Kerr Black Holes from multi-loop Scattering Amplitudes

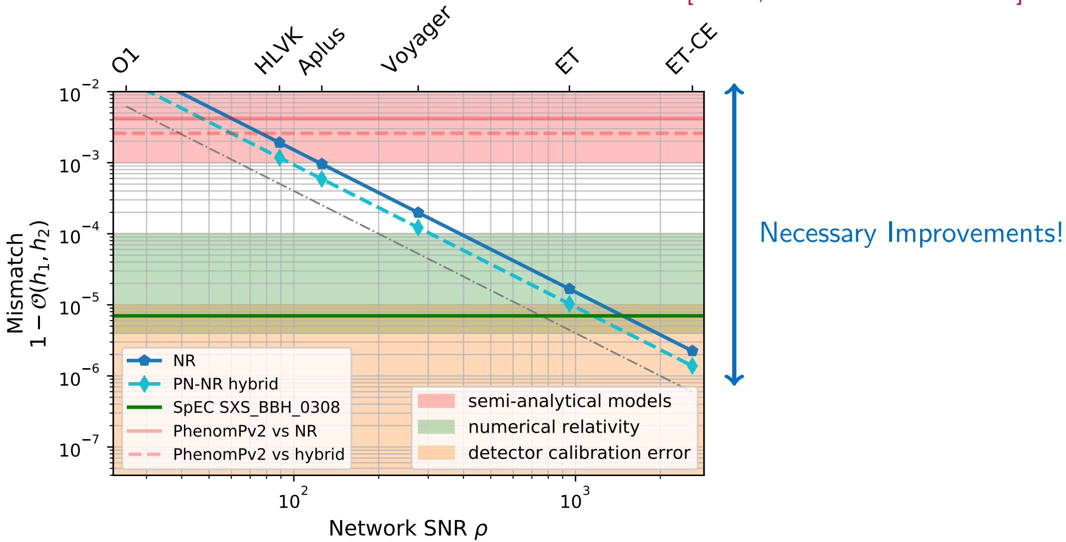
Manfred Kraus

F. Febres Cordero, G. Lin, M. S. Ruf, M. Zeng

ICTP-SAIFR São Paulo
Gravitational Waves meet Amplitudes in the Southern Hemisphere
21. August 2023

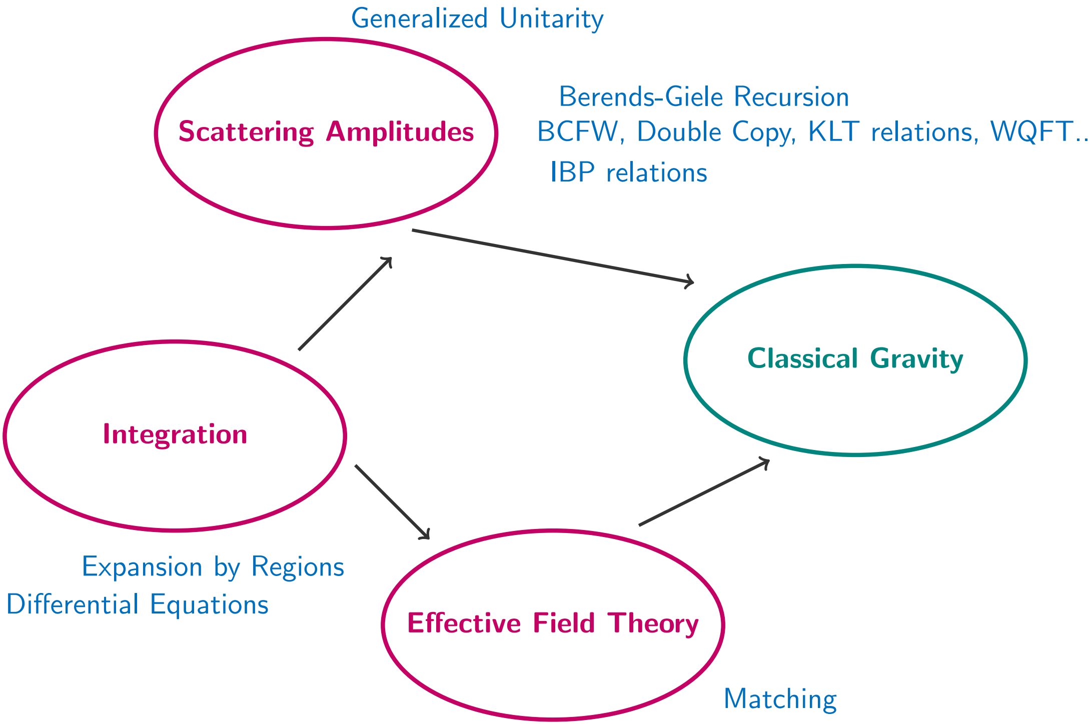
Precision requirements for Waveforms

[Pürrer,Haster arXiv:1912.10055]



See also Maria Haneys talk last Monday!

The Landscape of methods



Context

Last week we had talks:

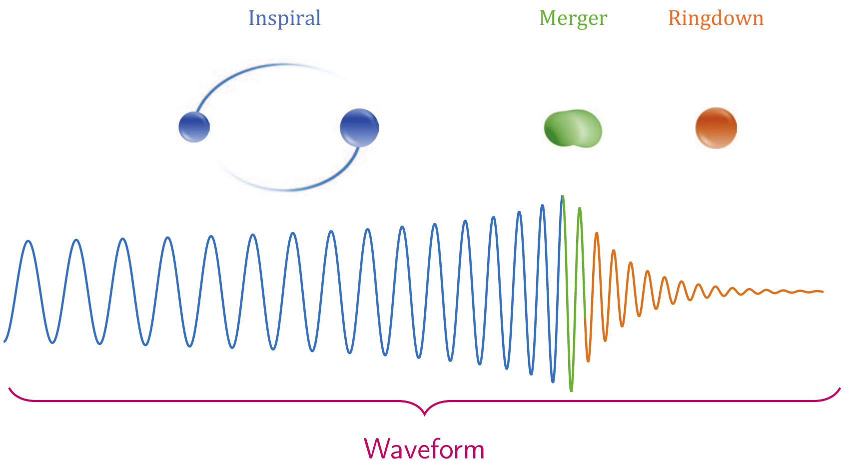
- Mao Zeng: Overview of Amplitudes methods
- Justin Vines: Spinning Black Holes to all orders in spin

Now:

Spinning Black Holes with finite Spin representation and their classical Hamiltonian at 3PM!

The Stages of a Binary Black Hole Merger

[Antelis,Moreno EPJP 132 (2017) 1, 10]



Inspiral — Weak-field approximation

$$r \gg r_s = \frac{2G(m_1+m_2)}{c^2}$$

Merger — Numerical Relativity - ab initio, but very computing intense

Ringdown — Black Hole Perturbation Theory

During the Inspiral the dynamics are described by classical Hamiltonians

$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2) + V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{L}\cdot\boldsymbol{S}}{\boldsymbol{r}^2}$$
$$+ V^{(3)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{p}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^2} + V^{(4)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{S}^2}{\boldsymbol{r}^2} + V^{(5)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{r}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^4} + \cdots$$

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0PN (1687)	1PN (1938)	2PN (1973)	3PN (2000)	4PN (2014)	5PN (2020)	6PN (2020)
Newton	Einstein,Infeld Hoffmann	Kimura et al	Damour et al Blanchet, Faye	Damour et al	Mastrolia et al Blümlein et al	Blümlein et al

$$\begin{aligned}
G(1 &+ v^2 + v^4 + v^6 + v^8 + v^{10} + \dots) \\
G^2(&1 + v^2 + v^4 + v^6 + v^8 + \dots) \\
G^3(&1 + v^2 + v^4 + v^6 + \dots) \\
G^4(&1 + v^2 + v^4 + \dots) \\
G^5(&1 + v^2 + \dots) \\
G^6(&1 + \dots)
\end{aligned}$$

$$\frac{v^2}{c^2} \sim \frac{GM}{r} \ll 1$$

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$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + \boxed{V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2)} + V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{L}\cdot\boldsymbol{S}}{\boldsymbol{r}^2}$$

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$$G(\,1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \ldots)$$ 1PM

$$G^2(\,1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$ 2PM (1985) Westphal

$$G^3(\,1 + v^2 + v^4 + v^6 + \ldots)$$ 3PM (2019) Bern, Cheung, Roiban Shen, Solon, Zeng

$$G^4(\,1 + v^2 + v^4 + \ldots)$$ 4PM (2021) Bern,Parra-Martinez Roiban,Ruf,Shen Solon, Zeng

$$G^5(\,1 + v^2 + \ldots)$$

$$G^6(\,1 + \ldots)$$

$$\frac{GM}{r} \ll 1$$

The Classical Hamiltonian

During the Inspiral the dynamics are described by classical Hamiltonians

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Spin Hamilitonian much less known!

Post Newtonian

- $\boldsymbol{S}$: 3PN $\mathcal{O}(G^4)$ [Levi et al arXiv:2208.14949, Mastrolia et al arXiv:2209.00611]
- $\boldsymbol{S}^2$: 5PN $\mathcal{O}(G^4)$ [Kim,Levi,Yin arXiv:2112.01509]
- $\boldsymbol{S}^3$: 4PN $\mathcal{O}(G^2)$ [Levi,Mougiakakos,Viera arXiv:1912.06276]
- $\boldsymbol{S}^4$: 5PN $\mathcal{O}(G^2)$ [Levi,Teng arXiv:2008.12280]

Post Minkowskian

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- $\boldsymbol{S}^4$: 2PM [Chen,Chung,Huang arXiv:2111.13639]
- $\boldsymbol{S}^5$: 2PM [Bern et al arXiv:2203.06202]
- $\boldsymbol{S}^\infty$: 2PM [Aoude,Haddad,Helset arXiv:2203.06197,arXiv:2205.02809]

The Classical Hamiltonian

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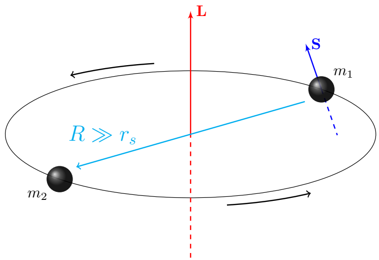
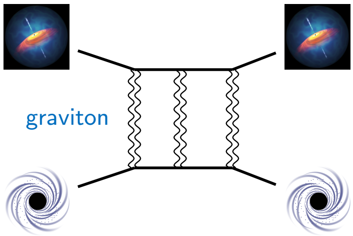
Point Particle Effective Field Theory



point particle EFT

[Goldberger,Rothstein'06]



Massive particles coupled to Gravity!

- Lagrangian:

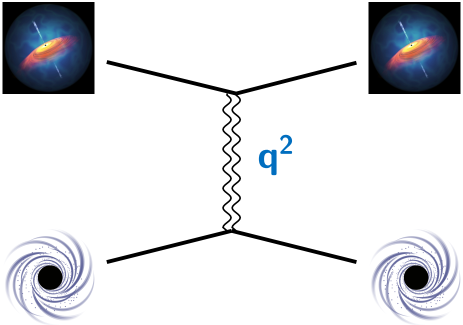
$$\mathcal{L} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{point-particle}} + \mathcal{L}_{\text{finite-size}}$$

- Finite-size effects via tidal operators

$$\mathcal{L}_{\text{finite-size}} \subset \sum c_{abc}\ \phi_i \nabla^a \phi_i \nabla^b R^c$$

- Spin d.o.f by including massive higher-spin fields

$$\phi_i \to \phi_i^{\mu_1 \dots \mu_N}$$



$$A^{\text{tree}} = \frac{4\pi\ G}{E_1 E_2}\ \frac{m_1^2 m_2^2(1-2\sigma^2)}{q^2} + \cdots$$

$$\sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

The classical potential is then given by

$$V^{(0)}(\boldsymbol{r}^2, \boldsymbol{p}^2) = \int \frac{d^3 q}{(2\pi)^3}\ e^{i\vec{q}\cdot\vec{r}} A^{\text{tree}} = -\frac{G m_1 m_2}{r}\left(\frac{m_1 m_2}{E_1 E_2}(2\sigma^2 - 1)\right)$$
$$= -\frac{G m_1 m_2}{r} + \mathcal{O}(\mathbf{v}^2)$$

- Scattering amplitude approach provides all-order $\mathbf{v}^2$ corrections!
- Loop amplitudes provide higher-order $\mathcal{O}(G^n)$ corrections!

- Consider Einstein-Hilbert gravity coupled to massive scalar and vector particles

$$S[\phi, A^\mu, g_{\mu\nu}] = \int d^4x \, \sqrt{-g}\Big[-\frac{2}{\kappa^2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} + \frac{1}{2}m_A^2 g^{\mu\nu}A_\mu A_\nu \Big]$$

- Expand spacetime metric around flat space

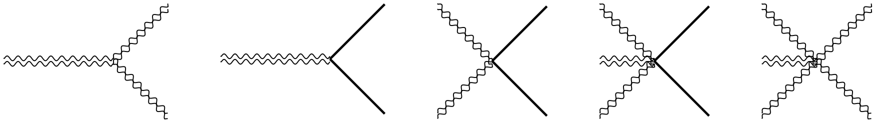
$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \,, \qquad \kappa = \sqrt{32\pi G}$$

- General relativity is non-linear

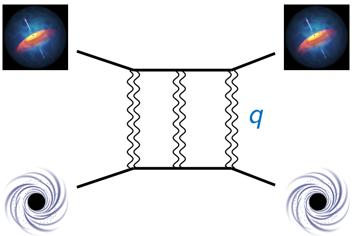
$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda}h^{\nu}{}_{\lambda} + \mathcal{O}(\kappa^3)$$

$$\sqrt{-g} = 1 + \frac{\kappa}{2}h + \frac{\kappa^2}{8}\left(h^2 - 2h_{\mu\nu}h^{\mu\nu}\right) + \mathcal{O}(\kappa^3)$$

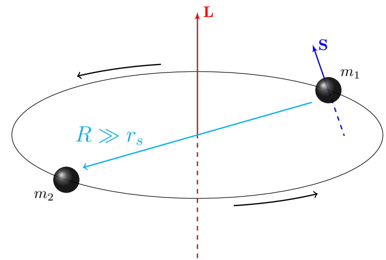
- Leads to quantum EFT of gravity with **infinite** many interaction terms



classical power counting



$\hbar \to 0$



$$m_\phi, m_A, \sqrt{s} \sim 1\,, \qquad q \sim \ell_i \sim \hbar\,, \qquad \mathbf{S}_{\rm cl} \sim \frac{1}{\hbar}$$

Expand Amplitude in $\boldsymbol{q} \sim \hbar$

$$\mathcal{M}^{\text{n-loop}} = G^{n+1}\left(\frac{c_0}{\boldsymbol{q}^2} + \frac{c_1}{|\boldsymbol{q}|} + c_2 \log(\boldsymbol{q}^2) + \dots\right)$$

Presence of finite Quantum-Spin complicates identification of classical terms

$$\mathbf{S}^2_{\rm cl} \sim \frac{1}{\hbar^2} \qquad \text{but} \qquad \mathbf{S}^2_{\rm QM} = s(s+1)\,\mathbb{1}$$

We come back to this later ...

What do we need to compute?

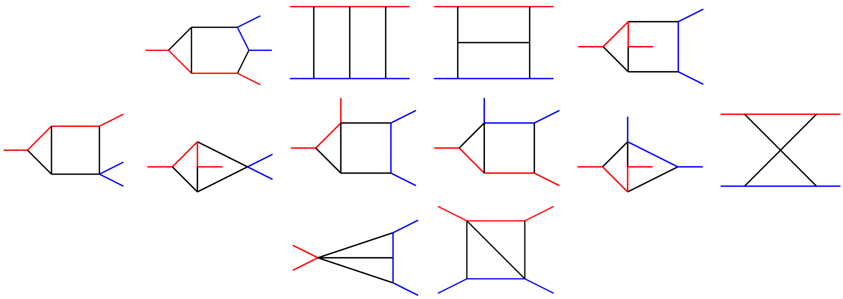
$$A(p_1, \epsilon_1) + \phi(p_2) \to \phi(p_3) + A(p_4, \epsilon_4)$$

- only subset of the full amplitude contributes to classical physics

  $|\ell_1| \sim |\ell_2| \sim |q| \sim \hbar \ll m_A, m_\phi, \sqrt{s}$ Spin: $\boldsymbol{S} \sim \frac{1}{\hbar}$ $q = p_4 - p_1$

- 1-loop:

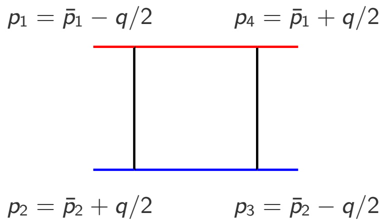
- 2-loop:



- Relate: Qunatum Spin $\Leftrightarrow$ Classical Spin

  Fernando will talk on Thursday about the explicit computation!

Structure of the Amplitude



Form factor decomposition:

$$\mathcal{M}_{\lambda_1\lambda_4} = \sum_{n=1}^{5} M_n \, \epsilon_\mu(p_1,\lambda_1)\, T_n^{\mu\nu} \epsilon_\nu^\star(p_4,\lambda_4)$$

with

$$T_1 = (\epsilon_1 \cdot \epsilon_4^\star) \qquad T_2 = (\epsilon_1 \cdot q)(\epsilon_4^\star \cdot q)$$

$$T_3 = q^2(\epsilon_1 \cdot \bar{p}_2)(\epsilon_4^\star \cdot \bar{p}_2) \qquad T_{4/5} = (\epsilon_1 \cdot \bar{p}_2)(\epsilon_4^\star \cdot q) \mp (\epsilon_1 \cdot q)(\epsilon_4^\star \cdot \bar{p}_2)$$

T_5 vanishes after IBP reduction!

Extraction of form factors

Solve System of Equations for each phase space point:

$$\begin{pmatrix} \mathcal{M}_{++} \\ \mathcal{M}_{+-} \\ \mathcal{M}_{-+} \\ \mathcal{M}_{--} \\ \mathcal{M}_{LL} \end{pmatrix} = \begin{pmatrix} T_1^{++} & T_2^{++} & T_3^{++} & T_4^{++} & T_5^{++} \\ T_1^{+-} & T_2^{+-} & T_3^{+-} & T_4^{+-} & T_5^{+-} \\ T_1^{-+} & T_2^{-+} & T_3^{-+} & T_4^{-+} & T_5^{-+} \\ T_1^{--} & T_2^{--} & T_3^{--} & T_4^{--} & T_5^{--} \\ T_1^{LL} & T_2^{LL} & T_3^{LL} & T_4^{LL} & T_5^{LL} \end{pmatrix} \times \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix}$$

- Expand relativistic $\epsilon^\mu(p)$ around rest-frame momenta

$$\epsilon^\mu_s(p) = \Lambda^\mu{}_\nu(p)\hat{\epsilon}^\nu_s \,, \qquad \Lambda^\mu{}_\nu(p) = \begin{pmatrix} \sqrt{1+v^2} & \boldsymbol{v}^T \\ \boldsymbol{v} & \delta^{ij} + \frac{1}{v^2}(\sqrt{1+v^2}-1)v^i v^j], \end{pmatrix}$$

  Expand Lorentzboost for $p = \bar{p}_1 \pm q/2$ for $q \sim \hbar \ll 1$

- Rest-frame Polarization vectors: $\hat{\epsilon}^\mu_s = (0, \hat{\boldsymbol{\epsilon}}_s)$ and $|1, s\rangle \equiv \hat{\boldsymbol{\epsilon}}_s$

$$|1,1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} \,, \qquad |1,0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,, \qquad |1,-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \,,$$

- Relate rest-frame vectors to Spin: $(\boldsymbol{S}_i)_{jk} = -i\epsilon_{ijk}$

$$\begin{aligned} \langle 1, m_2, |\mathbb{1}|1, m_1\rangle &= (\hat{\boldsymbol{\epsilon}}_{m_1} \cdot \hat{\boldsymbol{\epsilon}}^\star_{m_2}) \,, \\ i\epsilon_{ijk}\langle 1, m_2|\boldsymbol{S}_k|1, m_1\rangle &= \hat{\boldsymbol{\epsilon}}^{\star\, i}_{m_2}\hat{\boldsymbol{\epsilon}}^j_{m_1} - \hat{\boldsymbol{\epsilon}}^i_{m_1}\hat{\boldsymbol{\epsilon}}^{\star\, j}_{m_2} \,, \\ \langle 1, m_2|\boldsymbol{S}_i\boldsymbol{S}_j|1, m_1\rangle &= \delta_{ij}(\hat{\boldsymbol{\epsilon}}_{m_1} \cdot \hat{\boldsymbol{\epsilon}}^\star_{m_2}) - \hat{\boldsymbol{\epsilon}}^i_{m_1}\hat{\boldsymbol{\epsilon}}^{\star\, j}_{m_2} \,, \\ \langle 1, m_2|\boldsymbol{S}^2|1, m_1\rangle &= 2(\hat{\boldsymbol{\epsilon}}_{m_1} \cdot \hat{\boldsymbol{\epsilon}}^\star_{m_2}) \end{aligned}$$

😬

How to match quantum spin onto classical spin? - II

After expansion of Lorentz Boost:

$$T_1 = -\langle O_1 \rangle + \frac{1}{m_A^2(\gamma_1+1)}\left[\langle O_2 \rangle - \langle O_3 \rangle + \frac{\langle O_4 \rangle}{2m_A^2(\gamma_1+1)}\right] + \mathcal{O}(\hbar^2)$$

$$T_2 = \langle O_3 \rangle + \mathcal{O}(\hbar^2)$$

$$T_3 = -\frac{E^2}{m_A^2}\langle O_4 \rangle + \mathcal{O}(\hbar^2)$$

$$T_4 = -\frac{E}{m_A}\langle O_2 \rangle + \frac{E-m_A}{m_A}\langle O_3 \rangle - \frac{E}{m_A^3(\gamma_1+1)}\langle O_4 \rangle + \mathcal{O}(\hbar^2)$$

with

$$O_1 = \mathbb{1}$$

$$O_2 = -i(\boldsymbol{q}\times\boldsymbol{p})\cdot\boldsymbol{S}$$

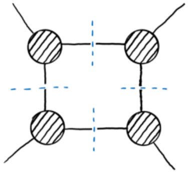
$$O_3 = \frac{1}{2}\boldsymbol{q}^2\boldsymbol{S}^2 - (\boldsymbol{q}\cdot\boldsymbol{S})^2$$

$$O_4 = \boldsymbol{q}^2\left(\frac{1}{2}\boldsymbol{p}^2\boldsymbol{S}^2 - (\boldsymbol{p}\cdot\boldsymbol{S})^2\right)$$

Spin multipoles!

- Computation performed in the framework of Numerical Multi-loop Unitarity

$$\lim_{\forall \rho_i \to 0} \mathcal{A}(\ell) = \prod_j \frac{1}{\rho_j} \times$$



Caravel

[extended for Masses & Gravity]

- Compute quantum Amplitude

$$\mathcal{M} = \sum_{i,j} c_{ij} I_i T_j \qquad I_i \text{ - Master integral} \qquad T_j \text{ - Polarization Tensor}$$

- Expand $\{c_{ij}, I_i, T_j\}$ in $\hbar$ to obtain asymptotic multi-pole expansion

$$\mathcal{M}^{\text{2-loop}} = G^3 \left[\left(\frac{c_0^{(1)}}{\boldsymbol{q}^2} + \frac{c_1^{(1)}}{|\boldsymbol{q}|} + c_2^{(1)} \log(\boldsymbol{q}^2) + \dots \right) \mathcal{O}_1 + \left(\frac{c_0^{(2)}}{\boldsymbol{q}^2} + \frac{c_1^{(2)}}{|\boldsymbol{q}|} + c_2^{(2)} \log(\boldsymbol{q}^2) + \dots \right) \mathcal{O}_2 + \dots \right]$$

The Amplitude

$$\mathcal{M}^{(2)} = -\left(\frac{\kappa}{2}\right)^6 \frac{1}{\boldsymbol{q}^2}\left(\frac{\mu^2}{q^2}\right)^{2\epsilon}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon}\sum_{n=1}^{4}\sum_{k=0}^{2} M_n^{(2,k)}\,|\boldsymbol{q}|^k\; T_n + \mathcal{O}(|\boldsymbol{q}|)\,,$$

Scalar coefficient:

$$M_1^{(2,0)} = \frac{m_A^4 m_\phi^4 (1-2\sigma^2)^2}{(8\pi)^2(\sigma^2-1)}\left[\frac{1-2\sigma^2}{\epsilon^2}+\frac{3}{\epsilon}\right] + \mathcal{O}(\epsilon^0)\,,$$

$$M_1^{(2,1)} = -i\pi\frac{3m_A^3 m_\phi^3(m_A+m_\phi)}{(16\pi)^2\sqrt{\sigma^2-1}}\left\{(1-7\sigma^2+10\sigma^4)\left[\frac{1}{\epsilon}-2\log(2)\right]-\frac{3+6\sigma^2-65\sigma^4+64\sigma^6}{3(\sigma^2-1)}\right\}+\mathcal{O}(\epsilon)\,,$$

$$M_1^{(2,2)} = \frac{1}{\epsilon^2}\frac{m_A^2 m_\phi^4(1-2\sigma^2)}{(16\pi)^2(\sigma^2-1)} + \frac{1}{\epsilon}\left[\frac{m_A^3 m_\phi^3(3+12\sigma^2-4\sigma^4)}{(8\pi)^2\sqrt{\sigma^2-1}}\operatorname{arccosh}(\sigma) + \frac{m_A^4 m_\phi^2(-3+4\sigma^2)(1-8\sigma^2+8\sigma^4)}{(16\pi)^2(\sigma^2-1)^2}\right.$$
$$\left. + \frac{m_A^2 m_\phi^4(-9+48\sigma^2-78\sigma^4+40\sigma^6)}{(16\pi)^2(\sigma^2-1)^2} + \frac{m_A^3 m_\phi^3\sigma(-59+156\sigma^2-162\sigma^4+68\sigma^6)}{6(8\pi)^2(\sigma^2-1)^2}\right] + \mathcal{O}(\epsilon^0)$$

Other coefficents M_2, M_3 and M_4 are similar in size!

Structure of Scattering Amplitudes in the classical limit

$$\mathcal{M}^{\text{tree}} = G\left(\frac{c_0^{\text{cl}}}{\boldsymbol{q}^2} + \dots\right)$$

$$\mathcal{M}^{\text{1-loop}} = G^2\left(\frac{c_1^{\text{scl}}}{\boldsymbol{q}^2} + \frac{c_1^{\text{cl}}}{|\boldsymbol{q}|} + c_1^{\text{Q}}\log(\boldsymbol{q}^2) + \dots\right)$$

$$\mathcal{M}^{\text{2-loop}} = G^3\left(\frac{c_2^{\text{sscl}}}{\boldsymbol{q}^2} + \frac{c_2^{\text{scl}}}{|\boldsymbol{q}|} + c_2^{\text{cl}}\log(\boldsymbol{q}^2) + \dots\right)$$

- **Classical** terms at *any* loop order
- **Super-classical** contributions have to be subtracted/cancelled
- **Quantum corrections** are suppressed by powers of $\hbar$

How to systematically extract a classical potential from that?

Effective Field Theory - I

[Cheung,Rothstein,Solon arXiv:1808.02489]

Non-relativistic, non-local, classical 3D EFT [Bern,Luna,Roiban,Shen,Zeng arXiv:2005.03071]

$$
L_{\mathrm{EFT}} = \int_{\boldsymbol{k}} \hat{\phi}^{\dagger}(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_\phi^2} \right) \hat{\phi}(\boldsymbol{k}) + \int_{\boldsymbol{k}} \hat{A}^{\dagger,i}(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_A^2} \right) \hat{A}^i(\boldsymbol{k})
$$

$$
- \int_{\boldsymbol{k},\boldsymbol{k}'} V_{ij}(\boldsymbol{k},\boldsymbol{k}') \hat{A}^{\dagger,i}(\boldsymbol{k}') \hat{A}^j(\boldsymbol{k}) \hat{\phi}^{\dagger}(-\boldsymbol{k}') \hat{\phi}(-\boldsymbol{k})
$$

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$$L_{\mathrm{EFT}} = \int_{\boldsymbol{k}} \hat{\phi}^{\dagger}(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_\phi^2} \right) \hat{\phi}(\boldsymbol{k}) + \int_{\boldsymbol{k}} \hat{A}^{\dagger,i}(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_A^2} \right) \hat{A}^i(\boldsymbol{k})$$

$$- \int_{\boldsymbol{k},\boldsymbol{k}'} V_{ij}(\boldsymbol{k},\boldsymbol{k}') \hat{A}^{\dagger,i}(\boldsymbol{k}') \hat{A}^j(\boldsymbol{k}) \hat{\phi}^{\dagger}(-\boldsymbol{k}') \hat{\phi}(-\boldsymbol{k})$$

$$V_{ij}(\boldsymbol{k},\boldsymbol{k}') \sim \sum_{n=1}^{4} \sum_{L}^{\infty} |\boldsymbol{q}|^{L-2} \left(\frac{\mu^2}{\boldsymbol{q}^2} \right)^{L\epsilon} c_L^{(n)}(\boldsymbol{k}^2)\, O_n^{ij}$$

Effective Field Theory - I

[Cheung,Rothstein,Solon arXiv:1808.02489]

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Matching coefficient

$$V_{ij}(\boldsymbol{k},\boldsymbol{k}') \sim \sum_{n=1}^{4} \sum_{L}^{\infty} |\boldsymbol{q}|^{L-2} \left(\frac{\mu^2}{\boldsymbol{q}^2} \right)^{L\epsilon} c_L^{(n)}(\boldsymbol{k}^2)\, O_n^{ij}$$

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$$L_{\rm EFT} = \int_{\boldsymbol{k}} \hat{\phi}^\dagger(-\boldsymbol{k})\left(i\partial_t - \sqrt{\boldsymbol{k}^2+m_\phi^2}\right)\hat{\phi}(\boldsymbol{k}) + \int_{\boldsymbol{k}} \hat{A}^{\dagger,i}(-\boldsymbol{k})\left(i\partial_t - \sqrt{\boldsymbol{k}^2+m_A^2}\right)\hat{A}^i(\boldsymbol{k})$$
$$- \int_{\boldsymbol{k},\boldsymbol{k}'} V_{ij}(\boldsymbol{k},\boldsymbol{k}')\hat{A}^{\dagger,i}(\boldsymbol{k}')\hat{A}^j(\boldsymbol{k})\hat{\phi}^\dagger(-\boldsymbol{k}')\hat{\phi}(-\boldsymbol{k})$$

Matching coefficient

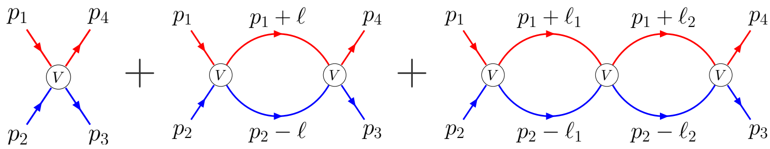
$$V_{ij}(\boldsymbol{k},\boldsymbol{k}') \sim \sum_{n=1}^{4}\sum_{L}^{\infty} |\boldsymbol{q}|^{L-2}\left(\frac{\mu^2}{\boldsymbol{q}^2}\right)^{L\epsilon} c_L^{(n)}(\boldsymbol{k}^2)\ O_n^{ij}$$

Spin Operators

$$O_1 = \mathbb{1}\ , \qquad O_2 = -i(\boldsymbol{q}\times\boldsymbol{p})\cdot\boldsymbol{S}\ ,$$
$$O_3 = \frac{1}{2}\boldsymbol{q}^2\boldsymbol{S}^2 - (\boldsymbol{q}\cdot\boldsymbol{S})^2\ , \qquad O_4 = \boldsymbol{q}^2\left(\frac{1}{2}\boldsymbol{p}^2\boldsymbol{S}^2 - (\boldsymbol{p}\cdot\boldsymbol{S})^2\right)$$

- EFT given by iterated Bubble diagrams

[Cheung,Rothstein,Solon arXiv:1808.02489]



- Form factor decomposition: $\mathcal{M}_{\rm EFT} = \sum_{n=1}^{4} M^{(n)}_{\rm EFT}\, O_n \to$ Projectors
- Follow stragegy:
  - integrate ℓ^0_n by contours

$$\Delta(\ell) = i\int \frac{d\ell^0}{2\pi} \frac{1}{(E_1+\ell^0)-\sqrt{E_1^2+Y_1}} \frac{1}{(E_2-\ell^0)-\sqrt{E_2^2+Y_1}}$$
$$= \frac{1}{E_1+E_2-\sqrt{E_1^2+Y_1}-\sqrt{E_2^2+Y_1}}$$

  - Expand around $Y_n \ll 1$

$$\Delta(\ell) = -\frac{2E_1E_2}{E_1+E_2}\frac{1}{Y_1} + \mathcal{O}(Y_1^0)$$

$Y_1 = (\boldsymbol{\ell}+\boldsymbol{p})^2 - \boldsymbol{p}^2$

- Perform Potential Region expansion: $|\boldsymbol{\ell}_i| \sim |\boldsymbol{q}| \sim \hbar \ll |\boldsymbol{p}|, m_A, m_\phi$
- Insert expressions for master integrals

[Parra-Martinez,Ruf,Zeng 2020]

- First time fully in Dimensional Regularization

- $O_1 = \mathbb{1}$ yields spinless Hamiltonian

$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\,, \qquad \text{with} \qquad V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2) = \sum_{i=1}^{3} c_i^{(1)}(\boldsymbol{k}^2)\left(\frac{G}{r}\right)^i$$

contains Newton!

$$c_1^{(1)}(\boldsymbol{k}^2) = \frac{m_A^2 m_\phi^2}{E_1 E_2}\left(1 - 2\sigma^2\right)\,, \qquad c_2^{(1)}(\boldsymbol{k}^2) = \frac{3(m_\phi + m_A) m_\phi^2 m_A^2}{4 E_1 E_2}(1 - 5\sigma^2)\,,$$

$$\begin{aligned} c_3^{(1)}(\boldsymbol{k}^2) = \frac{m_A^2 m_\phi^2}{E_1 E_2}\Bigg[&- \frac{2}{3} m_A m_\phi \left(\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \left(-12\sigma^4 + 36\sigma^2 + 9\right) + 22\sigma^3 - 19\sigma \right) \\ &- 2(m_\phi^2 + m_A^2)\left(6\sigma^2 + 1\right) \Bigg] + \frac{3 E m_A^2 m_\phi^2}{4 E_1 E_2}(m_A + m_\phi) \frac{(1 - 2\sigma^2)(1 - 5\sigma^2)}{(\sigma^2 - 1)} \\ &- \frac{3 m_A^4 m_\phi^4}{E_1 E_2 \boldsymbol{k}^2}\,, \end{aligned}$$

Agrees with [Bern, Cheung, Roiban, Shen, Solon, Zeng arXiv:1901.04424]

Conservative Potential: Spin-Orbit Term

- $O_2 = -i(\boldsymbol{q} \times \boldsymbol{p}) \cdot \boldsymbol{S}$ yields the spin-orbit coupling.

$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2) \frac{\boldsymbol{L} \cdot \boldsymbol{S}}{\boldsymbol{r}^2} \,, \quad \text{with} \quad V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2) = \sum_{i=1}^{3} c_i^{(2)}(\boldsymbol{k}^2) \left(\frac{G}{r}\right)^i$$

$$c_i^{(2)}(\boldsymbol{k}^2) = c_{i,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) + c_{i,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) + \frac{c_i^{(1)}(\boldsymbol{k}^2)}{m_A^2(\gamma_1 + 1)}$$

$$c_{1,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = -\frac{2\sigma m_\phi}{E\xi} \,, \qquad c_{2,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = \frac{m_\phi(4m_A + 3m_\phi)\sigma(5\sigma^2 - 3)}{4E\xi(\sigma^2 - 1)} \,,$$

$$\begin{aligned} c_{3,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = \frac{m_\phi}{E\xi(\sigma^2-1)^2} \Big[& -2m_A^2\sigma(3-12\sigma^2+10\sigma^4) - \left(\tfrac{83}{6}+27\sigma^2-52\sigma^4+\tfrac{44}{3}\sigma^6\right) m_A m_\phi - m_\phi^2\sigma\left(\tfrac{7}{2}-14\sigma^2+12\sigma^4\right) \\ & + \frac{(4m_A+3m_\phi)E}{4}\sigma(2\sigma^2-1)(5\sigma^2-3) + 4m_A m_\phi \sigma(\sigma^2-6)(2\sigma^2+1)\sqrt{\sigma^2-1}\,\mathrm{arccosh}(\sigma) \Big] \,, \end{aligned}$$

$$c_{1,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) = 0 \,, \qquad c_{2,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) = E\xi c_1^{(2)} \frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2} + c_1^{(1)} \left(E\xi \frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2} + \frac{c_1^{(2)}\left(\frac{2E^2\xi}{\boldsymbol{k}^2} + \frac{1}{\xi} - 3\right)}{2E} \right) \,,$$

$$\begin{aligned} c_{3,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) = & \left(c_1^{(1)}\right)^2 \left(-\frac{2}{3}E^2\xi^2 \frac{\partial^2 c_1^{(2)}}{\partial(\boldsymbol{k}^2)^2} + \left(\xi \left(3 - \frac{E^2\xi}{\boldsymbol{k}^2}\right) - 1 \right) \frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2} + c_1^{(2)} \left(\frac{\frac{1}{2\xi} - 2}{E^2} + \frac{3\xi - 1}{\boldsymbol{k}^2} \right) \right) \\ & + c_1^{(1)} \left(c_1^{(2)} \left(\left(-\frac{3E^2\xi^2}{\boldsymbol{k}^2} + 6\xi - 2 \right) \frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2} - \frac{4}{3}E^2\xi^2 \frac{\partial^2 c_1^{(1)}}{\partial(\boldsymbol{k}^2)^2} \right) \right. \\ & \left. + \frac{4}{3}E\xi \left(\frac{\partial c_2^{(2)}}{\partial \boldsymbol{k}^2} - 2E\xi \frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2} \frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2} \right) + \frac{E^2\xi^2 \left(c_1^{(2)}\right)^2}{2\boldsymbol{k}^2} + c_2^{(2)} \left(\frac{\frac{2}{3\xi} - 2}{E} + \frac{E\xi}{\boldsymbol{k}^2} \right) \right) - \frac{1}{6}E^2\xi^2 \left(c_1^{(2)}\right)^3 \\ & + c_1^{(2)} \left(\frac{2}{3}E\xi \left(\frac{\partial c_2^{(1)}}{\partial \boldsymbol{k}^2} - 2E\xi \left(\frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2} \right)^2 \right) + \frac{c_2^{(1)}\left(\frac{3E^2\xi}{\boldsymbol{k}^2} + \frac{1}{\xi} - 3 \right)}{3E} \right) + \frac{2}{3}E\xi c_2^{(1)} \frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2} + \frac{4}{3}E\xi c_2^{(2)} \frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2} \,. \end{aligned}$$

- Coefficients of O_3 and O_4 much more involved!
- We determined the classical Hamiltonian up to $\mathcal{O}(G^3)$

$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2) + V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{L}\cdot\boldsymbol{S}}{\boldsymbol{r}^2}$$
$$+ V^{(3)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{p}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^2} + V^{(4)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{S}^2}{\boldsymbol{r}^2} + V^{(5)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{r}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^4}$$

With our Hamilton we reproduced observables at 3PM computed in world-line QFT formalism by [Jakobsen,Mogull arXiv:2201.07778]

Spin Universality

Spin Universality

- fixed spin-s amplitudes have access to spin multipoles up to $\boldsymbol{S}^{2s}$

$$\mathcal{A}(\phi_1\phi_2 \to \phi_1\phi_2) \sim \mathbb{1} \qquad \mathcal{A}(\phi\psi \to \phi\psi) \sim \{\mathbb{1}, \boldsymbol{L}\cdot\boldsymbol{S}\}$$
$$\mathcal{A}(\phi V \to \phi V) \sim \{\mathbb{1}, \boldsymbol{L}\cdot\boldsymbol{S}, (\boldsymbol{r}\cdot\boldsymbol{S})^2, (\boldsymbol{p}\cdot\boldsymbol{S})^2\}$$

- **Spin Universality:** (conjecture!)
  Classical multipole coefficients are independent of the nature of the *test particle*!
- For arbitrary spin there should be a fifth contribution from

$$\mathcal{O}_5 = \boldsymbol{q}^2\boldsymbol{S}^2 \sim \frac{\hbar^2}{\hbar^2} = 1 \qquad \text{but} \qquad \mathcal{O}_5^{\text{fixed-spin}} = \boldsymbol{q}^2\boldsymbol{S}^2 = \boldsymbol{q}^2 s(s+1)\,\mathbb{1}$$

- 1-Loop example: the classical contributions live at $1/|\boldsymbol{q}|$

$$\mathcal{M}_{cl} = \frac{1}{|\boldsymbol{q}|}\left[c_0\mathbb{1} + c_5(\boldsymbol{q}^2\boldsymbol{S}^2) + \ldots\right] = \frac{1}{|\boldsymbol{q}|}\left[c_0 + c_5\boldsymbol{q}^2 s(s+1)\right]\,\mathbb{1} + \ldots$$

  - $\Rightarrow$ The $\boldsymbol{S}^2$ contribution lives for fixed-spin at subleading $\boldsymbol{q}^2$ with respect to $\mathbb{1}$
  - $\Rightarrow$ Mixes with quantum corrections to the $\mathbb{1}$ operator at $\mathcal{O}(|\boldsymbol{q}|)$!
  - $\Rightarrow$ How to disentangle this mess?!

Observation

- Compute $\hbar$ expansion of scalar-scalar Amplitude

$$\mathcal{A}(\phi_1\phi_2 \to \phi_1\phi_2) \sim \mathbb{1}\left(\frac{a^{\rm scl}}{|\boldsymbol{q}|^2} + \frac{a^{\rm cl}}{|\boldsymbol{q}|} + a_0 \log(\boldsymbol{q})^2 + a_1|\boldsymbol{q}|\right)$$

- Compute $\hbar$ expansion of scalar-vector Amplitude of $O_1 = \mathbb{1}$

$$\mathcal{A}(\phi_1 V \to \phi_1 V) \sim \mathbb{1}\left(\frac{b^{\rm scl}}{|\boldsymbol{q}|^2} + \frac{b^{\rm cl}}{|\boldsymbol{q}|} + b_0 \log(\boldsymbol{q})^2 + b_1|\boldsymbol{q}|\right) + \dots$$

- Spin-Universality:
  Superclassical and classical terms are identical $\to$ $a^{\rm scl} = b^{\rm scl}$ and $a^{\rm cl} = b^{\rm cl}$
- a_0 and b_0 are pure quantum corrections
- a_1 is pure quantum correction, b_1 is mixed quantum/classical

$$b_1 \equiv a_1 + s(s+1)c_5 \qquad \text{Why?!}$$

- After subtracting the scalar quantum corrections $c_5\mathcal{O}_5$ agrees with the literature!

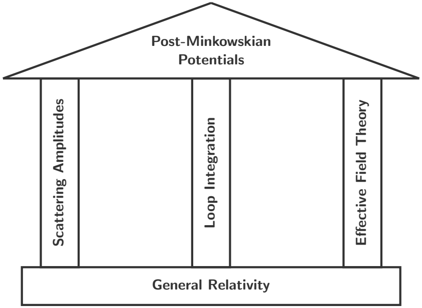
$$s(s+1)c_5|\boldsymbol{q}|\ \mathbb{1} = \frac{c_5}{|\boldsymbol{q}|}\boldsymbol{q}^2 s(s+1)\ \mathbb{1} = \frac{c_5}{|\boldsymbol{q}|}(\boldsymbol{q}^2\boldsymbol{S}^2) \qquad \text{classical!}$$

Summary & Outlook

Summary

Recent years has seen amazing synergies between Particle Physics and General Relativity!

- Scattering Amplitudes: obtain analytical results using a cocktail of methods



- computation of classical Hamiltonian up to $\boldsymbol{S}^2$ terms at $\mathcal{O}(G^3)$

$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2) + V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2) \frac{\boldsymbol{L} \cdot \boldsymbol{S}}{\boldsymbol{r}^2}$$
$$+ V^{(3)}(\boldsymbol{r}^2, \boldsymbol{p}^2) \frac{(\boldsymbol{p} \cdot \boldsymbol{S})^2}{\boldsymbol{r}^2} + V^{(4)}(\boldsymbol{r}^2, \boldsymbol{p}^2) \frac{\boldsymbol{S}^2}{\boldsymbol{r}^2} + V^{(5)}(\boldsymbol{r}^2, \boldsymbol{p}^2) \frac{(\boldsymbol{r} \cdot \boldsymbol{S})^2}{\boldsymbol{r}^4} + \cdots$$

Outlook

Our current Framework allows to study various aspects:

- higher spin contributions (massive higher-spin representations)

$$V_1 V_2 \to V_1 V_2 \qquad \text{spin-spin interactions}$$

$$\phi H^{\mu\nu} \to \phi H^{\mu\nu} \qquad \text{higher spin multipoles}$$

- finite size effects (higher dim. operators)

$$\Delta\mathcal{L} = \frac{c}{8} R_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \qquad \text{NS equation-of-state}$$

- Radiation effects (external gravitons)
- more loops
- Three-Body potentials? [Jones,Solon, arXiv:2208.02281]

$$\phi_1\phi_2\phi_3 \to \phi_1\phi_2\phi_3$$

Conceptually:

- Extracting systematically subleading $\boldsymbol{S}^2$ operators from finite-spin amplitudes?
- Outlined procedure checked so far only at one-loop order