Introduction to Supersymmetry

1) Fermions vs. Bosons (fermion ≠ half-integer spin)

$$\kappa^2 = \bar\kappa^2 = \{\kappa, \bar\kappa\} = 0 \quad , \quad \kappa_1 \cdots \kappa_N \quad \{\kappa_i, \kappa_j\} = 0$$

$$\int d\kappa_1 \cdots d\kappa_N \, (\underbrace{c + c^j \kappa_j + \cdots + c^{1\ldots N} \kappa_N \cdots \kappa_1}_{2^N \text{ terms}}) = c^{1\ldots N}$$

$$e^{\sum \kappa_j \bar\kappa_j} = (1+\kappa_1\bar\kappa_1)(1+\kappa_2\bar\kappa_2)\cdots(1+\kappa_N\bar\kappa_N)$$

$$\int d\bar\kappa_1 d\kappa_1 \, d\bar\kappa_2 d\kappa_2 \cdots d\bar\kappa_N d\kappa_N \, e^{\sum \kappa_j \bar\kappa_j} = 1 \quad , \quad \delta(\kappa_1) = \kappa_1$$

$$\int d\kappa_1 \, \delta(\kappa_1) = 1 \quad , \quad \int d\bar\kappa_1 d\kappa_1 \cdots d\bar\kappa_N d\kappa_N \, \kappa_j \, e^{\sum \kappa_l \bar\kappa_l} = 0$$

$$\int d\bar\kappa_1 \cdots d\kappa_N \, \kappa_j \bar\kappa_k \, e^{\sum \kappa_l \bar\kappa_l} = \delta_{jk}$$

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2) Supersymmetry is a symmetry exchanging bosons and fermions which ~~exten~~ generalizes ~~the~~ Poincaré symmetry (unique)

$m = 0 \ldots d-1$

$$P_m \, , \, M_{mn} \Rightarrow [P_m, M_{np}] = \eta_{mn} P_p - \eta_{mp} P_n$$

$$[M_{mn}, M_{pq}] = \eta_{mp} M_{nq} - \eta_{mq} M_{np} + \eta_{nq} M_{mp} - \eta_{np} M_{mq}$$

$$[P_m, P_n] = 0 \quad (= r M_{mn} \text{ in curved space})$$

$d = 2 + 8k$

"spinor"

$$\alpha = 1 \ldots 2^{\frac{d}{2}(-1)} \qquad \{Q_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} P_m \quad , \quad [Q_\alpha, P_m] = 0 \ \text{(flat space)}$$

$$[Q_\alpha, M_{mn}] = (\gamma_{mn})_\alpha{}^\beta Q_\beta$$

Extended SUSY: $J = 1$ to $\mathcal{N}$

$$\{Q^J_\alpha, Q^K_\beta\} = \gamma^m_{\alpha\beta} P_m \delta^{JK} \quad , \quad [Q^J_\alpha, P_m] = 0$$

$$[Q^J_\alpha, M_{mn}] = (\gamma_{mn})_\alpha{}^\beta Q^J_\beta$$

$\boxed{\gamma^m_{\alpha\beta} , (\gamma_{mn})_\alpha{}^\beta \text{ are constant matrices (lecture 3)}}$

a) $d=1$: "supersymmetric quantum mechanics"

Relativistic particle with spin $\frac{1}{2}$ and gyromagnetic ratio $g=2$

b) $d=2$: worldsheet of superstring (first example of SUSY in literature)

c) $d=4$: $\mathcal{N}=1$ ~~super Yang-Mills~~ "minimally supersymmetric standard model"

Predicts unification of coupling constants (EM, weak, strong) and light Higgs boson. ~~Up to now,~~ But also predicts light susy partners (squark, gluino, etc) that have not been found. "Spontaneous" breaking of SUSY is not well understood.

Supergravity might explain small cosmological constant since unbroken SUSY $\Rightarrow$ cosmological constant $= 0$

$$\langle 0|P_0|0\rangle = \langle 0|Q_\alpha Q_\alpha|0\rangle = 0 \quad \text{if } Q_\alpha|0\rangle = 0$$

Extended susy ($\mathcal{N}=2$ or $\mathcal{N}=4$) has interesting "duality" symmetries

$\mathcal{N}=4$ super-YM with coupling constant $g \equiv \mathcal{N}=4$ super-YM with coupling constant $1/g$.

d) $d=10$ and $d=11$: Largest dimensions in which supersymmetry is possible ~~preserves unitarity~~ with massless spins ≤ 1 and ≤ 2

Consistency of quantum gravity (perturbatively) appears to require spacetime supersymmetry to avoid quantum-mechanical divergences.

(describe spin $\frac{1}{2}$ particle)

4) SUSY in $d=1$ (~~relativistic particle~~ $\quad \{Q, Q\} = \frac{\partial}{\partial t}$

a) First consider worldline action of particle with mass M and charge e in an electromagnetic background $A_m(x)$

$$S = \int d\tau \left[cM\sqrt{\dot{x}^2} + \frac{e}{c}\dot{x}^m A_m(x) \right] \qquad \dot{x}^m = \frac{d}{d\tau}x^m$$

Eq. of motion $\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial \dot{x}^m}\right) = \frac{\partial L}{\partial x^m}$: $Mc\frac{\partial}{\partial\tau}\left(\frac{\dot{x}_m}{\sqrt{\dot{x}^2}}\right) + \frac{e}{c}\frac{\partial}{\partial\tau}A_m = \frac{e}{c}\dot{x}^n\partial_m A_n$

$$\Rightarrow Mc\frac{\partial}{\partial\tau}\left(\frac{\dot{x}_m}{\sqrt{\dot{x}^2}}\right) = \frac{e}{c}\dot{x}^n F_{mn}$$

In gauge $\sqrt{\dot{x}^2} = c$, $M\ddot{x}_m = \frac{e}{c}\dot{x}^n F_{mn} \xrightarrow[v \ll c]{} M\vec{a} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$

(local reparam inv)

Could have started with $S = \int d\tau \left[\frac{M}{2}\dot{x}^2 + \frac{e}{c}\dot{x}^m A_m(x) \right]$

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b) What action reproduces eqn. of motion for spin $\frac{1}{2}$?

$$M\vec{a} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} + \frac{ge}{2Mc}\vec{\nabla}(\vec{B}\cdot\vec{S}), \quad \dot{\vec{S}} = \frac{ge}{2Mc}\vec{S}\times\vec{B}$$

boson $x^m(\tau)$

fermion $\psi^m(\tau)$

$$S = \int d\tau \left[\frac{M}{2}\left(\dot{x}^m\dot{x}_m + i\dot{\psi}^m\psi_m\right) + \frac{e}{c}\left(\dot{x}^m A_m(x) + \frac{i}{2}\psi^m\psi^n F_{mn}(x)\right) \right]$$

$$\left(\int d\tau\, i\dot{\psi}^m\psi_m\right)^\dagger = -i\int d\tau\, \psi_m\dot{\psi}^m = +i\int d\tau\,\dot{\psi}^m\psi_m$$

Translation generator P_m: $x^m \to x^m + i\Lambda\dot{x}^m$, $\psi^m \to \psi^m + i\Lambda\dot{\psi}^m$

Susy generator Q: $x^m \to x^m + i\kappa\psi^m$, $\psi^m \to \psi^m + \kappa\dot{x}^m$

$$\delta_Q(\dot{x}^2 + i\dot{\psi}^m\psi^m) = 2i\kappa\dot{x}^m\dot{\psi}_m + i\kappa\ddot{x}^m\psi_m + i\dot{\psi}^m\kappa\dot{x}^m = i\kappa\frac{\partial}{\partial\tau}(\dot{x}^m\psi_m)$$

$\boxed{\text{Ex: } \delta_Q\left(\dot{x}^m A_m + \frac{i}{2}\psi^m\psi^n F_{mn}\right) = \frac{3}{2}\, i\kappa\frac{\partial}{\partial\tau}(\,?\,)}$ $\Rightarrow \delta_Q S = i\kappa\int d\tau\frac{\partial}{\partial\tau}[\;] = 0$

(Super-reparam inv action $\to S$ in gauge $\sqrt{\dot{x}^2 + i\dot{\psi}^m\psi^m} = c$, $\dot{x}^m\psi_m = 0$)

$\cancel{\{Q,Q\}}$

$$(\delta_{Q_1}\delta_{Q_2}-\delta_{Q_2}\delta_{Q_1})x^m = \delta_{Q_1}(i\kappa_2\psi^m)-\delta_{Q_2}(i\kappa_1\psi^m) = -2i\kappa_1\kappa_2\dot{x}^m = 2\delta_p x^m$$

$$(\delta_{Q_1}\delta_{Q_2}-\delta_{Q_2}\delta_{Q_1})\psi^m = \delta_{Q_1}(\kappa_2\dot{x}^m)-\delta_{Q_2}(\kappa_1\dot{x}^m) = -2i\kappa_1\kappa_2\dot{\psi}^m = -2\delta_p\psi^m \qquad \boxed{\Lambda = \kappa_1\kappa_2}$$

$\Rightarrow \cancel{\{Q,Q\}}$

$$\delta_Q = \kappa Q\ ,\ \delta_p = \Lambda P \rightarrow [\kappa_1 Q, \kappa_2 Q] = -2\kappa_1\kappa_2 Q^2 = -\kappa_1\kappa_2(2P) \Rightarrow \{Q,Q\} = 2P$$

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c) Angular momentum $M_{mn} = L_{mn} + S_{mn}$ ~~Noether current~~

(orbital) (spin)

$$= M\left(\dot{x}_{[n}x_{m]} + i\psi_n\psi_m\right)$$

gauge $\dot{x}^m\psi_m = 0 \Rightarrow$ nonrel. limit $\psi_0 = 0$ and $\vec{S} = iM(\psi^2\psi^3, \psi^3\psi^1, \psi^1\psi^2) = \epsilon^{jkl}S_{kl}$

Canonical Quantization $\Rightarrow \{\psi^j, \psi^k\} = -\frac{\hbar}{M}\delta^{jk} \Rightarrow [S_j, S_k] = i\epsilon_{jkl}S_l\ ,\ |\vec{S}|^2 = \frac{3}{4}\hbar^2 \Rightarrow s = \frac{1}{2}$

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d) Eq. of motion: $M\ddot{x}_m + \frac{e}{c}\dot{A}_m = \frac{e}{c}\left(\dot{x}^n\partial_m A_n + \frac{i}{2}\psi^n\psi^p\partial_m F_{np}\right)$

$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial\dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \Rightarrow \qquad \boxed{M\ddot{x}_m = \frac{e}{c}\left(\dot{x}^n F_{mn} + \frac{i}{2}\psi^n\psi^p\partial_m F_{np}\right)}$

$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial\dot{\psi}}\right) = \frac{\partial L}{\partial\psi} \qquad \frac{i}{2}M\dot{\psi}_m = \frac{ie}{c}F_{mn}\psi^n - \frac{i}{2}M\dot{\psi}_m \Rightarrow \boxed{M\dot{\psi}_m = \frac{e}{c}F_{mn}\psi^n}$

$$|v| \ll c \Rightarrow \dot{x}^m = (c, \vec{v})\ ,\ \psi^m = (0, \vec{\psi})$$

$$\Rightarrow M\vec{a} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} + \frac{e}{Mc}\vec{\nabla}(\vec{B}\cdot\vec{S})$$

$$\dot{\vec{S}} = iM\dot{\vec{\psi}}\times\vec{\psi} = \frac{ie}{c}(\vec{\psi}\times\vec{B})\times\vec{\psi} = \frac{ie}{2c}(\vec{\psi}\times\vec{\psi})\times\vec{B} = \frac{e}{Mc}\vec{S}\times\vec{B}$$

$(\Rightarrow g = 2)$

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e) Action in "superspace" $X^m(\tau,\kappa) = x^m(\tau) + i\kappa\psi^m(\tau)$

$$S = -i\int d\tau\int d\kappa\left[\frac{M}{2}DX^m\dot{X}_m + \frac{e}{c}DX^m A_m(X)\right] \quad \text{where } D = \frac{\partial}{\partial\kappa} + i\kappa\frac{\partial}{\partial\tau}$$