Ex 1 : Lindblad–Keldysh functional integral

We fix some details of derivation.

a) The Trotter step.

- Recall $\hat{\rho}(t) = e^{\mathcal{L}(t-t_0)}\hat{\rho}(t_0) = \lim_{N\to\infty}\left(e^{\hat{\mathcal{L}}\delta t}\right)^N \hat{\rho}(t_0) = \lim_{N\to\infty}(\hat{\mathbb{1}} + \hat{\mathcal{L}}\delta t)\hat{\rho}(t)$
- For $t_n = t_0 + n\cdot\delta t$, $\hat{\rho}_n = \hat{\rho}(t_n)$, represent

$$\hat{\rho}_n = \int \prod_{\sigma=\pm} d(\phi^*_{\sigma,n}, \phi_{\sigma n})\, e^{-\sum_\sigma \phi^*_{\sigma n}\phi_{\sigma n}} \underbrace{\langle \phi_{+n}|\hat{\rho}_n|\phi_{-n}\rangle}_{\rho_n \text{ matrix element}} |\phi_{+n}\rangle\langle\phi_{-n}|$$

- For definiteness, work with a single degree of freedom,

$$\mathcal{L}[\hat{\rho}] = -i[\omega_0 \hat{a}^+\hat{a}, \hat{\rho}] + \gamma\left(2\hat{a}\rho\hat{a}^+ - \{\hat{a}^+\hat{a}, \hat{\rho}\}\right)$$

for bosonic creation/annihilation operators $\hat{a}^+, \hat{a}$. This will illustrate the main points (*).

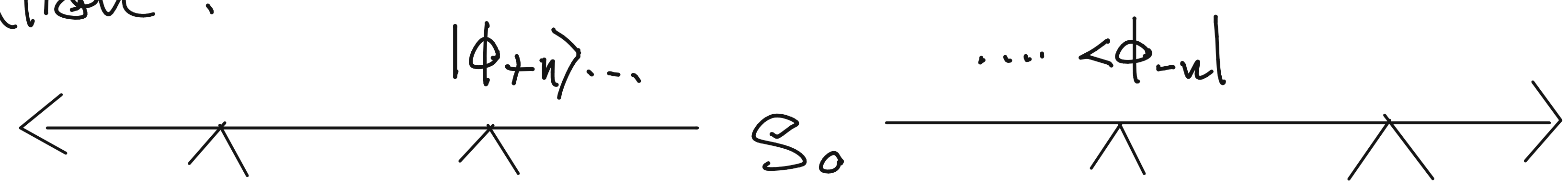
- Find the matrix element $\rho_{n+1} = \langle \phi_{+n+1}|\hat{\rho}_{n+1}|\phi_{-n+1}\rangle$ appearing in the coherent state representation of $\hat{\rho}_{n+1}$.

  Convince yourself that the Lindblad structure is reproduced.

b) Complete the steps to a contour functional integral.

(*) Comment: There is a subtlety w/o counterpart in Hamilton functional integrals: Even if $\hat{H}$, $\hat{L}$ are normal ordered, this need not be the case for L^+L (example?). This has to be accounted for by a <u>point splitting</u> in time, inserting a resolution of $\mathbb{1}$ between time steps n, $n+1$.

Solution :



• The matrix element involves

$$\langle \phi_{+,n+1} | \, |\phi_{+n}\rangle\langle\phi_{-n}| + \delta_t \mathcal{L}\left[|\phi_{+,n}\rangle\langle\phi_{-,n}|\right] |\phi_{-,n+1}\rangle \, e^{-\phi^*_{+n}\phi_{+n}} e^{-\phi^*_{-n}\phi_{-n}}$$

$$= \langle\phi_{+,n+1}|\phi_{+n}\rangle\langle\phi_{-n}|\phi_{-n+1}\rangle +$$

$$\delta_t \Big(\langle\phi_{+,n+1}| -i\omega_0 \underbrace{\hat a^\dagger \hat a}_{\phi^*_{+n+1}\phi_{+n}} - \gamma \underset{\phi^*_{+n+1}\phi_{+n+1}}{\hat a^\dagger \hat a} \; |\phi_{+,n}\rangle\langle\phi_{-n}|\phi_{-n+1}\rangle$$

$$+ \langle\phi_{+n+1}|\phi_{+n}\rangle\langle\phi_{-n}| +i\omega_0 \underbrace{\hat a^\dagger\hat a}_{\phi^*_{-n}\;\phi_{-n+1}} - \gamma a^\dagger a|\phi_{-n+1}\rangle$$

$$+2\gamma\langle\phi_{+,n+1}| \underset{\phi_{+n}}{a} |\phi_{+,n}\rangle\langle\phi_{-n}| \underset{\phi^*_{-n}}{a^\dagger}|\phi_{-n+1}\rangle$$

$$= e^{-(\phi^*_{+n}\phi_{+n} - \phi^*_{+n+1}\phi_{+n})} \, e^{-(\phi^*_{-n}\phi_{-n} - \phi^*_{-n}\phi_{-n+1})}$$

$$\times\Big(1+\delta_t\big[-i\omega_0\phi^*_{+n+1}\phi_{+n} + i\omega_0\phi^*_{-n}\phi_{-n+1}$$

$$+ 2\gamma\phi_{+n}\phi^*_{-n} - \gamma\phi^*_{+n+1}\phi_{+n+1} - \gamma\phi^*_{-n}\phi_{-n+1}\big]\Big)$$

$$\overset{1+\delta x\approx e^{\delta x}}{=} e^{i\delta t\left(i\frac{(\phi^*_{+n}-\phi^*_{+n+1})}{\delta_t}\phi_{+n} + i\phi^*_{-}\frac{(\phi_{-n}-\phi_{-n+1})}{\delta_t} - i[\dots]\right)}$$

$$\xrightarrow[\text{only keep } \mathcal{O}(\delta_t)]{\delta_t\to 0} e^{i\,dt\left(-i(\partial_t\phi^*_+)\phi_+ - i\phi^*_-\partial_t\phi_- - i\left\{-i(\underbrace{\omega_0\phi^*_+\phi_+}_{\equiv H_+} - \underbrace{\omega_0\phi^*_-\phi_-}_{\equiv H_-}) + 2\gamma\phi_+\phi^*_- - \gamma\phi^*_+\phi_+ - \gamma\phi^*_-\phi_-\right\}\right)}$$

• many time steps $\sum\delta_t \to \int_{-t_0\to-\infty}^{t_f\to\infty} dt$; in particular, $\int_{-\infty}^{\infty} dt\, i(\partial_t\phi^*_+)\phi_+$

$$= -\int_{-\infty}^{\infty} dt\, i\phi^*_+\partial_t\phi_+$$

$$\Rightarrow S = \int_{-\infty}^{\infty} dt\left[+i\phi^*_+\partial_t\phi_+ - i\phi^*_-\partial_t\phi_- - i\mathcal{L}\right]$$

$$\mathcal{L} = -i(H_+ - H_-) + \gamma(2L_+L^*_- - L^*_+L_+ - L^*_-L_-), \quad L=a$$

Ex. 2: Mean field dynamics in operator and Keldysh theory

We study the Lindblad ϕ^4 workhorse model.

a) - Compute the evolution of the field operator expectation value

$$\partial_t \langle \hat{\phi}(\vec{x}) \rangle(t) = \partial_t \, \mathrm{tr}(\hat{\phi}(\vec{x}) \hat{\rho}(t)) = \mathrm{tr}(\hat{\phi}(\vec{x}) \hat{\mathcal{L}}[\hat{\rho}])$$

Hint: Use cyclic invariance to bring to a form containing only commutators $[\hat{H}, \hat{\phi}(\vec{x})]$, $[\hat{L}, \hat{\phi}(\vec{x})]$.

- Make the ansatz $\hat{\rho} = \prod_{\vec{x}} \hat{\rho}(\vec{x})$, $\hat{\rho}(\vec{x}) = |\phi(\vec{x})\rangle\langle\phi(\vec{x})|$ with coherent states $|\phi(\vec{x})\rangle$. You obtain a dissipative Gross-Pitaevskii eq.
- Further simplify with ansatz $\phi(\vec{x}, t) = \phi(t)$, and solve for the stationary state with $\phi(t) = \sqrt{\rho_0}\, e^{i\nu t}$. Determine ν, ρ_0 as function of $\gamma_e - \gamma_p$ (assuming all other parameters positive).

b) Now derive the same equation in the deterministic limit of Keldysh field theory: $S[\Phi_+, \Phi_-]$

- Write the Lindblad-Keldysh action for the workhorse ϕ^4 Lindblad eq.
- Perform a Keldysh rotation and expand to first order in Φ_q

$$S[\Phi_+ = (\Phi_c + \Phi_q)/\sqrt{2}, \Phi_- = (\Phi_c - \Phi_q)/\sqrt{2}]$$

- Compare the dissipative Gross-Pitaevskii equation to the deterministic equation of motion $\left.\frac{\delta S}{\delta \phi_c^*}\right|_{\phi_q = 0} = 0$.

Solution 2:

• generally for operator $\hat{A}$:

$$\partial_t\langle\hat{A}\rangle = \mathrm{tr}\left[\hat{A}\left(-i[\hat{H},\hat{\varrho}] + \sum_\alpha \gamma_\alpha\left(\hat{L}_\alpha \hat{\varrho}\hat{L}_\alpha^\dagger - \{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{\varrho}\}\right)\right)\right]$$

$$= \left\langle +i[\hat{H},\hat{A}] + \sum_\alpha \gamma_\alpha\left(\hat{L}_\alpha^\dagger \hat{A}\hat{L}_\alpha - \{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{A}\}\right)\right\rangle$$

$$= \left\langle i[\hat{H},\hat{A}] + \sum_\alpha \gamma_\alpha\left([\hat{L}_\alpha^\dagger,\hat{A}]\hat{L}_\alpha + \hat{L}_\alpha^\dagger[\hat{A},\hat{L}_\alpha]\right)\right\rangle$$

• Lindblad ϕ^4, $\hat{A} = \hat{\phi}(\vec{x})$, $\phi(\vec{x},t) = \langle\hat{\phi}(\vec{x})\rangle(t)$: (*)

$$i\partial_t\phi(\vec{x},t) = \left[-\frac{1}{2m}\nabla^2 - \mu - i(\underbrace{\gamma_e}_{2} - \gamma_p)\right]\phi(\vec{x},t) + (\lambda - i\kappa)\langle\hat{\phi}^\dagger(x)\hat{\phi}^2(x)\rangle(t)$$

etc.

• $\hat{\varrho}(t) = \prod_x \hat{\varrho}(x,t)$, $\hat{\varrho} = |\phi(x)\rangle\langle\phi(x)|$ w/ $\hat{\phi}(\vec{x})|\phi(x)\rangle = \phi(x)|\phi(x)\rangle$

$\mathrm{tr}\,\hat{\varrho}(x) = 1$ $\quad \vec{x},t \quad$ $\langle\phi(\vec{x})|\hat{\phi}^\dagger(\vec{x}) = \langle\phi(x)|\phi^*(x)$

$$\Longrightarrow i\partial_t\phi(x,t) = \left[-\frac{1}{2m}\nabla^2 - \mu - i(\gamma_e - \gamma_p) + (\lambda - i\kappa)|\phi(\vec{x},t)|^2\right]\phi(\vec{x},t)$$

• homogeneous ansatz: $\phi(\vec{x},t) = \phi(t)$

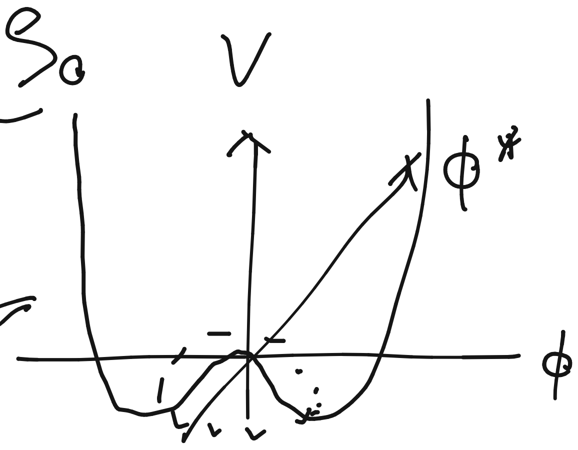
$$i\partial_t\phi = \left(-\mu - i(\gamma_e - \gamma_p) + (\lambda - i\kappa)|\phi(t)|^2\right)\phi(t)$$

• $\phi(t) = e^{i\nu t}\sqrt{\varrho_0}$

$$-\nu\sqrt{\varrho_0} = \left[(-\mu + g\varrho_0) - i\underbrace{\left(+(\gamma_e - \gamma_p) + \kappa\varrho_0\right)}_{=\frac{\partial V}{\partial\phi}}\right]\sqrt{\varrho_0}$$

$\gamma_e - \gamma_p > 0$: $\varrho_0 = 0$, ν free

$\gamma_e - \gamma_p < 0$: $\varrho_0 = 0$, $\nu = \mu - g\varrho_0$



(*) e.g. $\left[\int d^dy\, \hat{\phi}^\dagger(y)\frac{-\nabla_y^2}{2m}\hat{\phi}(y), \hat{\phi}(x)\right] = \int d^dy\, \delta(x-y)\frac{-\nabla_y^2}{2m}\hat{\phi}(y) = \frac{-\nabla_x^2}{2m}\hat{\phi}(x)$ ✓

b) • Use action in c, q basis in the lecture notes:

$$S[\Phi_c, \Phi_q] = \int_x \Big[\phi_q^* \big(i\partial_t + \frac{\nabla^2}{2m} + \mu + i(\gamma_e - \gamma_p) \big) \phi_c - \frac{1}{2}(\lambda - i\kappa)\phi_q^* \left(|\phi_c|^2 \phi_c + \phi_q^2 \phi_c^* \right) + \text{c.c.} \Big] + i \int_x \left(2(\gamma_e + \gamma_p) + 2\kappa |\phi_c|^2 \right) |\phi_q|^2$$

• deterministic limit

$$S[\Phi_c, \phi_q] \simeq \int_x \Big[\phi_q(x) \frac{\delta S}{\delta \phi_q(x)} + \phi_q^* \frac{\delta S}{\delta \phi_q^*(x)} \Big]$$

• equation of motion

$$0 = \frac{\delta S}{\delta \phi_q^*(x)} \implies i\partial_t \phi_c = \Big[-\frac{\nabla^2}{2m} - \mu - i(\gamma_e - \gamma_p) + \frac{\lambda - i\kappa}{2} |\phi_c|^2 \Big] \phi_c$$

$$\langle \hat{\phi} \rangle = \frac{\langle \phi_c \rangle}{\sqrt{2}}$$

Ex. 3 : Keldysh theory for the noisy quantum oscillator

Consider the Lindblad equation (*)

$$\partial_t \hat{\varrho} = -i[\omega_0 \hat{a}^\dagger \hat{a}, \hat{\varrho}] + \gamma_e(2\hat{a}\varrho\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a}, \varrho\}) + \gamma_p(2\hat{a}^\dagger\varrho\hat{a} - \{\hat{a}\hat{a}^\dagger, \hat{\varrho}\})$$

a) Find the Keldysh action ($\pm$ basis, time domain)

b) Perform a Keldysh rotation to q, c basis and frequency domain (Fourier convention $\phi(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\phi(\omega)$)

c) Find the Green functions $G^K(\omega), G^R(\omega)$

d) Optional : Fourier transform back the last result to the time dom. using residue theorem.

Hint for c) :

- Write the theory in matrix form (discrete indices c, q and continuous frequency)

$$S = \sum_{ab} \phi_a^* (G^{-1})_{ab} \phi_b$$

- compute the sourced partition function using the Gaussian identity

$$\int d(\phi^*, \phi)\, e^{-\sum_{ab} \phi_a^* (G^{-1})_{ab} \phi_b - i\sum_a \phi_a^* j_a + j_a^* \phi_a} = N e^{-\sum_{ab} j_a^* G_{ab} j_b}, \quad N = \det(G^{-1})$$

- Compute the Green function as the second variation of the sourced partition function.

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* This is an example where $L^\dagger L = a a^\dagger$ is not normal ordered, but for quadratic theories we may ignore the issue.

Solution 3

a,b) $$S = \int \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\left(a_c^*(\omega),\, a_q^*(\omega)\right) G^{-1}(\omega,\omega') \begin{pmatrix} a_c(\omega') \\ a_q(\omega') \end{pmatrix}$$

$\equiv P^A(\omega)$

$$G^{-1}(\omega,\omega') = \begin{pmatrix} 0 & \omega-\omega_0 - i(\gamma_e-\gamma_p) \\ \underbrace{\omega-\omega_0+i(\gamma_e-\gamma_p)}_{\equiv P^R(\omega)} & \underbrace{2i(\gamma_e+\gamma_p)}_{\equiv P^K} \end{pmatrix}\delta(\omega-\omega')$$

$$\Longrightarrow G(\omega,\omega') = \begin{pmatrix} -G^R_{(\omega)} P^K G^A_{(\omega)} & G^R_{(\omega)} \\ G^A_{(\omega)} & 0 \end{pmatrix}\delta(\omega-\omega') \qquad G^{R/A} = P^{R/A^{-1}}$$

Gaussian integral:

$$Z[j_c, j_q] = e^{\,i\int_{\omega,\omega'}\left(j_q^*(\omega),\, j_c^*(\omega)\right) G(\omega,\omega')\begin{pmatrix} j_q(\omega') \\ j_c(\omega')\end{pmatrix}}$$

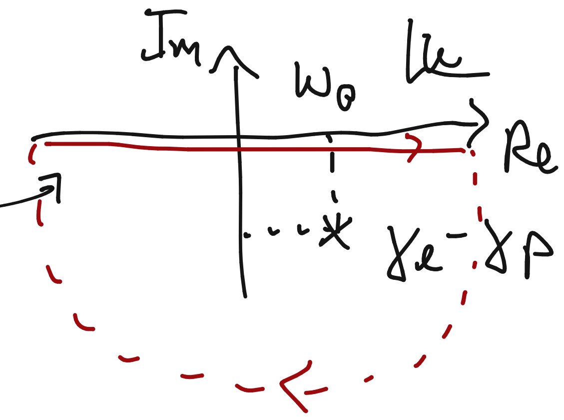
$$= e^{\,i\int_{\omega}\left(j_q^*(\omega),\, j_c^*(\omega)\right) G(\omega,\omega')\begin{pmatrix} j_q(\omega) \\ j_c(\omega')\end{pmatrix}}$$

$$\Longrightarrow \begin{pmatrix} \langle \phi_c(\omega)\phi_c^*(\omega')\rangle & \langle \phi_c(\omega)\phi_q^*(\omega')\rangle \\ \langle \phi_q(\omega)\phi_c^*(\omega')\rangle & \langle \phi_q(\omega)\phi_q^*(\omega')\rangle \end{pmatrix} = -\begin{pmatrix} \frac{\delta^2 Z}{\delta j_q^*(\omega)\delta j_q} & \cdots \\ \cdots & \cdots \end{pmatrix}$$

see lect. notes!

$$= i\, G(\omega)\,\delta(\omega-\omega') \quad \checkmark$$

• $$G(\omega) = \begin{pmatrix} \frac{-i(\gamma_e+\gamma_p)}{(\omega-\omega_0)^2+(\gamma_e-\gamma_p)^2} & \frac{1}{\omega-\omega_0+i(\gamma_e-\gamma_p)} \\ \frac{1}{\omega-\omega_0-i(\gamma_e-\gamma_p)} & 0 \end{pmatrix}$$



$$\overset{t-t'}{\underset{\|}{}}$$

$$G(\tau) = \int \frac{d\omega}{2\pi} e^{-i\omega\tau}\, G(\omega)$$

$$= \begin{pmatrix} e^{-i\omega_0\tau}\, e^{-(\gamma_e-\gamma_p)|\tau|}\,\frac{\gamma_e+\gamma_p}{\gamma_e-\gamma_p} & \Theta(\tau)\, e^{-i\omega_0 t}\, e^{-(\gamma_e-\gamma_p)\tau} \\ \Theta(-\tau)\, e^{i\omega_0 t}\, e^{(\gamma_e-\gamma_p)\tau} & 0 \end{pmatrix}$$

Appendix : Doing the functional integral for the deterministic limit

$$Z \cong \int \mathcal{D}(\Phi_q, \Phi_c)\, e^{i \int \left(\phi_q \frac{\delta S}{\delta \phi_c} + \phi_q^* \frac{\delta S}{\delta \phi_c^*} \right)}$$

- We want to perform integration over Φ_q, which appears linearly in the exponent. It will produce a δ-constraint on $\frac{\delta S}{\delta \phi_c}$, $\frac{\delta S}{\delta \phi_c^*}$.
- A quick & dirty way is noticing the analogy to Fourier transform,

$$\int dp\, e^{ipx} \quad \delta(x)$$

- A cleaner argument uses a regularization and Gaussian integration. We focus on a single degree of freedom, i.e. $\int = \int dt \equiv \int_t$

$$\int \mathcal{D}\Phi_q\, e^{i \int_t \left[\phi_q(t) \underbrace{\frac{\delta S}{\delta \phi_c(t)}}_{w(t)} + \phi_q^*(t) \underbrace{\frac{\delta S}{\delta \phi_c^*(t)}}_{\tilde{w}(t)} \right]} \quad ; \int \mathcal{D}\Phi_q = \int \prod_t d\Phi_q(t)$$

$$= \lim_{\tau \to 0} \int \mathcal{D}\Phi_q\, e^{i \int_t \left[\phi_q(t) \underbrace{\frac{\delta S}{\delta \phi_c(t)}}_{w(t)} + \phi_q^*(t) \underbrace{\frac{\delta S}{\delta \phi_c^*(t)}}_{\tilde{w}(t)} \right] - \int_{tt'} \phi_q^*(t)\, C(t,t')\, \phi_q(t')}$$

with $C(t-t') = \frac{\tau}{2} \underbrace{\delta(t-t')}_{\text{unit matrix}} \Rightarrow C^{-1}(t,t') = \frac{2}{\tau}\delta(t-t')$

$$\underset{\text{integral}}{\overset{\text{Gauss}}{=}} \lim_{\tau \to 0} \left(\det C^{-1}\right) e^{-\int_{tt'} \tilde{w}(t)\, C(t,t')\, w(t')} = \lim_{\tau \to 0} \left(\det C^{-1}\right) \underbrace{e^{-\frac{2}{\tau} \int_t w(t)\, \tilde{w}(t)}}_{= \prod_t e^{-\frac{2}{\tau} w(t) \tilde{w}(t)}}$$

$$= \delta(w)\, \delta(\tilde{w})$$

with functional δ-constraint $\delta(w) \equiv \prod_t \delta(w(t))$

The last equality follows because in the limit $\tau \to 0$, the Gaussian distribution becomes very narrow and centered around $w(t) = \tilde{w}(t) = 0 \quad \forall t$.