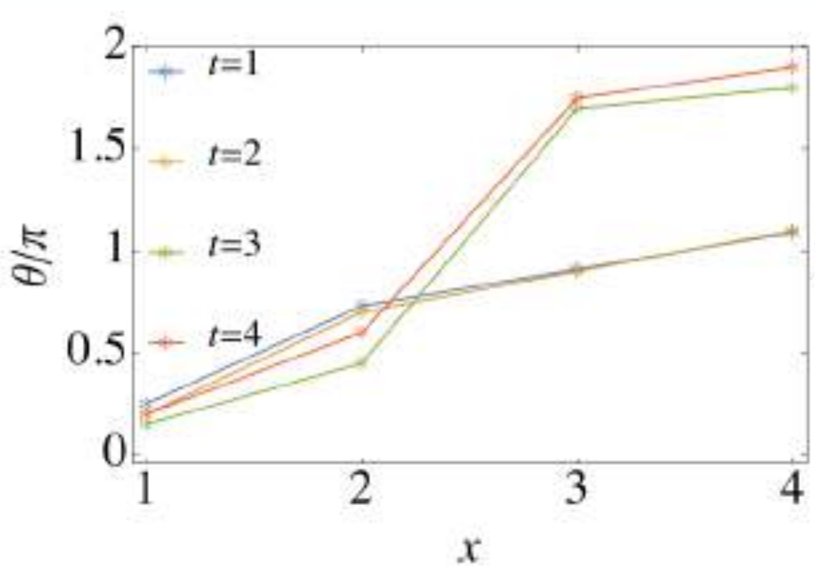
algebraic exponential

➡ no spatial vortices in 1D —> what causes the emergent length scale beyond KPZ?

Space-time vortices in 1D XP condensate

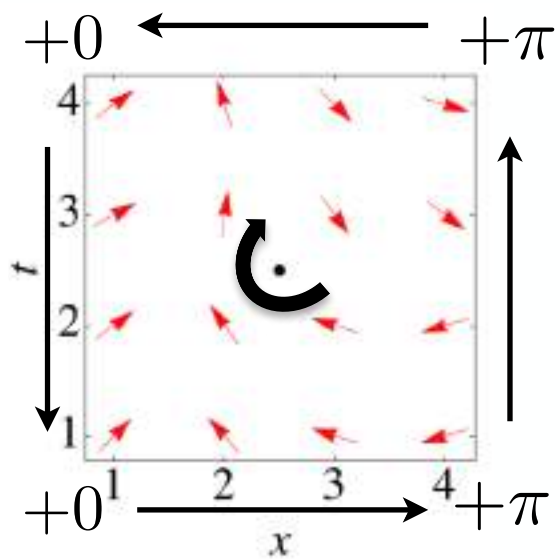
- Physical origin: compactness of phase field

spatial phase slip



$\Leftrightarrow$

vortex in space-time plane



round trip
$\Delta\varphi = 2\pi$

topologically nontrivial phase field configurations on (1+1)D space-time plane

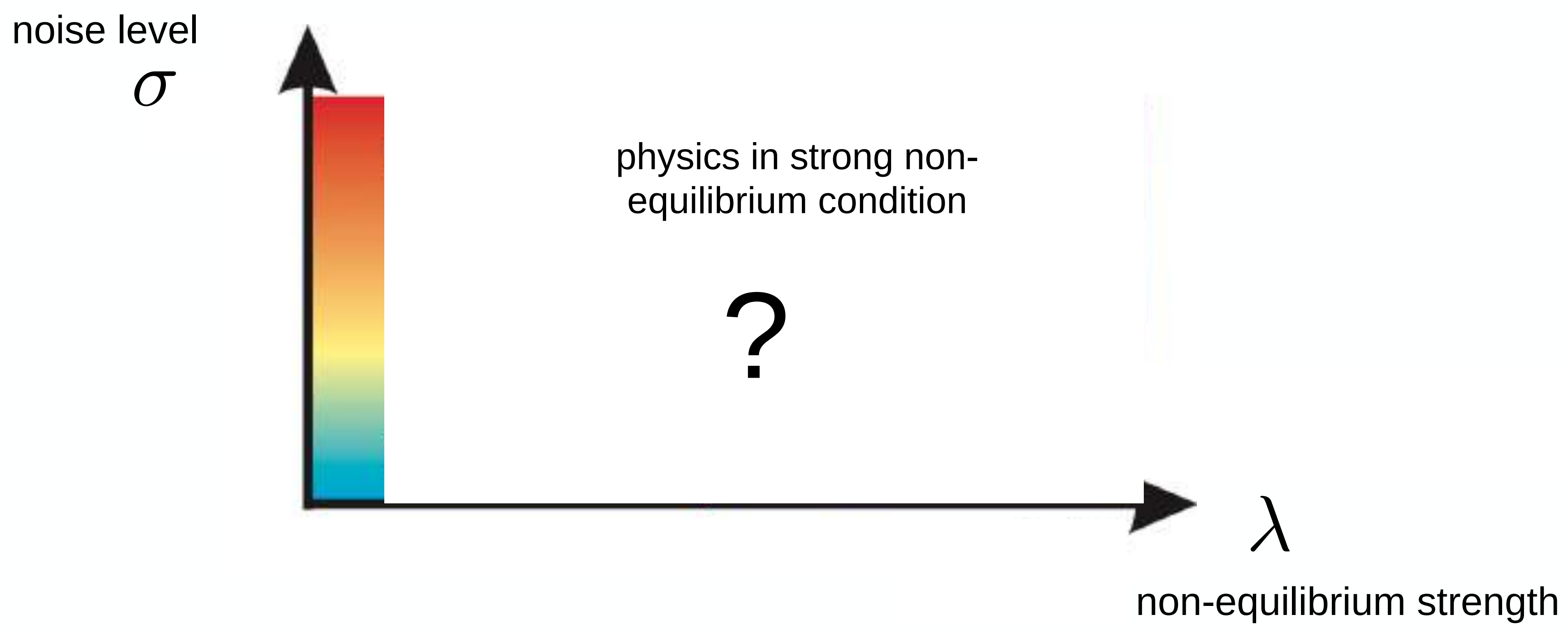
- unbound at infinitesimal noise level (weak non-equilibrium)
  - interaction potential: $(\partial_t + D\partial_x^2)^{-1} \sim (Dt)^{-1/2} e^{-x^2/(4Dt)}$ cf. 2D static equilibrium: $\nabla^{-2} \sim \log(|\mathbf{x}|)$

- explains qualitative features
  1. temporal scaling (random uncorrelated charges) $\langle \psi^*(x,t')\psi(x,t)\rangle \sim e^{-c|t-t'|}$
  2. noise level dependence of crossover scale $T_v \sim e^{E_c/\sigma}$ (mapping to static 2D active smectic A liquid crystal)

Toner and Nelson, PRB (1984)

Strong non-equilibrium: Compact KPZ vortex turbulence

- In search of the phase diagram for XP condensates



Strong non-equilibrium: Compact KPZ vortex turbulence

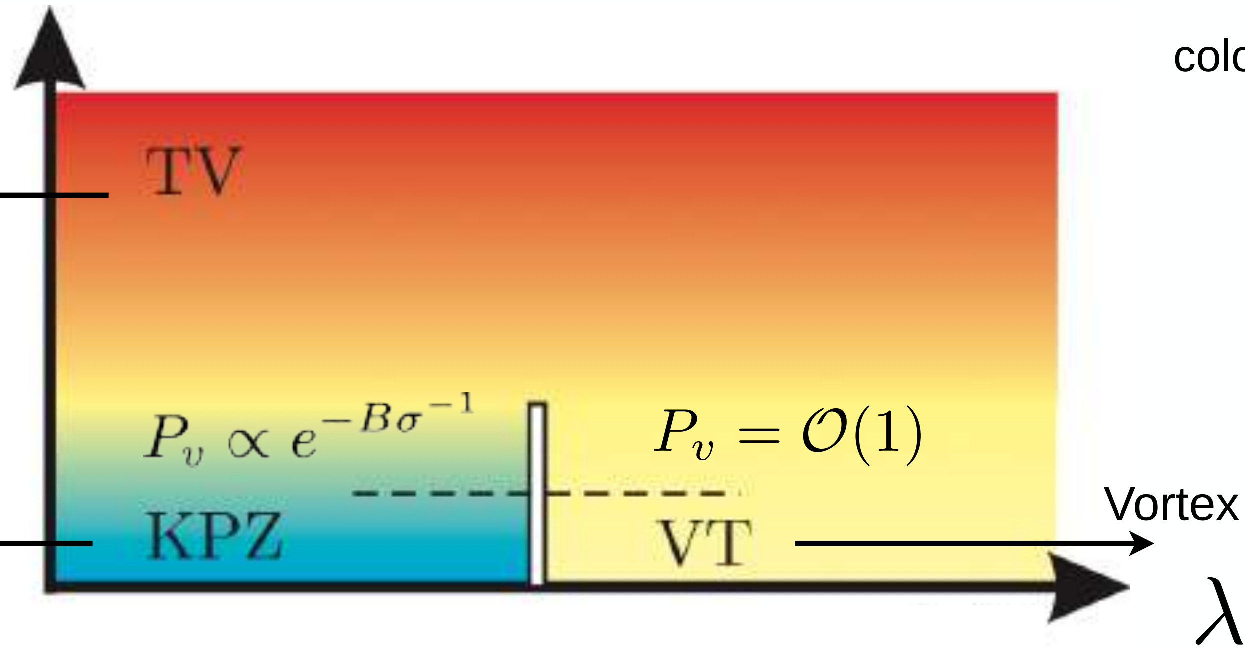
- In search of the phase diagram for XP condensates: 1+1 dimensions

L. He, L. Sieberer, SD PRL (2017)

noise level
σ

Noise activated vortices (TV)

KPZ / equilibrium dominated physics



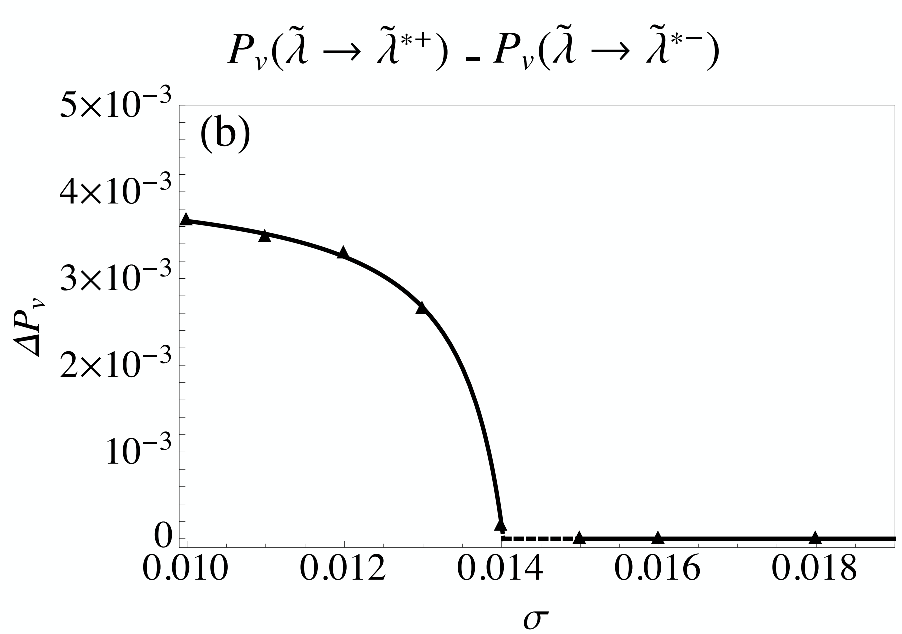
color code: vortex density on space-time plane

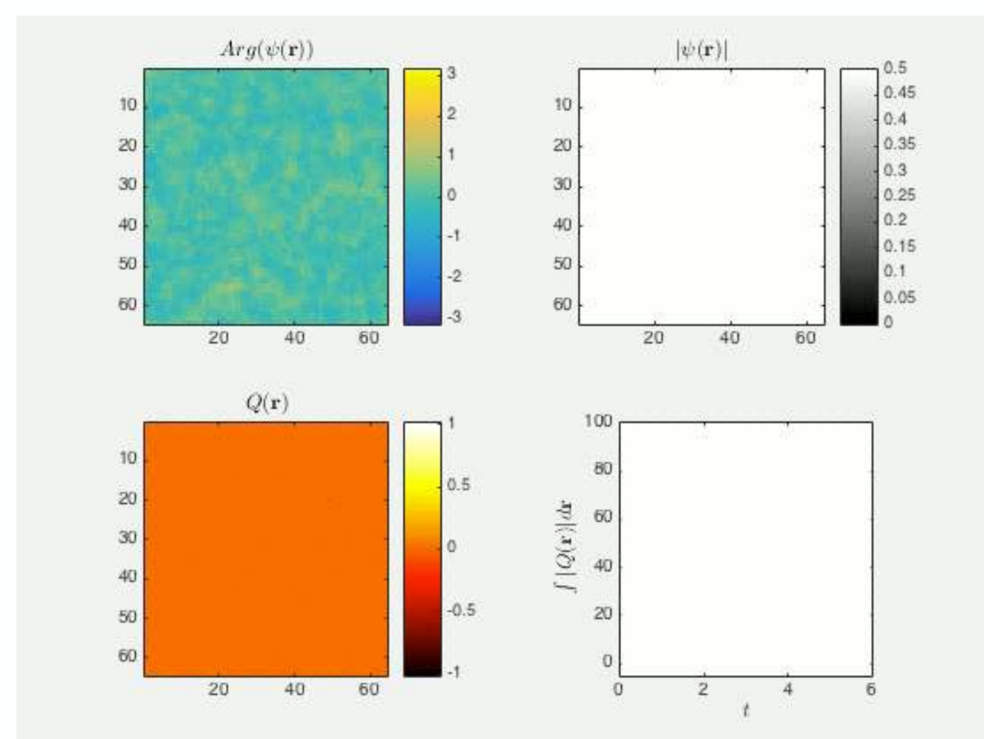
P_v

Vortex turbulence (VT)

first order non-equilibrium phase transition

non-equilibrium strength



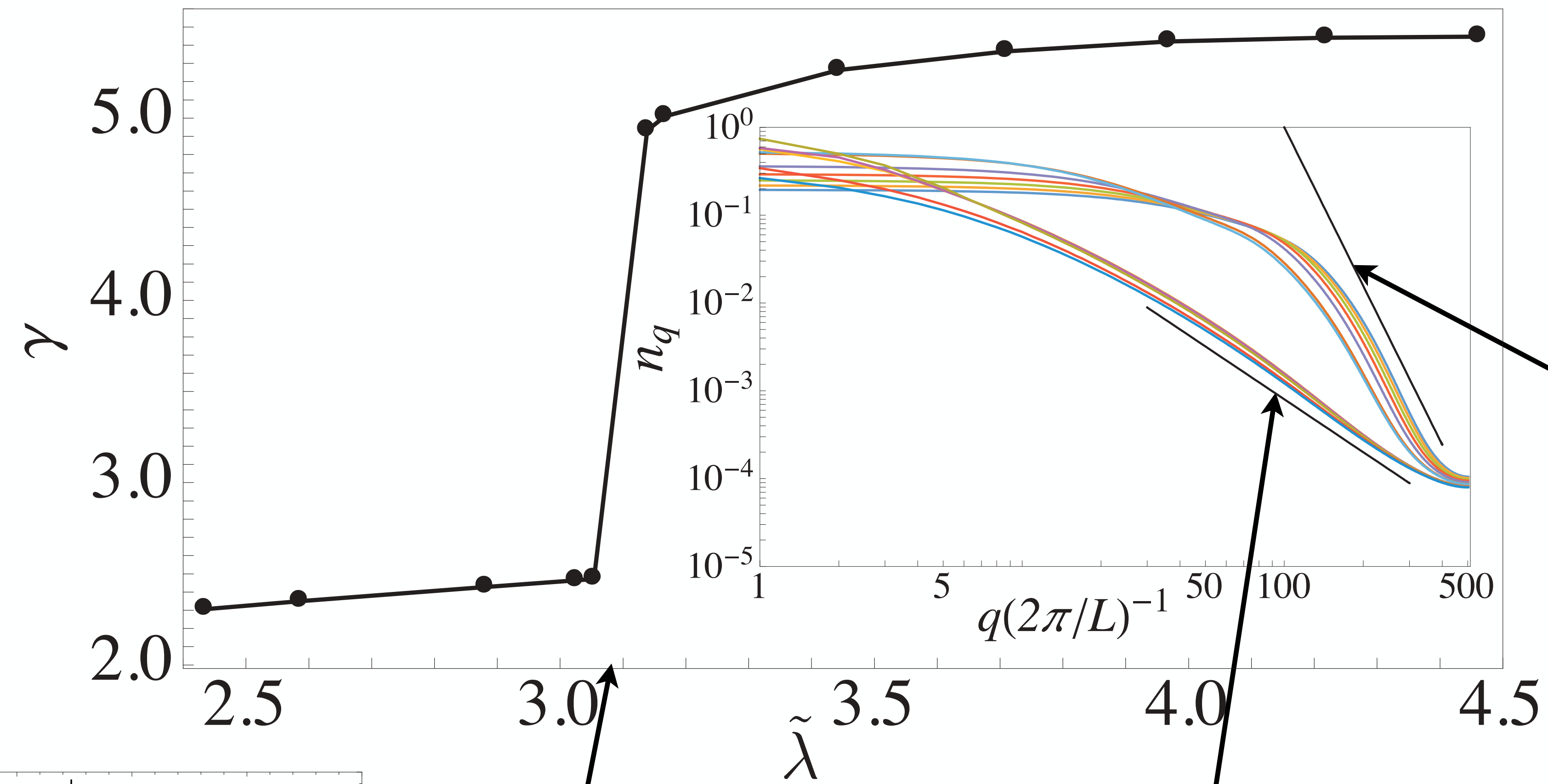


➡ deterministic limit: how does the system generate its own noise?

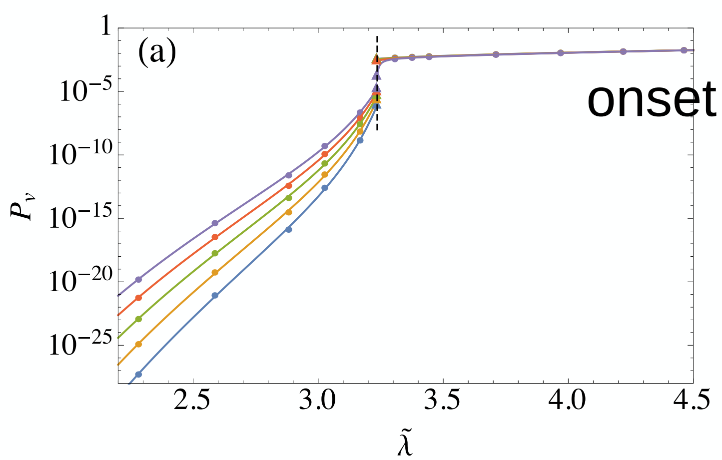
Compact KPZ vortex turbulence: Signatures

- scaling of the momentum distribution at intermediate momenta (full stochastic GPE)

$$n_q = \langle \psi^*(q)\psi(q)\rangle \sim q^{-\gamma}$$



scaling due to turbulent vortices: $\gamma \approx 5$



onset of vortex turbulence

scaling due to thermally activated vortices: $\gamma \approx 2$

diffusion constant

coherent propagation, inverse effective polariton mass

- experiments: vortex turbulence favored in systems with strong diffusion, $\lambda \sim K_d / K_c$
- flat band of 1D Lieb lattice realized with micropillar cavity arrays F. Baboux et al. PRL (2016)

Phase transition driven by non-equilibrium strength

- qualitative reason: competition in lattice regularized model (compact KPZ)

$$\partial_t \vec{\chi} = \underbrace{S\vec{V}_E[\vec{\chi}]}_{\text{equilibrium}} + \underbrace{A\vec{V}_N[\vec{\chi}]}_{\text{non-equilibrium}} + \vec{\xi}$$

lattice site

$$\chi_i = \nabla\theta_i = \theta_{i+1} - \theta_i$$

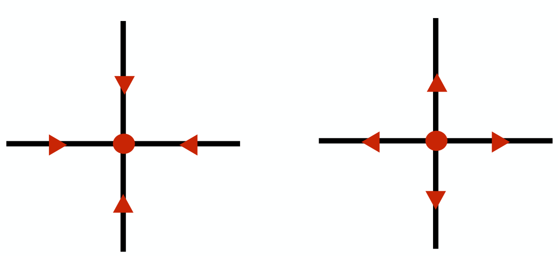
$$V_{E/N,i} = \frac{\delta}{\delta\chi_i}\mathcal{V}_{E/N}$$

$$\mathcal{V}_E = K\sum_j \cos\chi_j \qquad \mathcal{V}_N = \lambda\sum_j \sin\chi_j$$

$$D_{ij} = \delta_{ij+1} - \delta_{ij}$$

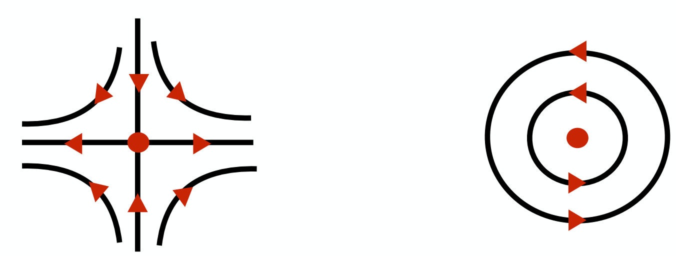
$$S = D + D^T = S^T$$

fixed point structures



attractive repulsive

$$A = D - D^T = -A^T$$



hyperbolic elliptic

=> emergent Liouville thm.

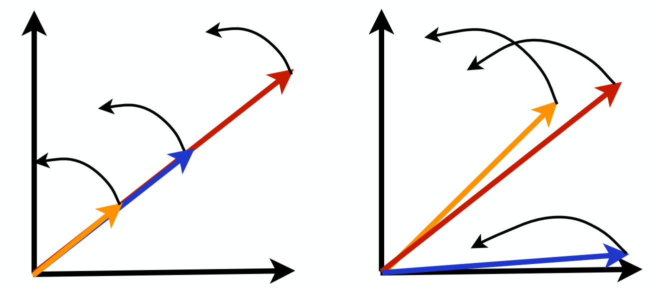
$$\mathrm{div}[A\vec{V}_N] = 0$$

➡ high dimensional configuration space

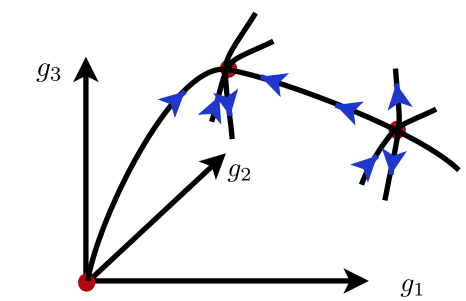
➡ transition to chaos

Further instances: Universality in driven open quantum systems

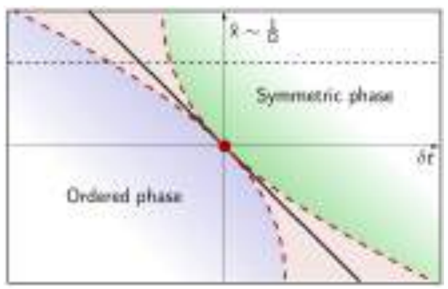
- criticality in non-equilibrium ϕ^4 model — Sieberer et al., PRL (2013)
  - ➡ dynamical fine-structure distinguishing eq. from non-eq.



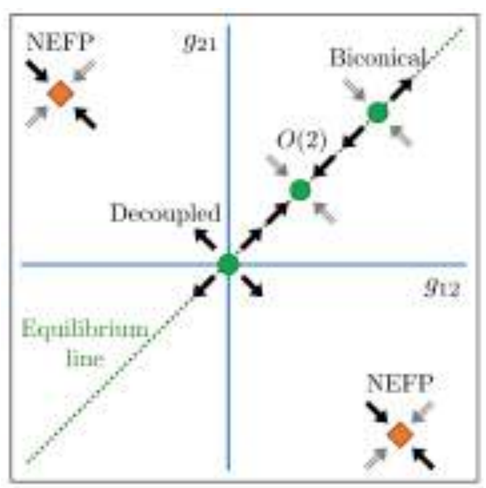
- Markovian quantum criticality — Marino, SD, PRL (2016)
  - ➡ new fixed point in dark state models with quantum scaling



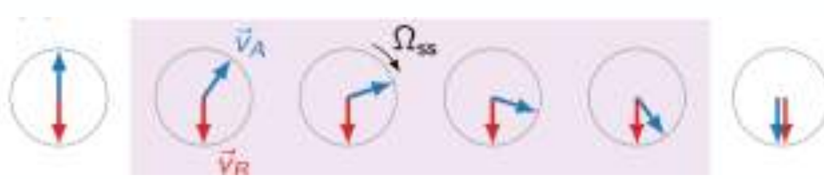
- phase transitions in open Floquet systems — Mathey, SD, PRL (2019)
  - ➡ absence of criticality (dual to Kibble-Zurek mechanism)



- coupled Ising models — Young et al., PRX (2020)
  - ➡ new fixed point for strong non-equilibrium drive



- phase transitions with exceptional points — Fruchart et al., Nature (2021)
  - ➡ universality yet to be discovered!



Summary lecture I & II

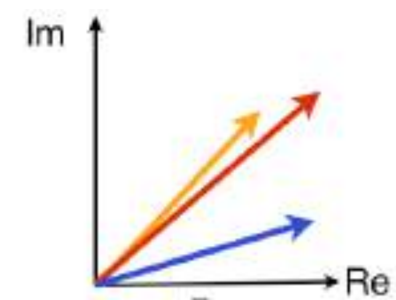
Review: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016)

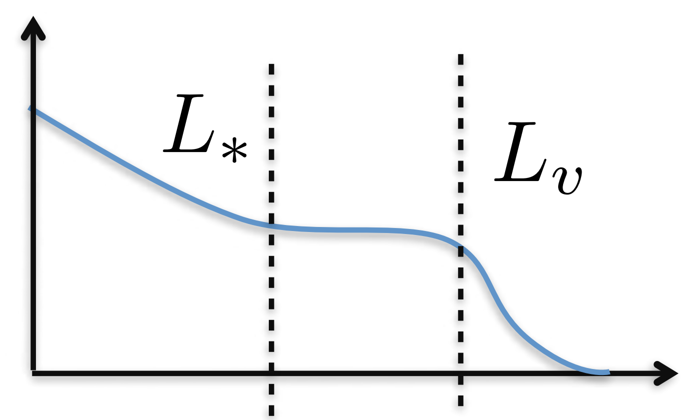
Driven open many-body systems: challenge to theory

microphysics → macrophysics

Many-Body Master Equation → 1-1 mapping → Keldysh functional integral

- mapping opens up QFT toolbox, today:
  - symmetries: eq. vs. non-eq.
  - control of IR fluctuations: understanding low dimensional gapless phases out of equilibrium
  - flexible choice of degrees of freedom: KPZ vs. vortices
- mapping opens up QFT toolbox, next lecture:
  - symmetries: ‘weak’ and ‘strong’
  - responses out of equilibrium: topological gauge theory





2D: $L_v \ll L_*$ 1D: $L_v \gg L_*$
(time scales)

School on Emergent Phenomena in Non-Equilibrium Many-Body Physics
July 03-05, 2023
Sao Paulo, Brasil

Lecture III

Lindblad-Keldysh 2.0: Measurement Induced Phase Transitions

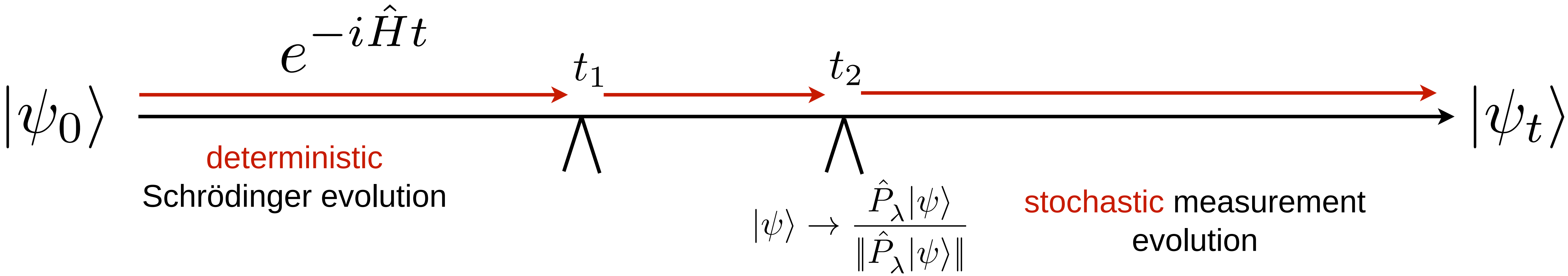
Sebastian Diehl

Institute for Theoretical Physics, University of Cologne

Introduction

Small quantum systems: Measurements

- two types of quantum dynamics



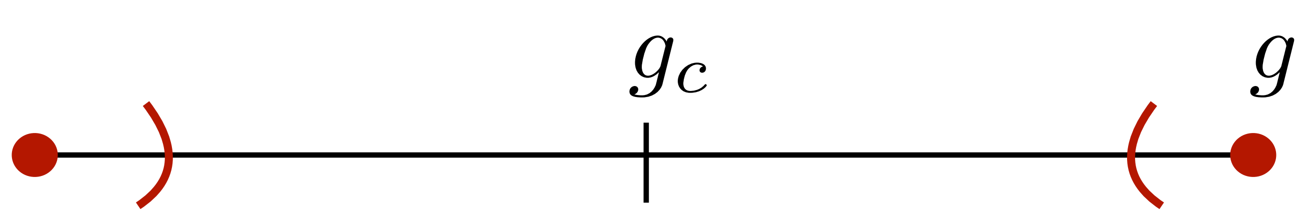
for measurement observable $\hat{M} = \sum_\lambda m_\lambda |\lambda\rangle\langle\lambda| \equiv \sum_\lambda m_\lambda \hat{P}_\lambda$

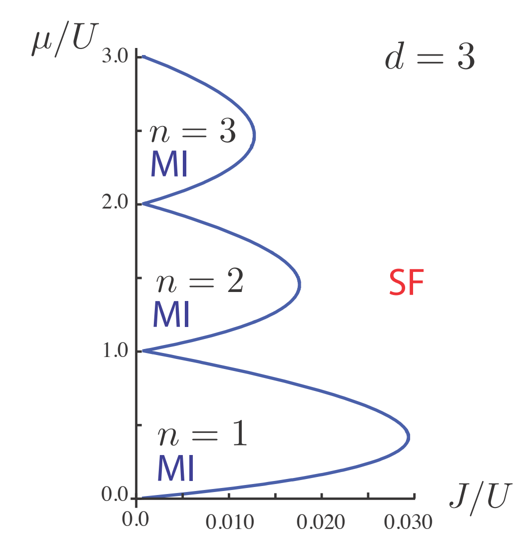
- dynamics non-trivial (eigenstates not shared) once $[\hat{H}, \hat{M}] \neq 0$

Many-body systems: Phase transitions

- non-commuting operators lead to (quantum) phase transitions

$$\hat{H} = \hat{H}_1 + g\hat{H}_2 \qquad [\hat{H}_1, \hat{H}_2] \neq 0$$



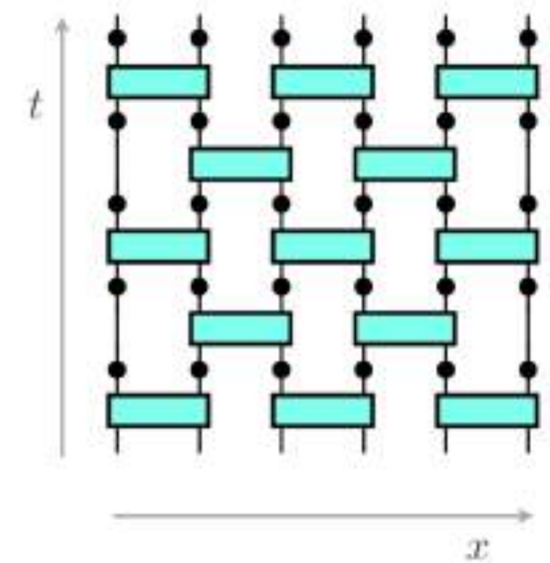


e.g. Mott-insulator to superfluid transition in cold atoms

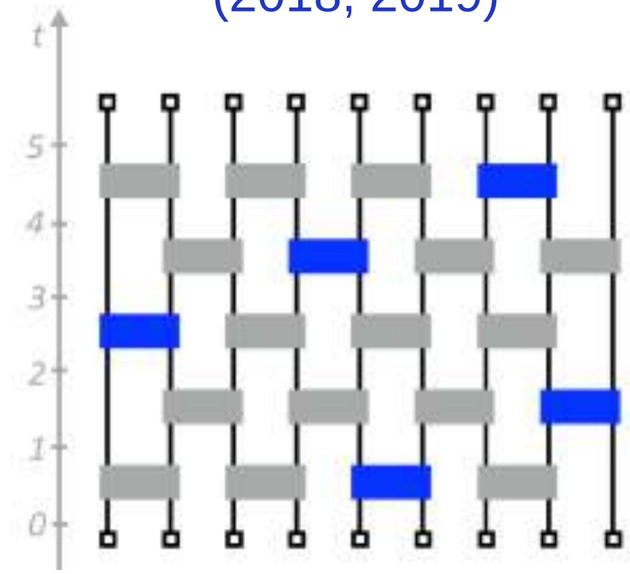
➡ combine measurement and many particles: similar scenario?

Entanglement Phase Transitions in Random Circuits

Skinner, Ruhman, Nahum PRX (2019)



Li, Chen, Fisher, PRB (2018, 2019)



- model and key ingredients:
  - randomly chosen local entangling unitary gates
  - projective local measurement of non-commuting observables

- basic picture: competition in many-body context (single trajectory, exactly local meas. of σ_i^z)

$g = 0$ $\qquad\qquad\qquad$ $g^{-1} = 0$ $\qquad g = \dfrac{\text{\# measurements/time}}{\text{\# unitaries/time}}$

chaotic / non-integrable dynamics

➡ entanglement growth

convergence to product state $|\{\sigma_i\}\rangle = \prod_i^{\sigma_i = \uparrow,\downarrow} |\sigma_i\rangle$

➡ entanglement saturation

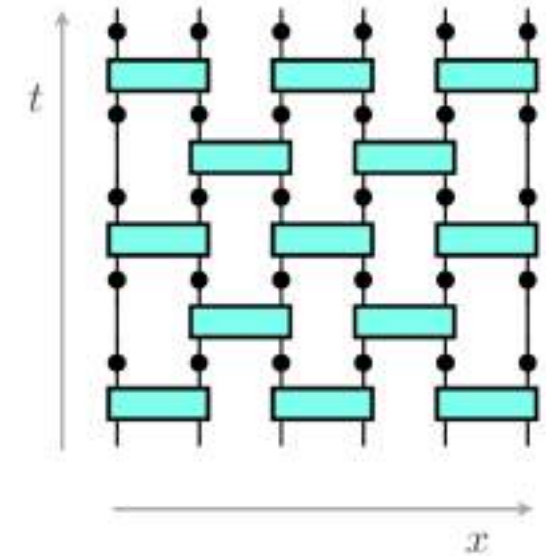
- Procedure
  - track single quantum trajectories (pure states)
  - compute the quantity of interest (e.g. entanglement entropy)
  - average over trajectory ensemble

non-commuting

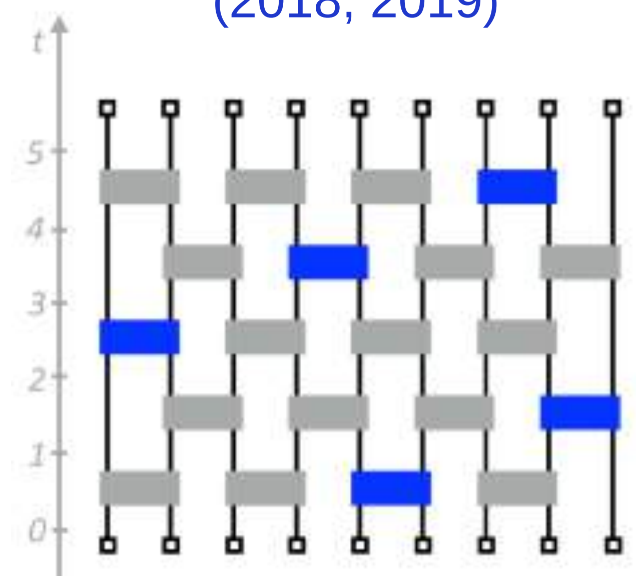
$\overline{|\{\sigma_i\}\rangle\langle|\{\sigma_i\}|} \sim \mathbf{1}$

Entanglement Phase Transitions in Random Circuits

Skinner, Ruhman, Nahum PRX (2019)



Li, Chen, Fisher, PRB (2018, 2019)



- model and key ingredients:
  - randomly chosen local entangling unitary gates
  - projective local measurement of non-commuting observables

- basic picture: competition in many-body context (single trajectory, exactly local meas. of σ_i^z)

$g = 0$ — g_c — $g^{-1} = 0$

$$g = \frac{\text{\# measurements/time}}{\text{\# unitaries/time}}$$

$S \sim L$ | $S \sim \text{const.}$

chaotic / non-integrable dynamics

➡ volume law

convergence to product state $|\{\sigma_i\}\rangle = \prod_i |\sigma_i\rangle$ $\quad \sigma_i = \uparrow, \downarrow$

➡ area law

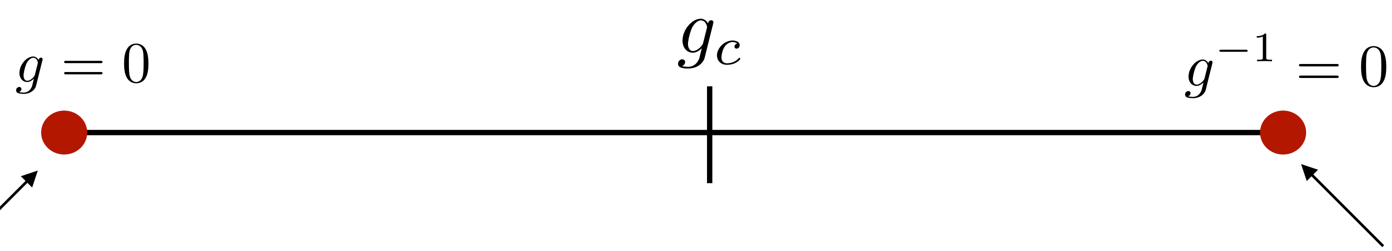
- Procedure
  - track single quantum trajectories (pure states)
  - compute the quantity of interest (e.g. entanglement entropy)
  - average over trajectory ensemble

non-commuting

$$\overline{|\{\sigma_i\}\rangle\langle|\{\sigma_i\}|} \sim \mathbf{1}$$

➡ Phase transition in entanglement growth at finite competition ratio g

Entanglement Phase Transitions: Physical pictures



$$g = \frac{\#\text{ measurements/time}}{\#\text{ unitaries/time}}$$

chaotic / non-integrable dynamics

convergence to product state

non-commuting unitary and measurement dynamics

- entanglement picture

  Skinner, Ruhman, Nahum PRX (2019) Li, Chen, Fisher, PRB (2018, 2019)

  scrambling -> extensive entanglement entropy

  disentangling evolution

- quantum error correction picture

  Choi, Bao, Qi, Altman, PRL (2020); Fan et al. arxiv (2020); Li Fisher PRB (2021)

  fast information spreading protects from errors (read-out by measurement)

  measurement as errors (extract information)

- purification picture

  Gullans, Huse, PRX (2020); Gopalakrishnan, Gullans arxiv (2020)

  initially mixed state remains mixed (thermal)

  initially mixed state purifies to product state

- statistical mechanics picture

  e.g. mapping to spin model partition function: Jian, You, Vasseur Ludwig, PRB (2020)

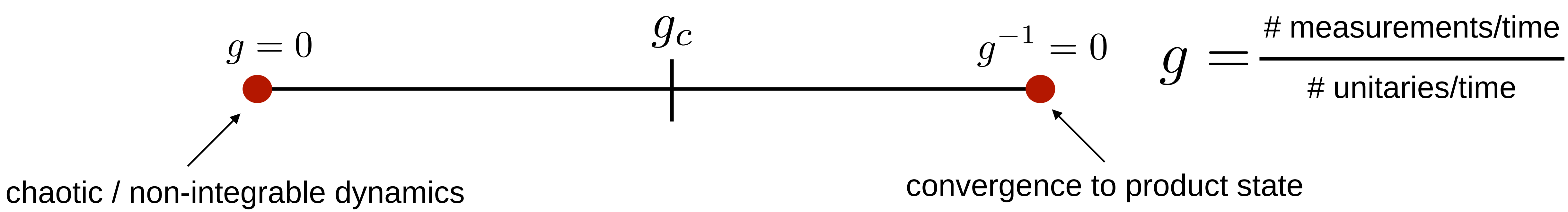
  field theory based on replica symmetry: Nahum, Roy, Skinner, Ruhman, PRX Quantum (2021)

  long ranged correlation functions? which ones?

  short ranged correlation functions? which ones?

➡ Here: non-equilibrium statistical mechanics approach to a monitored fermion chain

Outline Lecture III

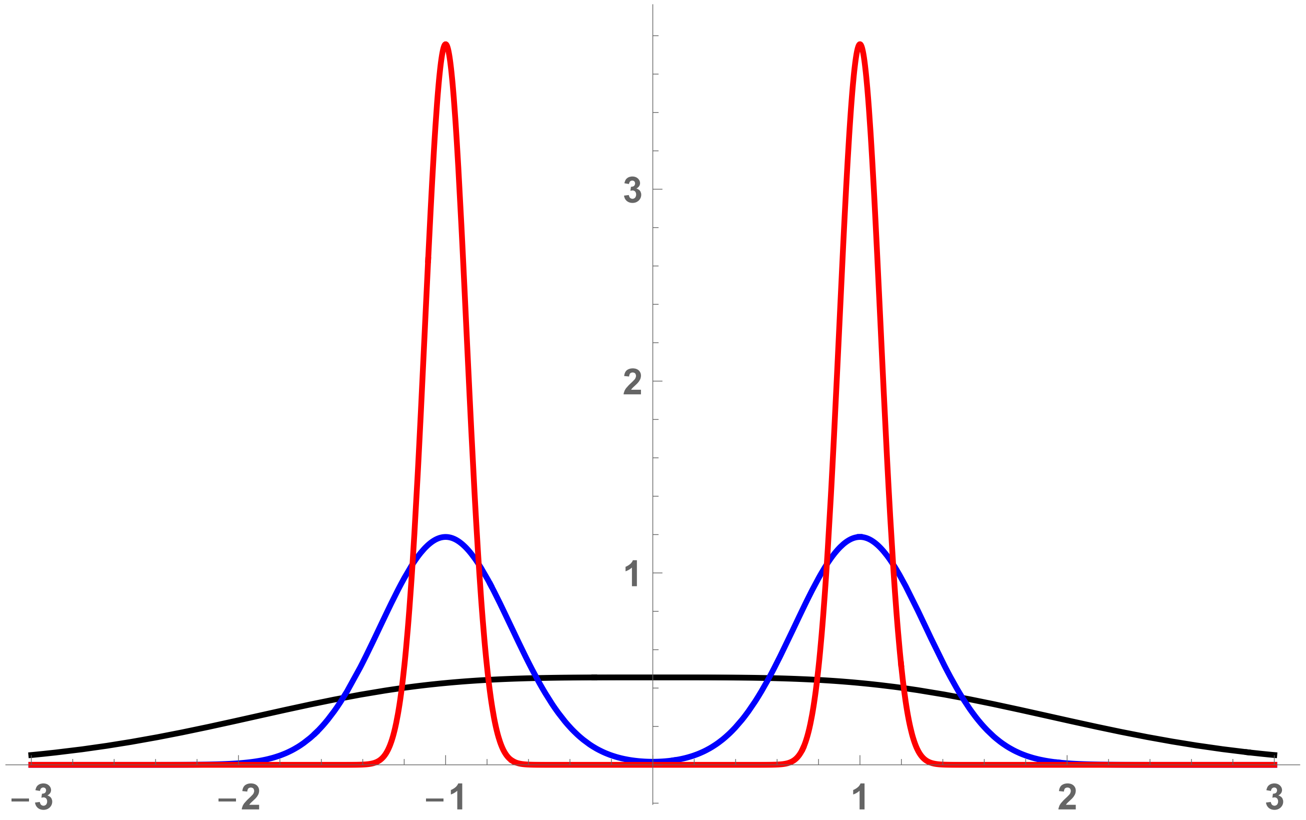


non-commuting unitary and measurement dynamics

- measurements: description?
  - general formulation
  - strong projective vs. weak continuous measurements
  - ‘observables’
- many-body problem: phase transitions?
  - continuously monitored lattice fermions
  - BKT type phase transition from critical to area law phase
  - signatures beyond entanglement entropy
- how to understand?
  - Lindblad-Keldysh 2.0:
  - replicated Lindblad equation
  - replicated Keldyhs field theory

➡ Quantum phase transition in trajectory wavefunction, revealed in non-linear-in-state observables

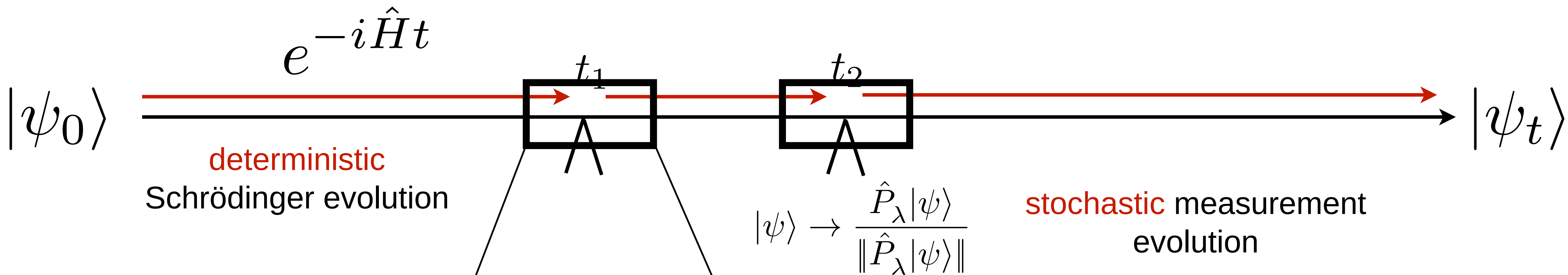
Theory Background:
Strong projective vs. weak continuous measurements



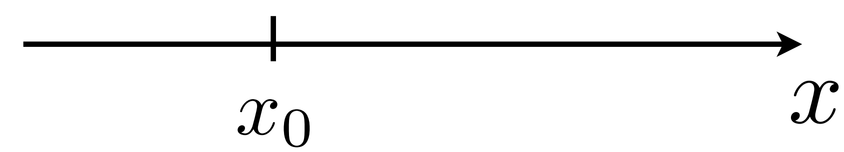
Projective vs. Weak Measurements

Review: Jacobs, Steck, Contemp. Phys. (2006)

- projective measurement: acquire full knowledge about observable



- example: projective position measurement $\lambda = x_0$



$$\hat{P}(x_0) = |x_0\rangle\langle x_0| = \int dx\, \delta(x - x_0)\, |x\rangle\langle x|$$

➡ know post-measurement position x_0 with certainty

- instantaneous collapse of wave function, but real measurements take time: continuous observation

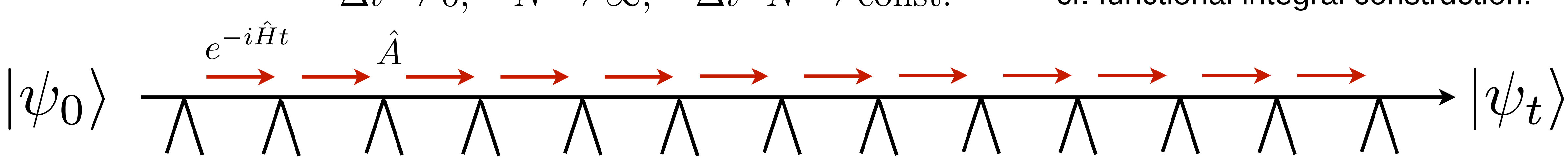
$$\hat{P}(x_0) \to \hat{A}(x_0) = \left(\tfrac{2\gamma\Delta t}{\pi}\right)^{1/4} \int dx\, e^{-\gamma\Delta t(x-x_0)^2}\, |x\rangle\langle x| \xrightarrow{\Delta t \to 0} \left(\tfrac{2\gamma\Delta t}{\pi}\right)^{1/4} e^{-\gamma\Delta t(\hat{x}-x_0)^2}$$

➡ strong projective measurement obtains for $\Delta t \to \infty$

➡ weak continuous measurement obtains for small Δt, but large number of repetitions N:

$$\Delta t \to 0, \quad N \to \infty, \quad \Delta t \cdot N \to \text{const.}$$

cf. functional integral construction!



Probabilistic Character of Weak Measurements

Review: Jacobs, Steck, Contemp. Phys. (2006)

- probability to measure x_0 on state $|\psi\rangle = \int dx \psi(x)|x\rangle$ in time $\Delta t \to 0$

$$p(x_0) = ||\hat{A}(x_0)|\psi\rangle|| = \mathrm{tr}[\hat{A}^\dagger(x_0)\hat{A}(x_0)|\psi\rangle\langle\psi|] \qquad \hat{A}(x_0) = (\tfrac{2\gamma\Delta t}{\pi})^{1/4} e^{-\gamma\Delta t(\hat{x}-x_0)^2}$$

- the measurement outcome x_0 is a Gaussian random variable with
  - expectation value $\langle\langle x_0\rangle\rangle \equiv \int dx_0 x_0 p(x_0) = (\frac{2\gamma\Delta t}{\pi})^{1/2}\int dx_0 \int dx x_0 e^{-2\gamma\Delta t(x-x_0)^2}|\psi(x)|^2 = \int dx x|\psi(x)|^2 = \langle\hat{x}\rangle$
  - variance $\langle\langle x_0^2\rangle\rangle - \langle\langle x_0\rangle\rangle^2 = 4\gamma\Delta t$

quantum mechanical expectation

- stochastic formulation (cf. Fokker-Planck vs. Langevin): parameterize

$$x_0 = \langle\hat{x}\rangle + \frac{\Delta W}{\Delta t}$$

Gaussian random variable

$$\overline{\Delta W} = 0$$

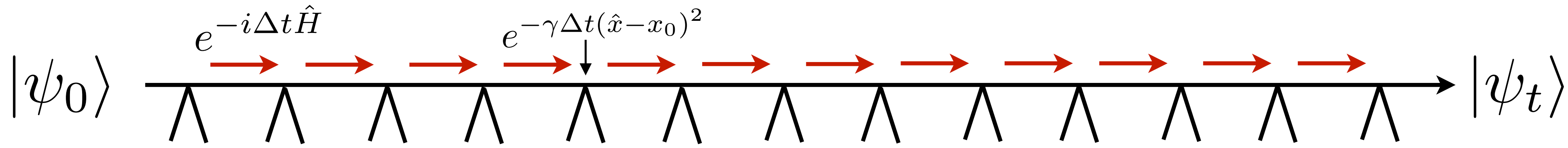
$$\overline{\Delta W \Delta W} = 4\gamma\Delta t \implies \Delta W \sim \sqrt{\Delta t}$$

- stochastic update of wave function: $|\psi_{t+\Delta t}\rangle = \dfrac{\hat{A}(x_0)|\psi_t\rangle}{||\hat{A}(x_0)|\psi_t\rangle||}$

- for short observation times: expand $|\psi_{t+\Delta t}\rangle = \hat{A}(x_0)|\psi_t\rangle$ to linear order $\Delta t \equiv dt \to 0$

$$d|\psi_t\rangle \equiv |\psi_{t+dt}\rangle - |\psi_t\rangle = \{[-\tfrac{1}{2}\gamma(\hat{x}-\langle\hat{x}\rangle)^2]dt + (\hat{x}-\langle\hat{x}\rangle)]dW\}|\psi\rangle$$

Weak Measurements: Stochastic Schrödinger Equation



- governed by stochastic Schrödinger equation for quantum trajectory $|\psi_t\rangle$ V. P. Belavkin (1987)

$$d|\psi_t\rangle = dt(-i\hat{H} - \frac{\gamma}{2}\sum_l (\hat{n}_l - \langle\hat{n}_l\rangle_t)^2|\psi_t\rangle + \underbrace{\sum_l dW_l\,(\hat{n}_l - \langle\hat{n}_l\rangle_t)\,|\psi_t\rangle}$$

$$\langle\hat{n}_l\rangle_t = \langle\psi_t|\hat{n}_l|\psi_t\rangle$$

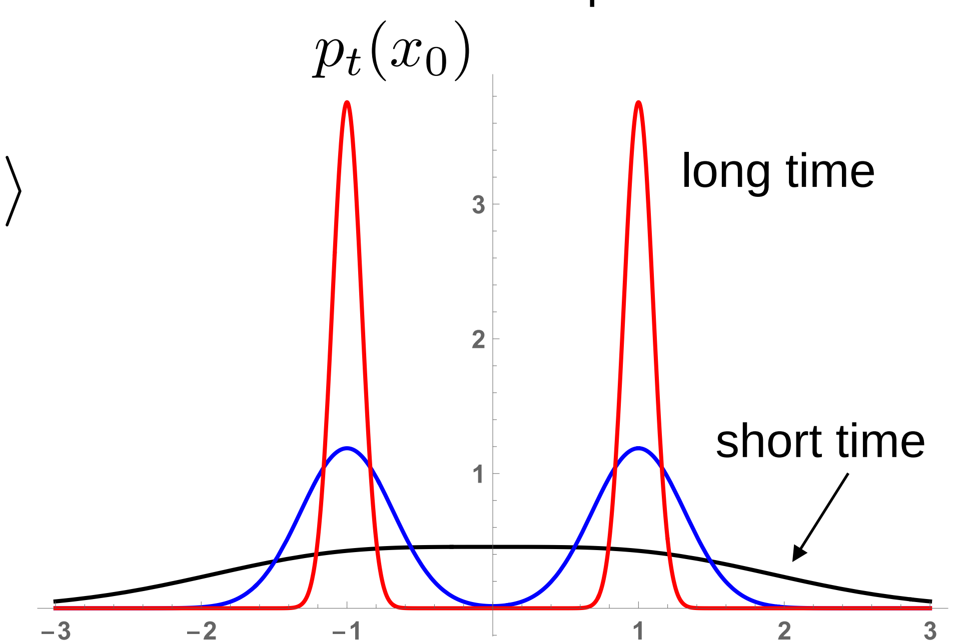
  - Hamiltonian added, many degree of freedom, $\hat{n}_l = \hat{n}_l^\dagger$
  - works for measurement operators with discrete spectrum (measurement record continuous, e.g. Stern-Gerlach)

- measurement only dynamics H =0: measurement dark states
  - dW 'multiplicative noise': inactive when $\hat{n}_l|\psi_t\rangle = \langle\hat{n}_l\rangle_t|\psi_t\rangle$
  - e.g. in eigenstate $\hat{n}_l|\psi_t\rangle = n|\psi_t\rangle$

dark state of measurement operator

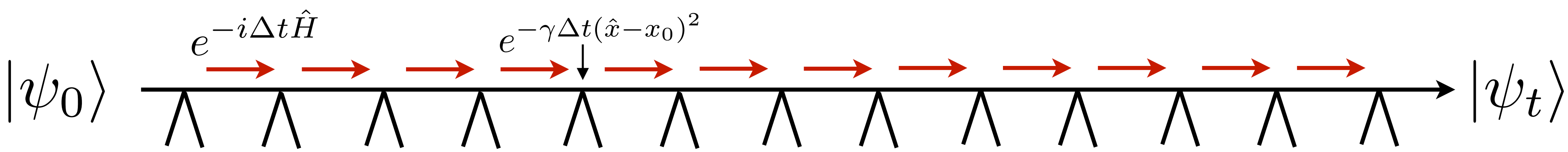
➡ continuous collapse: convergence to measurement eigenstate for long times (more gen. for) $[\hat{H},\hat{n}_l] = 0$

measurement of a spin 1/2



measurement record $x_0 = \langle\hat{x}\rangle + \frac{\Delta W}{\Delta t}$

Weak Measurements: Stochastic Schrödinger Equation



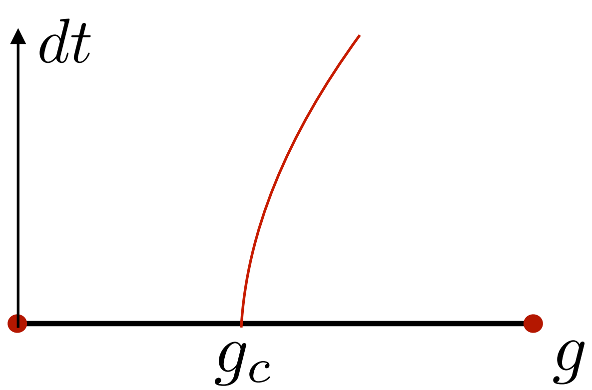
- governed by stochastic Schrödinger equation for quantum trajectory $|\psi_t\rangle$ Belavkin (1987); Gisin, Percival (1993)

$$d|\psi_t\rangle = dt(-i\hat{H} - \frac{\gamma}{2}\sum_l (\hat{n}_l - \langle \hat{n}_l\rangle_t)^2 |\psi_t\rangle + \sum_l \underbrace{dW_l \left(\hat{n}_l - \langle \hat{n}_l\rangle_t\right)|\psi_t\rangle}$$

$$\langle \hat{n}_l\rangle_t = \langle \psi_t|\hat{n}_l|\psi_t\rangle$$

  - Hamiltonian added, many degree of freedom, $\hat{n}_l = \hat{n}_l^\dagger$
  - works for measurement operators with discrete spectrum (measurement record continuous, e.g. Stern-Gerlach)

- Can we expect a measurement induced phase transition similar to projective measurements? Yes!
  - no competition => no phase transition $\hat{A}(x_0) = (\frac{2\gamma\Delta t}{\pi})^{1/4} e^{-\gamma\Delta t(\hat{x}-x_0)^2} \overset{\Delta t \to \infty}{=} |x_0\rangle\langle x_0|$
  - Demonstrated numerically Szyniszewski, Romito, Schomerus, PRB (2019)



➡ expect no phase transition upon taking temporal continuum limit

➡ continuum limit useful for analytical approach measurement induced phase transitions (e.g. Keldysh field theory approach)

Monitored Dynamics: Extracting Information

- stochastic Schrödinger equation for projector $\hat{\rho}_t = |\psi_t\rangle\langle\psi_t|$ $\longleftarrow$ random variable

$$d|\psi_t\rangle = dt(-i\hat{H} - \tfrac{\gamma}{2}\sum_l (\hat{n}_l - \langle\hat{n}_l\rangle_t)^2|\psi_t\rangle + \sum_l dW_l\,(\hat{n}_l - \langle\hat{n}_l\rangle_t)\,|\psi_t\rangle$$

➡ statistical analysis: consider trajectory ensemble

- usual observables:

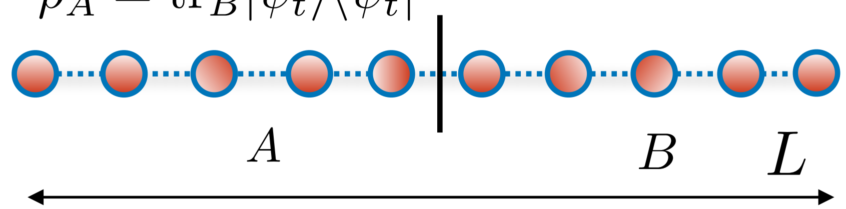
$$\underbrace{\langle\hat{O}\rangle = \langle\psi_t|\hat{O}|\psi_t\rangle}_{\text{quantum average}} \xrightarrow[\text{e.g. trajectories}]{\text{statistical average}} \overline{\langle\hat{O}\rangle} = \overline{\langle\psi_t|\hat{O}|\psi_t\rangle} = \mathrm{tr}[\hat{O}\overline{\hat{\rho}_t}]$$

- Problem: Hermitian measurement operators => $\overline{\hat{\rho}_t} \sim \mathbf{1}$

➡ use state-dependent observables $\overline{\langle\hat{O}(|\psi\rangle)\rangle} = \mathrm{tr}\overline{\hat{O}(\hat{\rho})\hat{\rho}}$

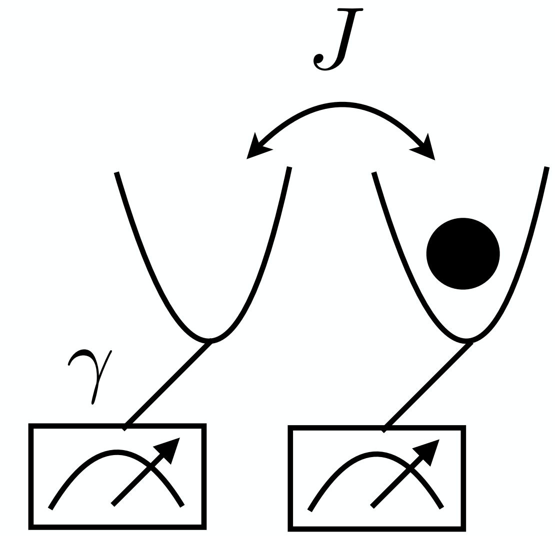
more promising, because in general $F(\overline{\hat{\rho}}) \neq \overline{F[\hat{\rho}]}$

$\rho_A = \mathrm{tr}_B|\psi_t\rangle\langle\psi_t|$



- examples:
  - von Neumann entropy $\overline{S_{vN}(l,L)} = \overline{\langle\log(\rho_A)\rangle}$ arbitrarily high power of state projector
  - correlation function $\overline{\langle\hat{n}_i\rangle\langle\hat{n}_j\rangle}$ quadratic in state projector

An example: TLS / one fermion on two sites



- toy model: trajectory evolution of single fermion on two sites

$$|\psi_{t+dt}\rangle = |\psi_t\rangle - idt\hat{H}_{\text{eff}}|\psi_t\rangle + \sum_{l=1}^{2} dW_l \left(\hat{n}_l - \langle \hat{n}_l \rangle_t\right) |\psi_t\rangle$$

$$\hat{H}_{\text{eff}} = \hat{H} - i\hat{K} \qquad \hat{H} = -J\left(c_1^\dagger c_2 + h.c.\right) \qquad \hat{K} = \frac{\gamma}{2}\sum_{l=1}^{2}\left(\hat{n}_l - \langle \hat{n}_l \rangle_t\right)^2$$

guiding physical picture:

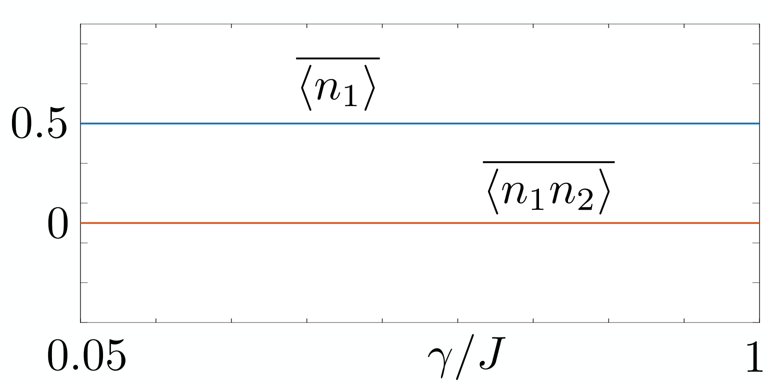
➡ thermodynamic limit: pinning quantum phase transition may happen at sharply defined point

➡ signalled in nonlinear-in-state ‘observable’, like the covariance matrix

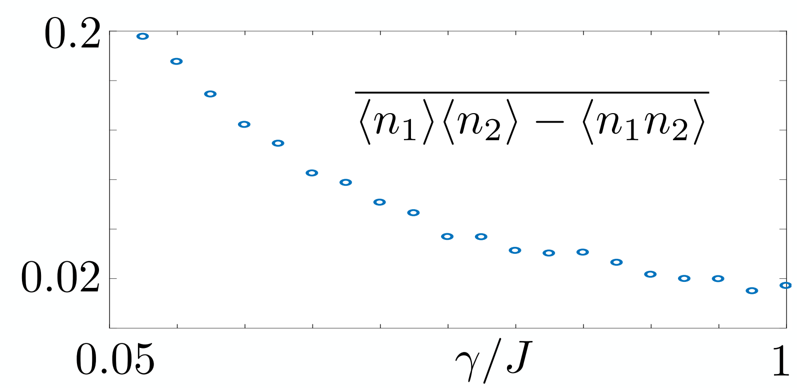
➡ pinning to measurement eigenstate

➡ vanishing time spent in eigenstate

- invisible in linear averages

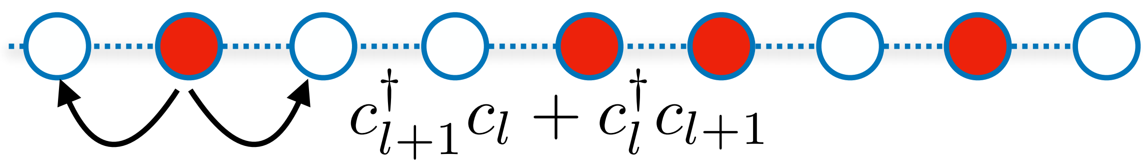


- seen in averaged trajectory covariance matrix



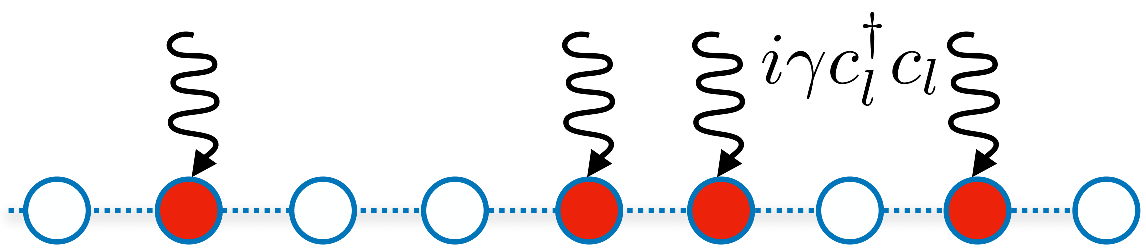
Entanglement Phase Transition in a Monitored Fermion Chain

Hamiltonian:



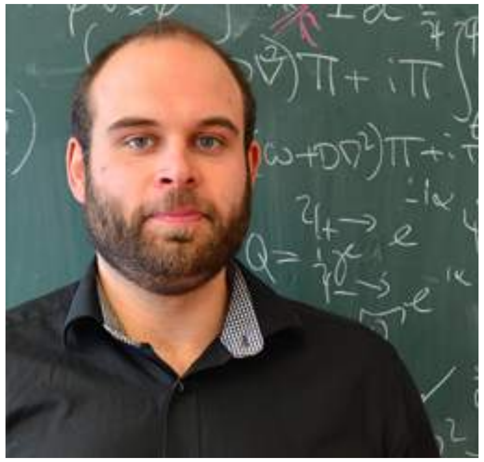
entanglement growth

Monitoring:

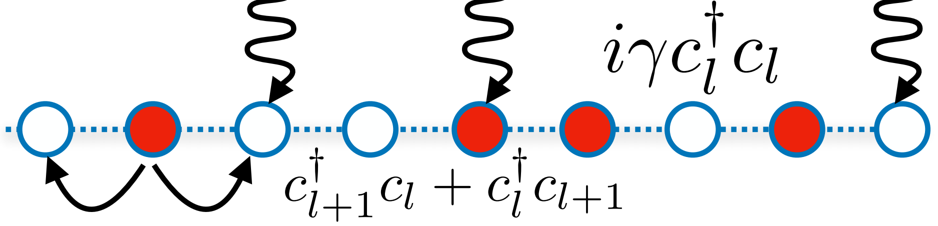


entanglement saturation

O. Alberton, M. Buchhold, SD, PRL (2021)



Monitored Fermion Dynamics



- Weak continuous measurements in many-body system

$$d|\psi_t\rangle = dt(-i\hat{H} - \tfrac{\gamma}{2}\sum_l \hat{M}_l^2|\psi_t\rangle + \sum_l dW_l \hat{M}_l|\psi_t\rangle$$

Gaussian white noise

- competition: $g = \dfrac{\gamma}{J}$

- unitary dynamics: hopping

$$H = -J\sum_l \left(c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l\right)$$

- measurement operators $\hat{M}_l = \hat{n}_l - \langle\hat{n}_l\rangle_t$
- H = 0: evolution stops after collapse into dark state

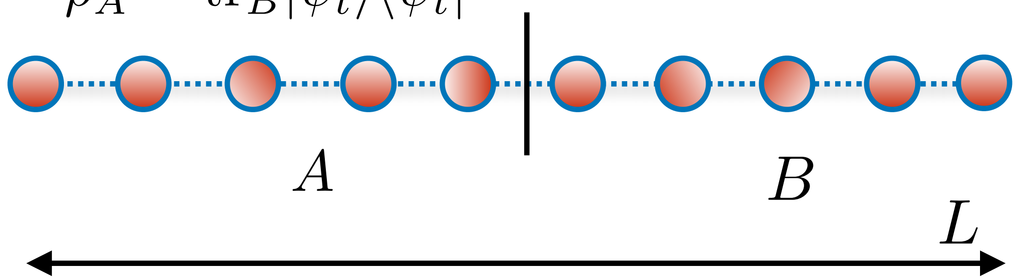
$$\hat{M}_l|\psi_t\rangle = 0 \quad \text{for} \quad \hat{n}_l|\psi_t\rangle = n_l|\psi_t\rangle$$

eigenstate of measurement operator

➡ volume law entanglement entropy

$$S_{vN}(L/2, L) = \mathrm{tr}\rho_A \log(\rho_A) \overset{t\to\infty}{\sim} L$$

$\rho_A = \mathrm{tr}_B|\psi_t\rangle\langle\psi_t|$



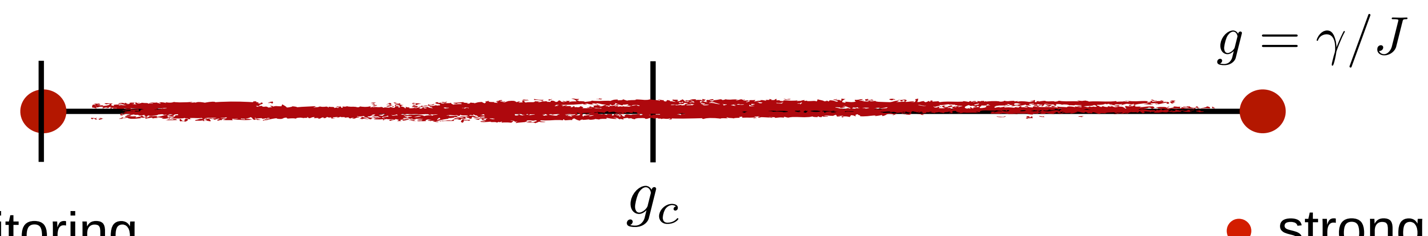
➡ area law entanglement entropy

$$S_{vN}(L/2, L) = s_0$$

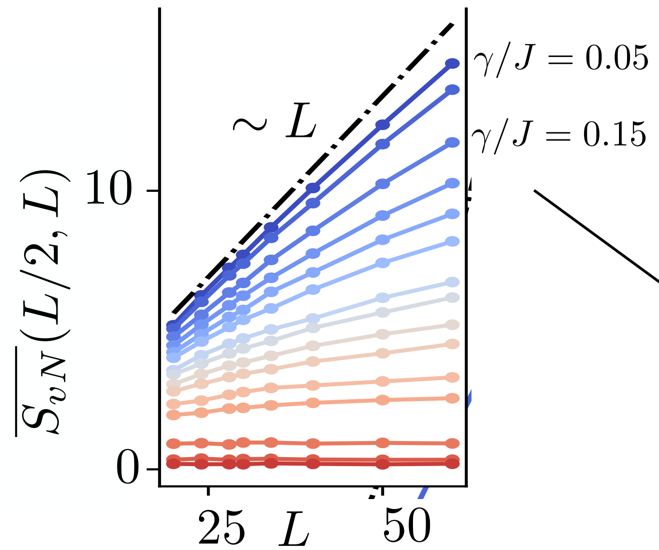
- caveat: $|\psi_t\rangle$ is a random variable
- binary measurement outcomes generate extensive configurational entropy

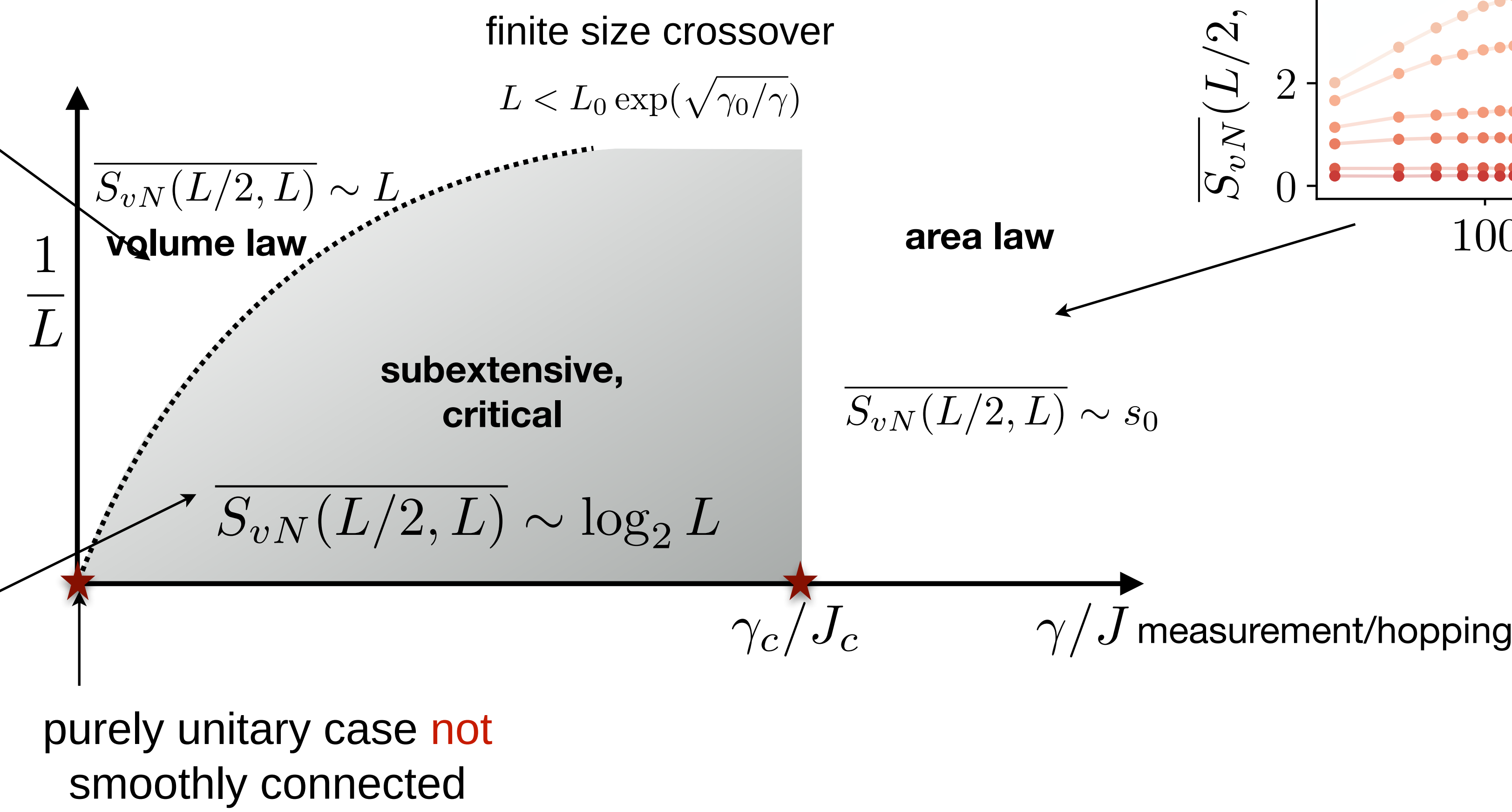
➡ 'observables': entanglement entropy, traj. averaged correlators

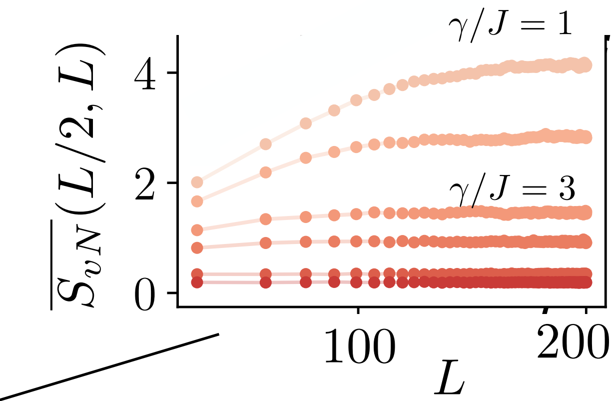
Trajectory Ensemble Phase Diagram: Entanglement Entropy

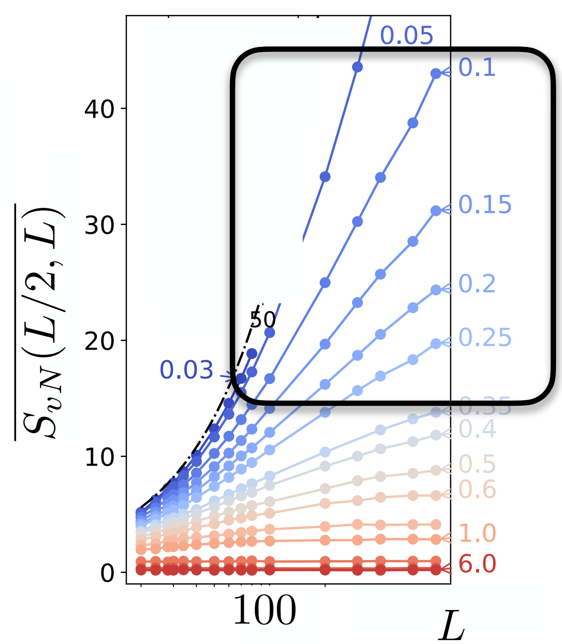


- weak monitoring
- strong monitoring









➡ extended subextensive, critical phase at intermediate monitoring

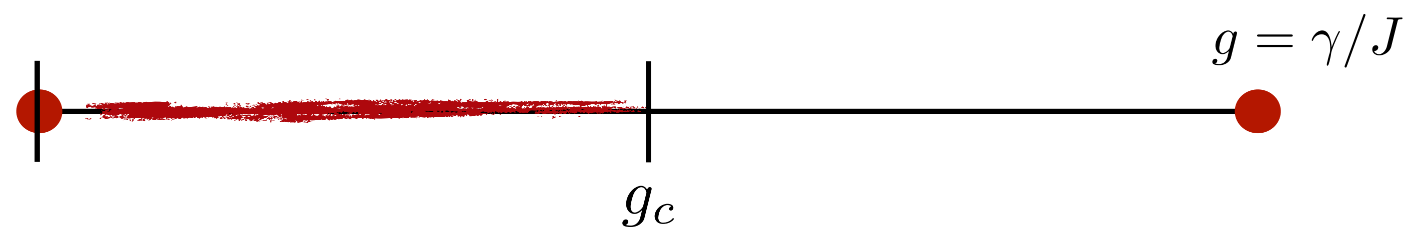
see also Chen, Li, Fisher, Lucas PRR (2020)

➡ new phase transition for physical measurement protocol

➡ consistent with Cao, Tilloy, De Luca SciPost (2019)

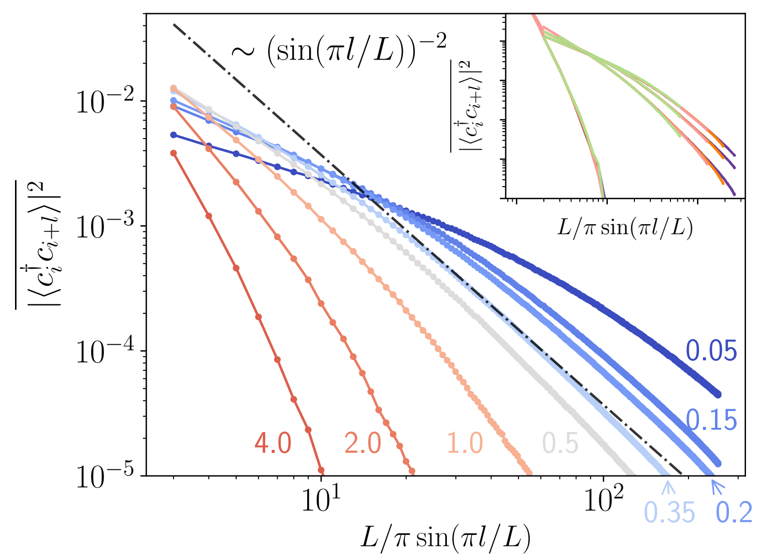
only excluding volume law in thermodynamic limit

Characterizing the Weak Monitoring Phase



- extended criticality: Connected correlation function

$$C_{i,i+l} = \overline{\langle \hat{n}_i \rangle \langle \hat{n}_{i+l} \rangle} - \overline{\langle \hat{n}_i \hat{n}_{i+l} \rangle}$$
$$\sim l^{-2}$$

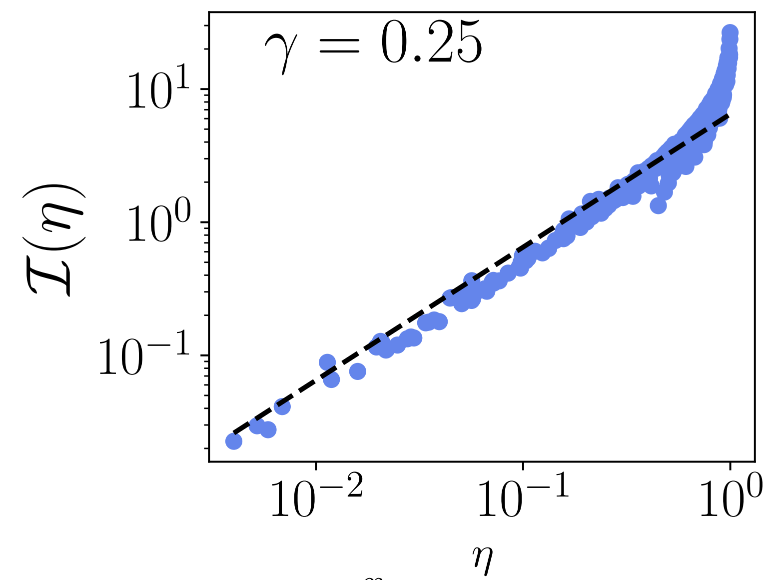


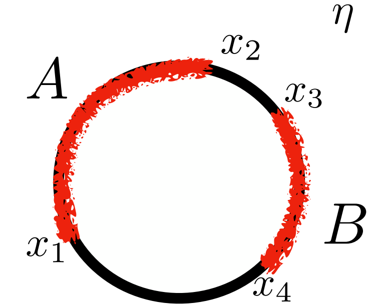
- captures all distinct phases:

$$C_{i,i+l} \sim \begin{cases} 0 & \text{for } H = 0 \\ \exp(-l/\xi) & \text{for } \gamma \gg J \\ l^{-2} & \text{for } \gamma \ll J \\ l^{-1} & \text{for } \gamma = 0 \end{cases}$$

- emergent conformality: Mutual information

$$\mathcal{I}(A,B) = \overline{S_{vN}(A)} + \overline{S_{vN}(A)} + \overline{S_{vN}(A \cup B)}$$



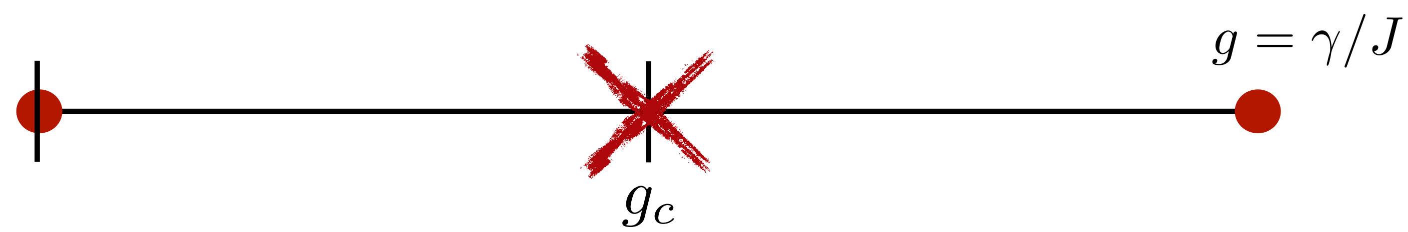


$$\eta = \frac{x_{12} x_{34}}{x_{13} x_{24}}$$

conformally invariant critical point: Nahum et al. PRX (2019); Li Chen Fisher PRB (2019); Jian et al. PRB (2020);

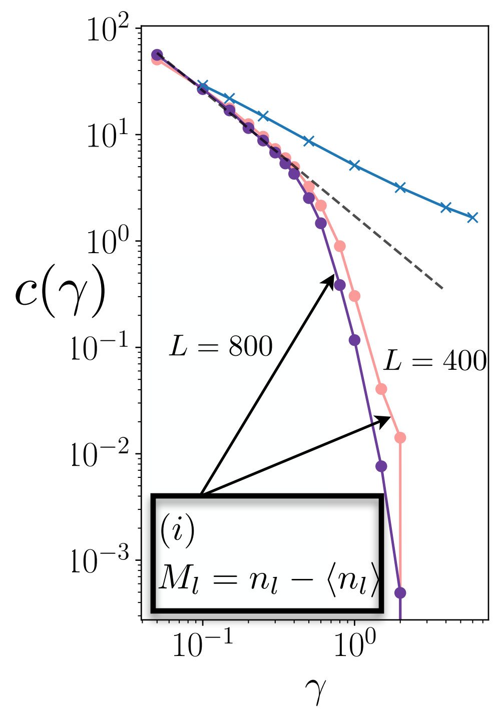
➡ emergent conformally invariant critical phase for weak monitoring

Characterizing the Phase Transition



- effective central charge $c(\gamma)$

$$\overline{S_{vN}(l,L)} = \frac{c(\gamma)}{3}\log_2\left[\frac{L}{\pi}\sin\left(\frac{\pi l}{L}\right)\right] + s(\gamma)$$

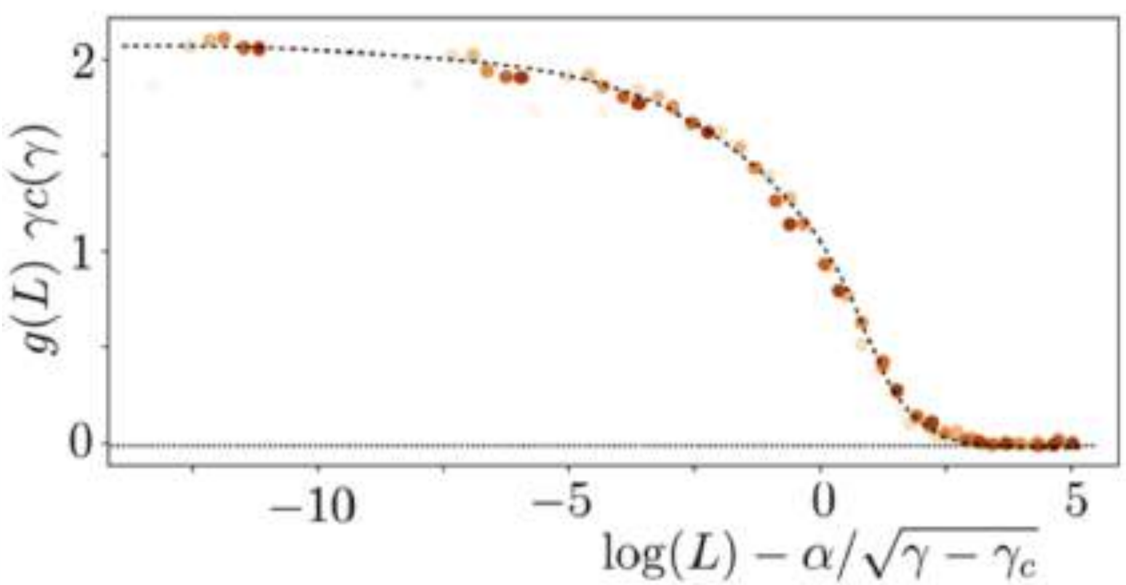


parameter dependent c

random systems: Cardy Jacobsen PRL (1997);
Refael, Moore PRL (2004)

➡ sudden jump reminiscent of BKT

- essential scaling of the central charge: scaling collapse above phase transition

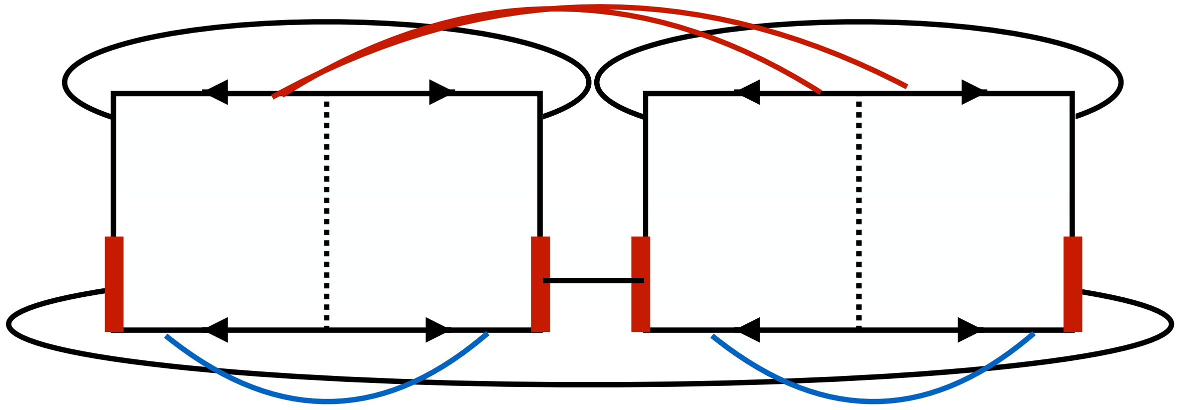


➡ BKT universal behavior

➡ establishes BKT type phase transition

➡ further: measurement protocol dependence, trajectory entanglement distribution as probe of transition…

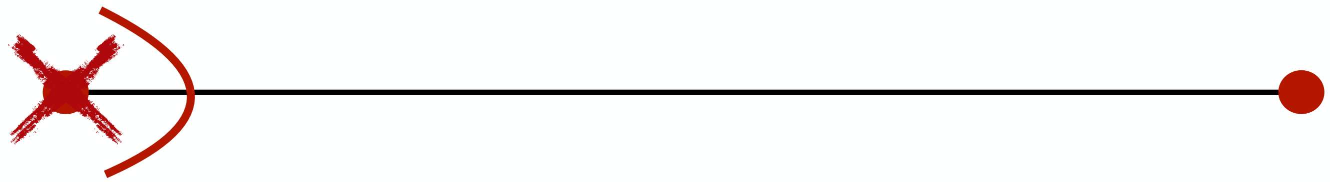
Replica Field Theory Approach
to
Measurement Induced Phase Transitions



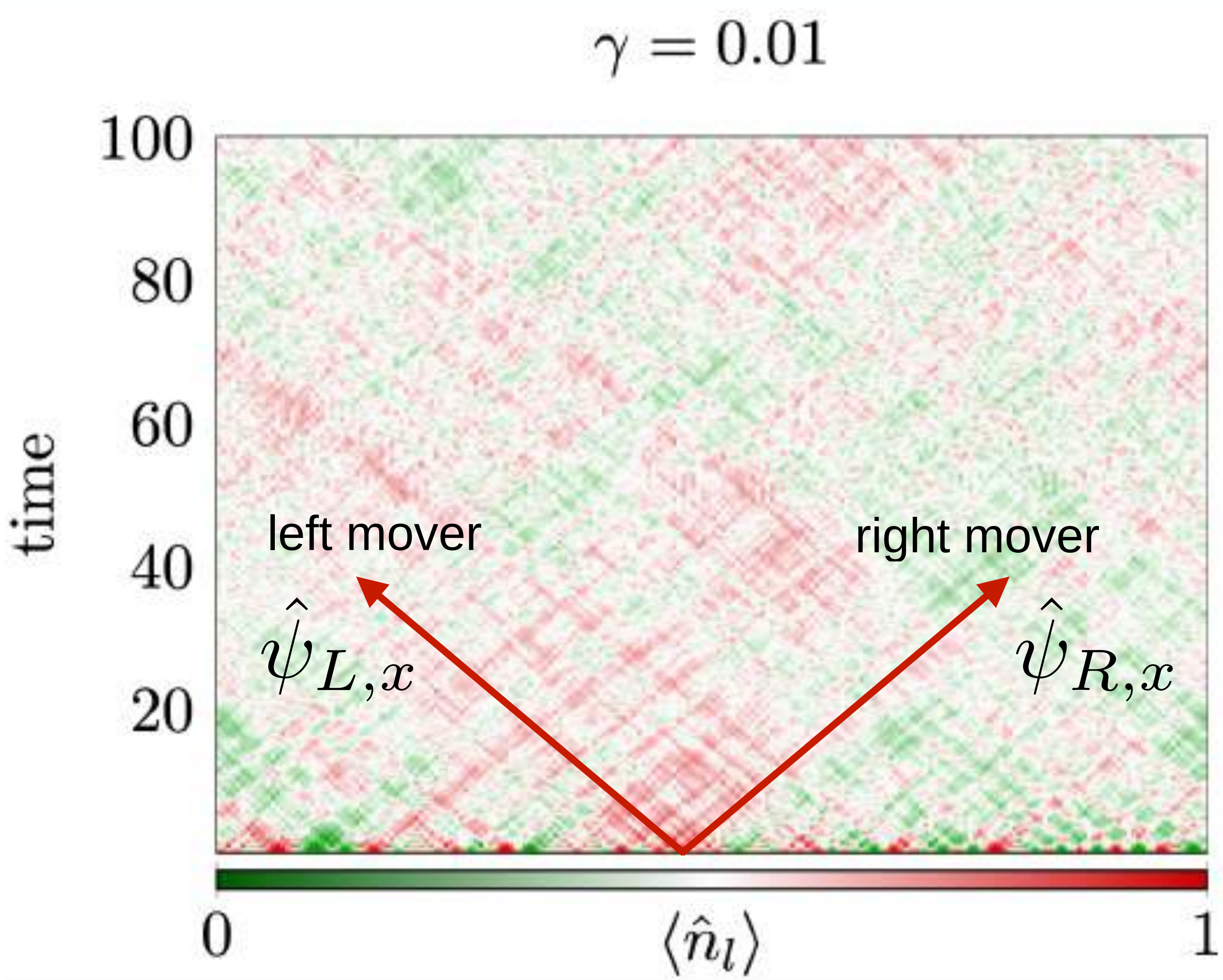
M. Buchhold, Y. Minoguchi, A. Altland, SD, PRX 11, 041004 (2021)

Continuum (1+1) dimensional Model

- approach problem from the weakly measured side

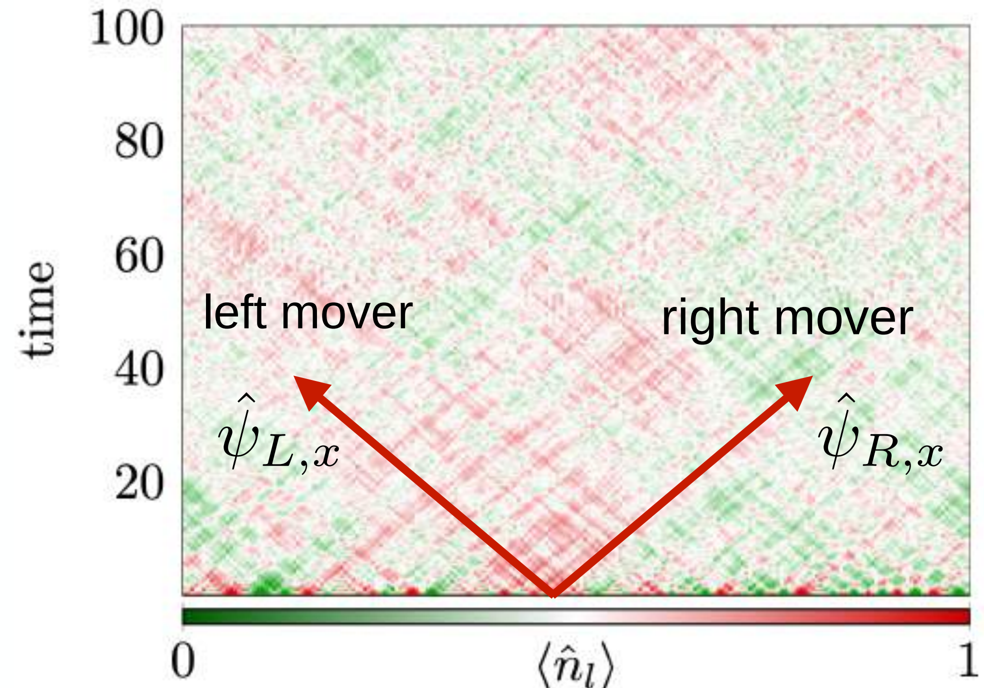


- model: phenomenologically motivated continuum limit for lattice fermions (weak measurement regime)



fermionic variant $\xrightarrow{\text{translation table}}$ bosonized variant

Continuum (1+1) dimensional Model



Luttinger liquid

- model: phenomenologically motivated continuum limit for lattice fermions (weak measurement regime)

- Hamiltonian: massless Dirac fermions $\hat\Psi_x = (\hat\psi_{R,x}, \hat\psi_{L,x})^T$

$$\hat H = iv \int_x \hat\Psi_x^\dagger \sigma_z \partial_x \hat\Psi_x \longrightarrow \hat H = \frac{v}{2\pi}\int_x [\underbrace{(\partial_x \hat\theta_x)^2}_{\text{phase}} + \underbrace{(\partial_x \hat\phi_x)^2}_{\text{density}}]$$

fermionic variant — bosonized variant

- measurement operators: local particle density —> current and vertex operators

rate γ_1 : $\hat O_{1,x} = \Psi_x^\dagger \Psi_x = \hat J_x^{(0)} \longrightarrow \hat O_{1,x} = -\frac{1}{\pi}\partial_x \hat\phi_x$ linear gapless

rate γ_2 : $\hat O_{2,x} = \Psi_x^\dagger \sigma_x \Psi_x \longrightarrow \hat O_{2,x} = \underbrace{m}_{\mathcal{O}(1)} \cos(2\hat\phi_x)$ nonlinear

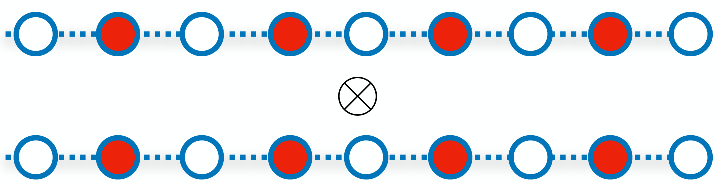
common eigenstates: $\hat\phi_x|\Psi_D\rangle = \phi_x|\Psi_D\rangle$

- stabilize product dark states: exactly local
- realize competition: do not commute with H (phase fluctuations)

Towards the Relevant Degrees of Freedom: Replica Approach (n=2)

- Access state-dependent observables, e.g. covariance matrix $C_{xy} = \overline{\langle \hat{n}_x \hat{n}_y \rangle} - \overline{\langle \hat{n}_x \rangle \langle \hat{n}_y \rangle}$

- Introduce replicas in Hilbert space $|\Psi_t\rangle = |\psi_t^{(1)}\rangle \otimes |\psi_t^{(2)}\rangle =$



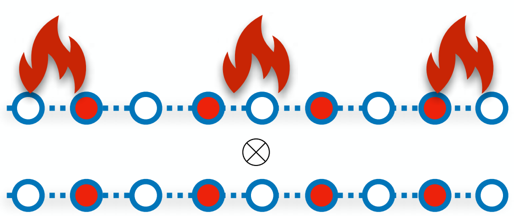
- All quadratic-in-state observables encoded in

$$\rho^{2\mathrm{R}} = \overline{|\Psi_t\rangle\langle\Psi_t|}$$

➡ linear statistical average of replica density matrix

- Quantum master equation (truncate coupling to ρ^{3R})

$\partial_t \rho^{2\mathrm{R}} =$



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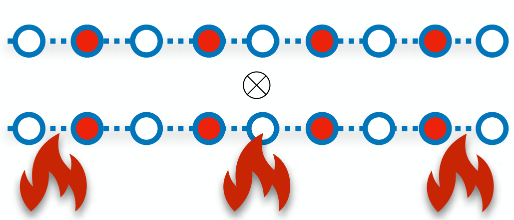

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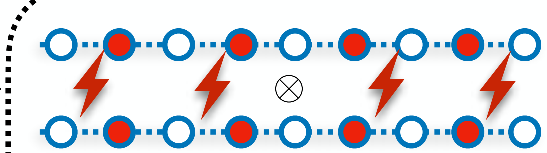


$$i[\rho^{2R}, H^{(\alpha)}] - \frac{\gamma}{2}[\hat{M}_x^{(\alpha)}, [\hat{M}_x^{(\alpha)}, \rho^{2R}]]$$

individual heating Lindbladians

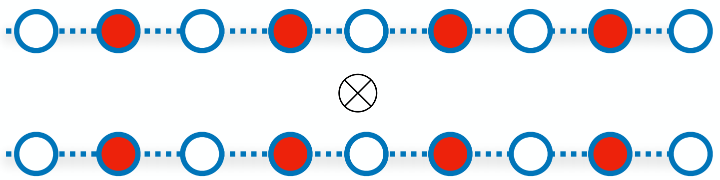
$$\gamma\{\hat{M}_x^{(1)}, \{\hat{M}_x^{(2)}, \rho^{2R}\}\}$$

replica coupling

➡ study structure of 2-replica theory

Towards the Relevant Degrees of Freedom: Replica Approach

- Q: What is the structure of $C_{xy} = \overline{\langle \hat{n}_x \hat{n}_y \rangle} - \overline{\langle \hat{n}_x \rangle \langle \hat{n}_y \rangle}$

- Introduce replicas in Hilbert space $|\Psi_t\rangle = |\psi_t^{(1)}\rangle \otimes |\psi_t^{(2)}\rangle =$ 

$$\hat{n}_x^{(1)} = \hat{n}_x \otimes \mathbf{1}$$

$$\hat{n}_x^{(2)} = \mathbf{1} \otimes \hat{n}_x$$

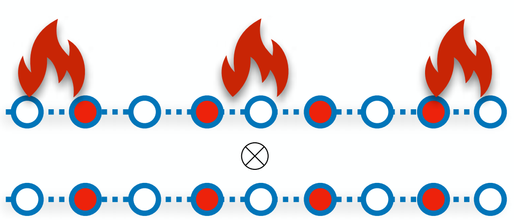
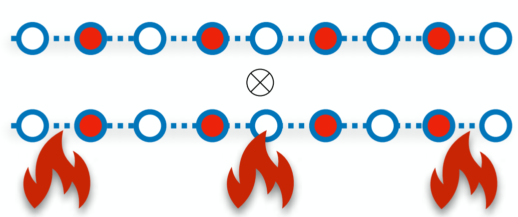
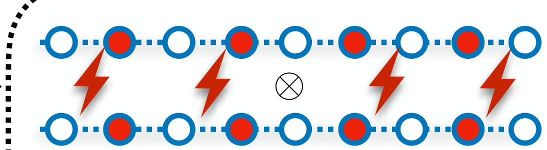
- Then $C_{xy} = \frac{1}{2} \mathrm{Tr} \left[\left(\hat{n}_x^{(1)} - \hat{n}_x^{(2)} \right) \left(\hat{n}_y^{(1)} - \hat{n}_y^{(2)} \right) \rho^{2R} \right]$

$$\rho^{2\mathrm{R}} = \overline{|\Psi_t\rangle\langle\Psi_t|}$$

➡ linear statistical average of replica density matrix

➡ correlations of relative replica coordinate

- Quantum master equation (truncate coupling to ρ^{3R})

$\partial_t \rho^{2\mathrm{R}} =$  $+$  $+$ 

$$i[\rho^{2R}, H^{(\alpha)}] - \frac{\gamma}{2}[\hat{M}_x^{(\alpha)}, [\hat{M}_x^{(\alpha)}, \rho^{2R}]]$$

individual heating Lindbladians

$$\gamma\{\hat{M}_x^{(1)}, \{\hat{M}_x^{(2)}, \rho^{2R}\}\}$$

replica coupling

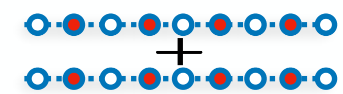
➡ study structure of 2-replica theory

Boson Replica Quantum Master Equation

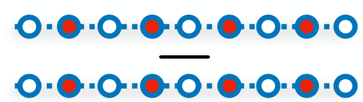
- Boson measurement $H^{(\alpha)} = \frac{\nu}{2\pi} \int_x (\partial_x \hat{\phi}^{(\alpha)})^2 + (\partial_x \hat{\theta}^{(\alpha)})^2$

$$\hat{O}^{(\alpha)}_{1,x} = -\frac{1}{\pi} \partial_x \hat{\phi}^{(\alpha)}_x \qquad \hat{O}^{(\alpha)}_{2,x} = 0$$ linear case first

- New degrees of freedom



: $\hat{\phi}^{(a)} = \hat{\phi}^{(1)} + \hat{\phi}^{(2)}$ average coordinate



: $\hat{\phi}^{(r)} = \hat{\phi}^{(1)} - \hat{\phi}^{(2)}$ replica fluctuations

➡ Master equation becomes separable

- Average coordinate: heating to infinite temperature (<—> unbounded growth of mode occupation)

$$\partial_t \rho^{(a)} = i[\rho^{(a)}, H^{(a)}] + \frac{2\gamma}{\pi} \sum_l \left(\partial_x \hat{\phi}^{(a)} - \overline{\langle \partial_x \hat{\phi}^{(a)} \rangle} \right) \rho^{(a)} \left(\partial_x \hat{\phi}^{(a)} - \overline{\langle \partial_x \hat{\phi}^{(a)} \rangle} \right)$$

only jump term!

- Relative coordinate: cooling/damping into dark state

$$\partial_t \rho^{(r)} = i[\rho^{(r)}, H^{(r)}] - \frac{\gamma}{\pi} \sum_l \left\{ (\partial_x \hat{\phi}^{(r)})^2, \rho^{(r)} \right\}$$

⟵ no jump term!

- further separable: $\partial_t |\psi_t^{(r)}\rangle = -i H_{\text{eff}} |\psi_t^{(r)}\rangle$

$$\boxed{H_{\text{eff}} = \frac{\nu}{2\pi} \int_x (\partial_x \hat{\theta})^2 + (1 - i\eta^2)(\partial_x \hat{\phi})^2} \qquad \eta^2 = \frac{2\gamma}{\nu}$$

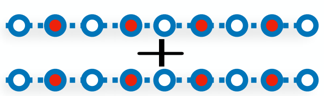
gapless non-Hermitean Hamiltonian

Boson Replica Quantum Master Equation

- Boson measurement $H^{(\alpha)} = \frac{\nu}{2\pi}\int_x (\partial_x \hat{\phi}^{(\alpha)})^2 + (\partial_x \hat{\theta}^{(\alpha)})^2$

$$\hat{O}^{(\alpha)}_{1,x} = -\frac{1}{\pi}\partial_x \hat{\phi}^{(\alpha)}_x \qquad \hat{O}^{(\alpha)}_{2,x} = m\cos(2\hat{\phi}_x)$$ general case

- $\hat{O}^{(\alpha)}_{2,x}$ couples relative and absolute degrees $\sim m\cos(\sqrt{2}(\hat{\phi}^{(a)}_x \pm \hat{\phi}^{(r)}_x))$

➡ how do  degrees enter? $\longrightarrow$ $\langle \hat{\phi}^{(a)}_x \hat{\phi}^{(a)}_x \rangle = \infty$ nonlinearity irrelevant for $\hat{\phi}^{(a)}_x$

➡ integrate out $\hat{\phi}^{(a)}_x$ in Gaussian approx. for $\hat{\rho}^{(a)}$

- Non-hermitian Schrödinger equation for relative coordinate

$$\partial_t |\psi^{(r)}_t\rangle = -iH_{\text{eff}}|\psi^{(r)}_t\rangle$$ ➡ cooling into dark state

$$\boxed{H_{\text{eff}} = \frac{\nu}{2\pi}\int_x (\partial_x\hat{\theta})^2 + (1 - i\eta^2)(\partial_x\hat{\phi})^2 \; -i\frac{\gamma m}{\pi}\underbrace{\int_x [1 - \cos(\sqrt{8}\hat{\phi}_x)]}_{\text{effect of non-linearity}}}$$

➡ non-Hermitian Sine-Gordon: pinning via cos term, depinning via theta term

➡ extract physics in path integral approach

Effective Non-Hermitian Hamiltonian and Path Integral

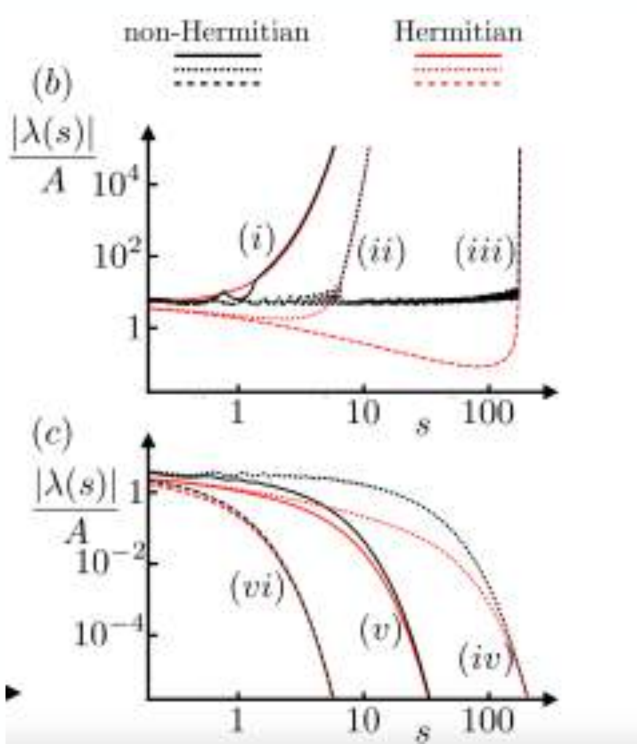
➡ Sine-Gordon action with complex coefficients

Fendley, Saleur, Zamolodchikov,
International Journal of Modern Physics (1993)

$$\boxed{S = \int_{t,x} \left\{ \frac{K}{16\pi} \left[\frac{1}{\eta} (\partial_t \phi)^2 - \eta (\partial_x \phi)^2 \right] - i\lambda \cos(\phi) \right\}}$$

- 'Wick rotation' brings free part to standard Euclidean (2+0) dimensional form $(x,t) \to (\eta^{\frac{1}{2}} x, i\eta^{-\frac{1}{2}} t)$

➡ RG flow: standard KT flow with complex K, λ



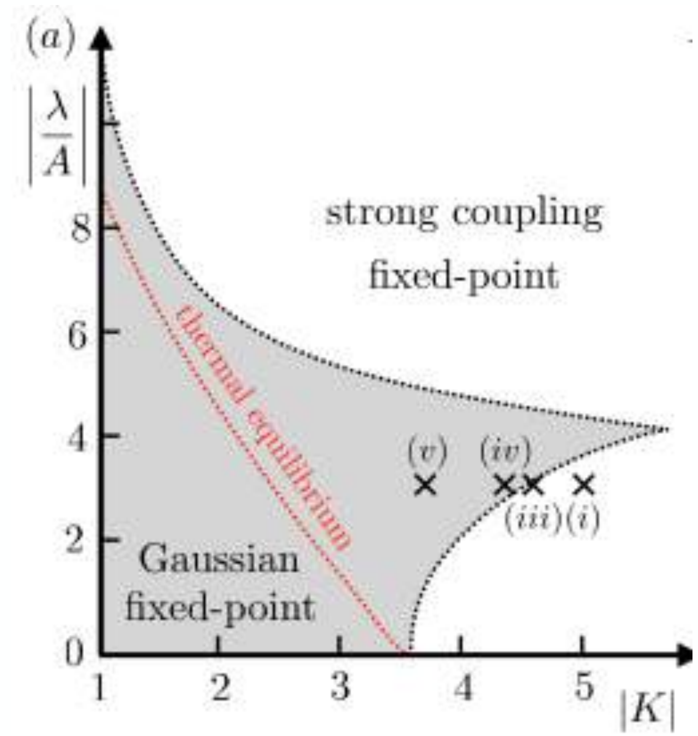
$$\partial_s \lambda = \left(2 - \frac{8\pi}{K}\right)\lambda,$$
$$\partial_s K = -\lambda^2$$

- UV flow modified

➡ shift of phase border

- IR flow reaches standard KT flow

➡ same long wavelength properties



➡ gapless generalized CFT phase with algebraic correlations and varying exponent

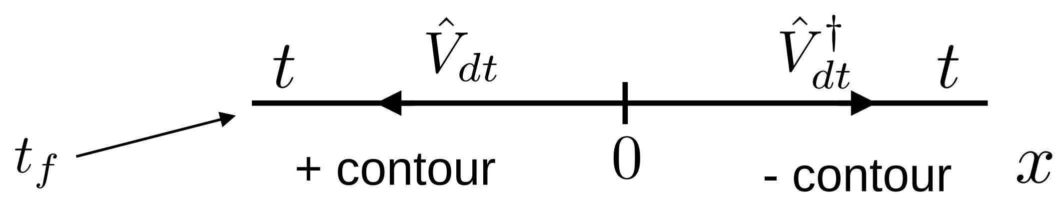
➡ phase transition in the BKT universality class in line with numerics

n-Replicas: Lindblad-Keldysh 2.0

n-replica Hamilton-Keldysh: Aleiner, Faoro, Ioffe, AoP (2016); Tsuji, Werner, Ueda, PRA (2017); Shenker, Stanford, JHEP (2015); Ansari, Nazarov, JETP (2016)

- Motivation:
  - Generalization of ‘hot’ and ‘cold’ modes?
  - Entanglement entropies?

- Lindblad-Keldysh construction for n replicas
  - evolution operator $\hat{V}_{dt} = \exp\left[-(i\hat{H} + \hat{M}_t^2)dt + \xi \hat{M}_t\right]$ expansion to second order: SSE
  - single replica



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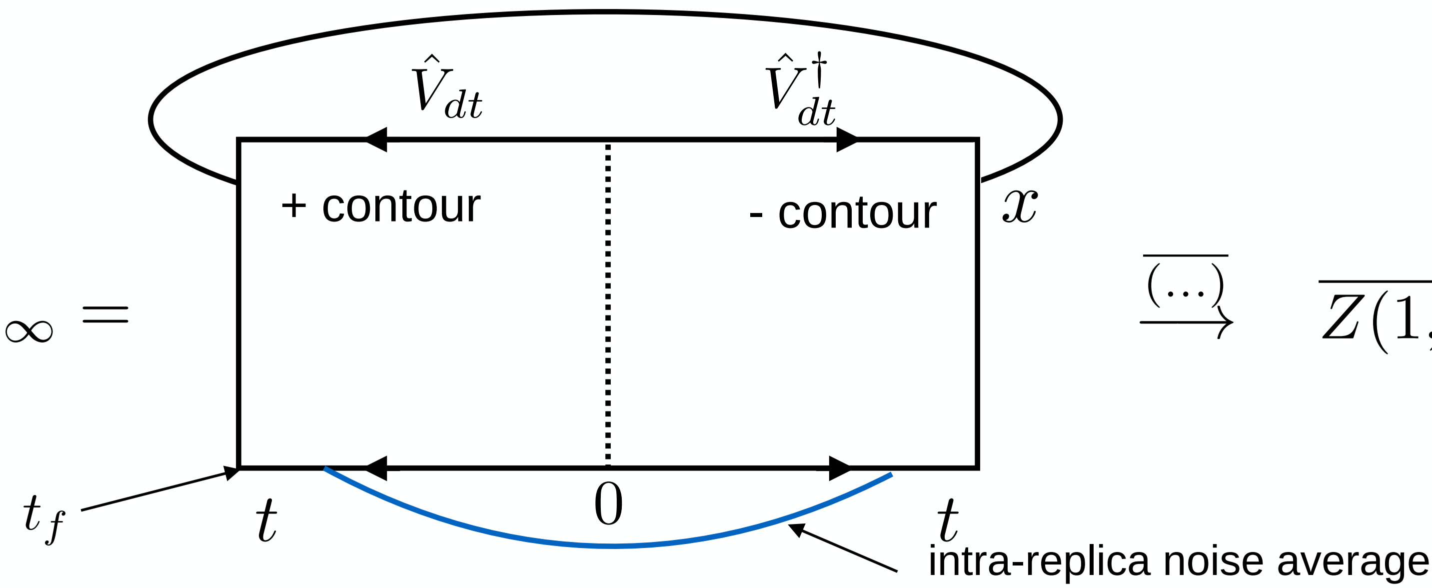
$$Z(1, \{\xi\}) = \mathrm{tr}\rho_{t_f \to \infty} =$$



$$\xrightarrow{\overline{(\ldots)}} \quad \overline{Z(1, \{\xi\})} \equiv Z(1)$$

$$Z(1, \{\xi\}) = \int \mathcal{D}[\Psi] \exp\left[i(S_{1,H}[\Psi] + S_{1,\xi}[\Psi])\right]$$

$$\Longrightarrow \quad Z(1) = \int \mathcal{D}\Psi \;\; \exp\left[i(S_{1,H}[\Psi] + S_{1,M}[\Psi])\right]$$

$$S_{1,H}[\Psi] = \sum_{\sigma=\pm} \sigma \int_t \left(\bar{\psi}_\sigma i\partial_t \psi_\sigma - H[\bar{\psi}_\sigma, \psi_\sigma]\right)$$

$$S_{1,\xi}[\Psi] = i \sum_{\sigma=\pm} \int_t \left[M_\sigma^2 - \xi M_\sigma\right]$$

$$S_{1,M}[\Psi] = i \int_t [M_+^2 + M_-^2 - \tfrac{1}{2}(M_+ + M_-)^2]$$

$$= -i \int_t [O_+ O_- - \tfrac{1}{2}O_-^2 + \tfrac{1}{2}O_-^2]$$

$$M = O - \bar{O}$$

➡ measurement expectation values cancel

➡ Lindblad-Keldysh functional integral reproduced (herm. Lindblads)

n-Replicas: Lindblad-Keldysh 2.0

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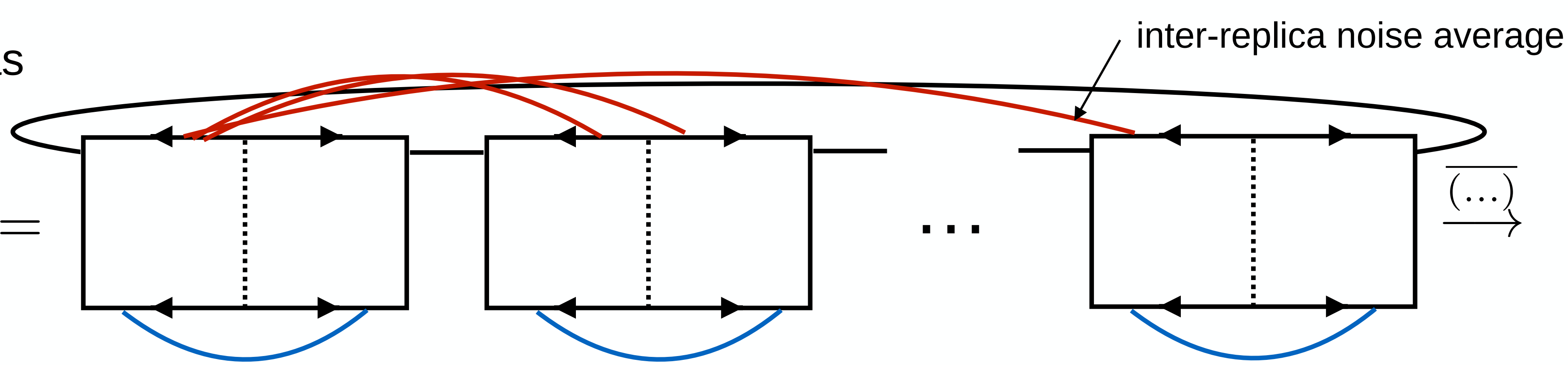
    expansion to second order: SSE

  - n replicas

$\overline{Z(n,\{\xi\})} =$



$\xrightarrow{\overline{(\dots)}} \quad \overline{Z(n,\{\xi\})} \equiv Z(n)$

$$Z(n,\{\xi\}) = \int \mathcal{D}[\Psi] \exp\left[i(S_{n,H}[\Psi] + S_{n,\xi}[\Psi])\right]$$

$$\Longrightarrow \quad Z(n) = \int \mathcal{D}\Psi \;\; \exp\left[i(S_{n,H}[\Psi] + S_{n,M}[\Psi])\right]$$

$$S_{n,H}[\Psi] = \sum_{\sigma=\pm} \sum_{l=1}^{n} \sigma \int_t \left(\bar{\psi}_\sigma^{(l)} i\partial_t \psi_\sigma^{(l)} - H[\bar{\psi}_\sigma^{(l)}, \psi_\sigma^{(l)}] \right)$$

$$S_{n,\xi}[\Psi] = i \sum_{\sigma=\pm} \sum_{l=1}^{n} \int_t \left[(M_\sigma^{(l)})^2 - \xi M_\sigma^{(l)} \right]$$

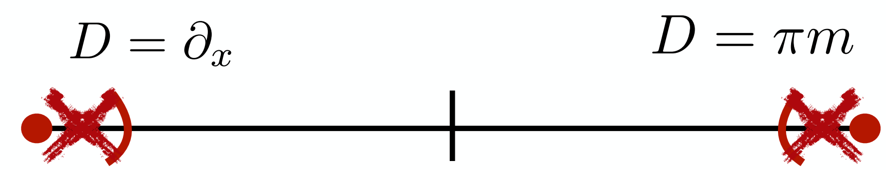
collective coupling to noise!

$$S_{n,M}[\Psi] = i \int_t \sum_{l=1}^{n} \left[\left([M_+^{(l)}]^2 + [M_+^{(l)}]^2 \right) - \tfrac{1}{2} \left(\sum_{l=1}^{n} M_+^{(l)} + M_-^{(l)} \right)^2 \right]$$

➡ collective coupling to noise

➡ structural simplification for linear measurement dynamics

n-Replicas: Decoupling of Gaussian theories



- practical importance: reduction to linear / Gaussian bosonic theory in limiting cases

- bosonized action in the presence of noise:

quadratic

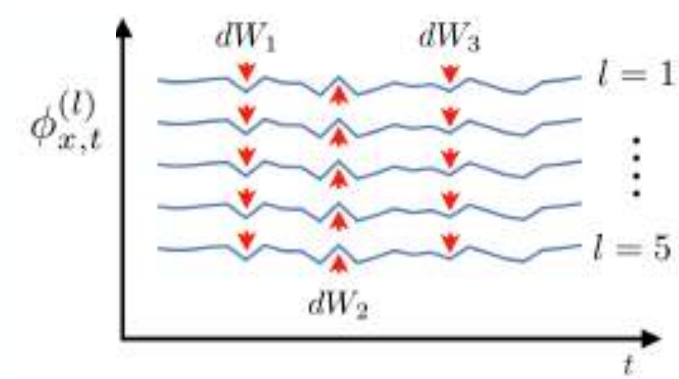
bosonic field

$$S[\phi] = S_{n,H}[\phi] + S_{n,\xi}[\phi]$$

$$S_{n,H}[\phi] = -\frac{1}{2\pi}\sum_{l=1}^{n}\sum_{\sigma=\pm}\sigma\int_{t,x}\phi_\sigma^{(l)}(\partial_t^2-\partial_x^2)\phi_\sigma^{(l)} \qquad S_{n,\xi}[\phi] = i\sum_{\sigma=\pm}\sum_{l=1}^{n}\int_{t,x}\left[(O_\sigma^{(l)}-\bar{O})^2-\xi(O_\sigma^{(l)}-\bar{O})\right]$$

$$O_\sigma^{(l)} = D\phi_\sigma^{(l)}, \quad D=\pi m, \partial_x$$

replica space picture



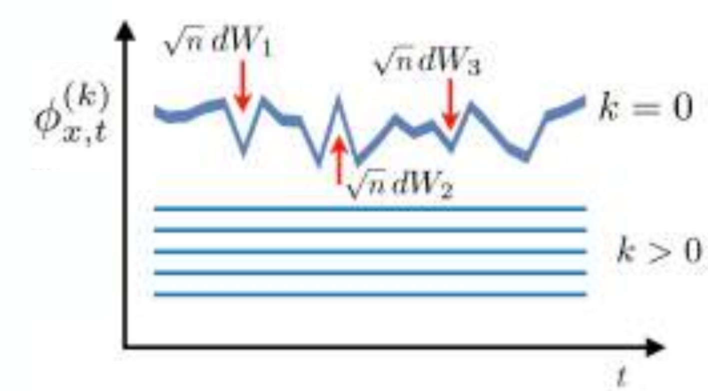
- decoupling of center-of-mass and relative modes
  - Fourier expansion in replica space

$$\phi_{\sigma,t,x}^{(l)} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}e^{-i\frac{2\pi kl}{n}}\phi_{\sigma,t,x}^{(k)}$$

  - equation of motion $\frac{\delta S}{\delta\phi}=0$

$$\boxed{\partial_t^2\phi_+^{(k)} = (\partial_x^2-\frac{\gamma i}{\pi v}D^2)\phi_+^{(k)} + \sqrt{n}\xi\delta_{k,0}}$$

reciprocal replica space picture



➡ no reference to measurement expectation value

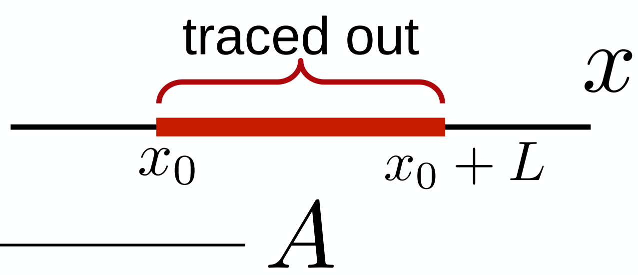
➡ exact decoupling into:

➡ 1 collective ‘hot’ mode, heating to infinite temperature

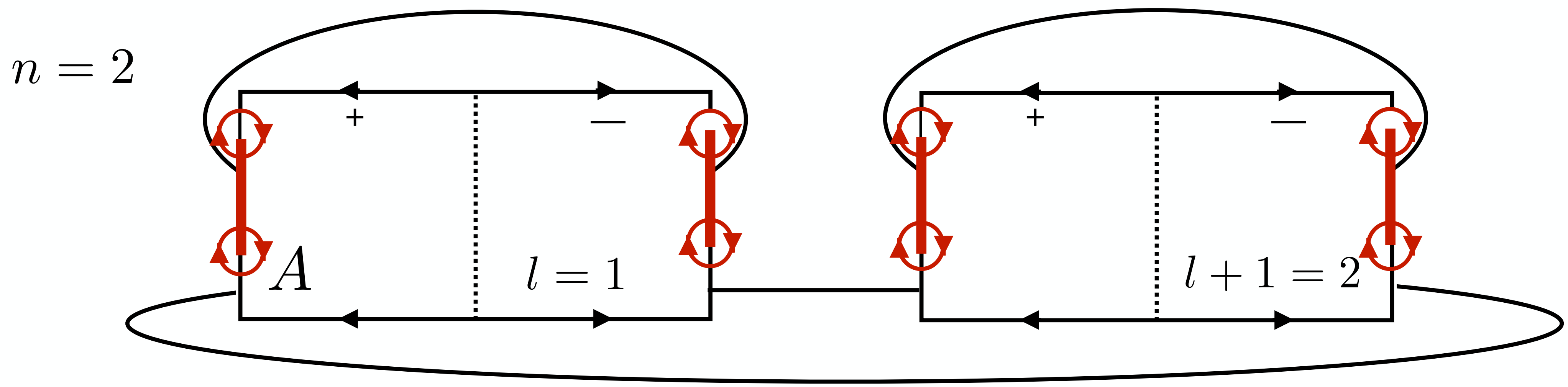
➡ (n-1) ‘cold’ modes, do not ‘see’ the noise, cool to ground state of non-hermitean Hamiltonian

Entanglement Entropies from Replica Approach

Calabrese, Cardy, JPA (2009)



- Keldysh path integral representation of $Z_A(n, \{\xi\}) = \mathrm{tr}\rho_A^n$



- formulate boundary condition via operator insertion (in the presence of noise):

$$Z_A(n), \{\xi\}) = \int \mathcal{D}[\Psi]\, T\, e^{iS_n[\Psi]} \qquad T\Psi^{(l)}_{+,x,t_f} = \begin{cases} -\Psi^{(l)}_{-,x,t_f} & \forall x \notin A \\ -\Psi^{(l+1)}_{-,x,t_f} & \forall x \in A \end{cases}$$

translation by one in replica space

- e.g. free massless Dirac fermions (after bosonization into equivalent Luttinger Hamiltonian): factorization

$$Z_A(n, \{\xi\}) = \langle \prod_{k=1}^{n} \exp\left(-\sqrt{2}i\frac{k}{n}\int dx[\delta(x - x_0) - \delta(x - (x_0 + L))]\phi^{(k)}_{c,x,t}\right)\rangle$$

ground states: Casini, Fosco, Huerta, J. Stat. Mech. (2005)

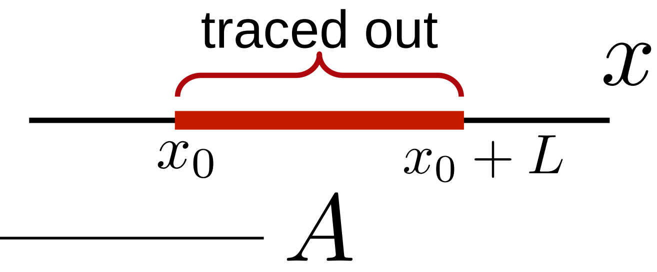
➡ boundary conditions appear as opposite charges

➡ $k = 0$ mode does not contribute!

➡ $k \neq 0$ independent of noise ξ !

Calabrese, Cardy, JPA (2009)

Entanglement Entropies from Replica Approach



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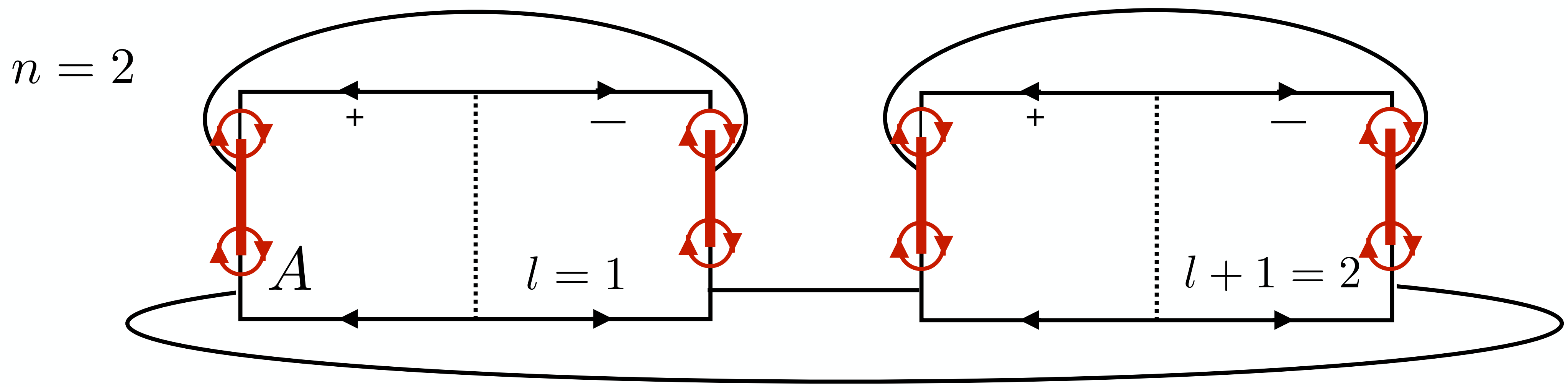


- n-th Rényi entropy

$$S_n = \frac{1}{1-n} \overline{\log Z_A(n, \{\xi\})}$$

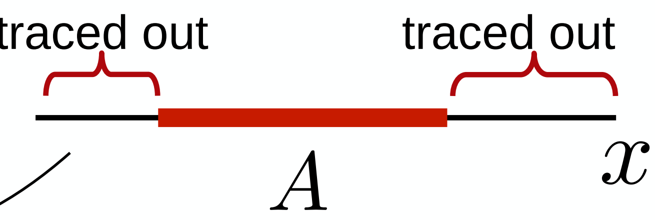
- von Neumann entropy

$$\boxed{S = \lim_{n \to 1} S_n = \frac{2}{3} \langle \phi_{c,x_0}^{(k>0)} \phi_{c,x_0+L}^{(k>0)} \rangle}$$

➡ correlator in the Gaussian dark state wave function

Construction summary

- Rényi entropy $S_n(L) = \frac{1}{1-n}\overline{\log Z_A(n, \{dW\})}, \; Z_A(n, \{dW\}) \equiv \mathrm{tr}[(\hat{\rho}_A^{(c)})^n]$

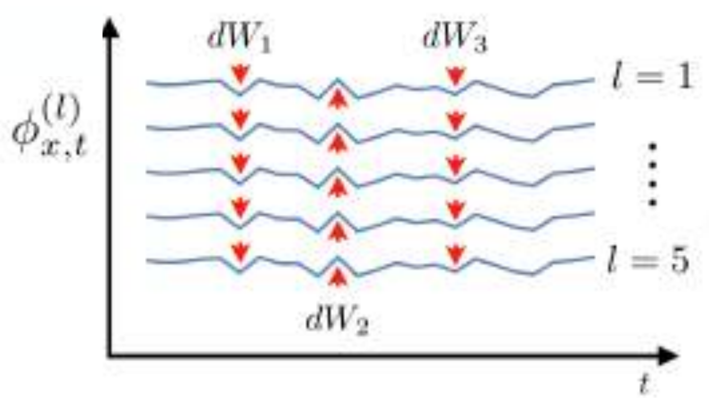


Calabrese, Cardy, JPA (2009)

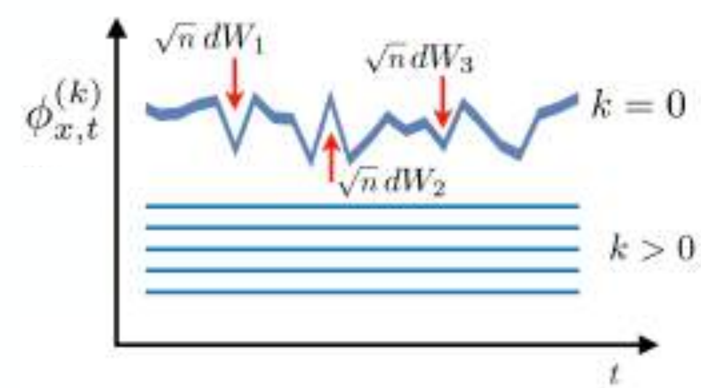
- von Neumann entropy: $n \to 1$
- in practice need theory for n Keldysh replicas

- decoupling of center-of-mass and relative modes (exact for Gaussian states -> good away from transition)

replica space picture



reciprocal replica space picture



- observation 1:
  - 1 mode heats up (noisy)
  - n-1 modes cool down (noiseless)
- observation 2:
  - noisy contribution A independent
  - all A dependence in noiseless modes!

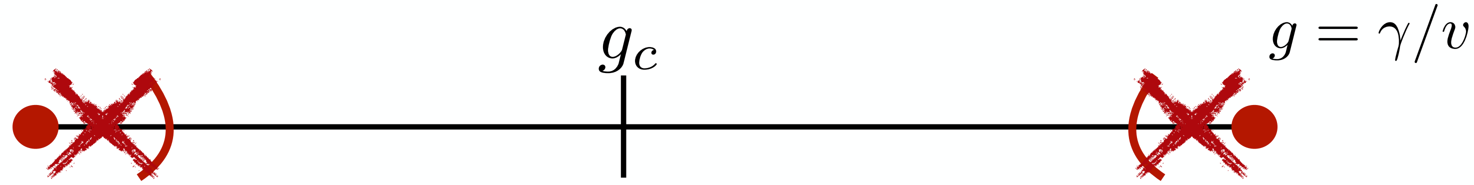
➡ entropy determined by noiseless modes

➡ rationalizes analogy to ground state phase transition

Entanglement Transition from Replica Approach

- focus on von Neumann entropy S in Gaussian limiting regimes

massless



massive

$$S = \frac{1}{3}c(\gamma)\log(L)$$

- sub-volume log-law
- $c(\gamma \to 0) \to 1$

➡ ground state entropy of massless Dirac

- in Gaussian state:

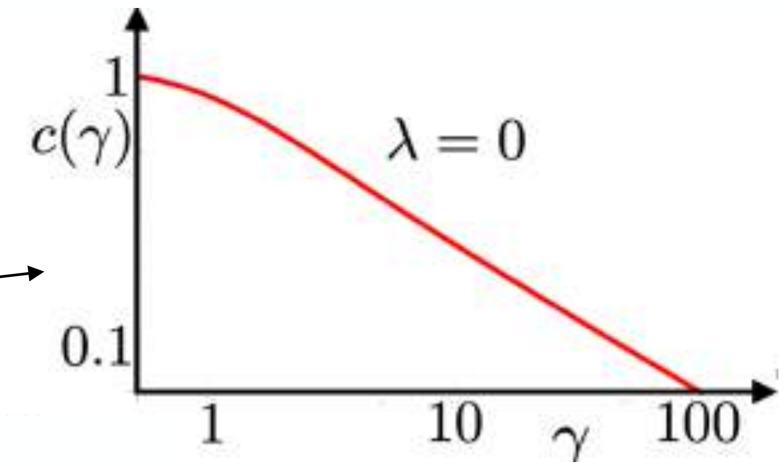
$$c(\gamma \to \infty) \sim \gamma^{-1/2}$$

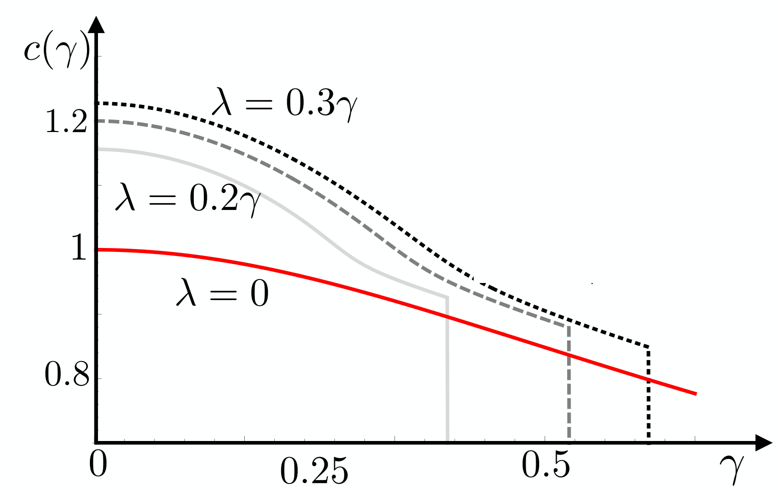
➡ compatible with numerics in critical phase

- with RG improvement, qualitatively similar to numerics

$$S = \tfrac{1}{3}c(\gamma)\log(m^{-1}) \sim L^0$$

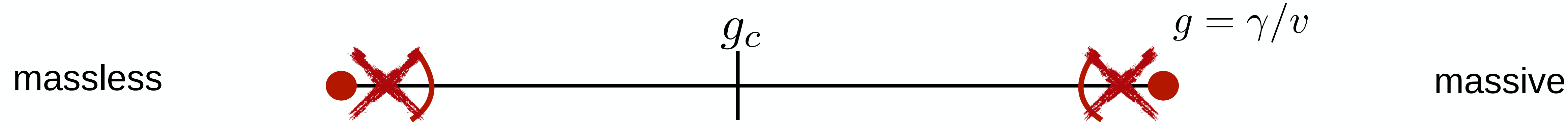
- saturation to area law





Entanglement Transition from Replica Approach

- focus on von Neumann entropy S in Gaussian limiting regimes



$$S = \frac{1}{3} c(\gamma) \log(L)$$

- sub-volume log-law
- $c(\gamma \to 0) \to 1$

➡ ground state entropy of massless Dirac

- in Gaussian state:

$$c(\gamma \to \infty) \sim \gamma^{-1/2}$$

➡ compatible with numerics in critical phase

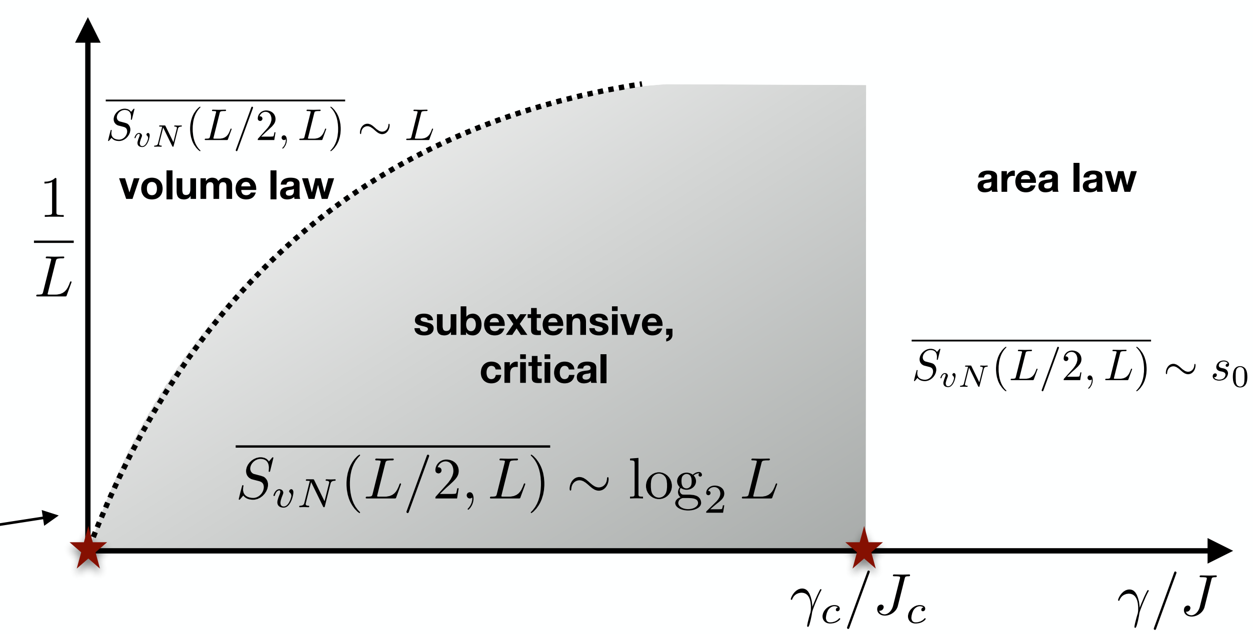
- non-commuting limit: $\gamma = 0$
  finite temperature initial state

$$S \sim L$$

➡ volume law <—> finite temperature massless Dirac

$$S = \tfrac{1}{3} c(\gamma) \log(m^{-1}) \sim L^0$$

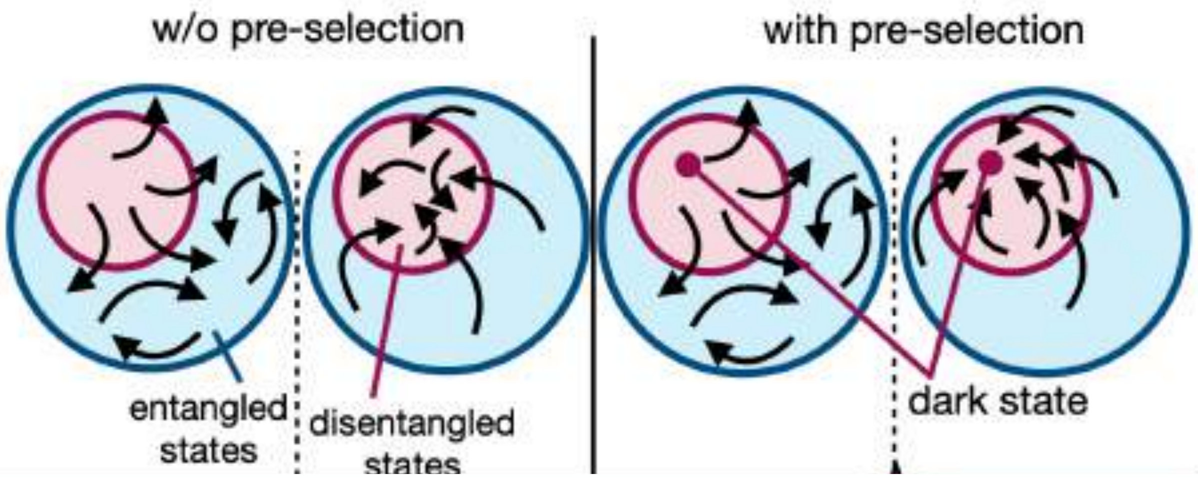
- saturation to area law

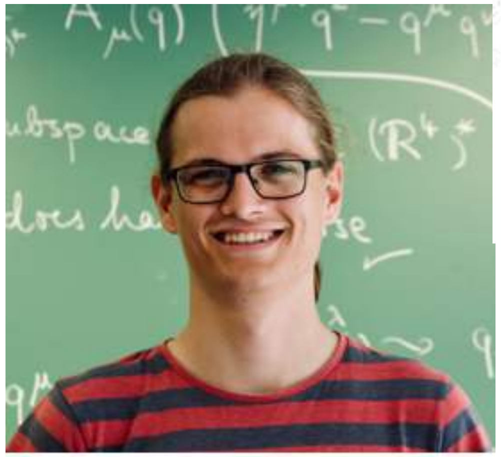


➡ underpins entanglement transition at finite critical g

➡ picture qualitatively in line with numerics

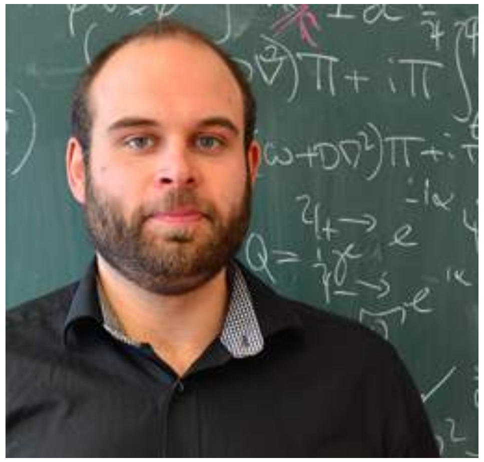
Revealing measurement-induced phase transitions by pre-selection





Thomas Müller

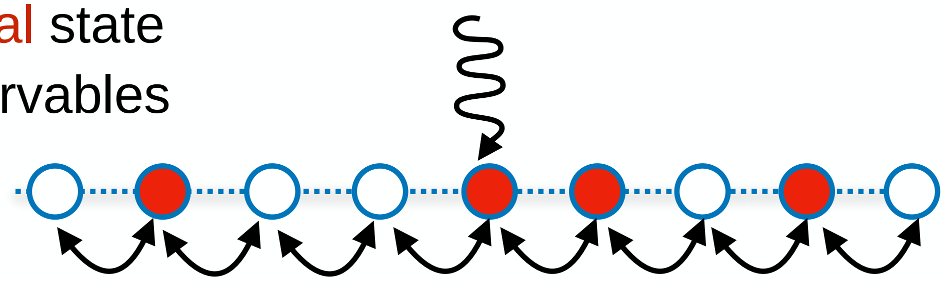
M. Buchhold, T. Müller, SD,
arxiv:2208.10506 (2022)



Michael Buchhold

Observability of the transition?

- Standard quantum mechanical observables: need to reproduce identical state
- Randomness of measurements outcomes masks transition in qm observables



- Post-selection problem:

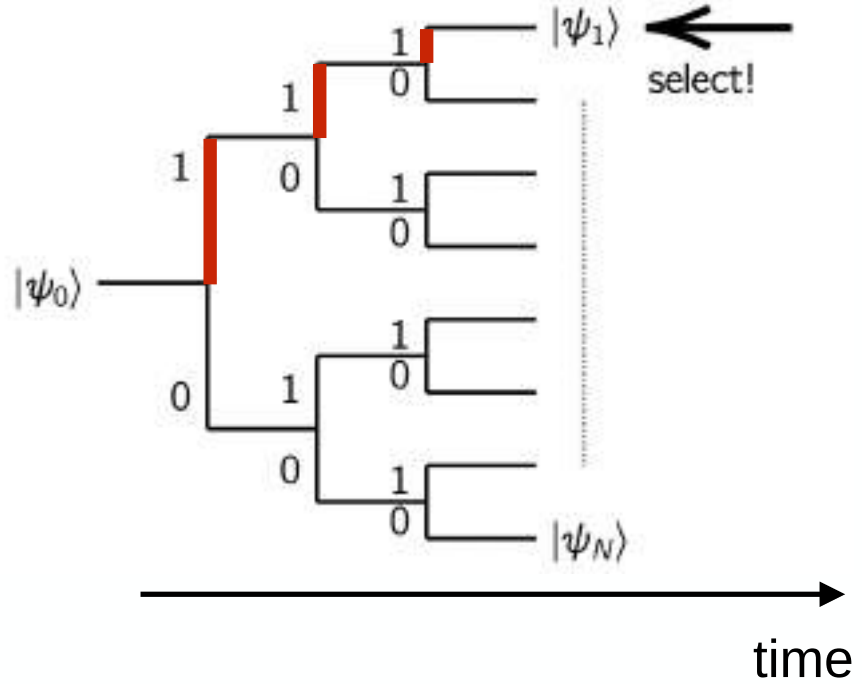
way out for Cliffords: Gullans, Huse, PRL (2020);
exp: C. Noel et al. Nat Phys. (2021)

  - Select one string of measurement outcomes (trajectory)
  - build an ensemble of identical states with same trajectory

➡ even for measuring single site, number of different outcomes

$$2^{\#\text{ of measurements}}$$

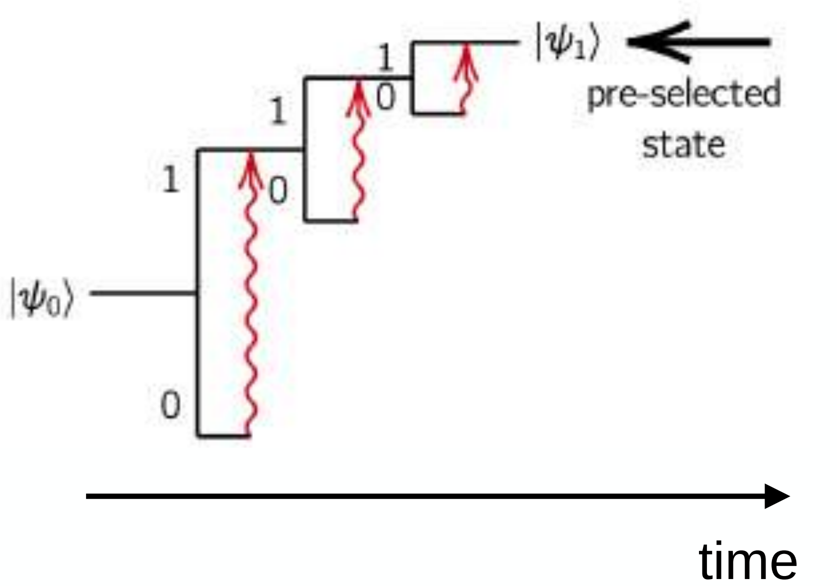
➡ post-selection exponentially costly



- Pre-selection solution:

  - break randomness in measurement outcomes
  - steer system into unique representative state in Hilbert space (dark state)

➡ can be done gently: no modification of entanglement structure & universal properties (as post-selection)
➡ no exponential overhead: observable by standard means in NISQ platforms



Pre-selection concept

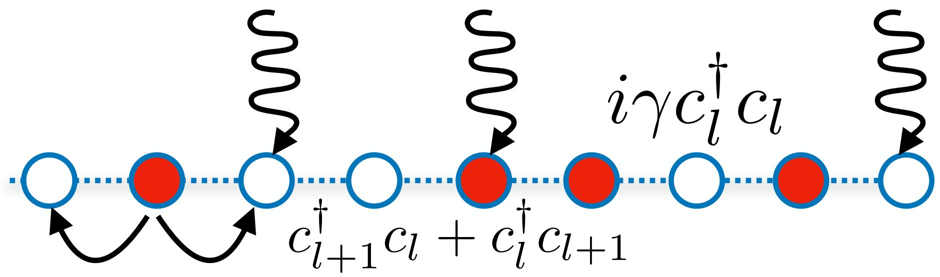
- pull MIPT to observable level <—> study measurement averaged density matrix

$$\hat{\rho}_{t+\delta t} = \hat{\mathcal{X}}[\hat{\rho}_t]\delta t$$ e.g. temporal continuum limt $$\hat{\mathcal{X}}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \sum_l \hat{L}_l \hat{\rho} \hat{L}_l^\dagger - \frac{1}{2}\{\hat{L}_l^\dagger \hat{L}_l, \hat{\rho}\}$$

- unmodified dynamics w/o preselection:

$\hat{\mathcal{X}}^{(0)}$ with $\hat{H}^{(0)}, \quad \hat{L}_l^{(0)} = (\hat{L}_l^{(0)})^\dagger$ hermitian

$\Longrightarrow \quad \hat{\mathcal{X}}^{(0)}[\hat{\mathbf{1}}] = 0$ fully mixed stationary state



ex.: monitored fermions

- modified dynamics with pre-selection:

$\hat{H}^{(0)}, \hat{L}^{(0)} \rightarrow \hat{H}, \hat{L}$ such, that

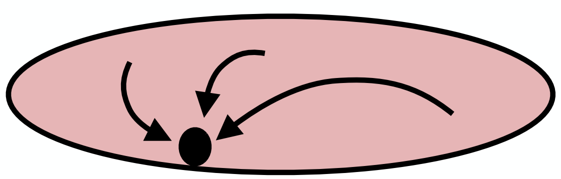
1. There exists a pure dark state representing one possible measurement outcome,

   $\hat{\rho}_D = |D\rangle\langle D|$ s.t. $\hat{\mathcal{X}}[\hat{\rho}_D] = 0$

   ➡ pre-select representative state (e.g. with order parameter)

2. The dark state is unique

   ➡ directionality in Hilbert space



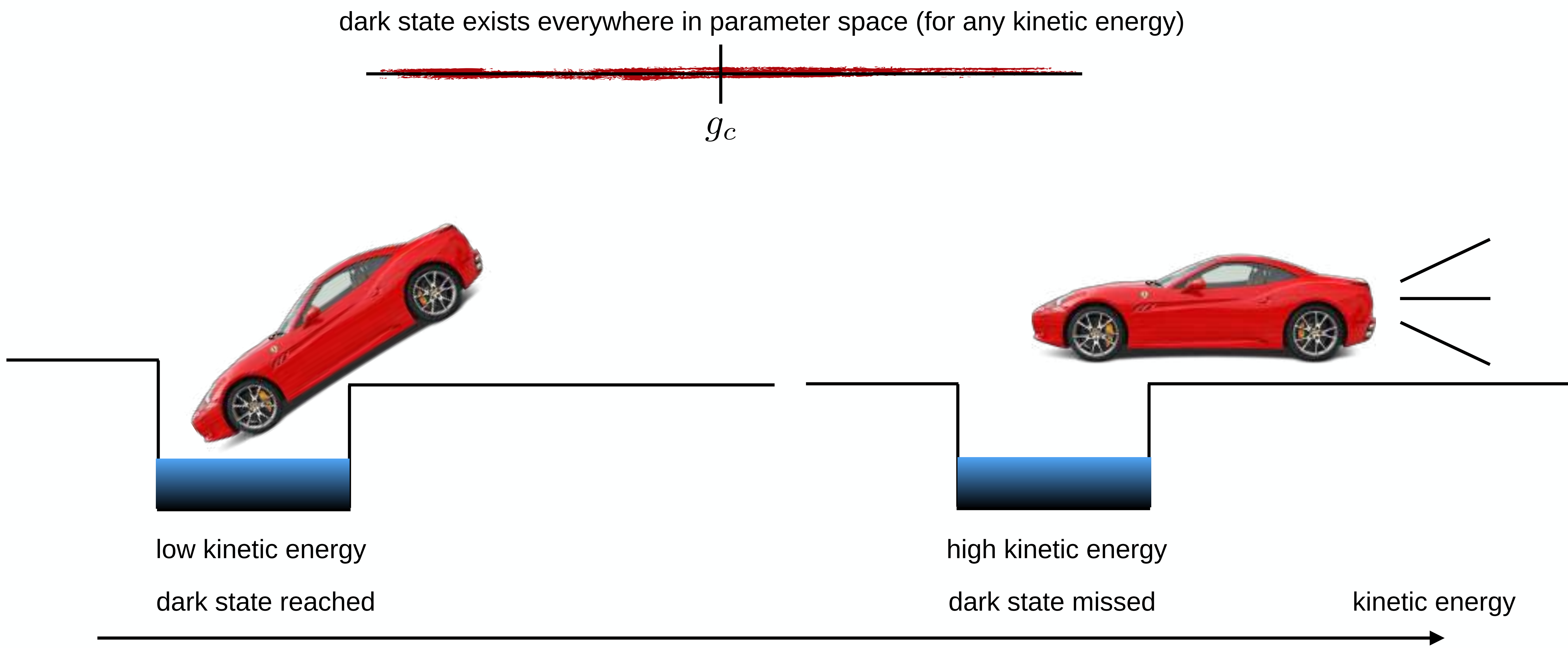
3. Modifications preserve symmetries and are irrelevant in the RG sense

   ➡ phases and phase transitions unmodified

Pre-selection concept: Existence of phase transition

- phase transition exists despite global dark state: quantum absorbing state transition

- global dark state and phase transition:

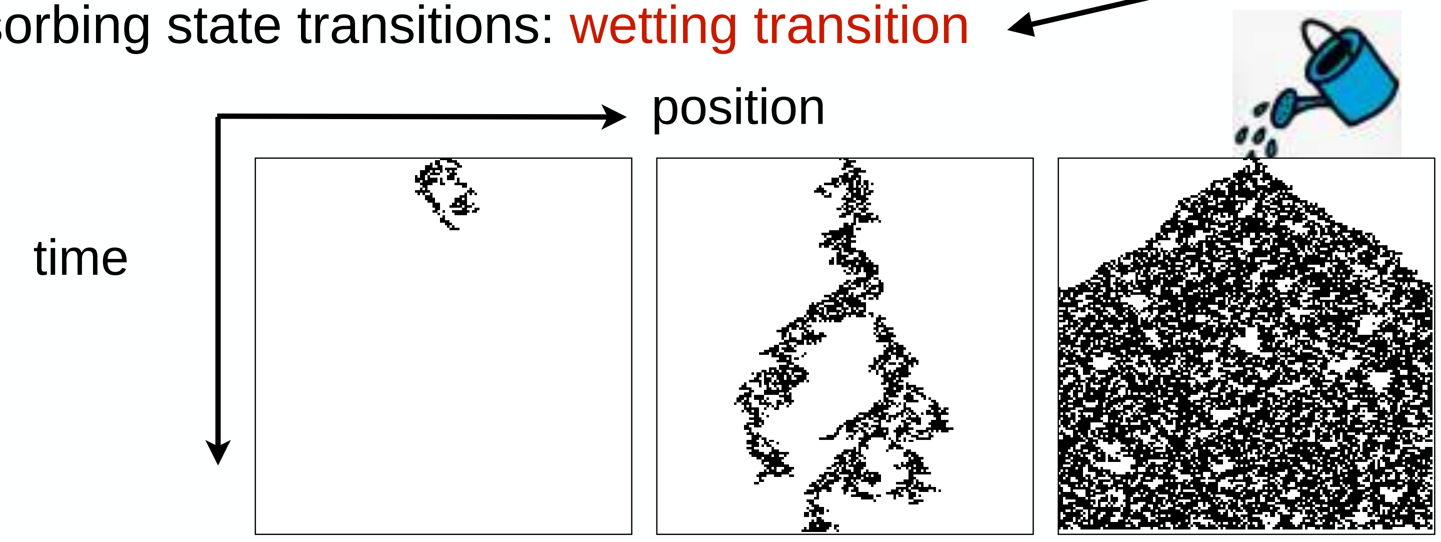


Pre-selection concept: Existence of phase transition

- phase transition exists despite global dark state: quantum absorbing state transition
- classical absorbing state transitions: wetting transition

classical absorbing state transitions: H. Hinrichsen, Adv. Phys. (2000); G. Odor, RMP (2004)

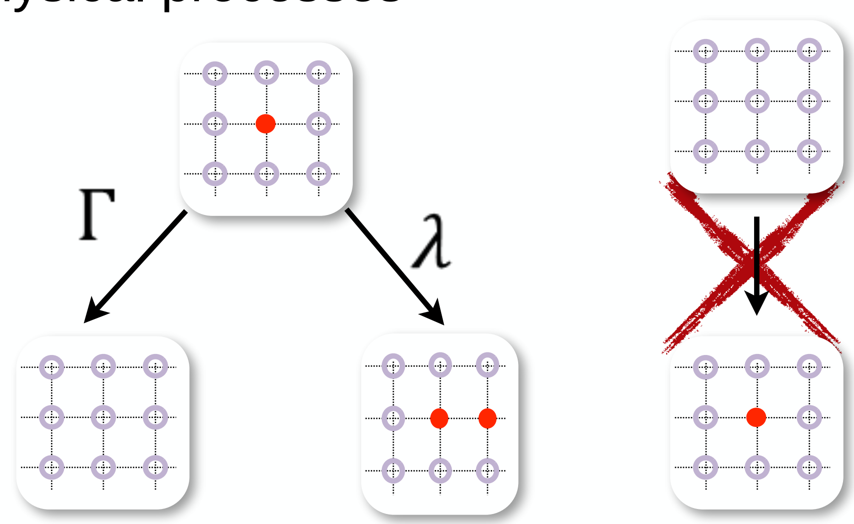
quantum absorbing state transitions: Marcuzzi, Buchhold, SD, Lesanovsky, PRL (20016); Carollo, Gillman, Weimer, Lesanonvsky, PRL (2019)



absorbing/dry state   critical state   wetted state

H. Hinrichsen, Adv. Phys. (2000)

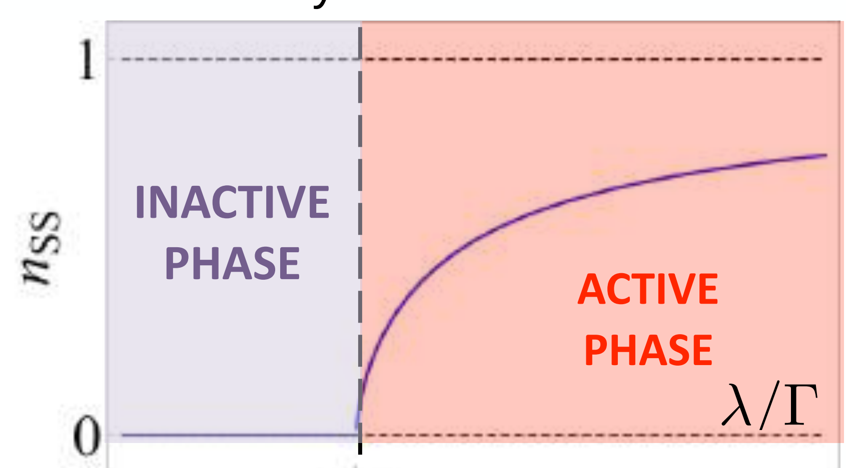
- physical processes



evaporation   branching   no offspring from vacuum
<-> global dark state

- phase transition:

density of active sites



➡ transition in directed percolation universality class

➡ true phase transition in 1+1 dim. possible

➡ dark state acts as attractor below threshold

➡ dark state missed above threshold (up to times $t \sim e^L$ via rare events)

Pre-selection: Analogy to absorbing state transitions

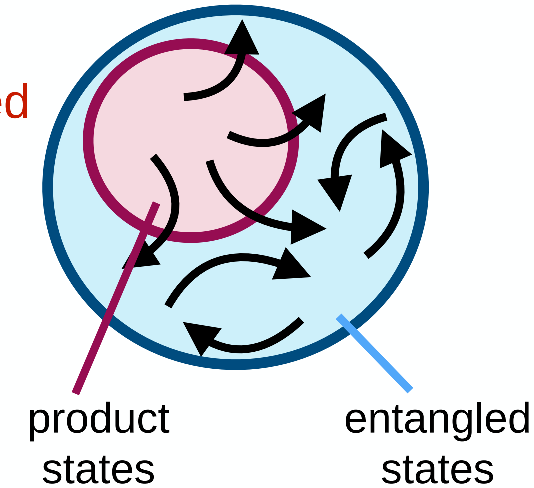
- dynamics $\partial_t \hat{\rho} = \mathcal{X}(\hat{\rho}) = \mathcal{X}_1(\hat{\rho}) + \gamma \mathcal{X}_2(\hat{\rho})$

unitary non-unitary

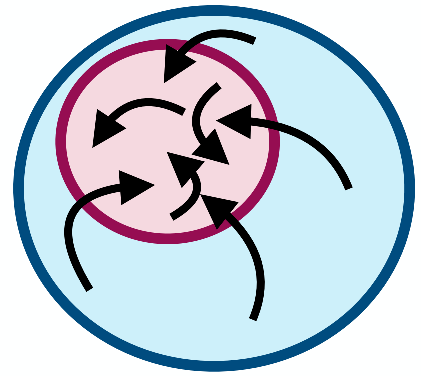
$\mathcal{X}_1(\hat{\rho})$: $\mathcal{X}_2(\hat{\rho})$:

- unmodified measurement dynamics:

prefers entangled



prefers product



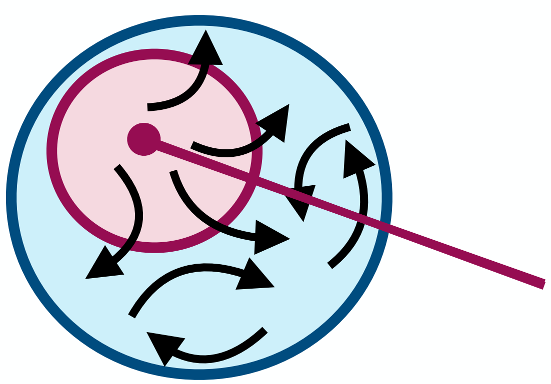
trajectory entanglement entropy:

$$\gamma \geq \gamma_c : \quad S(\hat{\rho}_A) \sim |A|^0$$

$$\gamma < \gamma_c : \quad S(\hat{\rho}_A) \sim |A|, \log(|A|)$$

but: extensive configuration entropy

- pre-selected dynamics w/ dark state:



$\hat{\rho}_D$

1. There exists a pure dark state representing one possible measurement outcome,
   $\hat{\rho}_D = |D\rangle\langle D|$ s.t. $\hat{\mathcal{X}}[\hat{\rho}_D] = 0$

   ➡ pre-select representative state (e.g. with order parameter)
2. The dark state is unique

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3. Modifications irrelevant in the RG sense

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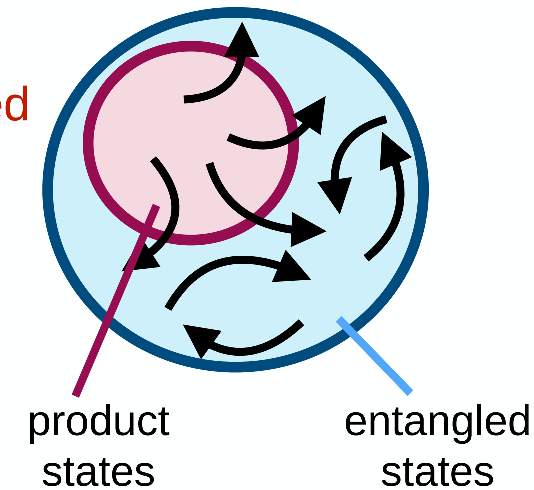
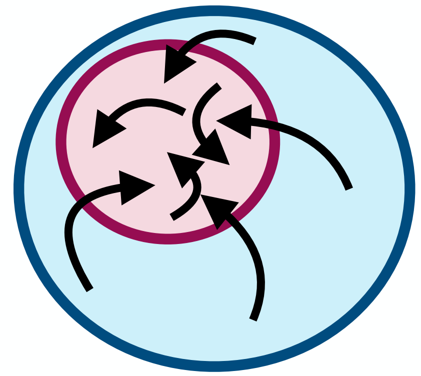
- unmodified measurement dynamics:

prefers entangled



prefers product



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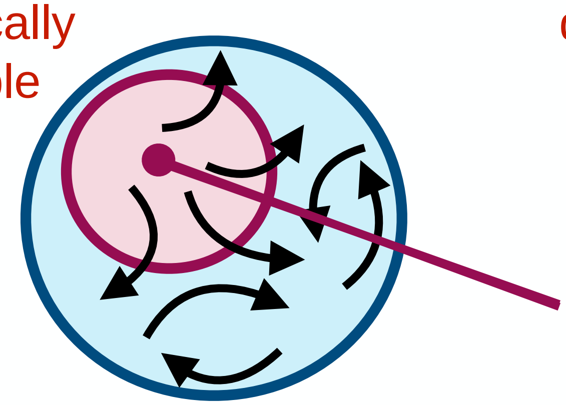
but: extensive configuration entropy

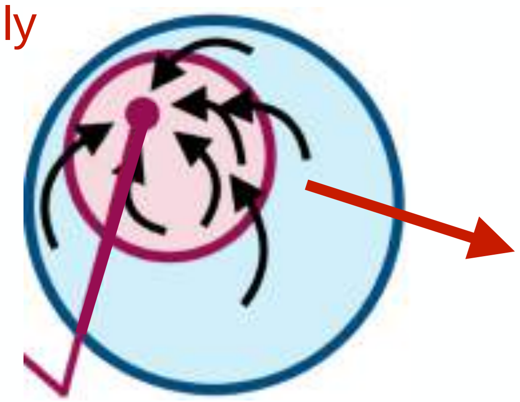
- pre-selected dynamics w/ dark state:

dynamically unstable



$\hat{\rho}_D$

dynamically stable



pre-selection succeeds

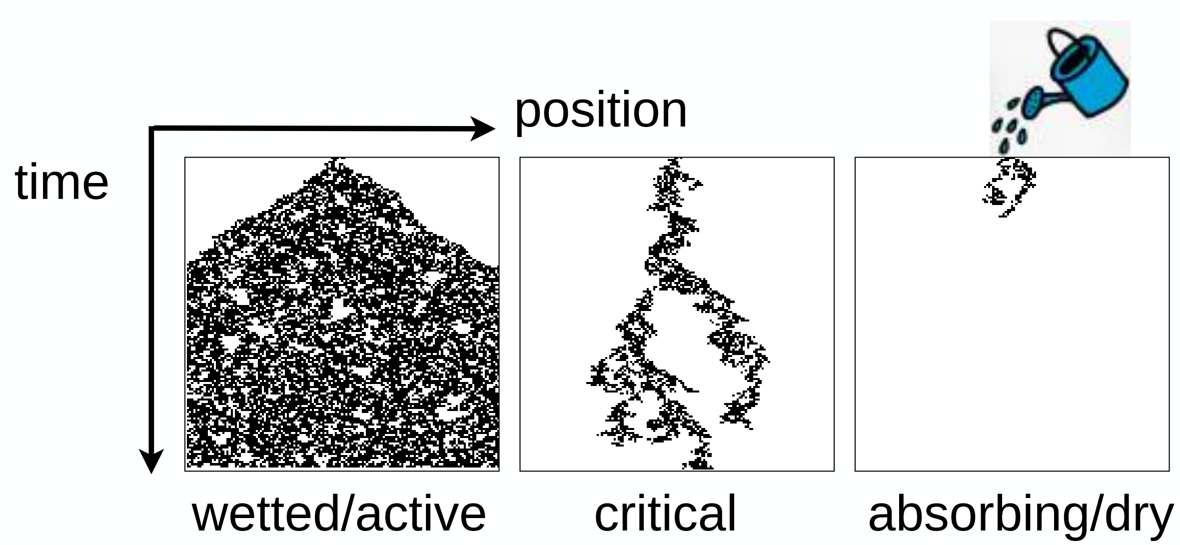
$$\gamma \geq \gamma_c : \quad \hat{\rho} \to \hat{\rho}_D$$

configuration entropy removed!

$$\gamma < \gamma_c : \quad \hat{\rho} \not\to \hat{\rho}_D$$

pre-selection fails

wetting transition

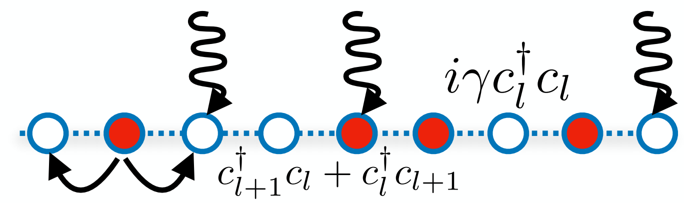


➡ transition in the dynamics

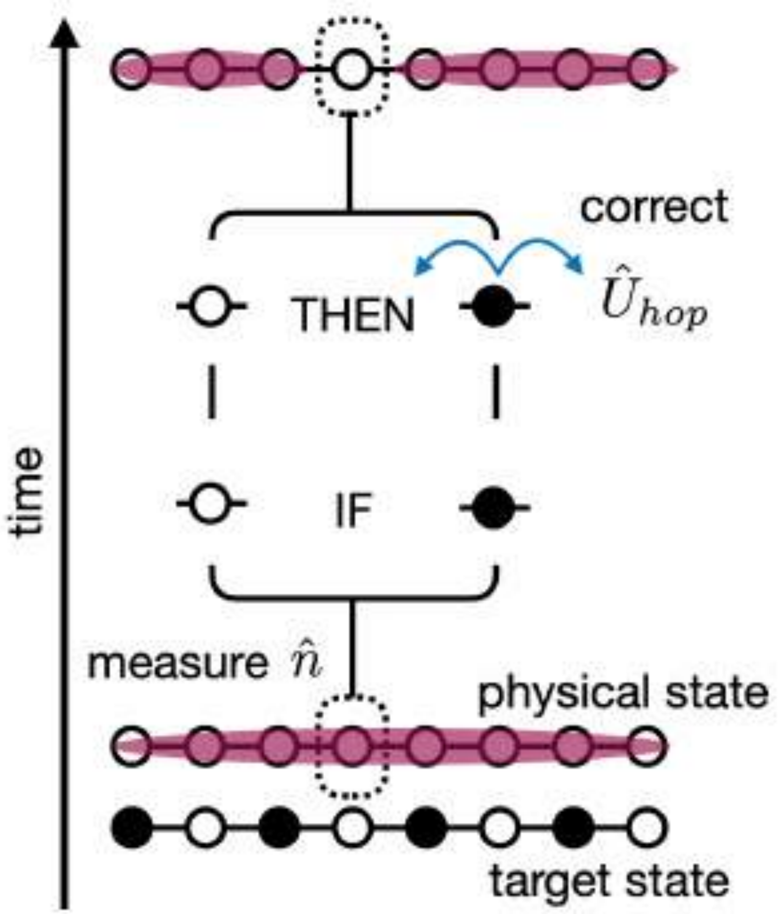
➡ structure of absorbing state transition in Hilbert space!

H. Hinrichsen, Adv. Phys. (2000);
G. Odor, RMP (2004)

Pre-selection strategies: modified dynamics



- consider monitored free fermions
- choose a Néel target state
- impose IF-THEN clause



$$|D\rangle = |101010...\rangle$$

Néel charge density wave

$$\hat{\rho}_D = |D\rangle\langle D|$$

- three variants:
  - 'classical' feedback: modify parameters - non-linear in state

    Wiseman, Milburn, PRL (1993)

  - 'quantum' feedback: modify operators - linear in state

    Santos, Iemini, Kamenev, Gefen, Nat. Comm. (2020)

  - quantum simulation: autonomous scheme w/o readout - linear

    SD, Micheli, Kantian, Kraus, Büchler, Zoller, Nat. Phys. (2008), Verstraete, Wolf, Cirac, Nat. Phys. (2009)

‘Classical’ pre-selection dynamics

- monitored fermions: Néel target dark state (half filing):

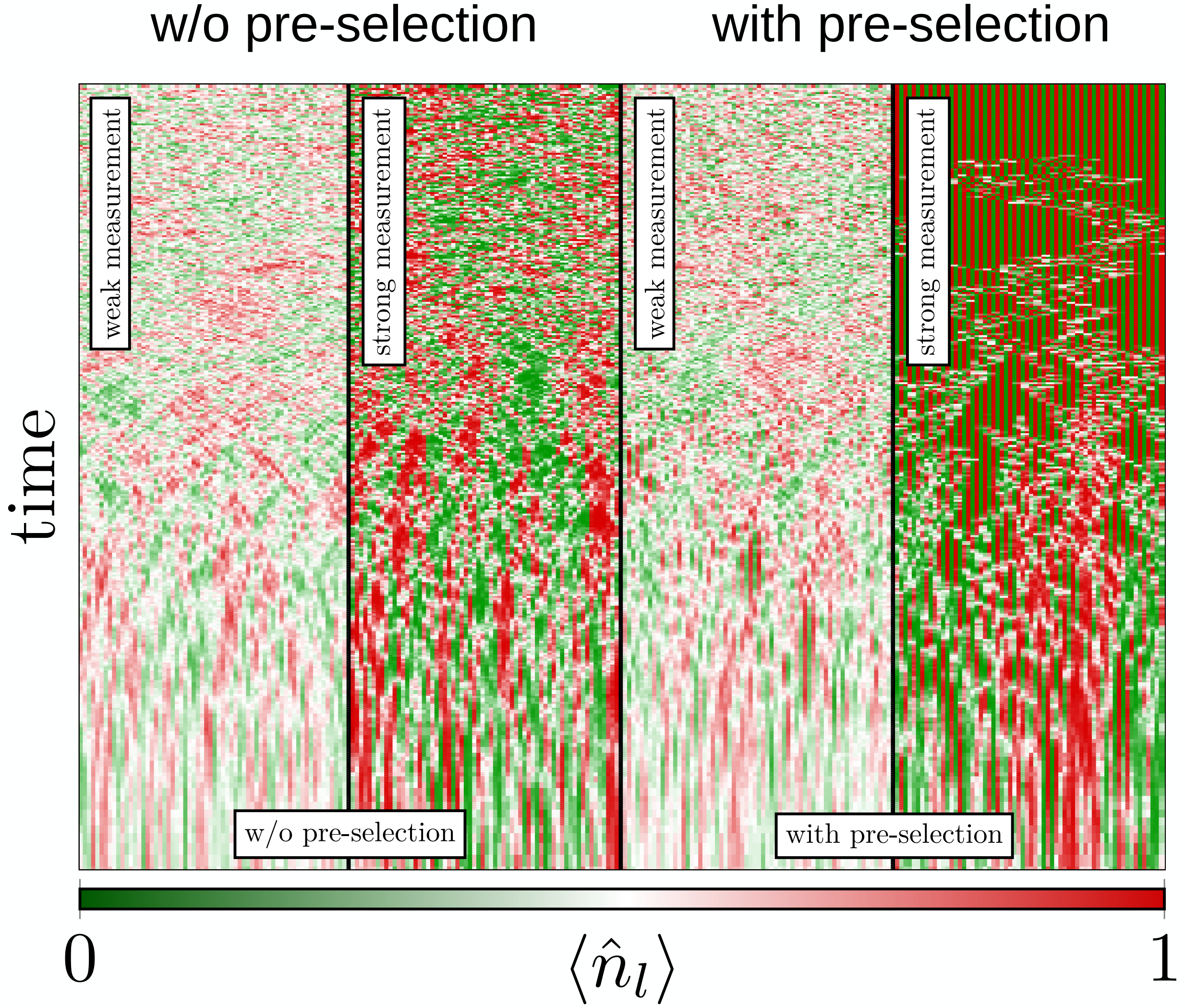
$$|D\rangle = |101010...\rangle \qquad \hat{\rho}_D = |D\rangle\langle D|$$

- condition hopping on site occupation rounded to nearest integer

$$J \to J_l[\hat{\rho}] = 2 - (\lfloor\langle\hat{n}_{l-1}\rangle\rfloor - \lfloor\langle\hat{n}_l\rangle\rfloor)^2 - (\lfloor\langle\hat{n}_{l+1}\rangle\rfloor - \lfloor\langle\hat{n}_{l+2}\rangle\rfloor)^2$$

$$\hat{H}^{(0)} \to \hat{H} = \hat{H}[\hat{\rho}] \quad \text{with} \quad \hat{H}[\hat{\rho}_D] = 0$$

- different phases distinguishable on the level of single trajectories



- intuitive in limiting cases:
- weak measurements
  - scrambling due to hopping:

  $$\langle\hat{n}_l\rangle = 1/2 \implies J_l[\hat{\rho}] \to 1$$

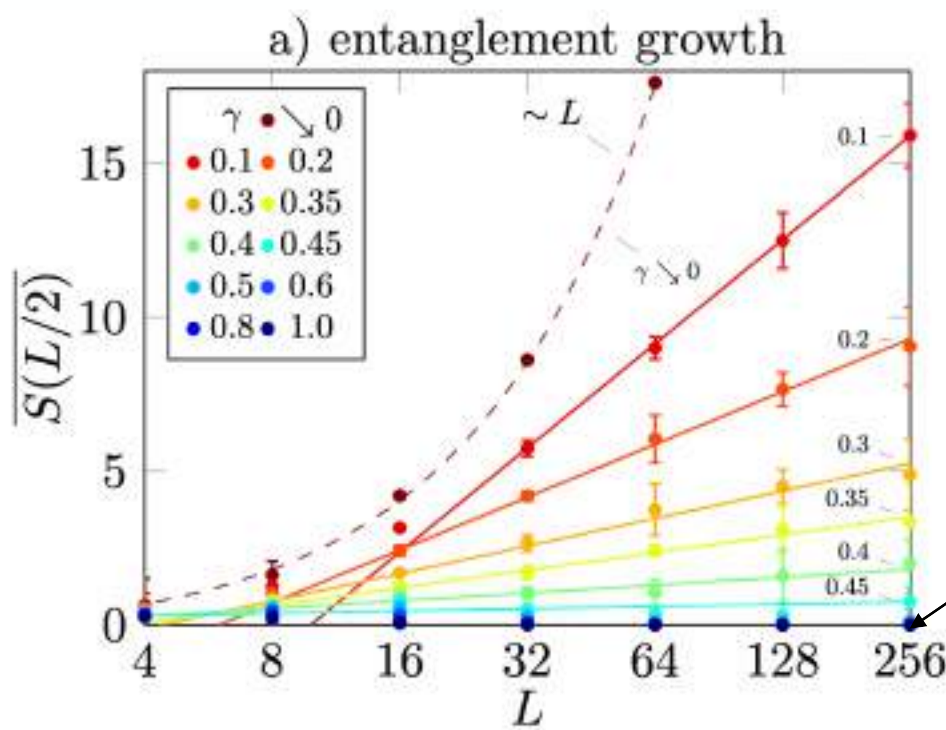
  ➡ effectively translation invariant dynamics as w/o pre-selection
- strong measurements
  - pinning due to local measurement:

  $$\langle\hat{n}_l\rangle\hat{\rho} \sim \hat{n}_l\hat{\rho} \implies J_l(\hat{\rho}) \to \hat{J}_l \qquad [\hat{J}_l, \hat{H}^{(0)}] \neq 0$$

  ➡ effectively non-Hermitian Hamiltonian steering to unique dark state

‘Classical’ pre-selection dynamics: Phases and phase transition

- properties of pre-selected system

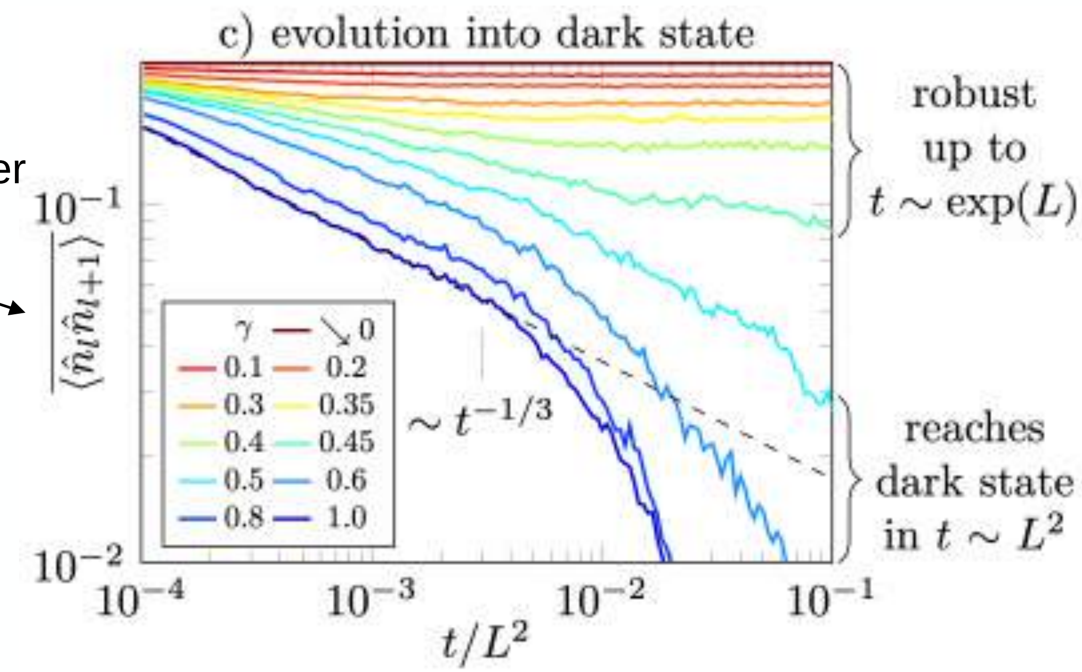


product dark state reached: no entanglement

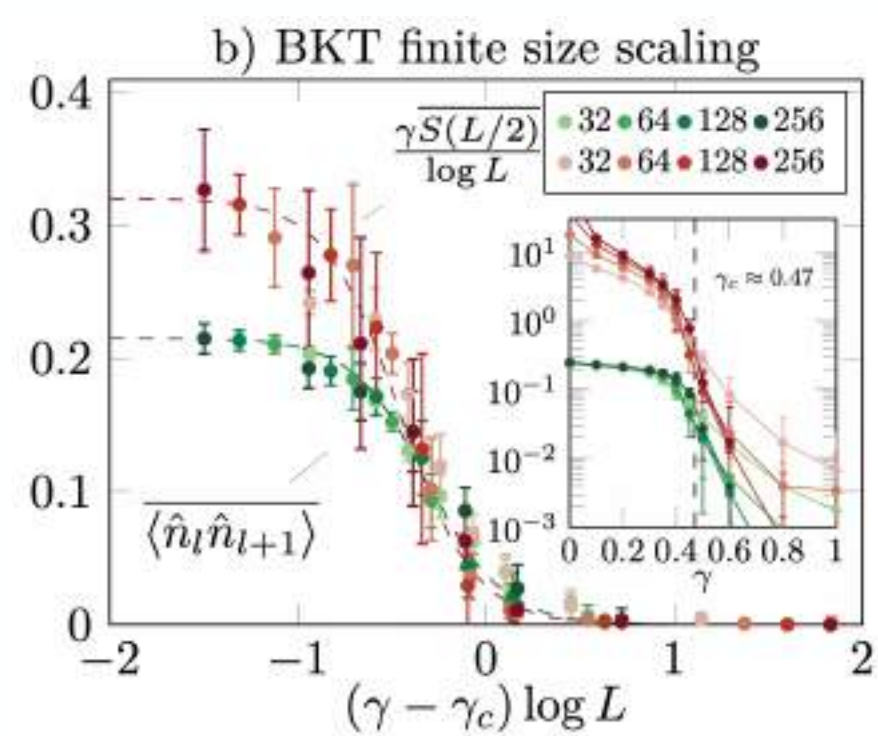
➡ log- to area law transition at critical value

$$\gamma_c \approx 0.47$$

observable Néel order parameter



➡ characteristic behavior of absorbing state transition w/ conservation law



➡ BKT scaling of entanglement entropy and Néel order parameter

➡ quantities that can be compared coincide: phases / phase transition unmodified

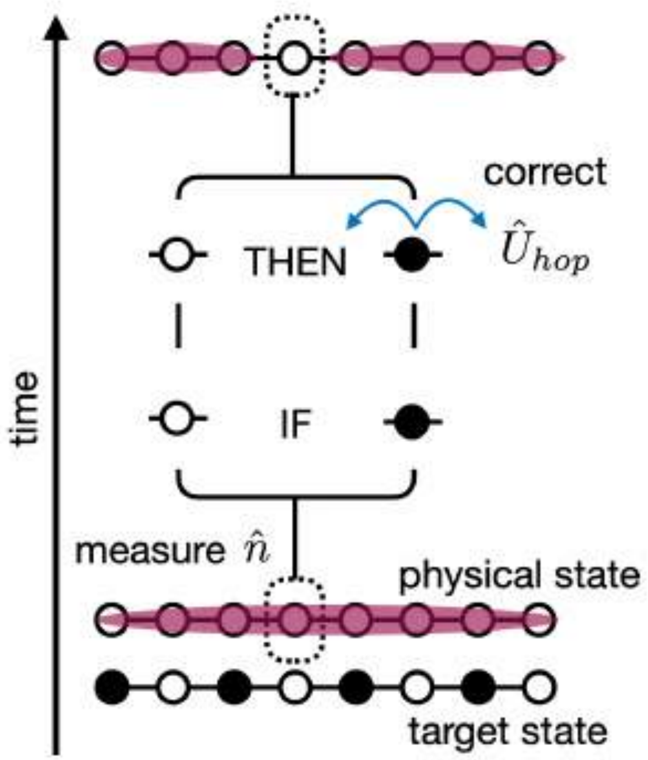
➡ but there exist standard qm observables witnessing it

Example: ‘quantum’ feedback

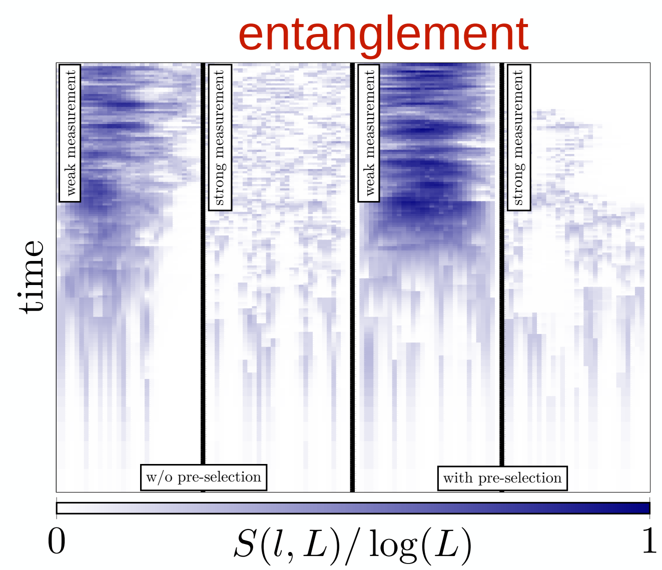
modified measurement operators

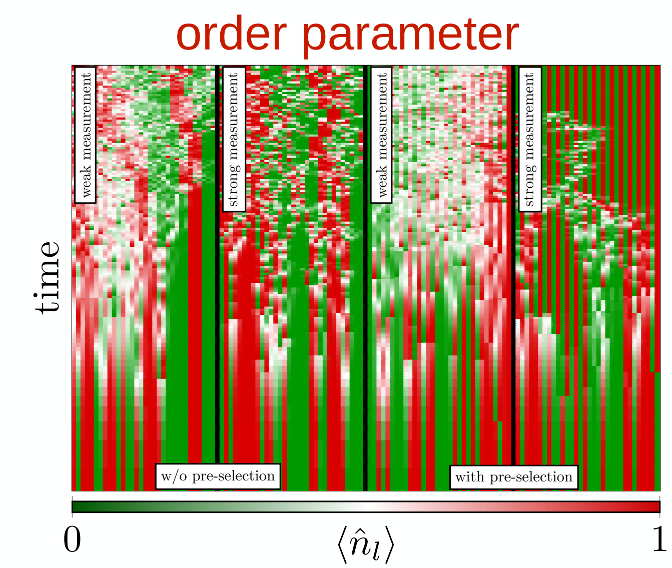
- Quantum schemes: linear in state $$\partial_t \rho = i[\rho, H] - \gamma \sum_l \{L_l^\dagger L_l, \rho\} - 2 L_l \rho L_l^\dagger$$

$$[\hat{H}, \hat{\rho}_D] = 0$$
$$\hat{L}_l \hat{\rho}_D = 0$$



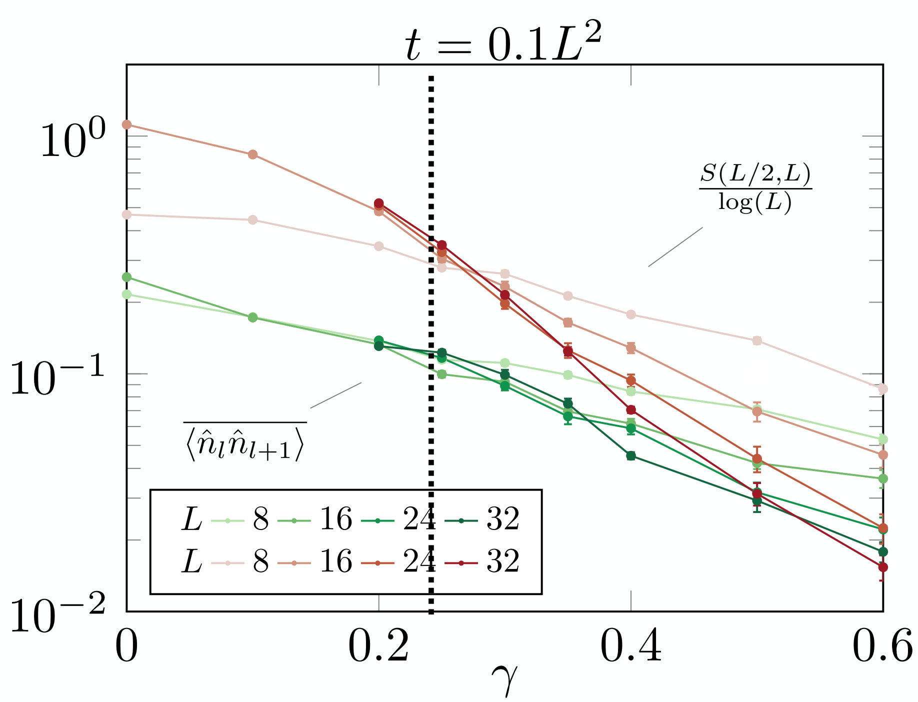
- measurement preselection:
  $\hat{L}_l = \hat{U}_l \hat{n}_l$
  $[\hat{U}_l, \hat{n}_l] \neq 0$
- non-Gaussian evolution





DMRG simulations, M. Buchhold, T. Müller, SD, in progress

- detect transition



- field theory

$$\hat{L}_l = \hat{n}_l \to \exp[i\pi(c_l^\dagger c_m + c_m^\dagger c_l)]\ \hat{n}_l$$

$$\partial_x \hat{\phi}_x + \cos(\hat{\phi}_s) \to \partial_x \hat{\phi}_x + \cos(\hat{\phi}_s) + i \partial_x \hat{\theta}_x$$

- Heisenberg-Langevin theory (or equivalent Keldysh)

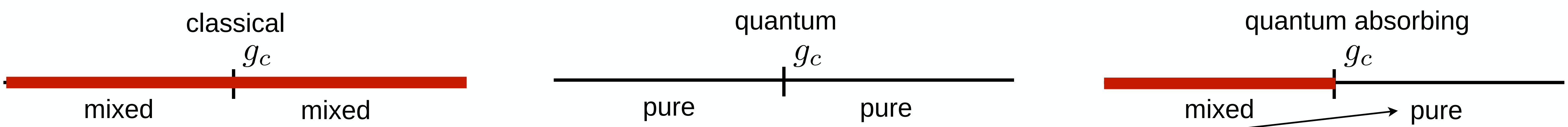
$$(\partial_t^2 - \partial_x^2)\hat{\phi}_x + \gamma \cos \hat{\phi}_x + \gamma \sin \hat{\phi}_x (\partial_t \hat{\phi}_x + \sin \hat{\phi}_x) = \hat{\xi}_x$$

➡ irrelevant in weak measurement regime: infinite T state

➡ becomes relevant at transition: pinning

Outlook: Quantum absorbing transitions beyond directed percolation?

- Grassberger-Janssen conjecture: No!   Janssen, Z. Phys. B (1981); Grassberger, Z. Phys. B (1982)
- Qualitative overview:



- possible pure state scenarios:

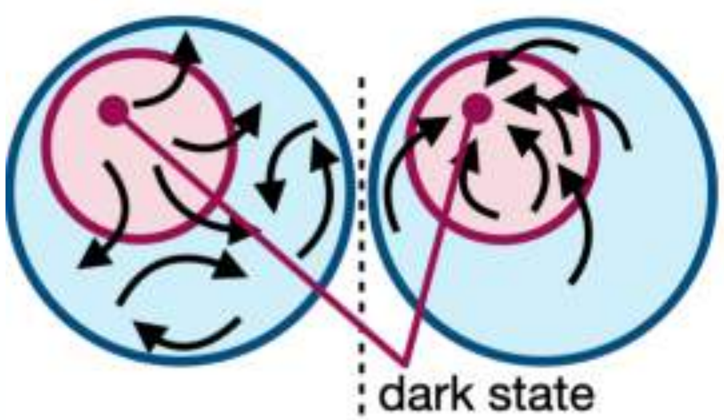
Sieberer, Buchhold, Diehl ROPP (2016)

  - general quantum dynamics w/ dark/absorbing state (or equivalent Keldysh)

$$\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}] + \gamma \sum_i \hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \tfrac{1}{2}\{\hat{L}_i^\dagger \hat{L}_i, \hat{\rho}\} \qquad \hat{H} = \sum_i \hat{h}_i \to \hat{H} = \sum_i \hat{h}_i \hat{L}_{i-1}^\dagger \hat{L}_{i-1}$$

repulsive fixed point

attractive fixed point



dark/absorbing state

$$\hat{L}_i |D\rangle = 0 \ \forall i$$

Outlook: Quantum absorbing transitions beyond directed percolation?

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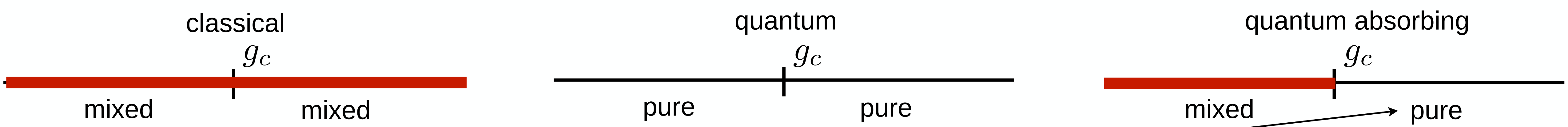


- possible pure state scenarios:

Sieberer, Buchhold, Diehl ROPP (2016)

  - general quantum dynamics w/ dark/absorbing state (or equivalent Lindblad-Keldysh functional integral)

$$\partial_t \hat\rho = -i[\hat H, \hat\rho] + \gamma \sum_i \hat L_i \hat\rho \hat L_i^\dagger - \tfrac{1}{2}\{\hat L_i^\dagger \hat L_i, \hat\rho\} \qquad \hat H = \sum_i \hat h_i \to \hat H = \sum_i \hat h_i \hat L_{i-1}^\dagger \hat L_{i-1}$$

$\hat L_i |D\rangle = 0 \ \forall i$ dark/absorbing state

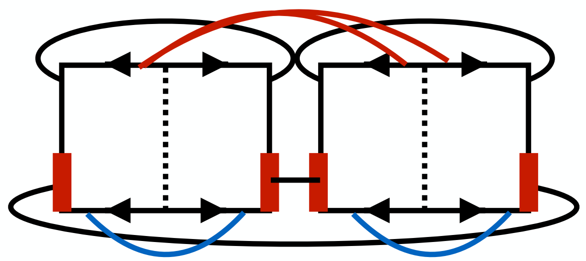
Dynamics / State

	product (w/o entanglement)	entangled ('non-trivial vacua')
w/o conservation law	directed percolation	$\lvert D\rangle =$ (BCS superfluid)
w/ conservation law	1+1: present problem, BKT d+1, d > 1: tbd	$\lvert D\rangle =$ (topological insulator)

Summary

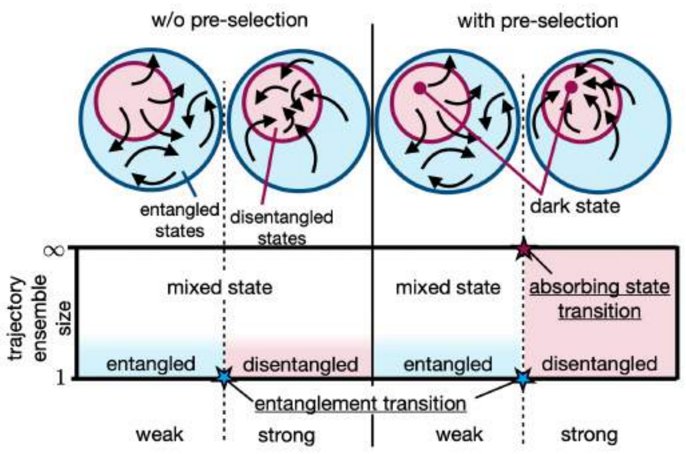
General non-unitary quantum dynamics hosts new types of phases and phase transitions

- here: critical to area law entanglement phase transition
- BKT transition revealed by Keldysh replica field theory



Observability via pre-selection

- like post-selection, but no exponential overhead
- various strategies for NISQ platforms
- link to quantum absorbing state transitions



Outlook

- general principles for observability?
- higher dimensions: novel quantum absorbing state transitions beyond directed percolation?
- relation of measurement vs. disorder problems? role of conservation laws? symmetry classification? Poboiko, Pöpperl, Gornyi, Mirlin, arXiv:2304.03138 (2023)
- emergent CFT behavior?