Relativistic Quantum Chaos: Neutrino Billiards
Central Conjectures of Quantum Chaos

• Nonrelativistic Quantum Systems / Quantum Billiards

• Berry-Tabor Conjecture:
The spectral fluctuation properties of generic integrable systems coincide with those of uncorrelated random numbers from a Poisson process

• Bohigas-Gianonni-Schmit Conjecture (1984) [Berry (1977), Casati et al. (1980)]:
The spectral fluctuation properties of generic classically chaotic systems coincide with those of random matrices from the Gaussian Ensembles

• Relativistic Quantum Systems / Quantum Billiards

→ Question: Do the conjectures apply to relativistic quantum billiards like neutrino billiards?
→ Problem: NBs do not have a well-defined classical limit
→ We use the semiclassical approach (length spectra, Husimi distributions)
Graphene

• Near each corner of the first hexagonal Brillouin zone the electron energy $\omega$ exhibits a linear dependence on the quasimomentum $q$.

• Close to the diabolical (‘Dirac’) points the band structure is described by the Dirac equation of massless fermions

$$\pm \begin{pmatrix} 0 & \partial_x - i \partial_y \\ \partial_x + i \partial_y & 0 \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = i \frac{\omega - \omega_D}{v_F} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}$$

• Independent contributions from $K_+$ and $K_-$ valleys $\Rightarrow$ 4D Dirac equation
Graphene Billiards

- **Tight-binding model**

\[ \hat{H}_{TB} = t_0 \delta_{ij} + t_1 \delta(\|r_i - r_j\| - d_0) + t_2 \delta(\|r_i - r_j\| - d_1) + \ldots \]

- Assumption: interaction of the graphene \( p_z \) orbitals non-negligible for 1\(^{\text{st}}\), 2\(^{\text{nd}}\) and 3\(^{\text{rd}}\) nearest neighbors
- Graphene billiard: Dirichlet BCs along the 1\(^{\text{st}}\) missing row of atoms outside sheet
DOS of Rectangular and Africa-Shaped GBs
• Ratio distribution of all 1656 (1823) resonance frequencies of rectangular (Africa) billiard agrees with Poisson (GOE)
• The same holds for the (k=1)-overlapping ratio distribution
Massless Spin-1/2 Particle in a Potential

Printed in Great Britain

Neutrino billiards: time-reversal symmetry-breaking without magnetic fields

By M. V. Berry, F.R.S., and R. J. Mondragon
H. H. Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, U.K.

• Dirac equation

\[
\begin{pmatrix}
  V & -i\hbar(\partial_x - i\partial_y) \\
  -i\hbar(\partial_x + i\partial_y) & -V \\
\end{pmatrix}
\begin{pmatrix}
  \psi_1(r) \\
  \psi_2(r) \\
\end{pmatrix}
= k
\begin{pmatrix}
  \psi_1(r) \\
  \psi_2(r) \\
\end{pmatrix}
\]

• For \( V \neq 0 \) the Dirac Hamiltonian is not invariant under \( \hat{T} = i\hat{\sigma}_y\hat{K} \)

→ chaotic systems with no geometric symmetries follow GUE statistics

Relativistic Quantum Chaos | 7
Dirac Equation for Neutrino Billiards

• Dirac equation for a free spin-1/2 particle with mass $m$

$$\hat{H}_D \psi = (c\vec{\sigma} \cdot \hat{p} + mc^2\hat{\sigma}_z) \psi = E\psi, \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

• BC requires that the outward current vanishes along boundary

$$n \cdot [\psi^\dagger \nabla_p \hat{H}_D \psi] = 0$$

• BC links the spinor components at the boundary

$$\psi_2(s) = ie^{i\alpha(s)}\psi_1(s)$$

• Coordinates in the complex plane: $w(s) = x(s) + iy(s)$

• Normal vector: $n(s) = [\cos \alpha(s), \sin \alpha(s)]$

• Polar coordinates: $x(r, \phi) + iy(r, \phi) = w(r, \phi) = R(r, \phi)e^{i\phi}$ $\phi \in [0, 2\pi)$, $r = [0, r_0]$

• Parameter $\phi \rightarrow$ arc length: $s(\phi) = \int_0^\phi |w'(\tilde{\phi})|d\tilde{\phi}$, $s \in [0, L)$, $ds = |w'(\phi)|d\phi$
Dirac Equation for Massive Neutrino Billiards

• The energy $E$ is given in terms of the free space wavevector $k$ as

$$E = \hbar c k E = \hbar c k \sqrt{1 + \beta^2}, \quad \beta = \frac{mc}{\hbar k}$$

• The nonrelativistic limit $k\beta \to \infty$ complies with the BC

• Define

$$\psi = \begin{pmatrix} \sqrt{\frac{1 + \sin \theta \beta}{2}} \tilde{\psi}_1 \\ \sqrt{\frac{1 - \sin \theta \beta}{2}} \tilde{\psi}_2 \end{pmatrix}$$

with

$$\sin \theta \beta = \frac{\beta}{\sqrt{1 + \beta^2}}.$$

nonrel. Limit $\Leftrightarrow \theta \beta \to \pi/2$

ultrarel. Limit $\Leftrightarrow \theta \beta \to 0$

$$\Rightarrow \quad k\tilde{\psi}(r) + i \hat{\sigma} \cdot \nabla \tilde{\psi}(r) = 0 \quad \text{with} \quad \tilde{\psi}_2(s) = ie^{i\alpha(s)} K^{-1} \tilde{\psi}_1(s) \quad K = \sqrt{\frac{1 - \sin \theta \beta}{1 + \sin \theta \beta}}$$

• The resulting Dirac equation has the same form as for the NB with $m=0$

• For $m \to 0$ the Dirac Hamiltonian is not time-reversal invariant

$\Rightarrow$ if it’s shape corresponds to that of a chaotic billiard, GUE is expected

• For $m \to \infty$ we have $\psi_2 \to 0$ and the BC for $\psi_1$ becomes Dirichlet
Boundary Integral Equation for NBs

• Combining the Dirac equation and the matrix equation satisfied by the associated free-space Green function yields the boundary integral

\[ 2i \oint_{\partial \Omega} ds \mathbf{n} \cdot \left[ \psi(r) \sigma \hat{G}_0(r, r') \right] = \begin{cases} \psi^\dagger(r') & , \ r' \in \Omega \setminus \partial \Omega \\ \frac{1}{2} \psi^\dagger(r') & , \ r' \in \partial \Omega \\ 0 & , \text{otherwise} \end{cases} \]

• Free-space Green function

\[ G^0(r, r') = -\frac{1}{4} \left( \begin{bmatrix} i k \\ \partial_x + i \partial_y \end{bmatrix} \frac{i k}{\partial_x - i \partial_y} \right) H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|). \]

• Incorporate the B.C. in the equation for \( r' \in \partial \Omega \)

\[ \tilde{\psi}_1^*(\phi') = \frac{i k}{2} \oint_{\partial \Omega} |w'(\phi)| d\phi \tilde{\psi}_1^*(\phi) \left\{ -K^{-1} H_0^{(1)}[k \rho(\phi, \phi')] + e^{-i \alpha(\phi)} e^{i \xi(\phi, \phi')} H_1^{(1)}[k \rho(\phi, \phi')] \right\} \]

\[ \tilde{\psi}_2^*(\phi') = \frac{i k}{2} \oint_{\partial \Omega} |w'(\phi)| d\phi \tilde{\psi}_2^*(\phi) \left\{ \mathcal{K} H_0^{(1)}[k \rho(\phi, \phi')] + e^{i \alpha(\phi)} e^{-i \xi(\phi, \phi')} H_1^{(1)}[k \rho(\phi, \phi')] \right\} \]

\[ e^{i \xi(\phi, \phi')} = \frac{w(\phi) - w(\phi')}{|w(\phi) - w(\phi')} , \ \rho(\phi, \phi') = |w(\phi) - w(\phi')| \]

• \( H_0 \) and \( H_1 \) have singularities at \( \rho = 0 \)
• For \( m=0 \) the singularities are removed with the replacements

\[
\Phi_1(n, s) = \left\{ \psi_1(n, s) + [-ie^{-i\alpha(s)}] \psi_2(n, s) \right\} / 2, \quad \Phi_1(n, s) \xrightarrow{n \to 0^-} \psi_1(s)
\]

\[
\Phi_2(n, s) = \left\{ \psi_2(n, s) + [ie^{i\alpha(s)}] \psi_1(n, s) \right\} / 2.
\]

in coordinates \((n, s)\) normal / tangential to boundary, and with the relation

\[
\int_{\partial\Omega} ds e^{i\Delta\Phi(s, s')/2} \tilde{\psi}_1^*(s) \cos \left( \frac{\Delta\Phi(s, s')}{2} \right) \frac{ik}{4} H_0^{(1)}(k\rho) = \int_{\partial\Omega} ds e^{i\Delta\Phi(s, s')/2} i\tilde{\psi}_1^*(s)
\]

\[
\times \left[ \sin \theta_\beta \sin \left( \frac{\Delta\Phi(s, s')}{2} \right) \frac{ik}{4} H_0^{(1)}(k\rho) - \cos \theta_\beta \sin \left( \frac{\alpha(s') + \alpha(s)}{2} - \xi(s, s') \right) \frac{ik}{4} H_1^{(1)}(k\rho) \right]
\]

• Boundary integral equation for the first spinor component

\[
(1 - \sin \theta_\beta) \tilde{\psi}_1^*(\phi') = \int_{0}^{2\pi} |w'(\phi)| d\phi Q_k(\phi', \phi) \tilde{\psi}_1^*(\phi)
\]

\[
Q_k(\phi', \phi) = \cos \theta_\beta \left[ e^{i(\alpha(\phi')-\alpha(\phi))} - 1 \right] H_0^{(1)}(k\rho)
\]

\[
+ \left\{ [1 - \sin \theta_\beta] e^{i(\xi(\phi, \phi')-\alpha(\phi))} + [1 + \sin \theta_\beta] e^{-i(\xi(\phi, \phi')-\alpha(\phi'))} \right\} H_1^{(1)}(k\rho).
\]

• In the integral \( \phi = \phi' \) is excluded, as it is accounted for on the lhs
Spectral Properties of Half-Circle QB With Robin BCs

- Robin BC
  \[
  \left( \tilde{\beta} + \frac{1}{2} \kappa(s) \right) \Phi_j(s) + \partial_n \Phi_j(n, s) \bigg|_{n \to 0^-} = 0 \quad \tilde{\beta} = \frac{mc}{\hbar}
  \]

- Shown are results for the symmetric solutions
Trace Formula for the Spectral Density of Massive Neutrino Billiards
BD & Z. Li, PRE102, 042214 (2020)

- Quantization condition deduced from the BIEs for Massive NBs

\[ u(s') = \hat{Q}(k)u(s) \implies \det \left( 1 - \hat{Q}(k) \right) = 0 \implies \rho^{\text{fluc}}(k) = \frac{1}{\pi} \Im \sum_{p=1}^{\infty} \frac{1}{p} \frac{d}{dk} \left[ \text{Tr} \left( \hat{Q} \right)^p(k) \right] \]

- Trace formula is a sum over the periodic orbits of the classical billiard

\[
\Im \frac{1}{p} \frac{d}{dk} \left[ \text{Tr} \left( \hat{Q} \right)^p(k) \right] = \Re \sum_{\gamma_p} \cos \left( \Phi_{\gamma_p} - p \frac{\pi}{2} \right) B_{\gamma_p}^{-} e^{i\Gamma_{\gamma_p}} A_{\gamma_p} e^{i\Theta_{\gamma_p}} + \Re \sum_{\gamma_p} \cos \left( \Phi_{\gamma_p} + p \frac{\pi}{2} \right) B_{\gamma_p}^{+} e^{i\Gamma_{\gamma_p}} A_{\gamma_p} e^{i\Theta_{\gamma_p}}
\]

- The factor \( B_{\gamma_p}^{(l)} e^{i\Gamma_{\gamma_p}} \) depends on \( m \) and the direction of propagation of the PO

\[
\rho^{\text{fluc}}(k; \tilde{\beta}) \xrightarrow{\tilde{\beta} \to \infty} \frac{1}{\pi} \Re \sum_{\gamma_p} A_{\gamma_p} e^{i\Theta_{\gamma_p}}
\]

\[
\rho^{\text{fluc}}(k; \tilde{\beta}) \xrightarrow{\tilde{\beta} \to 0} \frac{1}{\pi} \Re \sum_{\gamma_p} (-1)^p \cos \left( \Phi_{\gamma_p} \right) \cos \left( p \frac{\pi}{2} \right) A_{\gamma_p} e^{i\Theta_{\gamma_p}} \implies \text{Only POs with even \# reflections } p \text{ contribute}
\]
<table>
<thead>
<tr>
<th>QB</th>
<th>NB</th>
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<tbody>
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<td>\psi</td>
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</table>
NBs with Shapes Exhibiting a Mirror Symmetry

- Coordinate (orthogonal) transformation $\Leftrightarrow$ unitary transformation in Dirac Hamiltonian
  
  $r' = \hat{R}r \quad \Rightarrow \quad \hat{H}_D(r') = \hat{H}'_D = \hat{U}^\dagger \hat{H}_D \hat{U}, \quad \bar{\psi}(r') = \hat{U}^\dagger \bar{\psi}(r)$

- Reflection at the $x$- or $y$-axis
  
  $\hat{R}_{x,y} = \pm \hat{\sigma}_z \quad \hat{H}'_D = \hat{H}_D(\hat{R}_{x,y} r)$

  \[ \hat{U}_x = \hat{\sigma}_x \quad \bar{\psi}_x^T(r') = [\bar{\psi}_2(r), \bar{\psi}_1(r)] \]
  \[ \hat{U}_y = i\hat{\sigma}_y \quad \bar{\psi}_y^T(r') = [-\bar{\psi}_2(r), \bar{\psi}_1(r)] \]

- The transformed spinor functions do not fulfill the boundary condition
  \[ \bar{\psi}_2(s) \not\in ie^{i\alpha(s)}K^{-1}\bar{\psi}_1(s) \]

  $\Rightarrow$ not classifiable according to their transformation properties under reflection at the mirror axis

- Reflection at the $x$- or $y$-axis of the Dirac equation yields
  
  \[ \hat{H}_{NB}(-x,y) = \hat{H}_{NB}^*(x,y) \quad \Rightarrow \quad [\psi_1(-x,y), \psi_2(-x,y)] = \pm [\psi_1^*(x,y), \psi_2^*(x,y)] \]
  \[ \hat{H}_{NB}(x,-y) = \hat{\sigma}_z \hat{H}_{NB}^*(x,y) \hat{\sigma}_z \Rightarrow [\psi_1(x,-y), \psi_2(x,-y)] = [\pm \psi_1^*(x,y), \mp \psi_2^*(x,y)] \]
**Wave Functions and Currents of Elliptic NB**

<table>
<thead>
<tr>
<th>$	ext{Re}\psi_1$</th>
<th>$	ext{Im}\psi_1$</th>
<th>$	ext{Re}\psi_2$</th>
<th>$	ext{Im}\psi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Wave Function 1" /></td>
<td><img src="image2" alt="Wave Function 2" /></td>
<td><img src="image3" alt="Wave Function 3" /></td>
<td><img src="image4" alt="Wave Function 4" /></td>
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<tr>
<td><img src="image5" alt="Wave Function 5" /></td>
<td><img src="image6" alt="Wave Function 6" /></td>
<td><img src="image7" alt="Wave Function 7" /></td>
<td><img src="image8" alt="Wave Function 8" /></td>
</tr>
<tr>
<td><img src="image9" alt="Wave Function 9" /></td>
<td><img src="image10" alt="Wave Function 10" /></td>
<td><img src="image11" alt="Wave Function 11" /></td>
<td><img src="image12" alt="Wave Function 12" /></td>
</tr>
</tbody>
</table>

$|u|$, $u$

\[ \hat{u} = \nabla_p \hat{H}_D = c\hat{\sigma}, \ u(r) = c\psi\hat{\sigma}\psi \]

- The ellipse has a reflection symmetry with respect to the $x$- and $y$-axis $\Rightarrow$ it has a **twofold** symmetry
- All wave functions are either symmetric or antisymmetric with respect to rotation by $\pi$
NBs with Shapes with $N$-fold Rotational Symmetry

- Counterclockwise rotation by $2\pi/N$

$$
\hat{R}_N = \begin{pmatrix}
\cos \left(\frac{2\pi}{N}\right) & -\sin \left(\frac{2\pi}{N}\right) \\
\sin \left(\frac{2\pi}{N}\right) & \cos \left(\frac{2\pi}{N}\right)
\end{pmatrix} \quad \hat{U}_N = \begin{pmatrix}
e^{i\frac{2\pi}{N}} & 0 \\
0 & e^{-i\frac{2\pi}{N}}
\end{pmatrix} \quad \tilde{\psi}(r') = \begin{pmatrix}
\tilde{\psi}_1(r') \\
\tilde{\psi}_2(r')
\end{pmatrix} = \begin{pmatrix}
e^{-i\frac{2\pi}{N}} \tilde{\psi}_1(r) \\
e^{i\frac{2\pi}{N}} \tilde{\psi}_2(r)
\end{pmatrix}
$$

- $N$-fold rotational symmetry:

$$w(s') = e^{i\frac{2\pi}{N}} w(s) \quad e^{i\alpha(s')} = e^{i\frac{2\pi}{N}} e^{i\alpha(s)}$$

- Symmetry group $G = C_N = \{e, g, \ldots, g^{N-1}\}$ $g$: rotation by $2\pi/N$, $\hat{U}(g) = e^{i\frac{2\pi}{N} \hat{L}}$

- Wave functions classifiable according to their transformation properties

$$\psi^{(l)}_{1,2}(g_l^{-\lambda} r) = \tilde{\psi}^{(l)}_{1,2}(r') = e^{i\lambda \frac{2\pi}{N}} \psi^{(l)}_{1,2}(r) \quad l=0,1,2,\ldots,N-1$$

- The spinor components transform differently under rotation

- For $\tilde{\psi}^{(l)}_2(r)$

$$\tilde{\psi}_2(r') = e^{-i(l-1)\frac{2\pi}{N}} \tilde{\psi}_2(r) \equiv \tilde{\psi}^{(l-1)}_2(r')$$

$$\tilde{\psi}_2(s') = i e^{i\alpha(s')} \mathcal{K}^{-1} \tilde{\psi}^{(l)}_1(s') = e^{-i(l-1)\frac{2\pi}{N}} \tilde{\psi}_2(s) \equiv \tilde{\psi}^{(l-1)}_2(s')$$

- Symmetry-projected eigenvalue equations with $n=n_1+n_2$, $m=n_1-n_2$

\[
\begin{align*}
\tan \left( \frac{\sqrt{3}}{2} k_x - \hat{n}_x \pi \right) &= 3 \frac{\mathcal{K} (\mathcal{K}^2 - 1) \cos \theta_k}{1 + \mathcal{K}^4 - 4 \mathcal{K}^2 - \mathcal{K} (1 + \mathcal{K}^2) \sin \theta_k + 4 \mathcal{K}^2 \sin^2 \theta_k} \\
\tan \left( \frac{k_y}{2} - l \frac{2\pi}{3} - \hat{m}_l \pi \right) &= \sqrt{3} \frac{\frac{\mathcal{K}^2}{2} + \mathcal{K} \sin \theta_k}{1 - \frac{\mathcal{K}^2}{2} + \mathcal{K} \sin \theta_k}
\end{align*}
\]
Symmetry-Projected BIEs for NBs  
Zhang & Dietz, PRB 104, 064310 (2021)

- Boundary-integral equation can be written in the form

\[ u(\phi') = \int_0^{2\pi} d\phi |w'(\phi)| Q(k; \phi, \phi') u(\phi) \quad \text{with} \quad u(\phi) = \psi_{1,2}(\phi) \]

- 3fold symmetry implies

\[ w \left( \phi + \frac{2\pi}{3} \right) = e^{i\lambda \frac{2\pi}{3}} w(\phi) \Rightarrow Q \left( k; \phi, \phi' - \frac{2\pi}{3} \right) = Q \left( k; \phi + \frac{2\pi}{3}, \phi' \right) , \phi, \phi' \in [0, \frac{2\pi}{3}] \]

- Use 3fold symmetry and define \( M_\lambda(k; \phi, \phi') = Q \left( k; \phi + \frac{2\pi}{3}, \phi' \right) \)

\[ u(\phi') = \int_0^{\frac{2\pi}{3}} d\phi \tilde{M}(k; \phi, \phi') u(\phi) \quad \text{with} \quad u(\phi) = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}(\phi) \quad \text{and} \quad \tilde{M}(k; \phi, \phi') = \begin{pmatrix} M_0 & M_1 & M_2 \\ M_2 & M_0 & M_1 \\ M_1 & M_2 & M_0 \end{pmatrix}(k; \phi, \phi') \]

- Symmetry-projected boundary-integral equations

\[ u^{(l)}(\phi') = \int_0^{\frac{2\pi}{3}} d\phi \tilde{M}^{(l)}(k; \phi, \phi') u^{(l)}(\phi) \quad \text{with} \quad \tilde{M}^{(l)}(k; \phi, \phi') = \sum_{\lambda=0}^{2} e^{i\frac{2\pi}{3} \lambda} M_\lambda(k; \phi, \phi') \]
Wave functions of the Massive Neutrino Billiard with $C_3$ Symmetry

<table>
<thead>
<tr>
<th>kr$_0$ ∈ [0,35]</th>
<th>Singlets $l=0$</th>
<th>Doublets $l=1$</th>
<th>Doublets $l=2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1500 levels,</td>
<td>$</td>
<td>\psi_1</td>
<td>$</td>
</tr>
<tr>
<td>m=0</td>
<td>$</td>
<td>\psi_2</td>
<td>$</td>
</tr>
<tr>
<td>m=20</td>
<td>$</td>
<td>\psi_1</td>
<td>$</td>
</tr>
<tr>
<td>m=100</td>
<td>$</td>
<td>\psi_1</td>
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</tr>
</tbody>
</table>

- For $m \to \infty$ the wave function components $\psi_{1,2}$ decouple and $|\psi_2 / \psi_1| \to 0$
- The nodal-line structure of the singlets becomes discernible because $\text{Im}(\psi_{1,2}) \to 0$
- The intensity distributions of the doublets become similar
- The wave functions approach those of the QB
Spectral Properties of Neutrino & Quantum Billiards

- Spectral properties for mass $m=0$, $m=50$, $m=100$ (100), and for the QB
- Non-generic orbits manifest themselves as slow oscillations in $N^{\text{fluc}}(k)$
- Spectral properties are close to GUE for $m=0$ and to that of the QB for $m \geq 100$
Lengths Spectra of Neutrino Billiards

- For $m=0$ POs with an odd number of reflections are missing
- For $m=20$ the length spectrum exhibits peaks at the lengths of all POs
- For the singlets some peaks disappear with increasing $m$
  \[ \Rightarrow \text{pseudo orbits} \]
• The lengths spectra differ for small $m$ and become similar for $m \approx 100$
• The occurrence of peaks / interference of POs depends on $m$ and the direction of propagation of the POs and thus on $l$
→ The doublets exhibit peaks at the lengths of different pseudo orbits
### Momentum Distributions, Wave Functions & Currents of a Square NB ($l=1$)

| $|\Psi_n|$ | $\text{Re}\psi_1$ | $\text{Re}\psi_2$ | $|u|$ |
|---------|-----------------|-----------------|---------|

Local current:

$$\hat{u} = \nabla_p \hat{H}_D = c \hat{\sigma}, \quad u(r) = c \psi^\dagger \hat{\sigma} \psi$$

Momentum distribution:

$$\tilde{\psi}_n(q_x, q_y) = \iint dxdy \psi_n(x, y) e^{-iqr}$$

- The square has a **fourfold** symmetry $\Rightarrow$ the spinor components can be classified according to their transformation properties under rotation by $\pi/2$
The square QB is an untypical system with integrable dynamics.
The spectral properties of the square NB agree with Poisson.
→ overshootings are due to accidental degeneracies.
The spectral properties of the symmetry-projected eigenstates with \( l=0, \ l=1, \ l=2, \ l=3 \) are close to Semi-Poisson.
• Generally, the length spectra of the NBs exhibit peaks only at lengths of POs with an even number of reflections at the boundary
• For the symmetry-projected cases peaks appear at lengths of pseudo orbits
Wave Functions and Currents of Rectangular NB

| Re$\psi_1$ | Im$\psi_1$ | Re$\psi_2$ | Im$\psi_2$ | $|u|$ |
|------------|------------|------------|------------|------|

Local current:
\[ \hat{u} = \nabla_p \hat{H}_D = c\hat{\sigma}, \quad u(r) = c\psi^\dagger \hat{\sigma} \psi \]

- The rectangle has a reflection symmetry with respect to the $x$- and $y$-axis \( \Rightarrow \) it has a twofold symmetry
- All spinor components are either symmetric or antisymmetric with respect to rotation by \( \pi \)
The spectral properties of the symmetry-projected eigenstates with $l=0$ and $l=1$ are close to Semi-Poisson for all ratios of $a/b$.

How do the spectral properties change when destroying the $N$-fold symmetry by cutting the NB along symmetry lines?
Momentum Distributions, Wave Functions & Currents of a Triangle NB

- Triangular billiards are obtained by cutting the rectangular billiards long the diagonal → symmetry-reduced
- The NB and QB exhibit similar scarred wave functions

\[ |\Psi_n| \quad \text{Re}\psi_1 \quad \text{Re}\psi_2 \quad |u| \]

\( b/a = GM \)
Some Characteristics of Right-Triangle QBs

- Rectangular and right-triangle billiards with inner angles $\alpha=\pi/n$ are integrable and exhibit untypical spectral properties.
- Right-triangle billiards with inner angles $\alpha=\pi m/n$ ($m \neq 1$, $n, m$ incommensurable) are pseudointegrable:
  - Exhibit intermediate statistics, i.e., levels repel each other and the nearest-neighbor spacing distribution decays exponentially.
- Right-triangle with irrational $\alpha/\pi$ may exhibit GOE-like behavior.
- We found quarter-Poisson for NBs with $b/a$ irrational if it is semi-Poisson for the corresponding QBs.

Atas et al., JPA 46, 355204 (2013):

- **Semi-Poisson**: delete every second number from a Poisson sequence.
- **Quarter-Poisson**: delete every second number from a semi-Poisson sequence.

\[
P_\nu(s) = \frac{\Gamma(\nu + 2)^{\nu+1}}{\Gamma(\nu + 1)^{\nu+2}} s^\nu \exp\left(-\frac{\Gamma(\nu + 2)}{\Gamma(\nu + 1)} s\right) \quad \mathcal{P}_\nu(r) = \frac{\Gamma(2\nu + 2)\Gamma^2(\nu + 2)}{(\nu + 1)^2\Gamma^4(\nu + 1)} \frac{r^\nu}{(1 + r)^{2\nu+2}}
\]
Spectral Properties of Triangular NBs with $b/a=GM$ and $b/a=2$

- The spectral properties of the triangle QB is close to semi-Poisson statistics $\rightarrow$ linear level repulsion
- The spectral properties of the triangle NB is close to quarter-Poisson statistics $\rightarrow$ cubic level repulsion
- For $b/a=2$ the QB / NB shows GOE / GUE. Deviations are due to scarred functions
Circle Sector QB

\[ \Omega = \{(r, \varphi) : 0 < r < a, |\varphi| < \frac{\alpha}{2}\} \]

- **Eigenvalues:**
  \[ J_{\frac{\pi}{\alpha}}(k\ell_n a) = 0 \]

- **Eigenfunctions:**
  \[ \psi_{\ell,n}(r, \varphi) = \sin \left[ \frac{\pi \ell}{\alpha} \left( \varphi + \frac{\alpha}{2} \right) \right] J_{\frac{\pi}{\alpha}}(k\ell_n r) \]
• At the band edges the spectral properties coincide with those of the corresponding quantum billiard for any shape
• The system is effectively described by the Schrödinger equation
• The spectral properties are intermediate between Poisson and GOE
• With increasing energy the lattice structure starts to prevail leading to the occurrence of the van Hove singularities in the spectral density and the Dirac point
Near the Dirac points the spectral properties coincide with those of chaotic systems with preserved time-reversal invariance (GOE).

Attributed to the edge structure implying different boundary conditions for the independent triangular lattices.

→ Spectral properties of relativistic quantum billiards?
Spectral Properties of the 3fold-Symmetric Solutions of the Circle NB and the $120^\circ$ Circle Sector NB

- Circle neutrino billiard:
  \[ \psi_1^{nl} = J_l(k_n r) e^{i l \phi}, \psi_2^{nl} = i J_{l+1}(k_n r) e^{i (l+1) \phi} \]

- Trace formula for diameter orbit:
  \[ J_m(k\mathcal{R}) = \mathcal{K} J_{m+1}(k\mathcal{R}). \]
  \[ \rho_{\text{diam}}(k; \tilde{\beta}) = \sqrt{\frac{k\mathcal{R}}{\pi}} \sum_{p \text{ even}} \cos \left( \frac{2kp\mathcal{R} - p\theta_\beta + \pi}{4} \right). \]
Spectral Properties of Massive Half-Circle NB

- Spectral properties for mass \( m=0 \), 10, 20, 50, 100, and for the QB
- Contributions of diameter orbit are extracted by employing its trace formula
- Deviations from Poisson due to intermingling of symmetry classes of the spinor components and discontinuity in the BCs at the corners
- NBs do not have a classical counterpart \( \rightarrow \) Semiclassical limit?

Relativistic Quantum Chaos | 37
Semiclassical Approach: Length Spectra & Quantum Poincaré Section Map

- Length spectrum: \( |\tilde{\rho}(l)| = \left| \int_0^{k_{\text{max}}} dk e^{ikl} \rho_{\text{fluc}}(k) \right| \)

- Semiclassical approach: semiclassical approximation for the fluctuating part of spectral density in terms of a sum over periodic orbits

- Husimi function on PSOS: Projection of boundary function onto a coherent state on the billiard boundary

- Husimi functions:
  \[
  H_j(s, p) = \frac{1}{2\pi k_j} \int_0^L ds' \frac{1}{\left| \partial_{n'} \psi(s') \right|^2} \left| \int_0^L ds' \partial_{n'} \psi(s') C^{\delta}_{(s', p)}(s'; k_j) \right|^2
  \]

- Coherent state:
  \[
  C^{\delta}_{(s, p)}(s'; k_j) = \left( \frac{k_j}{\pi \delta^2} \right)^{1/4} \sum_{m=-\infty}^{\infty} \exp \left( ipk_j (s' - s + mL) - \frac{k_j^2}{2\delta^2} (s' - s + mL)^2 \right)
  \]

- Definition of Birkhoff coordinates

- The Poincaré map is defined in terms of the arclength \( s_n \) and the momentum \( p_n = |p| \sin \theta_n \) at \( n \) th bounce with the boundary
Trace Formula for the Spectral Density of Massive Neutrino Billiards
BD & Z. Li, PRE102, 042214 (2020)

• Quantization condition deduced from the BIEs for Massive NBs

\[ u(s') = \hat{Q}(k)u(s) \quad \Rightarrow \quad \det \left( \mathbb{1} - \hat{Q}(k) \right) = 0 \quad \Rightarrow \quad \rho_{\text{fluc}}(k) = \frac{1}{\pi} \Im \sum_{p=1}^{\infty} \frac{1}{p} \frac{d}{dk} \left[ \text{Tr} \left( \hat{Q} \right)^p (k) \right] \]

• Trace formula is a sum over the periodic orbits of the classical billiard

\[
3 \frac{1}{p} \frac{d}{dk} \left[ \text{Tr} \left( \hat{Q} \right)^p (k) \right] = \Re \sum_{\gamma_p} \cos \left( \Phi_{\gamma_p} - p \frac{\pi}{2} \right) B_{\gamma_p}^{(\gamma_p)} e^{i \Gamma_{\gamma_p}} A_{\gamma_p} e^{i \Theta_{\gamma_p}}
+ \Re \sum_{\gamma_p} \cos \left( \Phi_{\gamma_p} + \frac{\pi}{2} \right) B_{\gamma_p}^{(\gamma_p)} e^{i \Gamma_{\gamma_p}} A_{\gamma_p} e^{i \Theta_{\gamma_p}}
\]

\[ A_{\gamma_p} = \frac{f_{(p)}^{(PO)}}{r_{PO} \sqrt{|\text{Tr} M_{PO}^{(p)} - 2|}} \]

\[ \Theta_{\gamma_p} = k f_{(p)}^{(PO)} - \frac{\pi}{2} \mu_{(p)}^{(PO)} \]

QB

Gutzwiller

• The factor \( B_{\gamma_p}^{(l)} e^{i \Gamma_{\gamma_p}^{(l)}} \) depends on \( m \) and the direction of propagation of the PO

\[
\rho_{\text{fluc}}(k; \beta) \xrightarrow{\beta \to \infty} \frac{1}{\pi} \Re \sum_{\gamma_p} A_{\gamma_p} e^{i \Theta_{\gamma_p}}
\]

\[
\rho_{\text{fluc}}(k; \beta) \xrightarrow{\beta \to 0} \frac{1}{\pi} \Re \sum_{\gamma_p} (-1)^p \cos \left( \Phi_{\gamma_p} \right) \cos \left( p \frac{\pi}{2} \right) A_{\gamma_p} e^{i \Theta_{\gamma_p}}
\]

Only POs with even # reflections \( p \) contribute
The length spectra of the NBs exhibit peaks only at length of POs.
Husimi Distribution of the First Wave Function Component of $1/2$-Circle NB

<table>
<thead>
<tr>
<th>Mass</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>m=0</td>
<td>![Image]</td>
</tr>
<tr>
<td>m=10</td>
<td>![Image]</td>
</tr>
<tr>
<td>m=20</td>
<td>![Image]</td>
</tr>
<tr>
<td>m=50</td>
<td>![Image]</td>
</tr>
<tr>
<td>QB</td>
<td>![Image]</td>
</tr>
</tbody>
</table>

- For $m \approx 50$ the Husimi distribution is close to that of the QB, for smaller mass it is similar to it for diameter-orbit like WFs
Spectral Properties of Massive $1/4$-Ellipse NB

- Spectral properties for mass $m=0$, 10, 20, 50, 100, and for the QB
- Contributions of diameter orbit are extracted by employing its trace formula
- Deviations from Poisson due to intermingling of symmetry classes of the spinor components and discontinuity in the BCs at the corners
For $m \approx 20$ the Husimi distribution is close to that of the QB, for smaller mass it is similar to it for diameter-orbit like WFs.
**Constant Width Billiards**


- Classical dynamics is **unidirectional**
- Change from clockwise to anti-clockwise motion **classically forbidden**
- In quantum billiard transition from $p>0$ to $p<0$ possible due to tunneling

\[
w(\phi) = -R_0ie^{i\phi} - ia_3 \left[ \frac{e^{i\phi} - 1}{4} + \frac{e^{-i2\phi} - 1}{2} \right] - ia_5 \left[ \frac{e^{i6\phi} - 1}{6} - \frac{e^{-i4\phi} - 1}{4} \right], \phi \in [0, 2\pi)
\]

\[
w'(\phi) = R(\phi)e^{i\phi}, \quad R(\phi) = R_0 + 2ia_3 \sin(3\phi) + 2a_5 \cos(5\phi), \quad R(\phi) = R^*(\phi)
\]

\[
t(\phi) = [\cos \phi, \sin \phi], \quad n(\phi) = \left[ \cos \left( \phi - \frac{\pi}{2} \right), \sin \left( \phi - \frac{\pi}{2} \right) \right]
\]
Spectra of the CW NB & QB

• The spectra of the QB & NB can be separated into singlets and nearly-degenerate doublets

• **Singlets** correspond to **diameter orbit** \((p=0)\) of circle NB & QB

• For the QB **dynamical tunneling** leads to a splitting of the degenerate eigenvalues of the circle billiard into **doublets of nearly degenerate ones**

• Doublets can be separated into **two independent sequences**

• The circle **NB** has no **degeneracies**

• For the **NB** the spectra can be separated into eigenstates corresponding to Husimi functions localized in the upper or lower part of the PSOS
Comparison of Wave Functions & Husimi Functions of the QB and Massless NB

QB

adjacent eigenvalues (nearly degenerate doublets)

NB

eigenvalues separated by \(\approx 4\) others

located on small islands!

• NB: \(H_n(q,p)\neq 0\) only in upper or lower half of the PSM like classical dynamics
Comparison of Wave Functions and Husimi Distributions of the QB and Massless NB

$$\mu = 1$$

$$\mu = -1$$

Located on small islands

Boundary condition:

$$K \tilde{\psi}_2(\phi) = e^{i\phi} \mu \tilde{\psi}_1(\phi)$$
• RMT model: \( H^{\text{mixed}} = \begin{pmatrix} \mathcal{H} & d_0 \tau_1 V_1^\dagger \\ d_0 \tau_1 V_1 & D_1 \end{pmatrix} \)  
  \( H : 300 \times 300 \) GUE matrix 
  \( D_1 : 100 \times 100 \) diagonal Poisson matrix 

• \( V_1 \) couples \( H \) (chaotic dynamics) to \( D_1 \) (regular dynamics) with the coupling \( \tau_1 \) measured in units of mean spacing \( d_0 \)
Length Spectra of the QB and Massless NB

• The ‘doublet partners’ of the relativistic constant-width billiard are separated by several mean spacings → there is no dynamical tunneling
• Length spectra exhibit peaks at the lengths of the periodic orbits
• Peak at $l \approx 79$ corresponds to regular island in chaotic sea (11 bounces)
Thank you for your attention
Husimi Distribution of the First Wave Function Component of Half-Circle NB

- Husimi functions
  \[ H_n(p, q) = \frac{1}{2\pi k_n} \int_0^L ds \left| \frac{1}{\langle \hat{n}(s), \vec{\nabla}\Psi_n(s) \rangle} \right|^2 \int_0^L ds \langle \hat{n}(s), \vec{\nabla}\Psi_n(s) \rangle C_{(p, q)}^0(s; k_n) \]^2 \]

- For \( m \approx 50 \) the Husimi distribution is close to that of the QB, for smaller mass it is similar to it for diameter-orbit like WFs

Relativistic Quantum Chaos | 51
Spectral Properties of Robin Sector Billiard for Symmetric Case

• Robin BC

\[
\tilde{\beta} + \frac{1}{2} \kappa(s) \Phi_j(s) + \partial_n \Phi_j(n, s) \big|_{n \to 0^-} = 0
\]

• Shown are the symmetric solutions
TBM Description for Graphene Billiards

- TBM for energies $\omega$ and eigenstates $|\Psi_{\vec{q}}(\vec{r})\rangle = \sum_j \left( C_A |\Phi_j^A\rangle + C_B |\Phi_j^B\rangle \right) e^{i\vec{q} \cdot \vec{R}_j}$

- Generalized eigenvalue problem
  \[ \mathcal{H}_{TB} |\Psi_{\vec{q}}(\vec{r})\rangle = \omega(\vec{q}) \mathcal{S}_{WO} |\Psi_{\vec{q}}(\vec{r})\rangle \]

- Tight-binding Hamiltonian
  \[ \mathcal{H}_{TB} = \begin{pmatrix} f_0 + t_2 f_2(\vec{q}) & t_1 f_1(\vec{q}) + t_3 f_3(\vec{q}) \\ t_1 f_1(\vec{q}) + t_3 f_3(\vec{q}) & f_0 + t_2 f_2(\vec{q}) \end{pmatrix} \]

- Wavefunction overlap matrix
  \[ \mathcal{S}_{WO} = \begin{pmatrix} 1 + s_2 f_2(\vec{q}) & s_1 f_1(\vec{q}) + s_3 f_3(\vec{q}) \\ s_1 f_1(\vec{q}) + s_3 f_3(\vec{q}) & 1 + s_2 f_2(\vec{q}) \end{pmatrix} \]

- $f_0$ is resonance frequency of an “isolated“ void
- Nearest-neighbour contribution $\sim t_1, s_1$
- Next-nearest neighbour contribution $\sim t_2, s_2$
- Second-nearest neighbour contribution $\sim t_3, s_3$