PROBLEM SET

Universal Prethermal Dynamics in Quantum Magnets and Fracton Fluids

Problem 1: Kinetic theory of bosons in the presence of a condensate.

Consider a gas of weakly-interacting bosons

$$H = \int d^d\boldsymbol{x} \frac{\nabla\psi^\dagger\nabla\psi}{2m} + g(\psi^\dagger\psi)^2. \tag{1}$$

A finite number of bosons is pumped at high energies at time $t = 0$.

(a) Show that, once enough particles cascade down in energy and start condensing, the fluctuations on top of the condensate have dispersion $\omega_k = k\sqrt{2\rho_0 + k^2}$, and the lowest order resonant interaction is three-wave scattering with matrix element:

$$V^{(3)} = \frac{\sqrt{\rho_0\omega_1\omega_2\omega_3}}{2(2\pi)^{d/2}}\left[\frac{12}{\sqrt{\alpha_1\alpha_2\alpha_3}} + \frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2\alpha_3} + \frac{\boldsymbol{k}_2\cdot\boldsymbol{k}_3}{\alpha_1k_2k_3} + \frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_3}{k_1\alpha_2k_3}\right]. \tag{2}$$

Here $\rho_0 = \psi_0^\dagger\psi_0$ is the condensate density, and α_k is defined as $\alpha_k = 2\rho_0 + k^2$.

(b) Keeping only the leading order term $V^{(3)} \approx \frac{\sqrt{k_1k_2k_3}}{\rho_0^{1/4}}$, find the scaling exponents α and β for $\langle\psi_{\boldsymbol{k}}^\dagger\psi_{\boldsymbol{k}}\rangle \sim t^\alpha f(t^\beta|\boldsymbol{k}|)$ describing fluctuations of the order parameter.

Problem 2: Energy-scaling approach to find coarsening laws.

Let's consider the simplest case of a coarsening law in a d-dimensional systems with an ordering field $\vec{\phi}$ with n components. Let's assume $n > d$ such that topological defects are not allowed. The equations of motion of the field $\vec{\phi}_{\boldsymbol{k}}$ is given by $\partial_t\vec{\phi}_{\boldsymbol{k}} = -\frac{\delta H}{\delta\vec{\phi}_{-\boldsymbol{k}}}$, and let's assume the structure factor obeys the simple scaling law $\langle\vec{\phi}_{-\boldsymbol{k}}\vec{\phi}_{\boldsymbol{k}}\rangle \sim L^d f(kL)$ containing only a single scaling parameter $L(t)$. Show that the rate of dissipated energy obeys the law:

$$\frac{d\varepsilon}{dt} = \int d^d\boldsymbol{k}\langle\frac{\delta H}{\delta\vec{\phi}_{\boldsymbol{k}}}\dot{\vec{\phi}}_{\boldsymbol{k}}\rangle \sim \frac{\dot{L}^2}{L^2} \tag{3}$$

In addition, show that if the 'elastic' energy stored in the ordering field is given by $\varepsilon = \int d^d\boldsymbol{x}\langle\nabla\vec{\phi}\cdot\nabla\vec{\phi}\rangle$, then $\varepsilon \sim 1/L^2$. By equating the decrease of energy of the ordering field and the dissipation rate, argue that $L \sim t^{1/2}$, or $\beta = 1/2$.

For more general growth laws in the presence of topological defects, see Bray, Advances in Physics **43**, 357 (1994).

Problem 3: Dynamic instabilities in a spin chain.

Consider the one-dimensional Heisenberg model

$$H = J \sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}, \tag{4}$$

where the initial state is a product state of spin spiraling in real space $\langle S_i^x \rangle = S\cos(qr_i)$, $\langle S_i^y \rangle = S\sin(qr_i)$, and $\langle S_i^z \rangle = 0$. By doing a linearized analysis in terms of Bosonic excitations or, alternatively, classical spins, find the most unstable modes that will govern the short time dynamics of the correlation function $\langle S^z_{-\boldsymbol{k}}(t) S^z_{\boldsymbol{k}}(t) \rangle$.

Problem 4: Hydrodynamics of a rotor model.

Consider the d-dimensional rotor model:

$$H = \sum_i \frac{p_i^2}{2I} - \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j), \tag{5}$$

Assume first that the initial state has energy larger than the critical temperature (or the BKT temperature if $d = 2$).

(a) How many conservation laws does the system have?

(b) Derive the hydrodynamic equations for the conserved quantities of the systems. Are there any ballistic modes?

(c) Assume now that the energy of the system is below the critical (or BKT) temperature. How would you modify the hydrodynamic equations?

Problem 5: Microscopic model with dipole conservation.

Consider the classical model

$$H = \sum_{i=1}^{N} \frac{(p_{i+1} - p_i)^2}{2} + V(x_i - x_{i+1}), \tag{6}$$

where p_i is the momentum of particle i, x_i is the position of particle i, and $V(x)$ is a generic polynomial $V(x) = V_2 x^2 + V_3 x^2 + \ldots$.

(a) Using the Poisson bracket $\{x_i, p_j\} = \delta_{i,j}$, show that the charges $Q = \sum_i 1$, $D = \sum_i x_i$, and $P = \sum_i p_i$, obey the classical multipole algebra $\{D, P\} = Q$, while all other Poisson brackets vanish.

(b) Show that Q, D, and P commute with H, thus we have a spatially local theory with charge, dipole and momentum conservation (also energy, but this can easily be removed by adding noise).