Ex 1 : Lindblad-Keldysh functional integral

We fix some details of derivation.

a) The Trotter step.

- Recall $\hat{\rho}(t) = e^{\hat{\mathcal{L}}(t-t_0)}\hat{\rho}(t_0) = \lim_{N\to\infty}\left(e^{\hat{\mathcal{L}}\delta t}\right)^N \hat{\rho}(t_0) = \lim_{N\to\infty}(\hat{\mathbb{1}} + \hat{\mathcal{L}}\delta t)\hat{\rho}(t)$
- For $t_n = t_0 + n\cdot\delta t$, $\hat{\rho}_n = \hat{\rho}(t_n)$, represent

$$\hat{\rho}_n = \int \prod_{\sigma=\pm} d(\phi^*_{\sigma,n}, \phi_{\sigma n})\; e^{-\sum_\sigma \phi^*_{\sigma n}\phi_{\sigma n}} \underbrace{\langle\phi_{+n}|\hat{\rho}_n|\phi_{-n}\rangle}_{\rho_n \text{ matrix element}} |\phi_{+n}\rangle\langle\phi_{-n}|$$

- For definiteness, work with a single degree of freedom,

$$\mathcal{L}[\hat{\rho}] = -i[\omega_0\hat{a}^+\hat{a}, \hat{\rho}] + \gamma\left(2\hat{a}\rho\hat{a}^+ - \{\hat{a}^+\hat{a}, \hat{\rho}\}\right)$$

for bosonic creation/annihilation operators $\hat{a}^+, \hat{a}$. This will illustrate the main points (*).

- Find the matrix element $\rho_{n+1} = \langle\phi_{+n+1}|\hat{\rho}_{n+1}|\phi_{-n+1}\rangle$ appearing in the coherent state representation of $\hat{\rho}_{n+1}$.

  Convince yourself that the Lindblad structure is reproduced.

b) Complete the steps to a contour functional integral.

(*) Comment: There is a subtlety w/o counterpart in Hamilton functional integrals: Even if $\hat{H}$, $\hat{L}$ are normal ordered, this need not be the case for L^+L (example?). This has to be accounted for by a <u>point splitting</u> in time, inserting a resolution of $\mathbb{1}$ between time steps n, $n+1$.

Ex. 2: Mean field dynamics in operator and Keldysh theory

We study the Lindblad ϕ^4 workhorse model.

a) - Compute the evolution of the field operator expectation value

$$\partial_t \langle \hat{\phi}(\vec{x}) \rangle (t) = \partial_t \operatorname{tr}(\hat{\phi}(\vec{x})\, \hat{\varrho}(t)) = \operatorname{tr}(\hat{\phi}(\vec{x})\, \hat{\mathcal{L}}[\hat{\varrho}])$$

Hint: Use cyclic invariance to bring to a form containing only commutators $[\hat{H}, \hat{\phi}(\vec{x})]$, $[\hat{L}, \hat{\phi}(\vec{x})]$.

- Make the ansatz $\hat{\varrho} = \prod_{\vec{x}} \hat{\varrho}(\vec{x})$, $\hat{\varrho}(\vec{x}) = |\phi(\vec{x})\rangle\langle\phi(\vec{x})|$ with coherent states $|\phi(\vec{x})\rangle$. You obtain a dissipative Gross-Pitaevskii eq.
- Further simplify with ansatz $\phi(\vec{x}, t) = \phi(t)$, and solve for the stationary state with $\phi(t) = \sqrt{\varrho_0}\, e^{i\nu t}$. Determine ν, ϱ_0 as function of $\gamma_e - \gamma_p$ (assuming all other parameters positive).

b) Now derive the same equation in the deterministic limit of Keldysh field theory: $S[\Phi_+, \Phi_-]$

- Write the Lindblad-Keldysh action for the workhorse ϕ^4 Lindblad eq.
- Perform a Keldysh rotation and expand to first order in Φ_q

$$S[\Phi_+ = (\Phi_c + \Phi_q)/\sqrt{2},\ \Phi_- = (\Phi_c - \Phi_q)/\sqrt{2}]$$

- Compare the dissipative Gross-Pitaevskii equation to the deterministic equation of motion $\left.\frac{\delta S}{\delta \phi_c^*}\right|_{\phi_q = 0} = 0$.

Ex. 3: Keldysh theory for the noisy quantum oscillator

Consider the Lindblad equation (*)

$$\partial_t \hat{\rho} = -i[\omega_0 \hat{a}^\dagger \hat{a}, \hat{\rho}] + \gamma_e(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a}, \rho\}) + \gamma_p(2\hat{a}^\dagger\rho\hat{a} - \{\hat{a}\hat{a}^\dagger, \hat{\rho}\})$$

a) Find the Keldysh action ($\pm$ basis, time domain)

b) Perform a Keldysh rotation to q, c basis and frequency domain (Fourier convention $\phi(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\phi(\omega)$)

c) Find the Green functions $G^K(\omega)$, $G^R(\omega)$

d) Optional: Fourier transform back the last result to the time dom. using residue theorem.

Hint for c):

- Write the theory in matrix form (discrete indices c, q and continuous frequency)
  $$S = \sum_{ab} \phi_a^*(G^{-1})_{ab}\phi_b$$
- compute the sourced partition function using the Gaussian identity
  $$\int d(\phi^*, \phi)\, e^{-\sum_{ab}\phi^*_a (G^{-1})_{ab}\phi_b - i\sum_a \phi^*_a j_a + j^*_a \phi_a} = N e^{-\sum_{ab} j^*_a G_{ab} j_b}, \quad N = \det(G^{-1})$$
- Compute the Green function as the second variation of the sourced partition function.

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* This is an example where $L^\dagger L = a a^\dagger$ is not normal ordered, but for quadratic theories we may ignore the issue.

Appendix : Doing the functional integral for the deterministic limit

$$Z \cong \int \mathcal{D}(\underline{\underline{\Phi_q}}, \Phi_c)\, e^{i\int\left(\underline{\underline{\phi_q}} \frac{\delta S}{\delta \phi_c} + \underline{\underline{\phi_q^*}} \frac{\delta S}{\delta \phi_c^*}\right)}$$

- We want to perform integration over Φ_q, which appears linearly in the exponent. It will produce a δ-constraint on $\frac{\delta S}{\delta \phi_c}$, $\frac{\delta S}{\delta \phi_c^*}$.
- A quick & dirty way is noticing the analogy to Fourier transform,

$$\int dp\, e^{ipx} = \delta(x)$$

- A cleaner argument uses a regularization and Gaussian integration. We focus on a single degree of freedom, i.e. $\int = \int dt \equiv \int_t$

$$\int \mathcal{D}\Phi_q\, e^{i\int_t \left[\phi_q(t) \underbrace{\frac{\delta S}{\delta \phi_c(t)}}_{w(t)} + \phi_q^*(t) \underbrace{\frac{\delta S}{\delta \phi_c^*(t)}}_{\tilde{w}(t)}\right]} \quad ; \int \mathcal{D}\Phi_q = \int \prod_t d\Phi_q(t)$$

$$= \lim_{\tau \to 0} \int \mathcal{D}\Phi_q\, e^{i\int_t \left[\phi_q(t) \underbrace{\frac{\delta S}{\delta \phi_c(t)}}_{w(t)} + \phi_q^*(t) \underbrace{\frac{\delta S}{\delta \phi_c^*(t)}}_{\tilde{w}(t)}\right] - \int_{tt'} \phi_q^*(t)\, C(t,t')\, \phi_q(t')}$$

with $C(t-t') = \frac{\tau}{2} \underbrace{\delta(t-t')}_{\text{unit matrix}} \Rightarrow C^{-1}(t,t') = \frac{2}{\tau}\delta(t-t')$

Gauss integral

$$= \lim_{\tau \to 0} (\det C^{-1})\, e^{-\int_{tt'} \tilde{w}(t)\, C(t,t')\, w(t')} = \lim_{\tau \to 0} (\det C^{-1}) \underbrace{e^{-\frac{2}{\tau}\int_t w(t)\tilde{w}(t)}}_{= \prod_t e^{-\frac{2}{\tau} w(t)\tilde{w}(t)}}$$

$$= \delta(w)\,\delta(\tilde{w})$$

with functional δ-constraint $\delta(w) \equiv \prod_t \delta(w(t))$

The last equality follows because in the limit $\tau \to 0$, the Gaussian distribution becomes very narrow and centered around $w(t) = \tilde{w}(t) = 0 \quad \forall t$.