Ex 1: Lindblad–Keldysh functional integral

We fix some details of derivation.

a) The Trotter step.
   - Recall $\hat{S}(t) = e^{i(t-t_0)} \hat{S}(t_0) = \lim_{N \to \infty} (e^{\frac{i}{N}\hat{S} t})^N \hat{S}(to) = \lim_{N \to \infty} (\hat{1} + \frac{i}{N}\hat{S} t)^N \hat{S}(t_0)$
   - For $t_m = t_0 + n \cdot \Delta t$, $\hat{S}_m = \hat{S}(bt_n)$, represent
     \[ \hat{S}_m = \int \frac{d\phi}{\sqrt{2\pi}} \left< \phi^* \mid \phi \right> e^{-\frac{\Delta}{2} \phi^* \phi} \left< \phi^* \mid \hat{S}_m \mid \phi \right> \]
   - For definiteness, work with a single degree of freedom,
     \[ \hat{S} = -i \left[ \omega, \hat{a}^+\hat{a} \right] \hat{S} \]
   - For bosonic creation/annihilation operators $\hat{a}^+, \hat{a}$. This will illustrate the main points.

b) Complete the steps to a continuum functional integral.

Comment: There is a subtlety w/o counterpart in Hamiltonian functional integrals. Even if $\hat{H}, \hat{L}$ are normal ordered, this need not be the case for $\hat{L} + \hat{H}$ (example?). This has to be accounted for by a point splitting in time, inserting a resolution of 1 between time steps $m, n+1$. 

exercises
Ex. 2: Mean field dynamics in operator and Keldysh theory

We study the Lindblad $\phi^4$ work horse model.

a) Compute the evolution of the field operator expectation value
\[
\langle \hat{\phi}(x) \rangle(t) = e^{-\hat{H}t} \langle \hat{\phi}(x) \hat{\phi}(x) \rangle = e^{\hat{H}t} \langle \hat{\phi}(x) \rangle \hat{\phi}(x)
\]

Hint: Use cyclic invariance to bring to a form containing only commutators $[\hat{H}, \hat{\phi}(x)]$, $[\hat{L}, \hat{\phi}(x)]$.

- Make the ansatz $\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \hat{\phi}(k) e^{ikx}$, $\hat{\phi}(k) = |\phi(k)| \phi(k)$ with coherent states $|\phi(k)|$. You obtain a dissipative Gross-Pitaevskii eq.

- Further simplify with ansatz $\phi(x,t) = \phi(0)t$, and solve for the stationary state with $\phi(0)t = \sqrt{\frac{C}{\omega}} e^{i\nu t}$. Determine $\nu$ as a function of $x$, $r$ (assuming all other parameters positive).

b) Now derive the same equation in the deterministic limit of Keldysh field theory:

- Write the Lindblad-Keldysh action for the work horse $\phi^4$ Lindblad eq.

- Perform a Keldysh rotation and expand to first order in $\Phi_0$

\[
S[I_+ + I_-] = \langle I_+(I_+ + I_-)/2 \rangle = (I_+ + I_-)/2
\]

- Compute the dissipative Gross-Pitaevskii equation to the deterministic equation of motion $\frac{d\phi_c}{dt} = 0$. 

exercises
Ex. 3: Keldysh theory for the noisy quantum oscillator

Consider the Lindblad equation

\[ \dot{\hat{\phi}} + \gamma_0 (\hat{\phi} \hat{\phi}^+ - \hat{\phi}^+ \hat{\phi}) + i \hbar \left( \hat{\phi} \hat{\phi}^+ - \hat{\phi}^+ \hat{\phi} \right) = 0 \]

\[ \dot{\hat{\phi}} = -i [H, \hat{\phi}] + \gamma_0 (\hat{\phi} \hat{\phi}^+ - \hat{\phi}^+ \hat{\phi}) \]

a) Find the Keldysh action (\( \pm \) basis, time domain)

b) Perform a Keldysh rotation to \( \hat{q}, \hat{c} \) basis and frequency domain (Fourier convention \( \phi(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \phi(\omega) \))

c) Find the Green functions \( G^K(w), G^R(w) \)

d) Optional: Fourier transform back the last result to the time domain using residue theorem.

Hint for (c):

- Write the theory in matrix form (discrete indices \( c, q \) and continuous frequency)

\[ S = \sum_{\omega, \omega'} \phi^*_\omega (G^{-1})_{\omega'\omega} \phi_{\omega'} \]

- Compute the sourced partition function using the Gaussian identity

\[ \int d\phi^* d\phi e^{-\frac{1}{2} \phi^* (G^{-1})_{\omega \omega'} \phi + i \sum_a \phi^*_a \delta_a + i \phi_a} \]

\[ = N \exp \left( -\frac{1}{2} \sum_a \delta_a^{\ast} G a \omega \delta_a \right), \quad N = \det(G^{-1}) \]

- Compute the Green function as the second variation of the sourced partition function.

* This is an example where \( \mathcal{L} = a \hat{a}^+ \) is not normal ordered, but for quadratic theories we may ignore the issue.
Appendix: Doing the functional integral for the deterministic limit

\[ Z = \int d(\Phi_{\alpha}, \Phi_{\bar{\alpha}}) e^{i\int \left( \frac{\phi_q}{8\Phi_c} + \phi_q^* \frac{8\Phi}{8\Phi_c^*} \right) } \]

- We want to perform integration over $\Phi_{\alpha}$, which appears linearly in the exponent. It will produce a $S$-constraint on $\frac{8\Phi}{8\Phi_c}$, $\frac{8\Phi}{8\Phi_c^*}$.

- A quick & dirty way is noticing the analogy to Fourier transform,

\[ \int d\phi e^{i\phi x} = S(x) \]

- A cleaner argument uses a regularization and Gaussian integration.

We focus on a single degree of freedom, i.e. $\int = \int dt \equiv \int_t^]\]

\[ \int d\Phi_{\alpha} e^{i\int \left[ \frac{\phi_q}{8\Phi_c} \frac{8\Phi}{8\Phi_c(t)} + \phi_q^* \frac{8\Phi}{8\Phi_c^*} \frac{8\Phi}{8\Phi_c^*} \right] } - \int d\Phi_{\alpha} = \int d\Phi_{\alpha} \left[ \frac{8\Phi}{8\Phi_c(t)} + \frac{8\Phi}{8\Phi_c^*} \right] \]

\[ = \lim_{t \to 0} \int d\Phi_{\alpha} e^{i\int \left[ \frac{\phi_q}{8\Phi_c} \frac{8\Phi}{8\Phi_c(t)} + \phi_q^* \frac{8\Phi}{8\Phi_c^*} \frac{8\Phi}{8\Phi_c^*} \right] } - \int d\Phi_{\alpha} \left[ \frac{8\Phi}{8\Phi_c(t)} + \frac{8\Phi}{8\Phi_c^*} \right] \]

with $C(t-t') = \frac{1}{2\pi} S(t-t') \implies C^{-1}(t,t') = \frac{2}{\pi} S(t-t')$.

Gauss

\[ = \lim_{\text{integral} \to \infty} \left( \det C^{-1} \right) e^{\int_{t_0}^t \tilde{W}(t) C(t,t') W(t') dt} = \lim_{\text{integral} \to \infty} \left( \det C^{-1} \right) e^{\int_{t_0}^t \tilde{W}(t) C(t,t') W(t') dt} \]

\[ = S(W) S(\tilde{W}) \]

with functional $S$-constraint $S(W) = \int \overline{S(W[\tilde{t}])}$

The last equality follows because in the limit $t \to 0$, the Gaussian distribution becomes very narrow and centered around $w(t) = \tilde{W}(t) = 0 \forall t$. 

exercises