# Self-force meets post-Minkowskian in the scattering regime

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Gravitational Waves meet Amplitudes in the Southern Hemisphere

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MAX-PLANCK-GESELLSCHAFT

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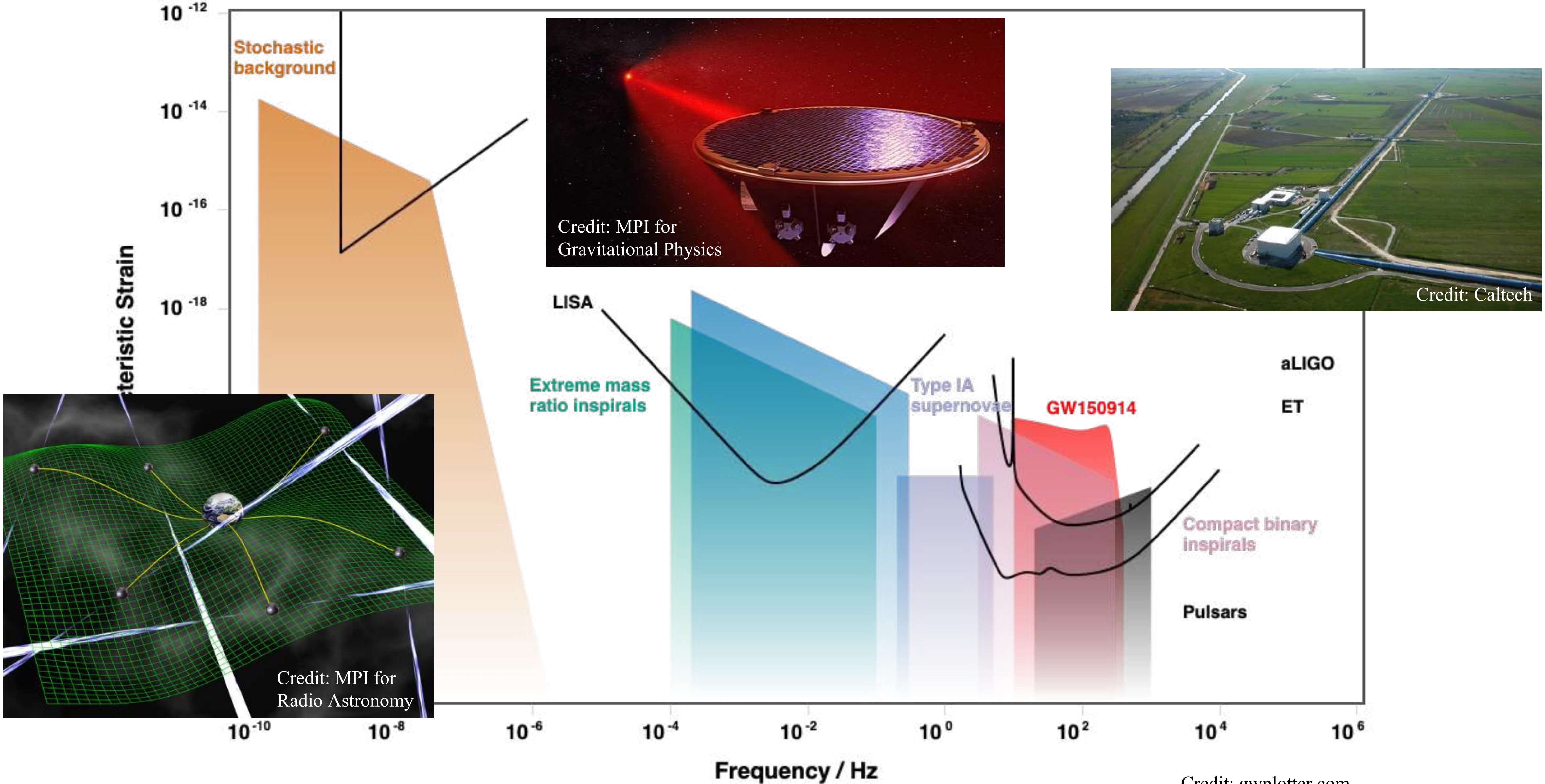
- Self-force introduction.
- Self-force scattering motivation.
- Scalar field toy model:
  - Mass change of the particle.
  - Self-force correction to the scattering angle.
- Interfaces with post-Minkowskian:
  - Weak field comparison.
  - Strong field resummation.
- Future steps.

# Frequencies ranges of gravitational wave sources




Credit: gwplotter.com

# Extreme-mass ratio inspirals

EMRI: compact object orbiting a supermassive BH with mass ratio:

$$q_m := \frac{m_1}{m_2} < 10^{-4}$$

$m_1$ : small BH $\qquad$ $m_2$ : big BH

Expected to be highly eccentric with precessing spins:

- Complex phenomenology.

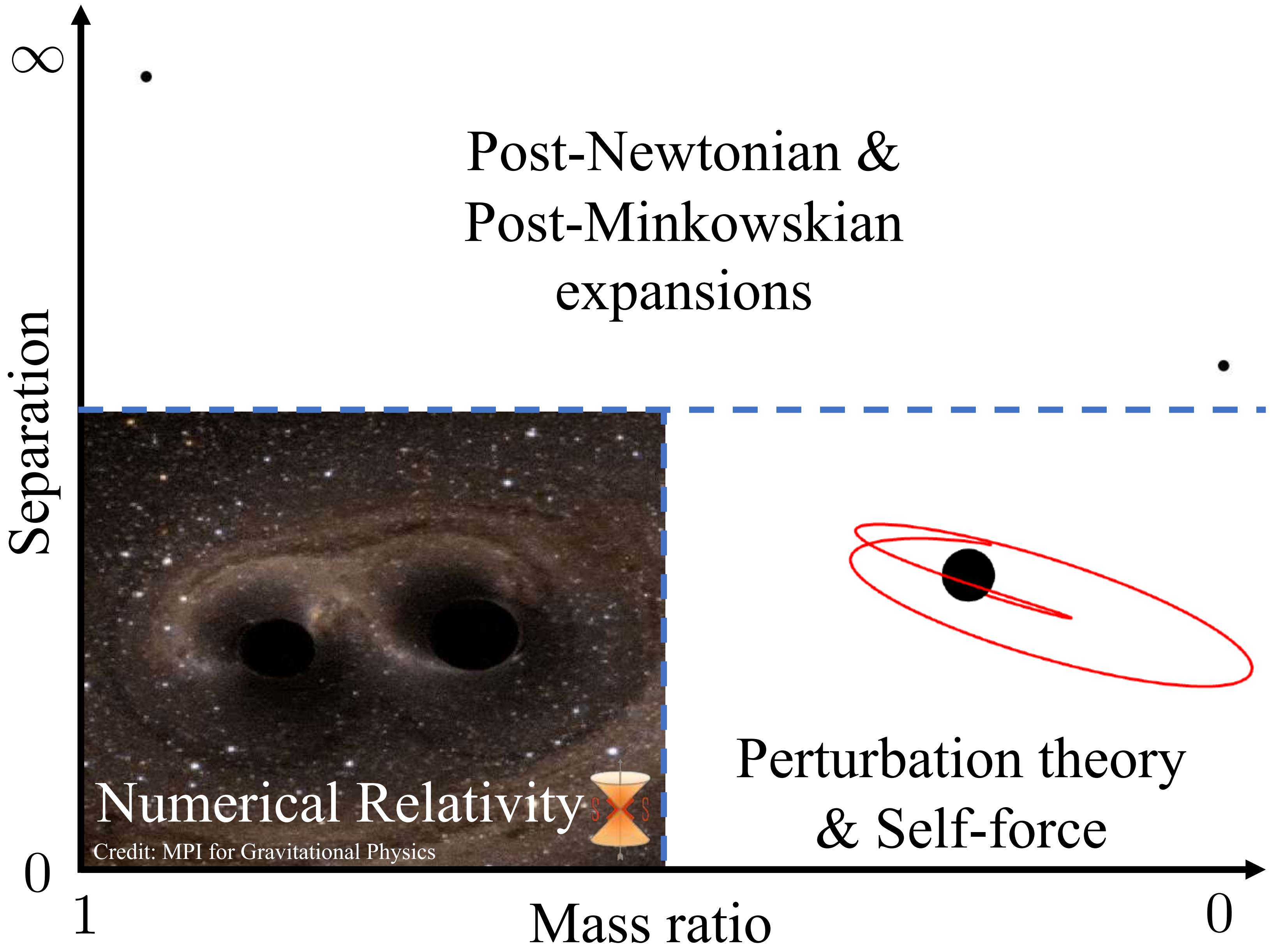
$10^5$ orbits within LISA band:

- A year of observation.

EMRI signals weaker than instrumentation noise:

- Large time within the LISA band leads to large cumulative signal-to-noise ratio.

# Two-body parameter space



# Self-force expansion

Expand in the mass ratio:

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + q_m h^{(1)}_{\alpha\beta} + q_m^2 h^{(2)}_{\alpha\beta} + \ldots \qquad q_m := \frac{m_1}{m_2} < 10^{-4}$$

Schwarzschild/Kerr

0SF:

- Orbits are fixed geodesics.
- No dissipation.
- Have full analytic solutions for bound and unbound in Kerr [BHPToolkit].

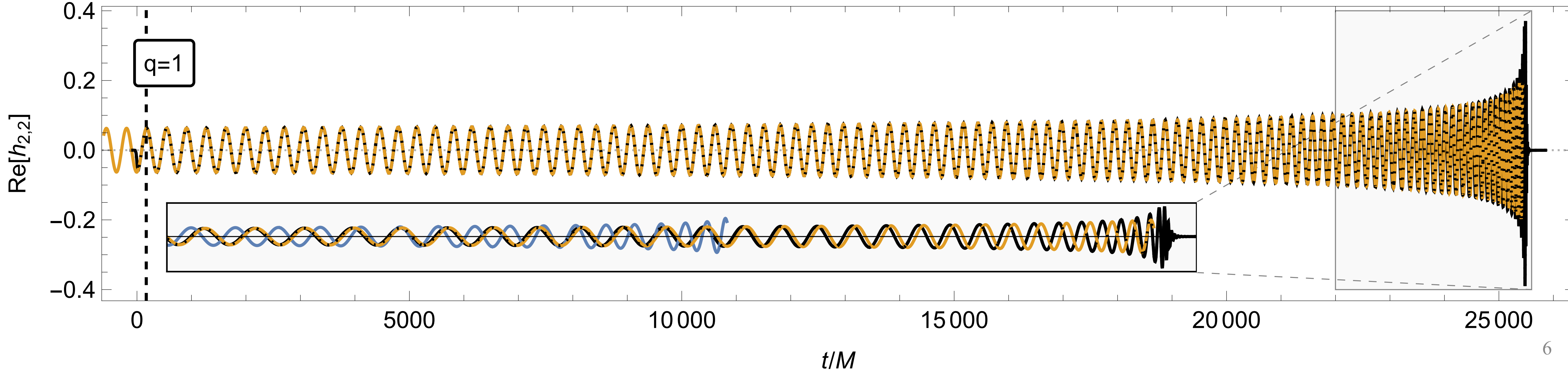
1SF:

- System is ***exactly*** described by point particle on fixed geodesic of background metric.
- All information in one SF calculation: conservative and dissipative.
- Formulated in time and frequency domains.
- Have generic numerical solutions for bound SF in Kerr [van de Meent '17].

# Self-force expansion

2SF:

- System is described by particle on fixed geodesic of 1SF metric.
  - Back-reaction from 1SF included.
- All information in one SF calculation: conservative and dissipative.
- Finite-size and secondary spin effects can be included.
- Only formulated in the frequency domain.
- Have quasi-circular non-spinning waveforms using 2SF fluxes [Wardell et al. '22].



# Self-force vs post-Minkowskian expansions

Two independent expansions of the same system:

| | 1PM | 2PM | 3PM | 4PM | $\dots$ | |
|---|---|---|---|---|---|---|
| 0SF | $G$ | $G^2$ | $G^3$ | $G^4$ | $\dots$ | $\rightarrow$ analytical |
| 1SF | $qG$ | $qG^2$ | $qG^3$ | $qG^4$ | $\dots$ | $\rightarrow$ numerical |
| 2SF | $q^2G$ | $q^2G^2$ | $q^2G^3$ | $q^2G^4$ | $\dots$ | $\rightarrow$ impossible (for now) |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | |

$n$SF contains all orders in $G$

$n$PM contains all orders in $q$

# SF scattering motivation: Informing PM

Post-Minkowskian expansion of momentum transfer of a scatter orbit:

$$|\mathbf{Q}| = \frac{2Gm_2m_1}{b}\left[Q^{1\text{PM}}_{\text{Schw}} + \frac{G}{b}Q^{2\text{PM}}_{\text{Schw}}(m_1+m_2) + \frac{G^2}{b^2}Q^{3\text{PM}}_{\text{Schw}}\left(m_1^2+m_2^2\right) + \frac{G^2m_2m_1}{b^2}Q^{3\text{PM}}_{1\text{SF}} + \dots\right]$$

Geodesic

1SF

Mass-ratio dependence of terms determined by exchange symmetry $m_1 \leftrightarrow m_2$.

Self-force information determines the complete 2-body Hamiltonian [Damour '20]:

$0\text{SF} \to 2\text{PM}$ $1\text{SF} \to 4\text{PM}$ $2\text{SF} \to 6\text{PM}$

Information from extreme-mass ratio scattering can determine bound motion across all mass ratios.

# SF scattering motivation: Informing EOB

Expand a Schwarzschild (or Kerr) Hamiltonian:

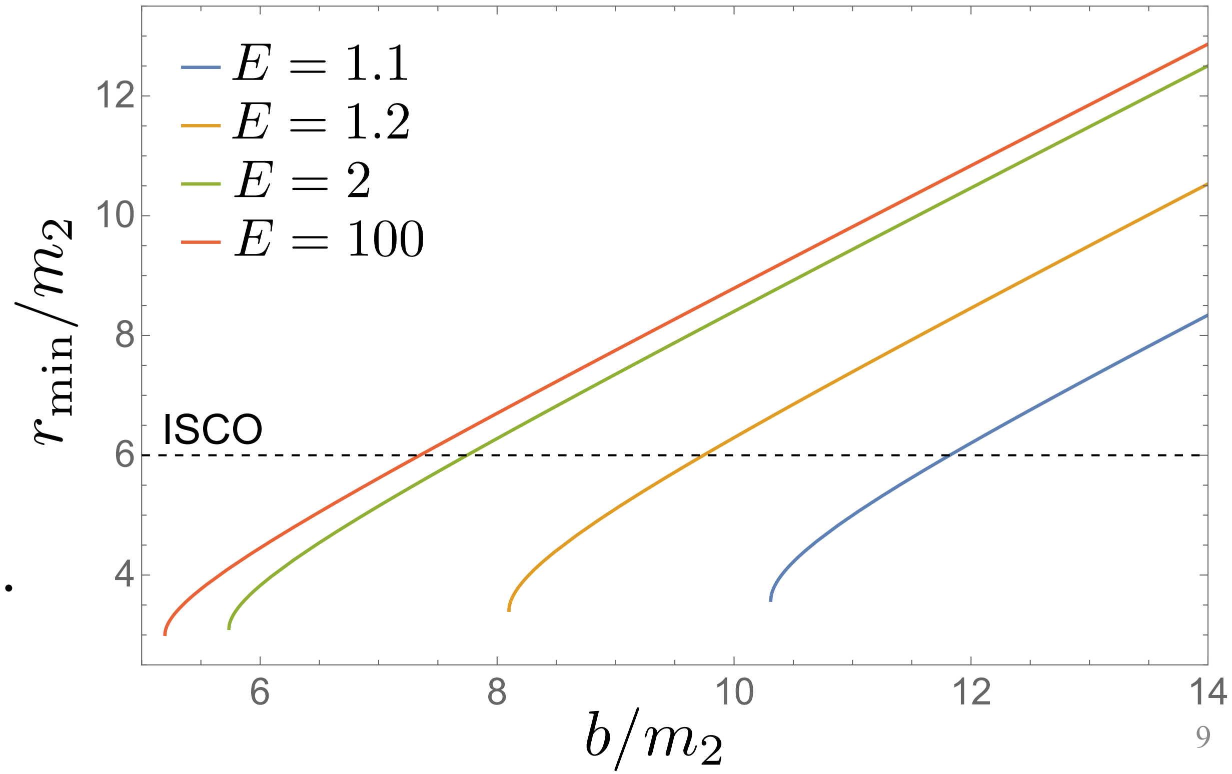
$$H = \sqrt{p_{r_*}^2 + A(r)\left[\frac{p_\varphi^2}{r^2} + \left(\frac{m_1 m_2}{(m_1+m_2)}\right)^2 + Q(r)\right]}$$

$A$ and $Q$ potentials expanded in PN (or PM).

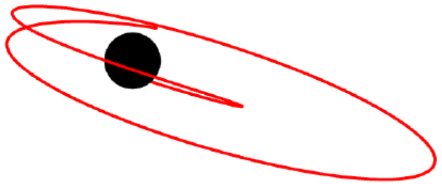
Some calibration terms:

- Comparable mass: calibrate to NR.
- Extreme mass-ratio: calibrate to SF fluxes.

Possible to calibrate to strong-field SF scattering.



# Bound vs unbound self-force



Periodic.

Distinct set of frequencies:

$$\sum_{\omega} e^{-i\omega t}$$

More naturally tackled as PDEs in the frequency domain.

PDEs are faster and more accurate.

No natural period.

Continuous spectrum of frequencies:

$$\int e^{-i\omega t}\, d\omega$$

More naturally tackled as ODEs in the time domain.

ODEs are more tractable.

# Scalar field toy model

Endow particle with a spin-0 scalar charge $Q$.

Keep inertial mass but ignore gravitational mass.

New small expansion parameter:

$$q_s := \frac{Q^2}{m_1 m_2}$$

Scalar field $\Phi$ obeys the Klein-Gordon equation:

$$\Box\Phi = Q \int_{-\infty}^{\infty} \delta^4 \left(x^\mu - x_p^\mu(\tau)\right) \, d\tau$$

Decompose into (time-domain) modes:

$$\Phi = \frac{Q}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \psi_{\ell m}(t,r) Y_{\ell m}(\theta,\varphi)$$

$$\psi_{,uv} + V(\ell;r)\psi = S_\psi(\ell; x_p^\mu)\, \delta\left(r - R\right)$$

$u, v$ : Eddington-Finkelstein coordinates

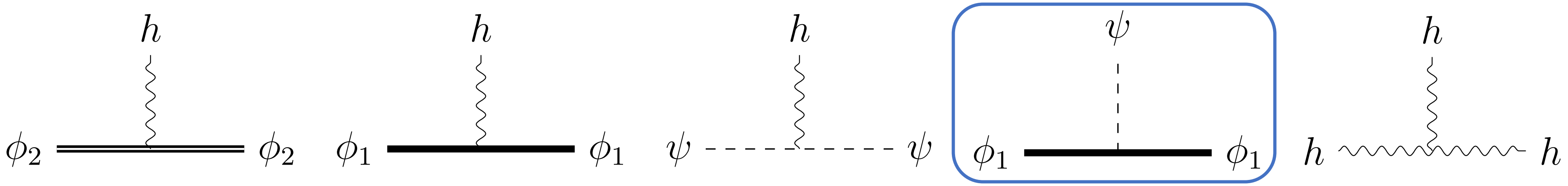
# Scalar self-force in terms of amplitudes

Lagrangian:

$$S = \int d^D x \sqrt{-g} \left[ -\frac{2}{\kappa^2} R + \frac{1}{2}\phi_1(\Box + m_1^2)\phi_1 + \frac{1}{2}\phi_2(\Box + m_2^2)\phi_2 + \frac{1}{2}\psi \Box \psi + \frac{1}{2} Q \psi \phi_1^2 \right]$$

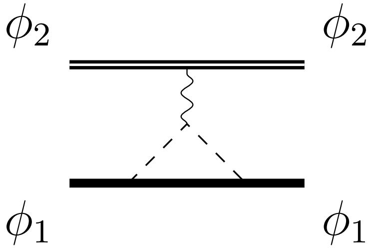
$\phi_{1,2}$ : black holes $\qquad\qquad$ $\psi$ : scalar field

Three-point interaction vertices:

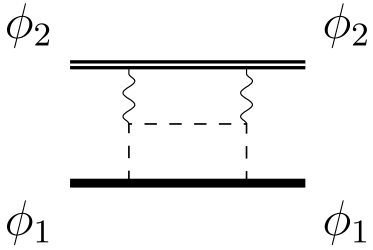



Only keep terms which are linear in mass-ratio and proportional to $Q^2$ :

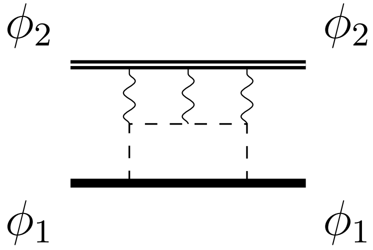
2PM :




3PM :




4PM :




[Cheung, Rothstein, Solon] [Bern, Cheung, Roiban, Shen, Solon, Zeng] [Bern, Cheung, Para-Martinez, Roiban, Ruf, Shen, Solon, Zeng]

# Scattering geodesics

Several parameterisations:

- Energy and angular momentum: $E > 1$ $L > L_c(E)$
- Eccentricity and semi-latus rectum: $e > 1$ $p > 6 + 2e$
- Velocity at infinity and impact parameter: $0 < v < 1$ $b > b_c(v)$

Geodesic equations:

$$\frac{dt}{d\tau} = \frac{E\,r}{r - 2M} \qquad \frac{d\varphi}{d\tau} = \frac{L}{r^2} \qquad \left(\frac{dr}{d\tau}\right)^2 = E^2 - V(L;r)$$

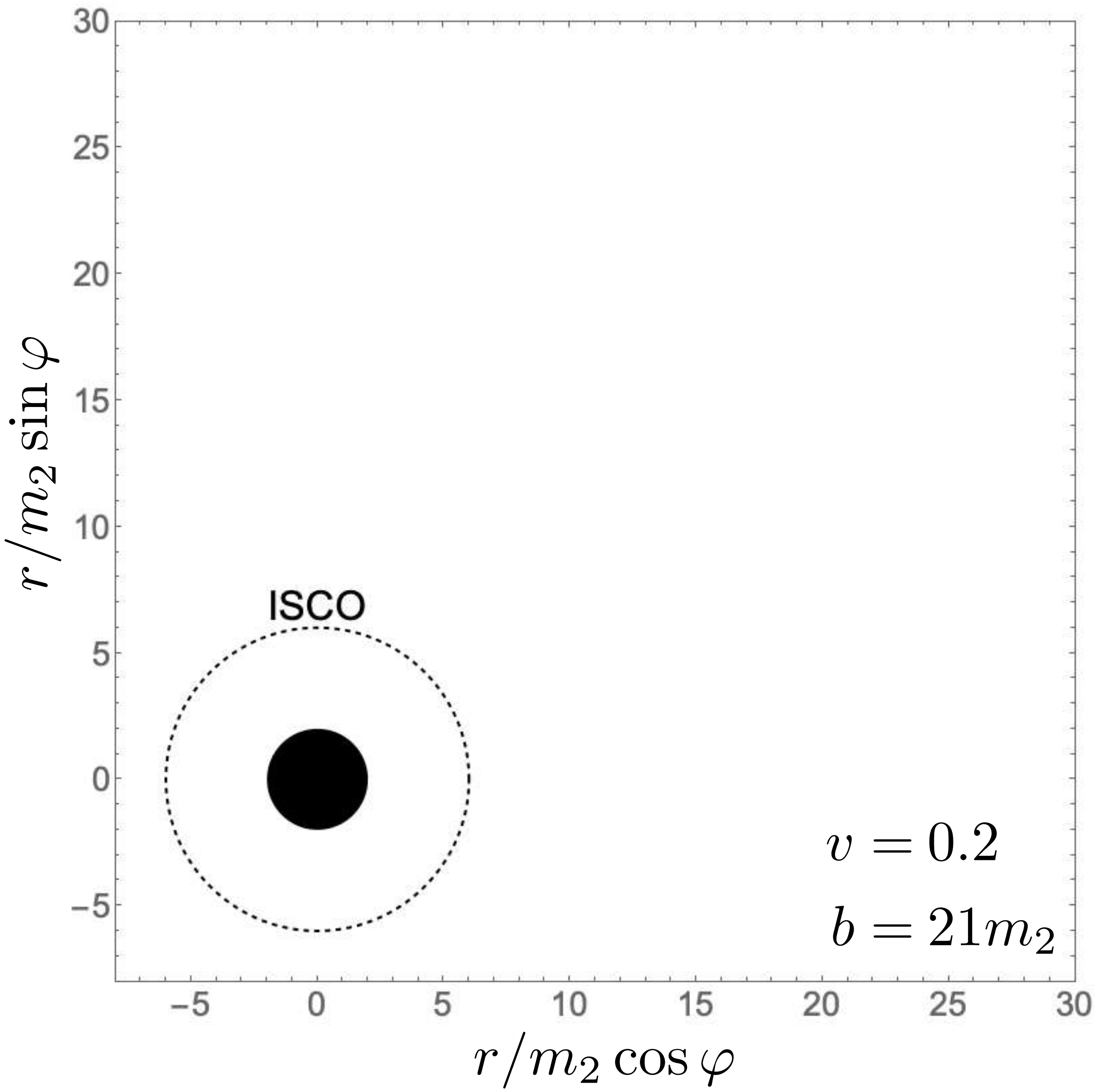
Analytic solutions:

$$r_p(\xi) = \frac{p m_2}{1 - e\cos\xi} \qquad \varphi_p(\xi) = \varphi_p(0) + k\sqrt{p/e}\,\mathsf{El}_1\left(\frac{\xi}{2}; -k^2\right) \qquad t_p(e,p;\xi) = \ldots$$

Relativistic anomaly

Elliptic integral

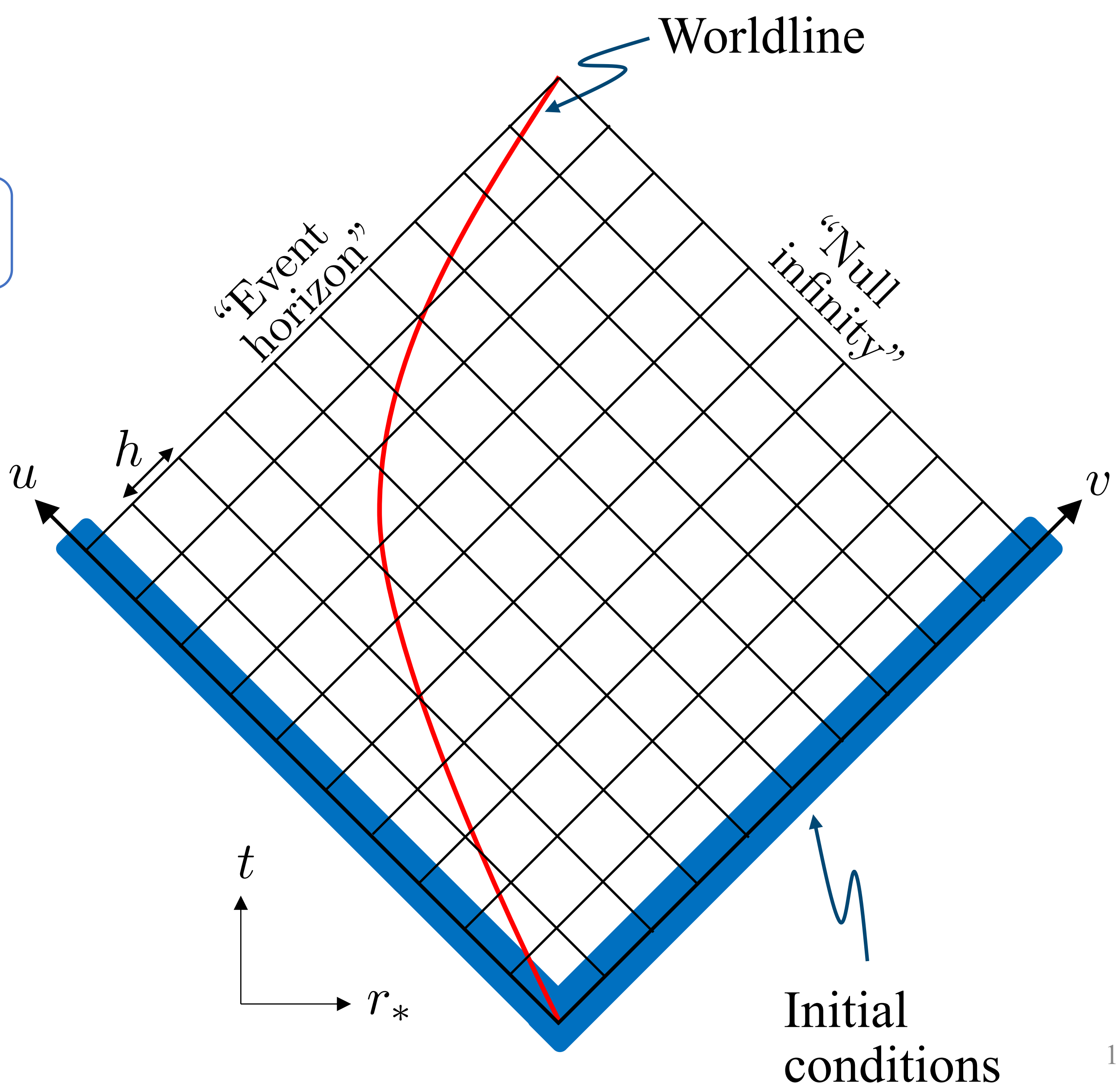
# Sample hyperbolic orbit



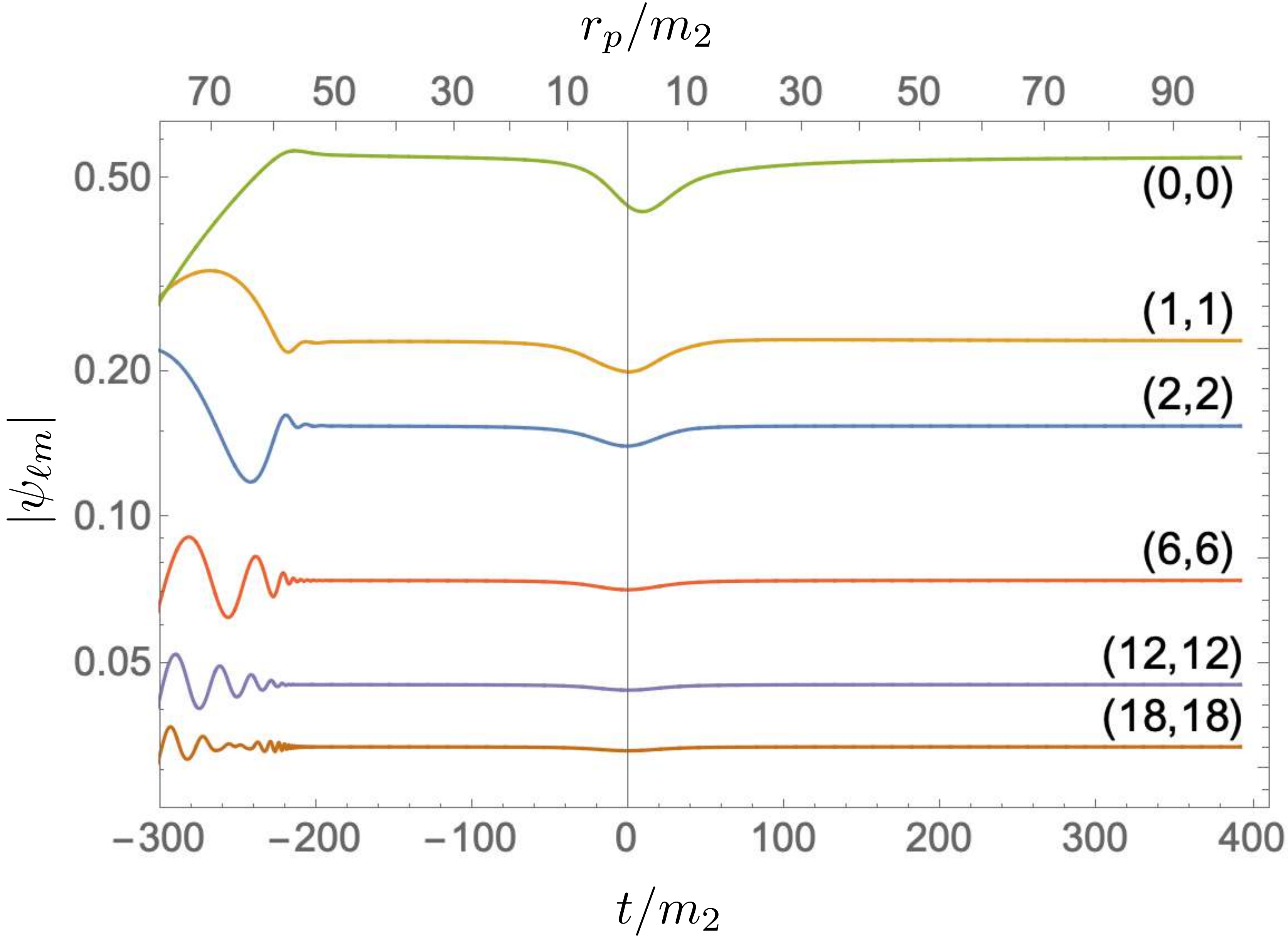
# 1+1D evolution scheme [Barack & OL ‘22]

$$\psi_{,uv} + V(\ell; r)\psi = S_\psi(\ell; x_p^\mu)\,\delta\,(r - R)$$

$u, v$ : Eddington-Finkelstein coordinates



# Scalar field modes for a scattering particle



# Scalar self-force and mass correction

Equation of motion:

$$\frac{d}{d\tau}(m_1 u^\alpha) = Q\nabla^\alpha\Phi^R$$

$\tau$ : Proper time
$u^\alpha$ : 4-velocity
$\Phi^R$ : Regular field

Tangent to $u^\alpha$ :

$$\frac{dm_1}{d\tau} = -Q\frac{d\Phi^R}{d\tau}$$

Integrate:

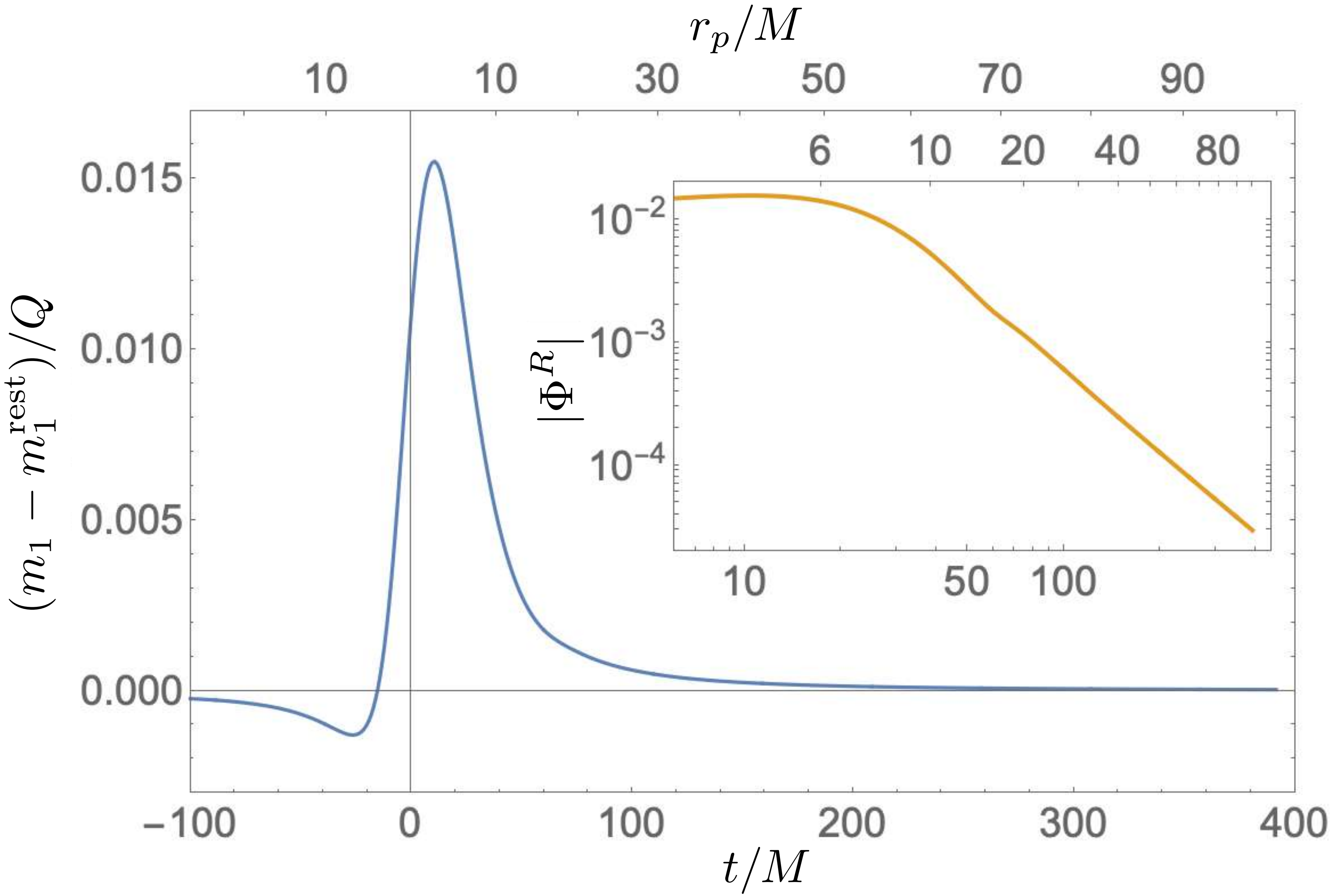
$$m_1(\tau) = m_1^{\text{rest}} - Q\Phi^R(\tau)$$

Orthogonal to $u^\alpha$ :

$$m_1\frac{du^\alpha}{d\tau} = Q(\delta^\alpha_\beta + u^\alpha u_\beta)\nabla^\beta\Phi^R =: m_1 q_s F^\alpha$$

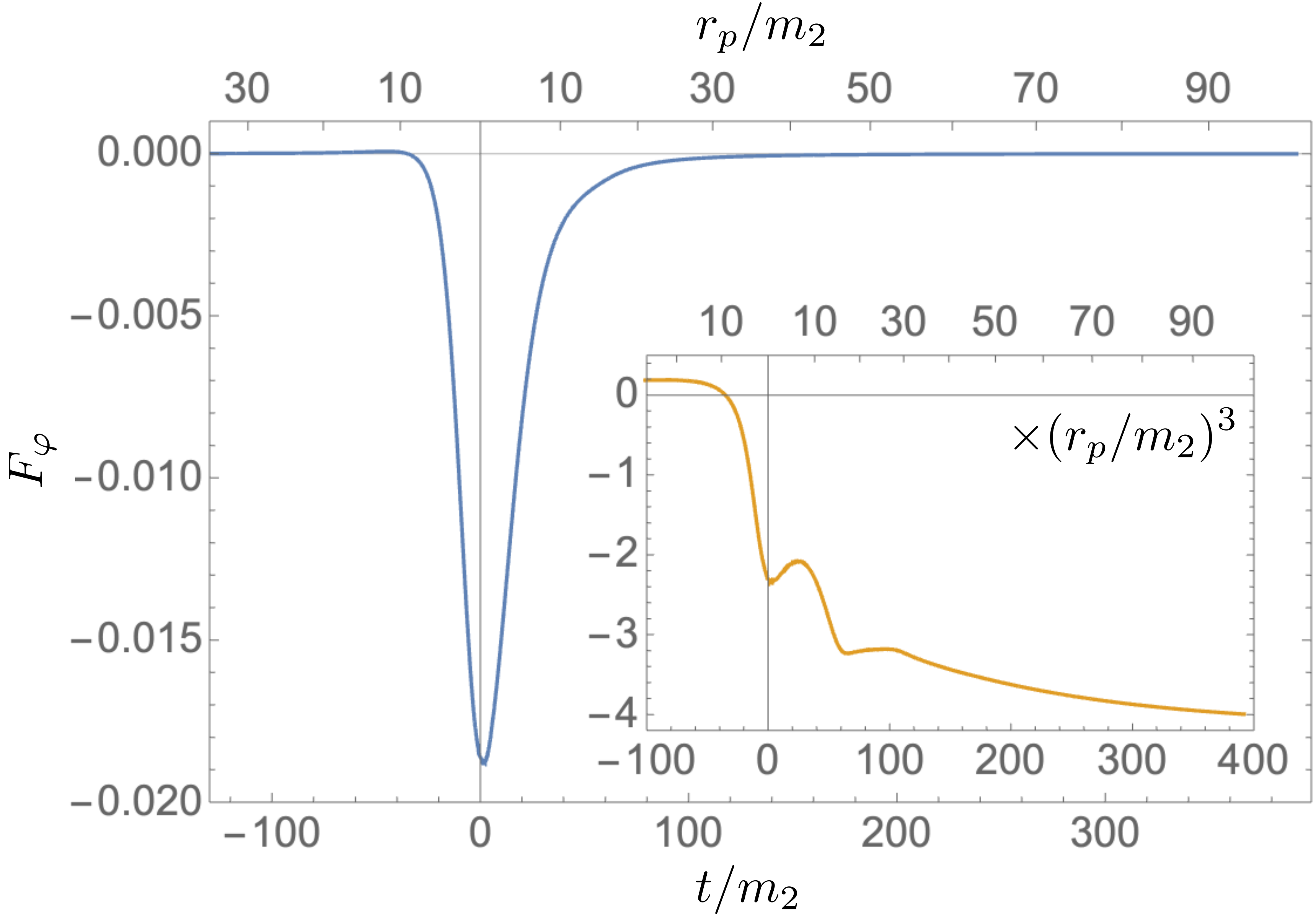
This defines the self-force.

# Change in mass results



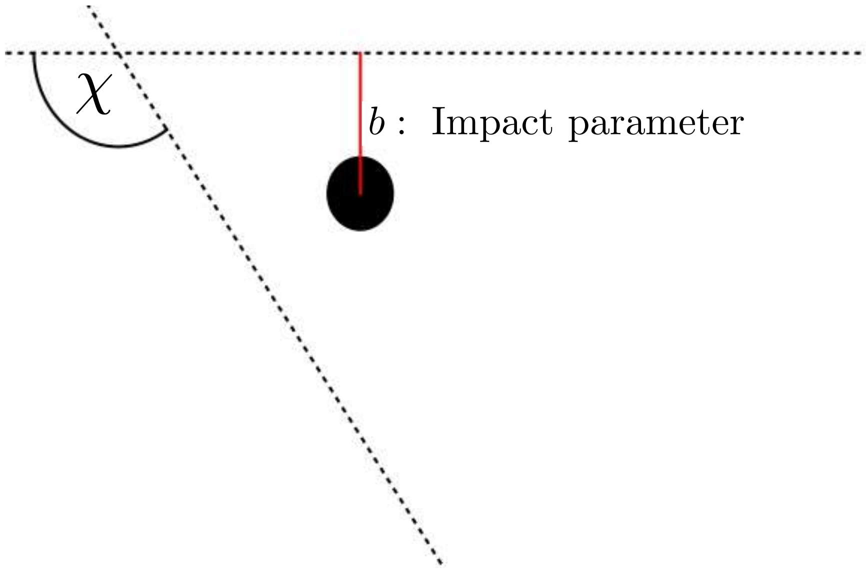

$$m_1 = m_1^{\text{rest}} - Q\Phi^R$$

# Self-force for a scattering particle



# Self-force correction to the scattering angle

Scattering angle [Barack & OL ‘22]:




$$\chi = \chi^{(0)} + q\,\delta\chi$$

$$\delta\chi = \sum_{\pm} \int_{r_{\min}}^{\infty} \left[\mathcal{G}_E^{\pm}(r) F_t^{\pm} - \mathcal{G}_L^{\pm}(r) F_\varphi^{\pm}\right] dr$$

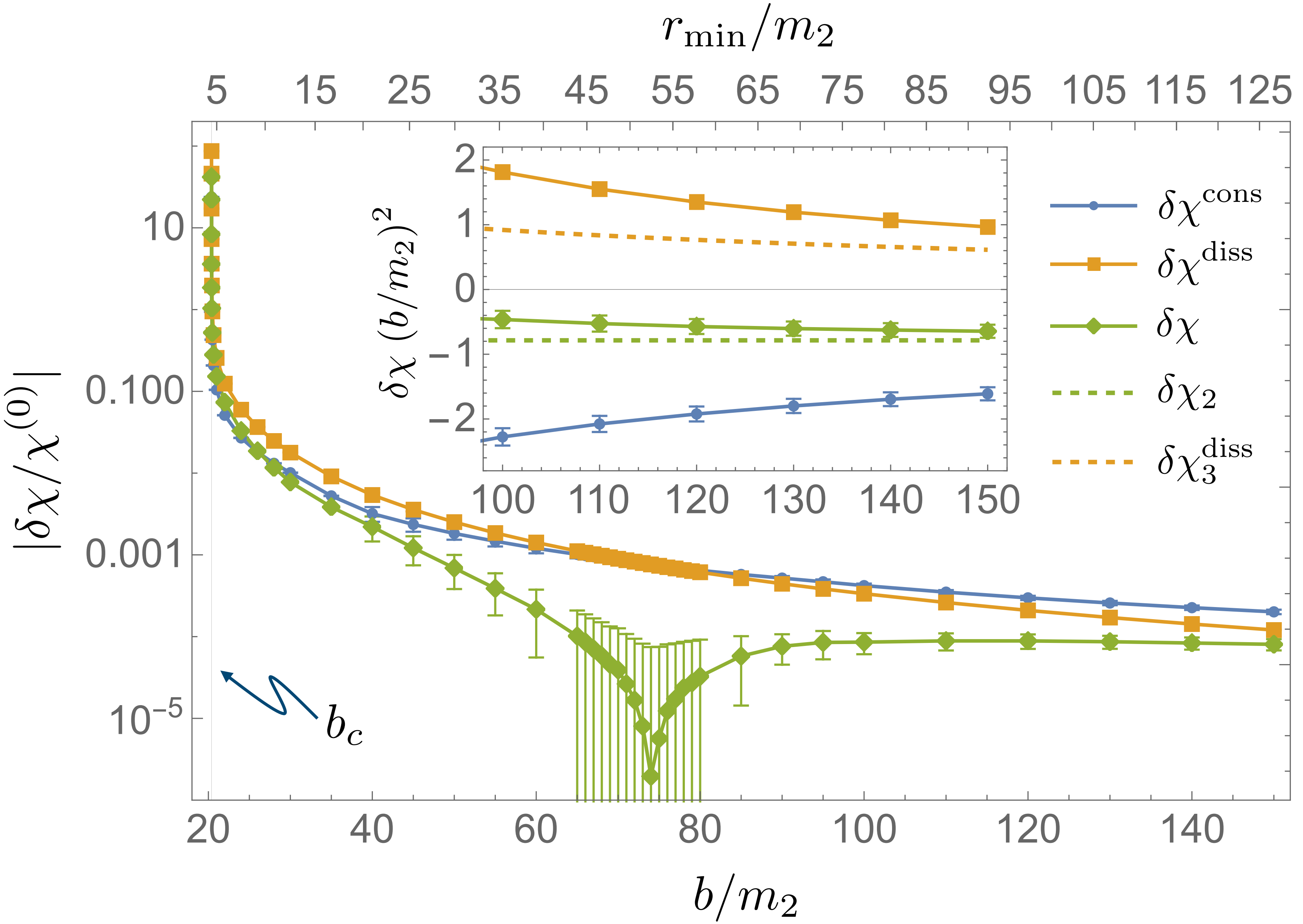
Incoming/outgoing leg

Functions of geodesics

Can split into conservative and dissipative pieces using symmetries about periastron:

$$F_\alpha^{\text{cons}}(r, \dot{r}) = -F_\alpha^{\text{cons}}(r, -\dot{r}) \qquad F_\alpha^{\text{diss}}(r, \dot{r}) = F_\alpha^{\text{diss}}(r, -\dot{r}) \qquad \alpha = t, \varphi$$

# Scattering angle results: $v = 0.2$



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| 2SF | $q^2G$ | $q^2G^2$ | $q^2G^3$ | $q^2G^4$ | $\dots$ | $\rightarrow$ | impossible (for now) |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | | |

$n$SF contains all orders in $G$

$n$PM contains all orders in $q$

# Scattering angle correction: PM expansion

Expansion around flat space:

$$\delta\chi^{\mathrm{PM}} = \sum_{i=0}^{\infty} \delta\chi_i \left(\frac{Gm_2}{b}\right)^i$$

2PM [Gralla & Lobo '22]:

$$\delta\chi_2^{\mathrm{cons}} = -\frac{\pi}{4}\left(\frac{m_2}{b}\right)^2$$

$$\delta\chi_2^{\mathrm{diss}} = 0$$

$v:$ Velocity at infinity

$b:$ Impact parameter

3PM:

$$\delta\chi_3^{\mathrm{cons}} = -\frac{4\left(3-v^2\right)}{3v^2\sqrt{1-v^2}}\left(\frac{m_2}{b}\right)^3$$

$$\delta\chi_3^{\mathrm{diss}} = \frac{2\left(v^2+1\right)^2}{3v^3\sqrt{1-v^2}}\left(\frac{m_2}{b}\right)^3$$

LO

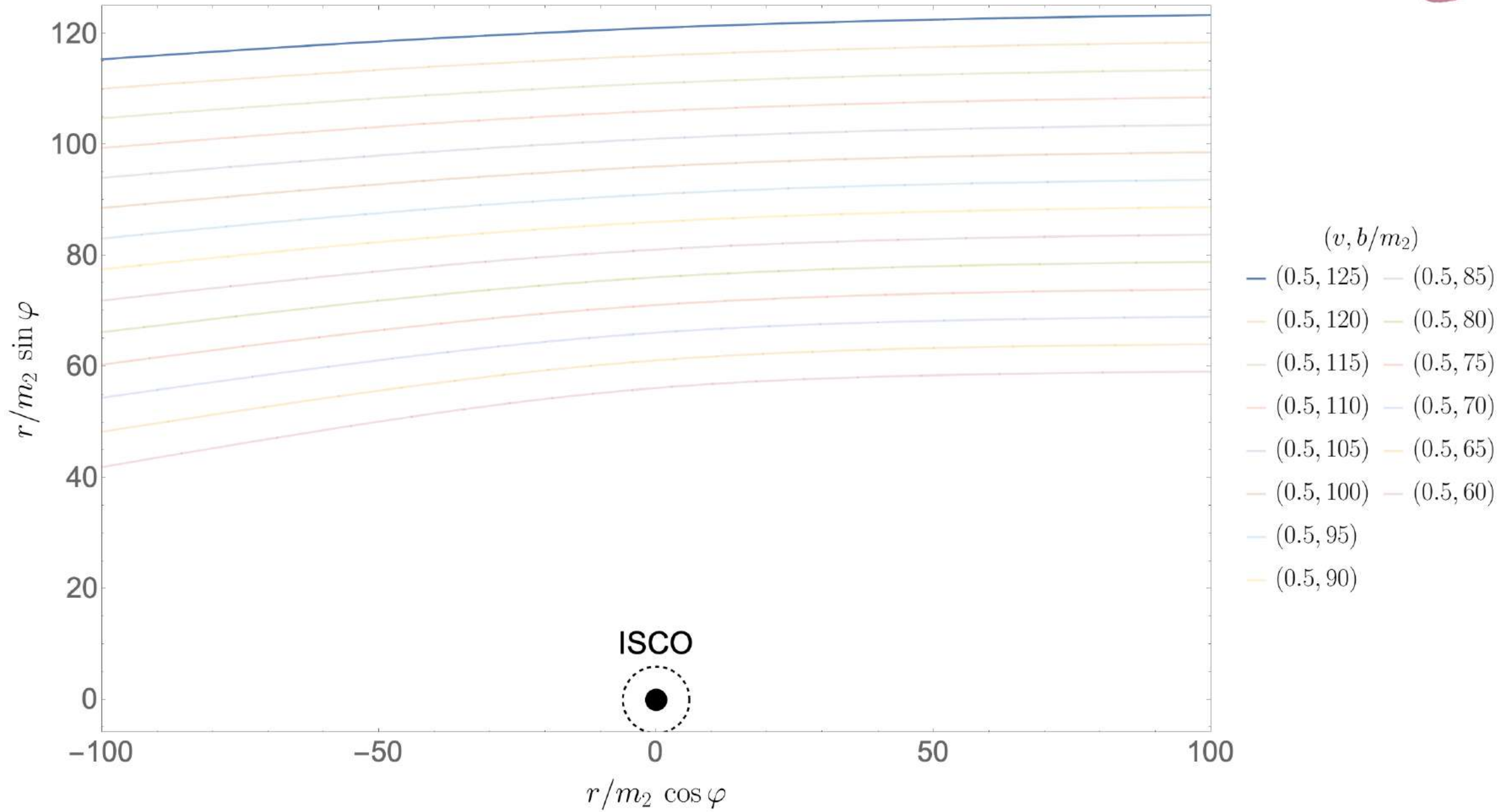
NLO

4PM dissipative:

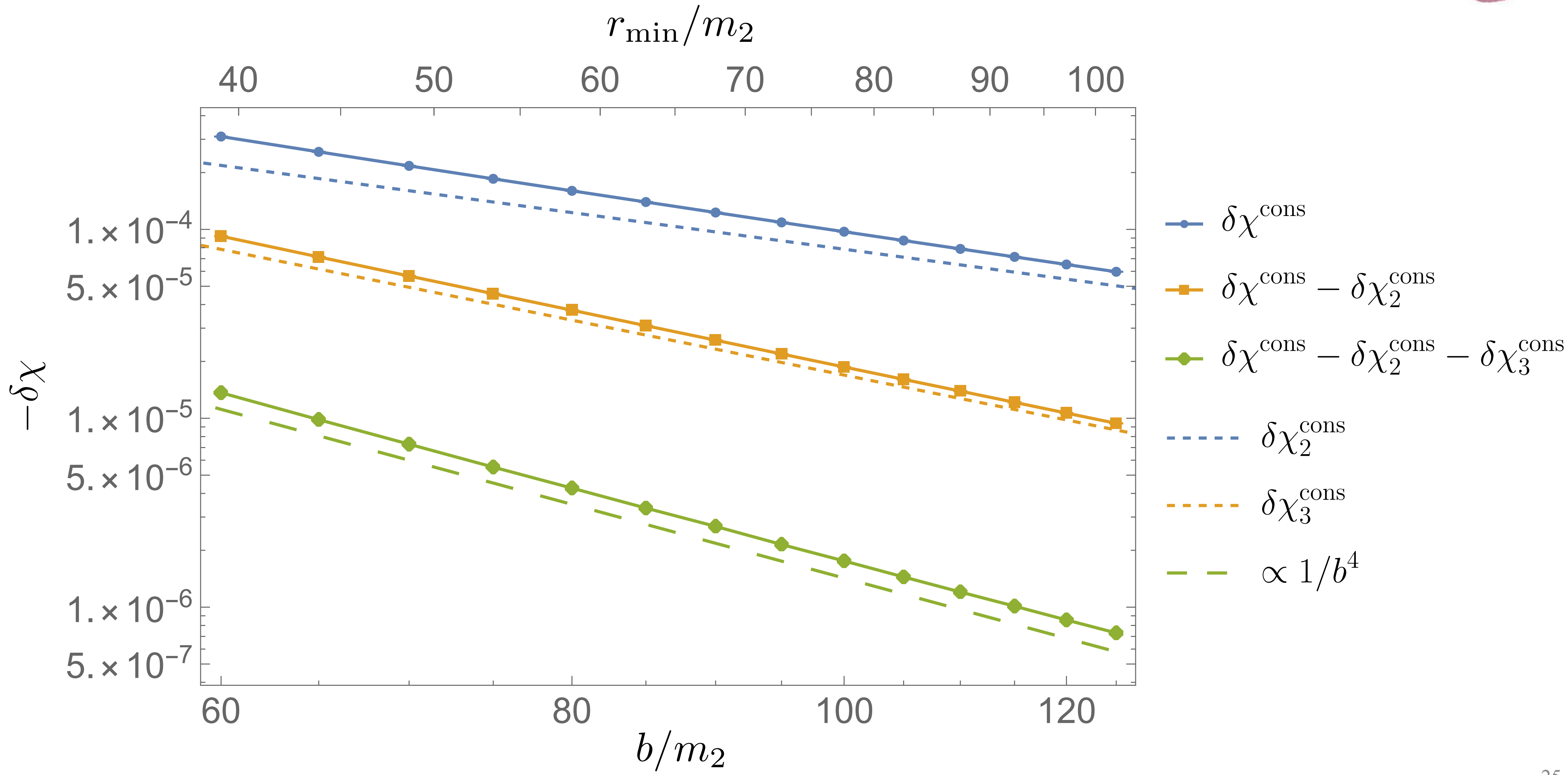
$$\delta\chi_4^{\mathrm{diss}} = \left(r_1 + r_2\,\mathrm{arcsech}\left(\sqrt{1-v^2}\right) + r_3 \log\left[\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right]\right)\left(\frac{m_2}{b}\right)^4$$

$r_i$ = rational coefficients

# Weak-field orbits



# Conservative: $v = 0.5$



# Extraction of high-order conservative PM results

PM expansion with free parameters:

$$\delta\chi^{\text{cons}} = \frac{a_2}{b^2} + \frac{a_3}{b^3} + \frac{a_4}{b^4} + \frac{a_5}{b^5} + \ldots$$

Up to 3PM can fit value or use **analytic value**.

| $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|
| $-1.0886$ | – | – | – |
| $-0.7535$ | $-21.77$ | – | – |
| $-0.7899$ | $-16.17$ | $-206.5$ | – |
| $-0.7803$ | $-18.49$ | $-25.0$ | $-4620$ |
| $\mathbf{-0.785398}$ | $-19.18$ | – | – |
| $\mathbf{-0.785398}$ | $-16.93$ | $-176.2$ | – |
| $\mathbf{-0.785398}$ | $-17.20$ | $-131.1$ | $-1793$ |
| $\mathbf{-0.785398}$ | $\mathbf{-16.9356}$ | $-175.9$ | – |
| $\mathbf{-0.785398}$ | $\mathbf{-16.9356}$ | $-174.4$ | $-107$ |
| $< 1\%$ | $\sim 1\%$ | $\sim -175$ | $< 0(?)$ |

# Scattering angle correction: 4PM conservative

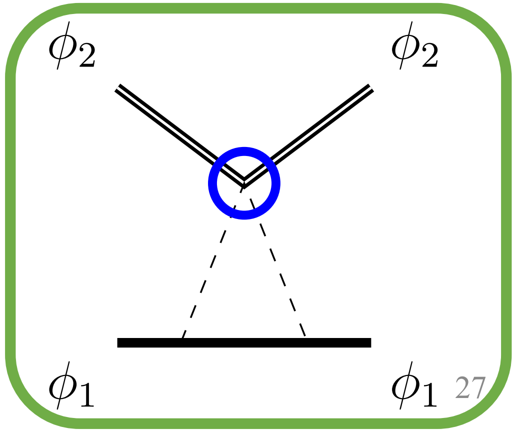
$$\delta\chi_4^{\text{cons}} = \Bigg(r_1 + r_2 \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right) + r_3 \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right)^2 + r_4 \mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2$$
$$+ r_5 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)\mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right) + r_6 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2$$
$$+ r_7 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) + r_8 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right)$$
$$+ r_9 \log\left(\frac{v}{2\sqrt{1-v^2}}\right)\log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{10}\log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right)$$
$$+ r_{11}\log^2\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{12}\,\alpha + r_{13}\frac{\beta}{v^2} + r_4\log(b)\Bigg)\left(\frac{m_2}{b}\right)^4$$

Elliptic integrals

Free coefficients

Log term

$r_i$ = rational coefficients



# Extraction of high-order conservative PM results

Subtract known analytic parts of conservative 4PM:

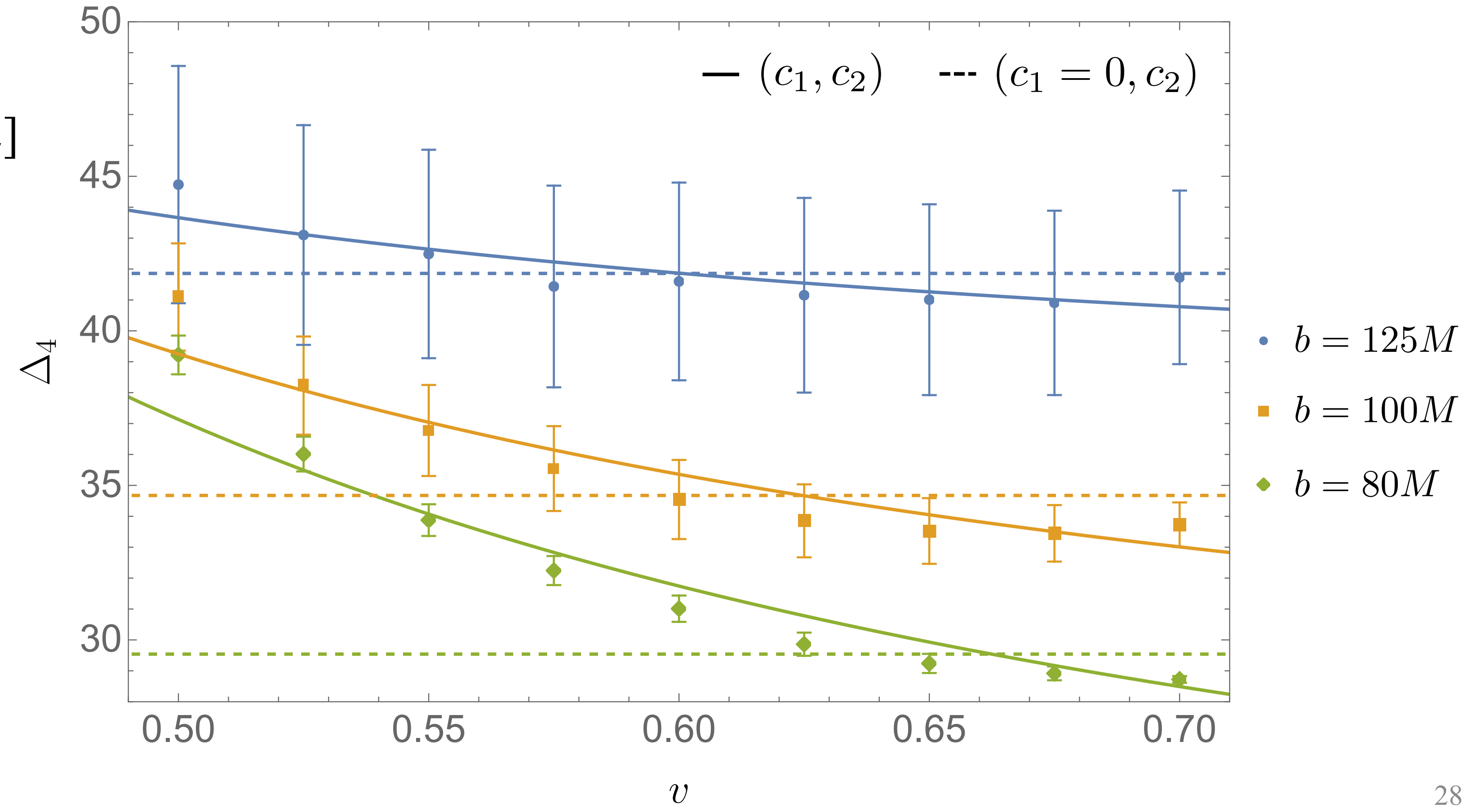
$$\Delta_4(v) := (\delta\chi_4^{\text{cons}} - \delta\chi_4^{\text{known}})b^4 = \boxed{\frac{3}{8}\pi m_2^4\left[c_2 + c_1(5 - 4/v^2)\right]} + \mathcal{O}(1/b)$$

$c_1$ & $c_2$ are Wilson coefficients.

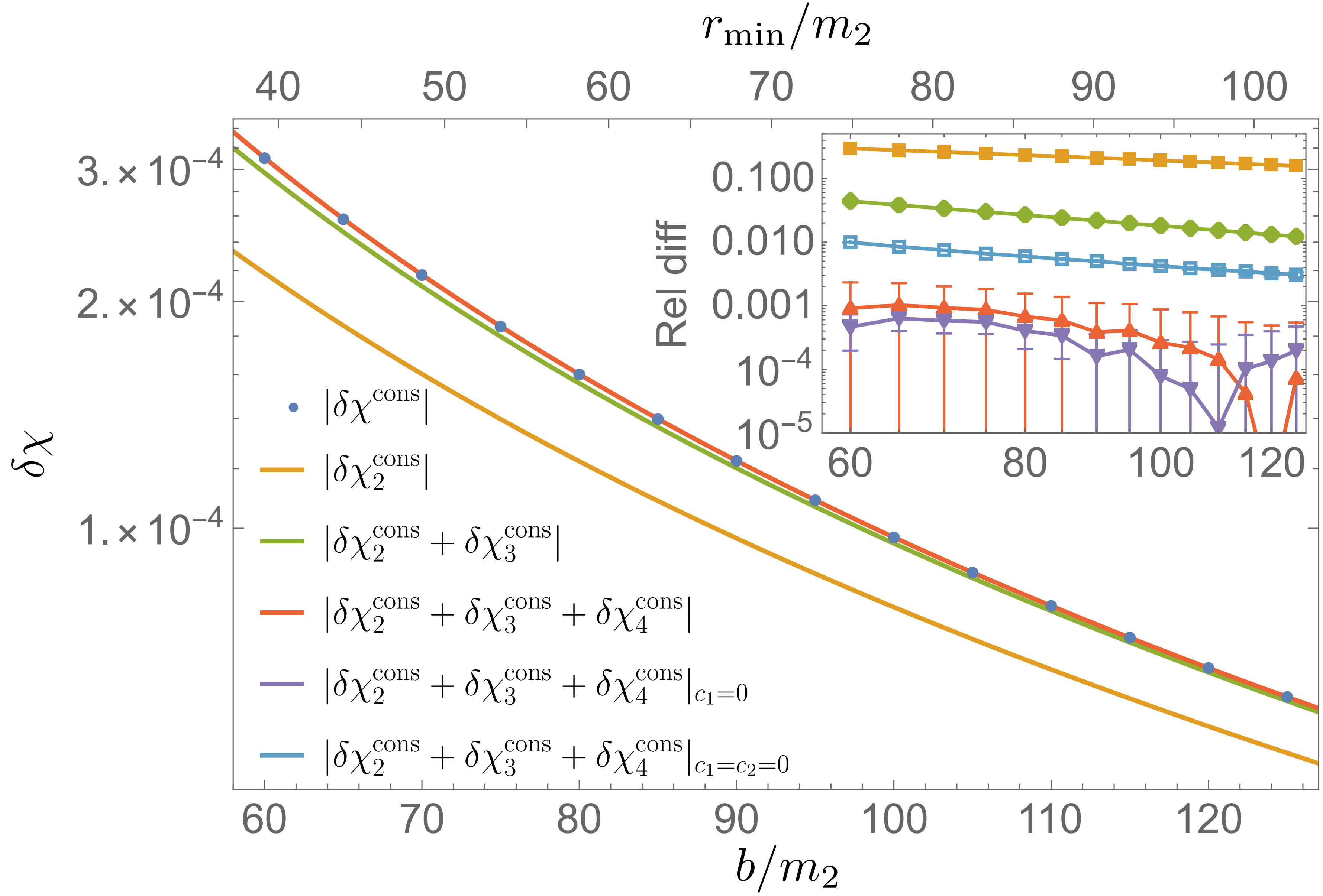
Expect $c_1 = 0$ [Ivanov & Zhou '22]

| $b/M$ | $c_1$ | $c_2$ |
|---|---|---|
| 80 | 0.94 | −21.2 |
| 100 | 0.68 | −25.9 |
| 125 | 0.31(±0.38) | −33.6 |
| 80 | **0** | −25.1 |
| 100 | **0** | −29.4 |
| 125 | **0** | −35.5 |

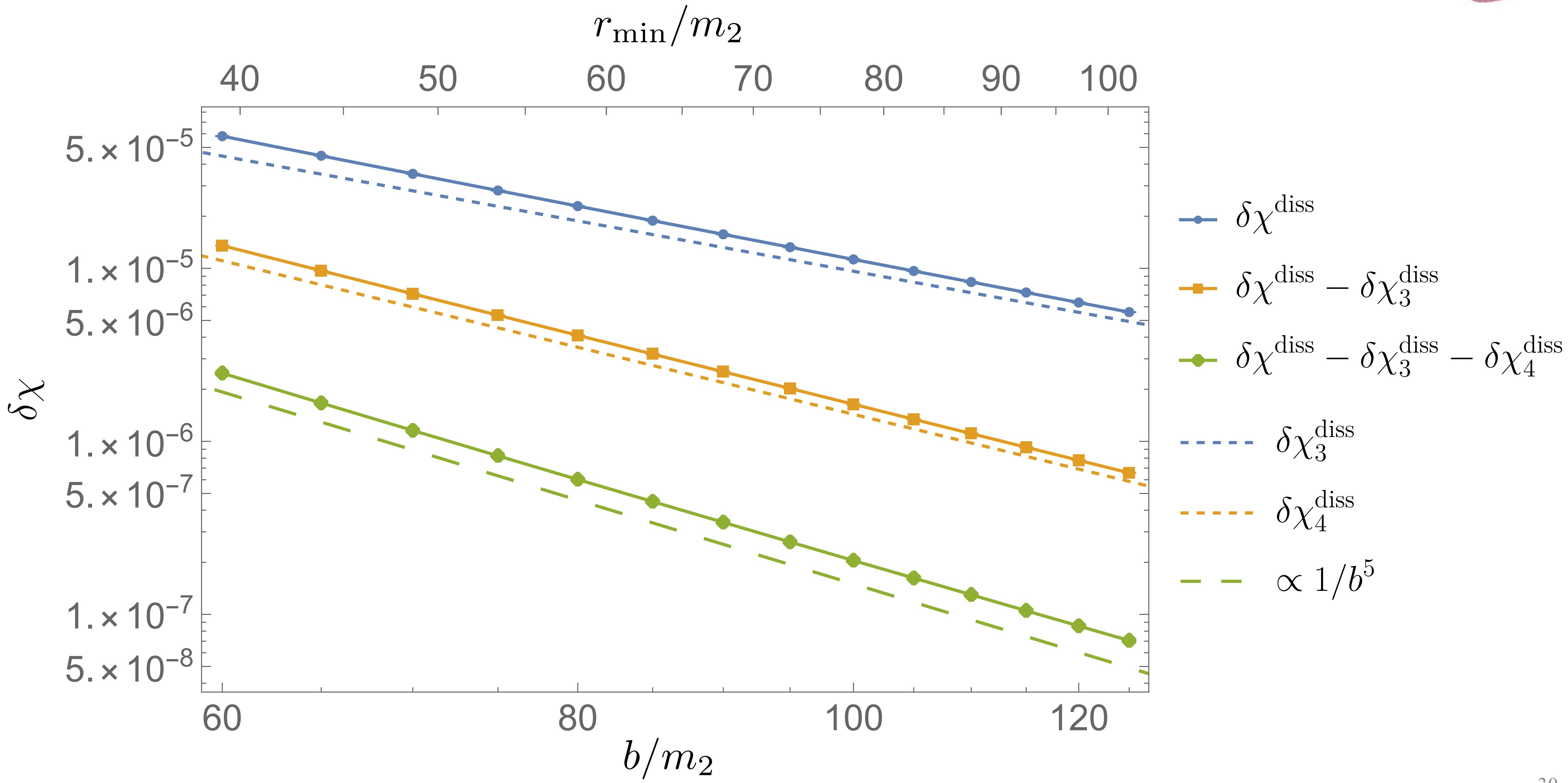
**Fixed** (Fitting error)



# PM comparison: Conservative $v = 0.5$



# Dissipative: $v = 0.5$



# Extraction of high-order dissipative PM results

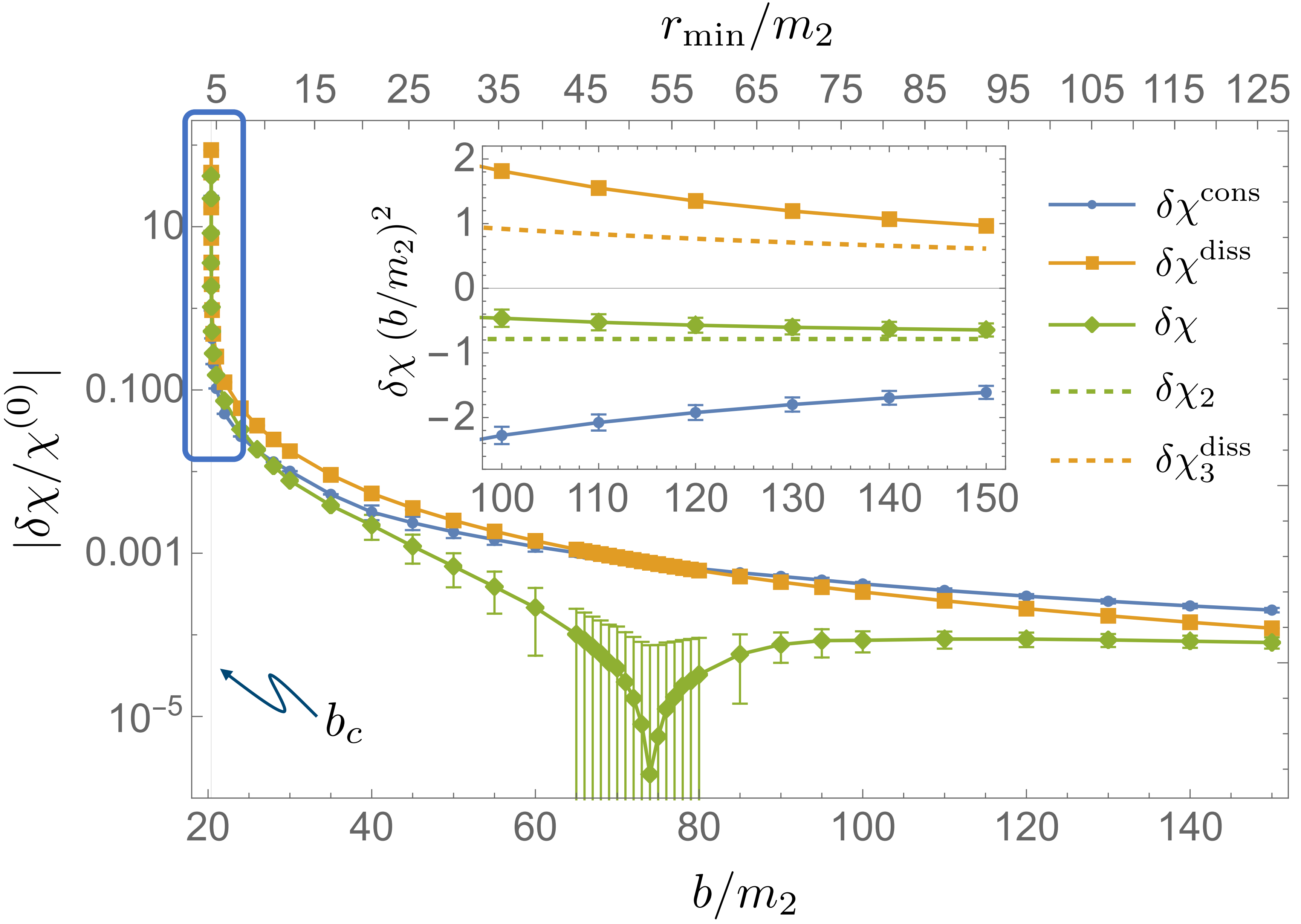
PM expansion with free parameters:

$$\delta\chi^{\rm diss} = \frac{\alpha_3}{b^3} + \frac{\alpha_4}{b^4} + \frac{\alpha_5}{b^5} + \frac{\alpha_6}{b^6} + \ldots$$

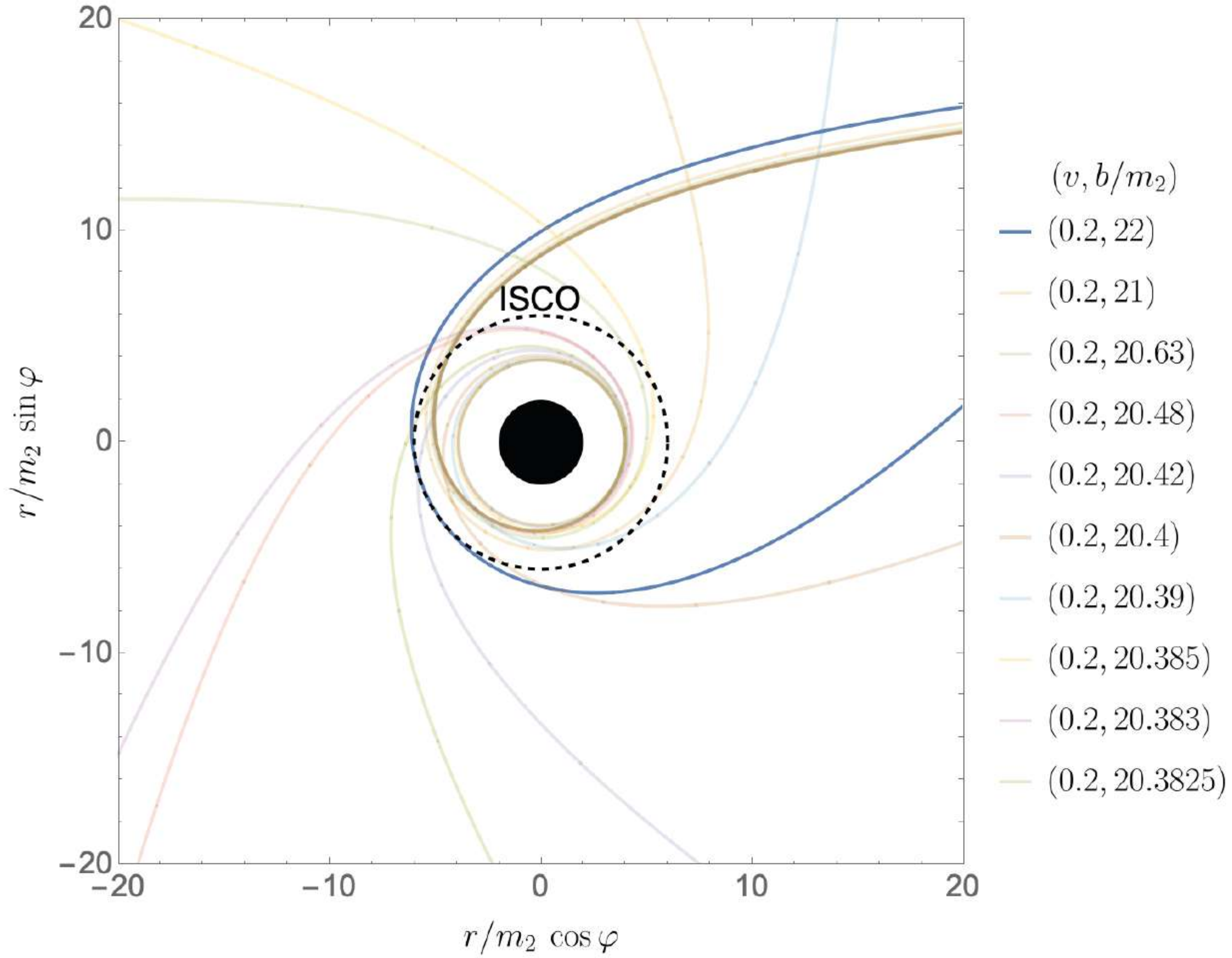
Up to 4PM can fit value or use **analytic value**.

| $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ |
|---|---|---|---|
| 11.19 | – | – | – |
| 9.44 | 188 | – | – |
| 9.64 | 142 | 1900 | – |
| 9.61 | 154 | 920 | 26615 |
| **9.6225** | 169 | – | – |
| **9.6225** | 147 | 1720 | – |
| **9.6225** | 149 | 1321 | 15859 |
| **9.6225** | **143.344** | 1965 | – |
| **9.6225** | **143.344** | 2248 | −20216 |
| $< 1\%$ | $\sim 1\%$ | $\sim 2000(?)$ | ??? |

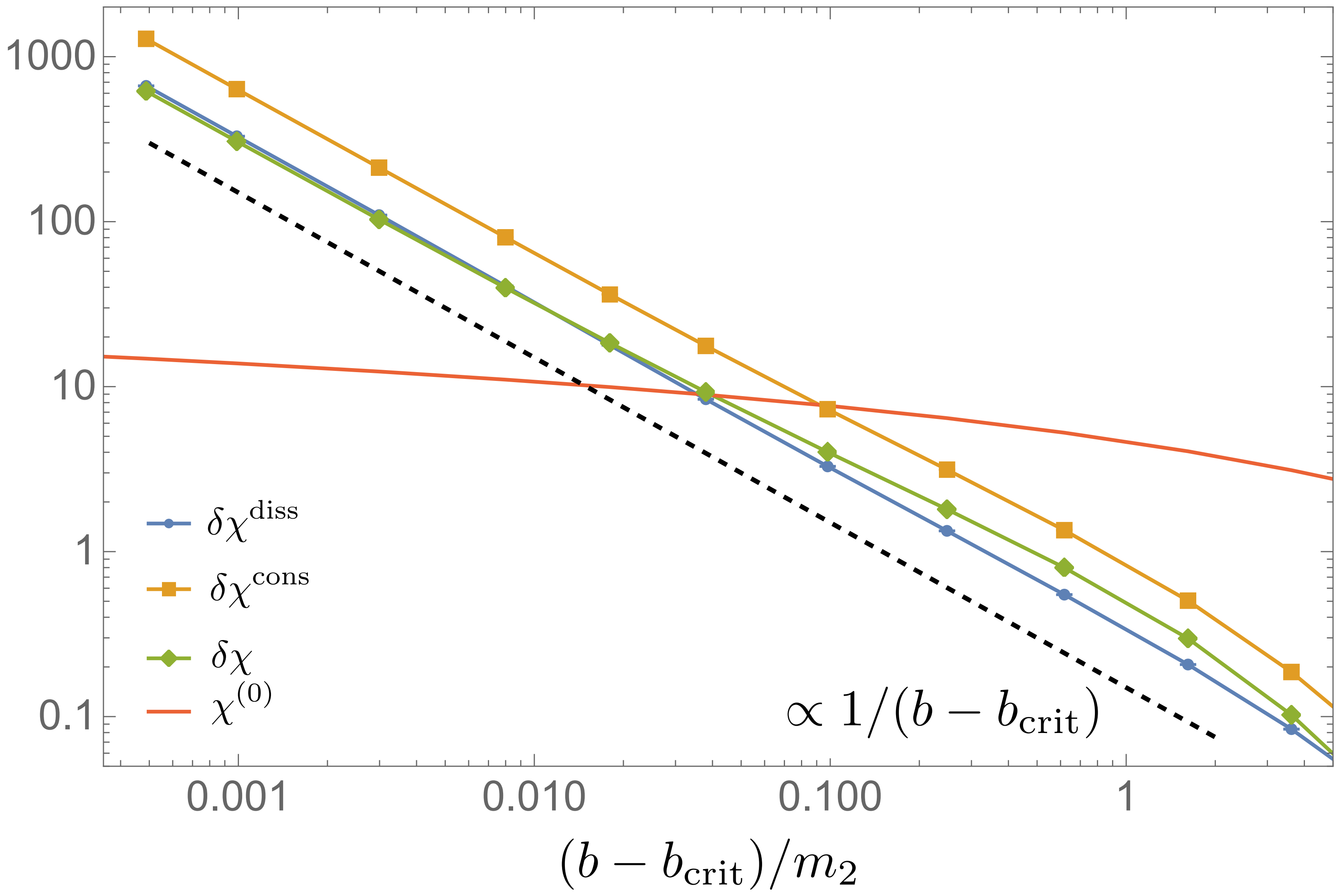
# Scattering angle results: $v = 0.2$



# Strong-field orbits



# Scattering angle results: Strong field $v = 0.2$



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Two independent expansions of the same system:

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| 2SF | $q^2G$ | $q^2G^2$ | $q^2G^3$ | $q^2G^4$ | $\dots$ | $\rightarrow$ | impossible (for now) |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | | |

$n$SF contains all orders in $G$

$n$PM contains all orders in $q$

# Geodesic resummation

Can use information of how the scattering angle diverges to resum PM expressions:

$$\chi_{0\mathrm{SF}} \overset{b\to b_c^+}{\approx} A(v)\ln\left(1-\frac{b_c(v)}{b}\right)+\mathrm{const}(v)+\ldots$$

Introduce the function:

$$\Psi_{0\mathrm{SF}}^{n\mathrm{PM}}(v,b) := A(v)\left[\ln\left(1-\frac{b_c(v)}{b}\right)+\sum_{k=1}^{n}\frac{1}{k}\left(\frac{b_c(v)}{b}\right)^k\right]$$

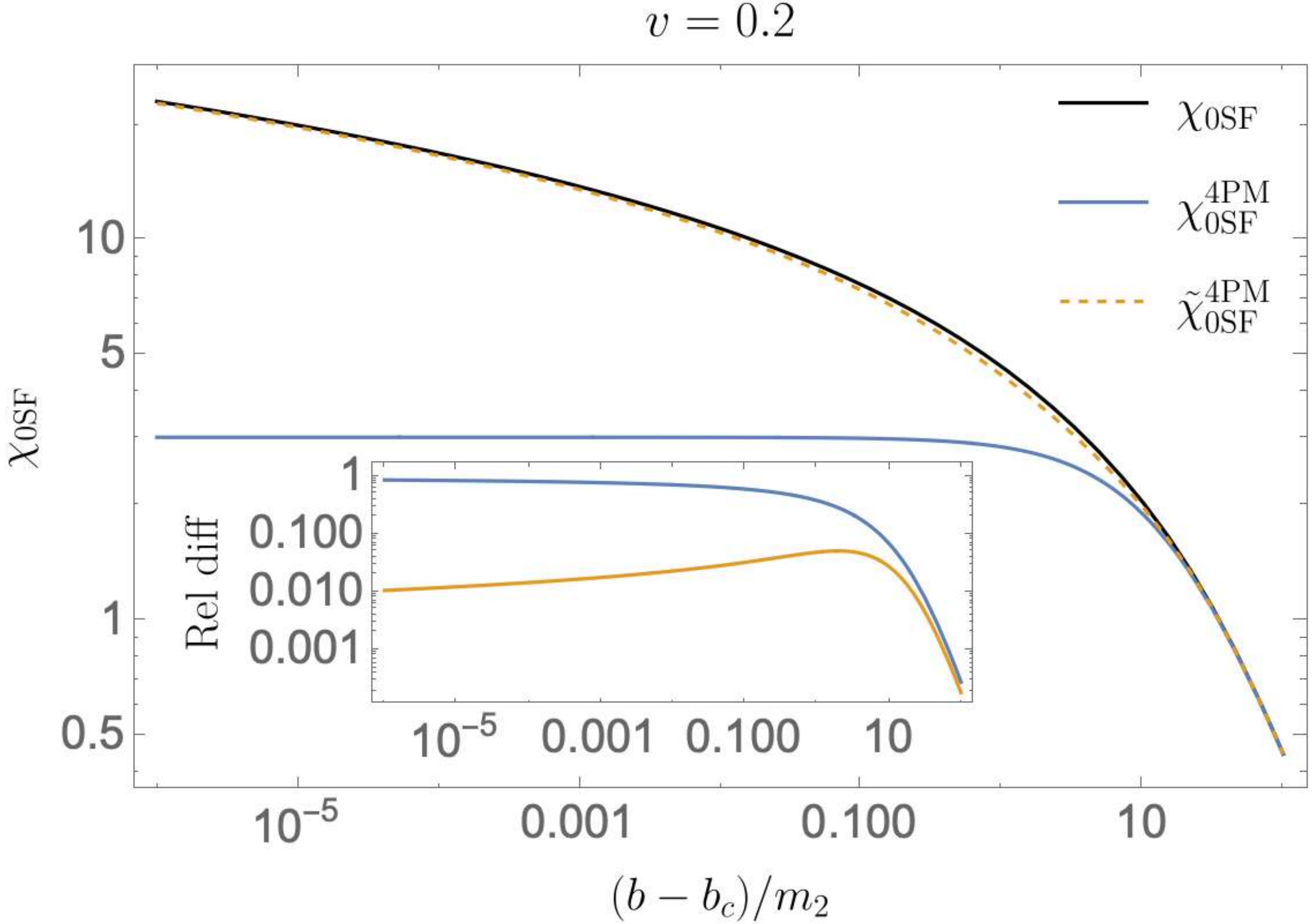
Resummed (geodesic) scattering angle:

$$\tilde{\chi}_{0\mathrm{SF}}^{n\mathrm{PM}}(v,b) := \chi_{0\mathrm{SF}}^{n\mathrm{PM}}(v,b)+\Psi_{0\mathrm{SF}}^{n\mathrm{PM}}(v,b)$$

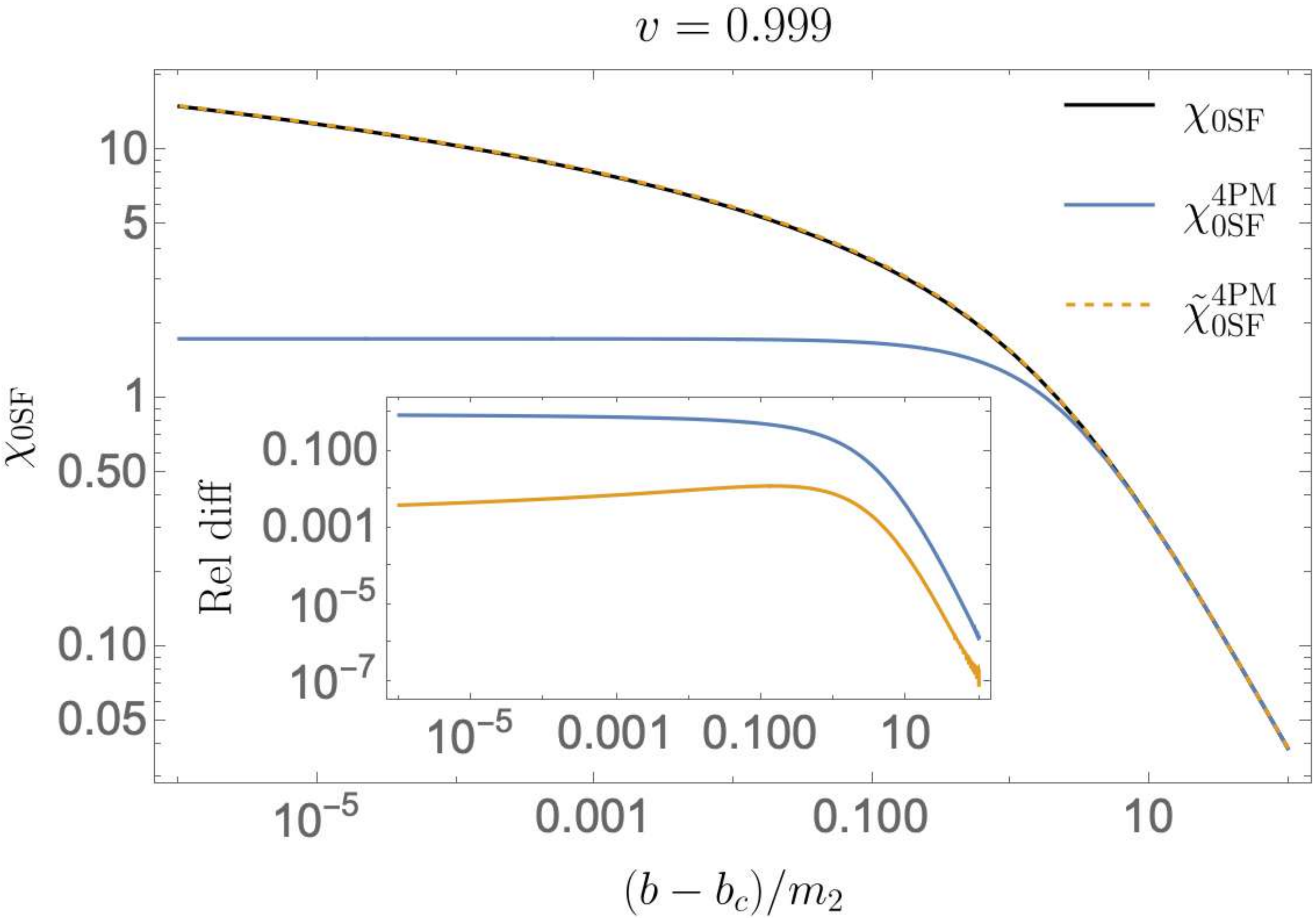
Properties:

- Identical to normal PM expression through $n$PM order in large-$b$ limit.
- Has the same logarithmic divergence near separatrix.

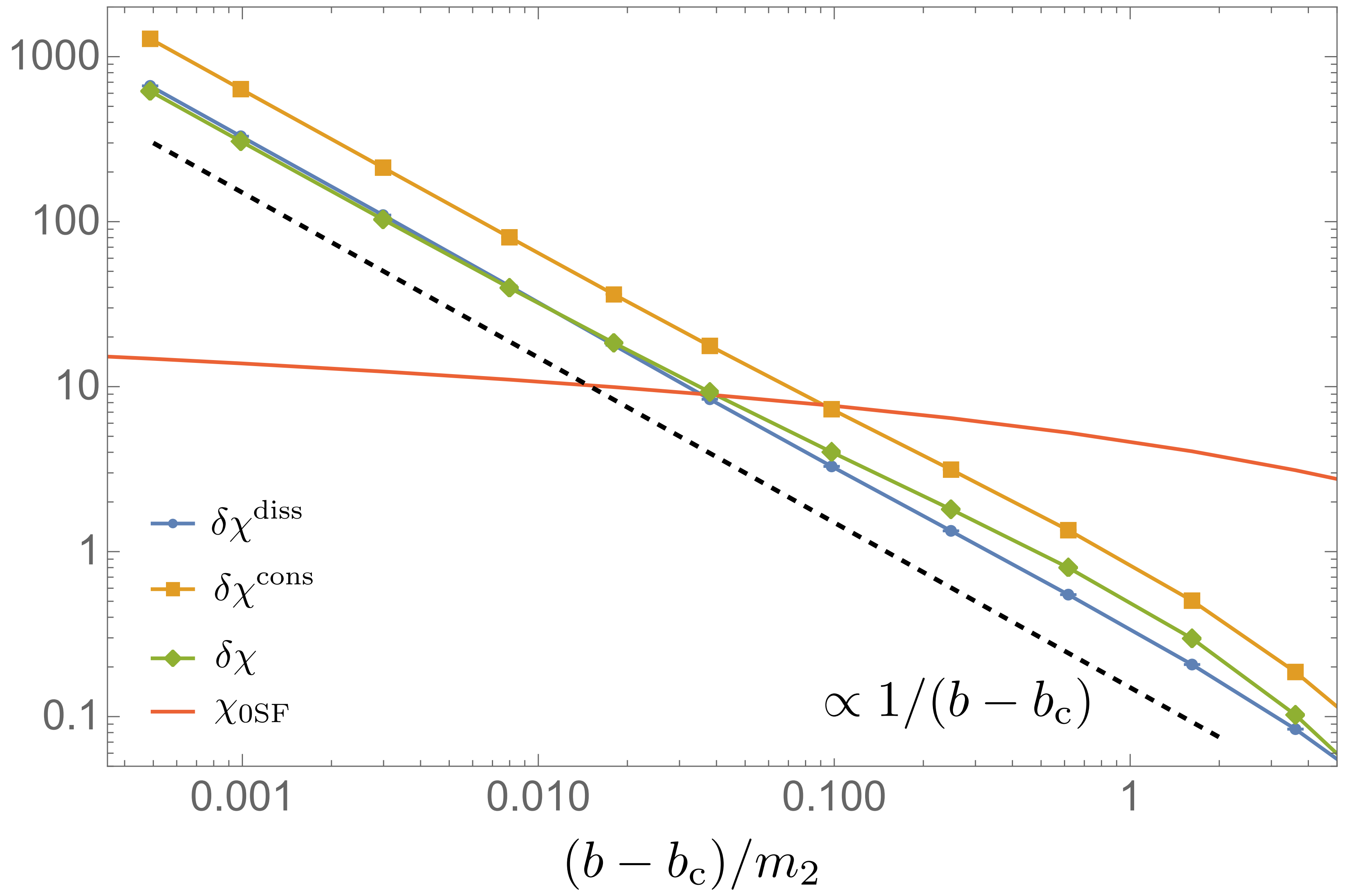
# Geodesic resummation results



# Geodesic resummation results



# Scattering angle results: Strong field $v = 0.2$



# Self-force vs post-Minkowskian expansions

Two independent expansions of the same system:

| | 1PM | 2PM | 3PM | 4PM | $\dots$ | |
|---|---|---|---|---|---|---|
| 0SF | $G$ | $G^2$ | $G^3$ | $G^4$ | $\dots$ | $\to$ analytical |
| 1SF | $qG$ | $qG^2$ | $qG^3$ | $qG^4$ | $\dots$ | $\to$ numerical |
| 2SF | $q^2G$ | $q^2G^2$ | $q^2G^3$ | $q^2G^4$ | $\dots$ | $\to$ impossible (for now) |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | |

$n$SF contains all orders in $G$

$n$PM contains all orders in $q$

# Self-force resummation

Divergence near separatrix:

$$\delta\chi_{1\mathrm{SF}} \overset{b\to b_c^+}{\approx} q_s B(v)\frac{b_c(v)}{b-b_c(v)}$$

Introduce a new function:

$$\Psi^{n\mathrm{PM}}(v,b) := A\left[\ln\left(1-\frac{b_c(v)(1-q_sB/A)}{b}\right)+\sum_{k=1}^{n}\frac{1}{k}\left(\frac{b_c(v)(1-q_sB/A)}{b}\right)^k\right]$$

Resummed scattering angle:

$$\tilde{\chi}^{n\mathrm{PM}}(v,b) := \chi^{n\mathrm{PM}}(v,b)+\Psi^{n\mathrm{PM}}(v,b)$$

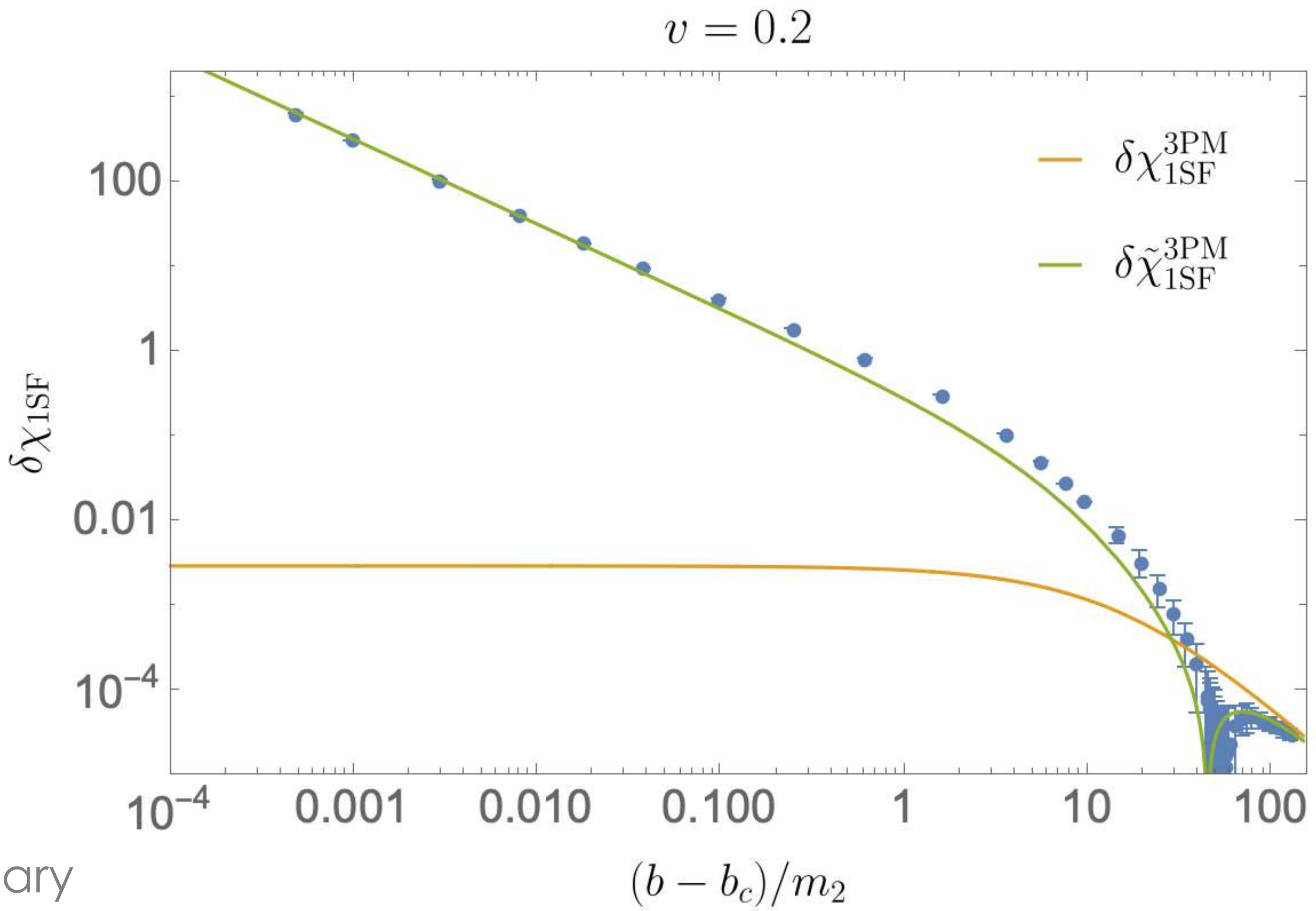
Properties:

- Identical to normal PM expression through $n$PM order in large-$b$ limit.
- Has the same logarithmic divergence near separatrix in the geodesic limit.
- At 1SF order it has the same divergence as the numerical data.

Preliminary

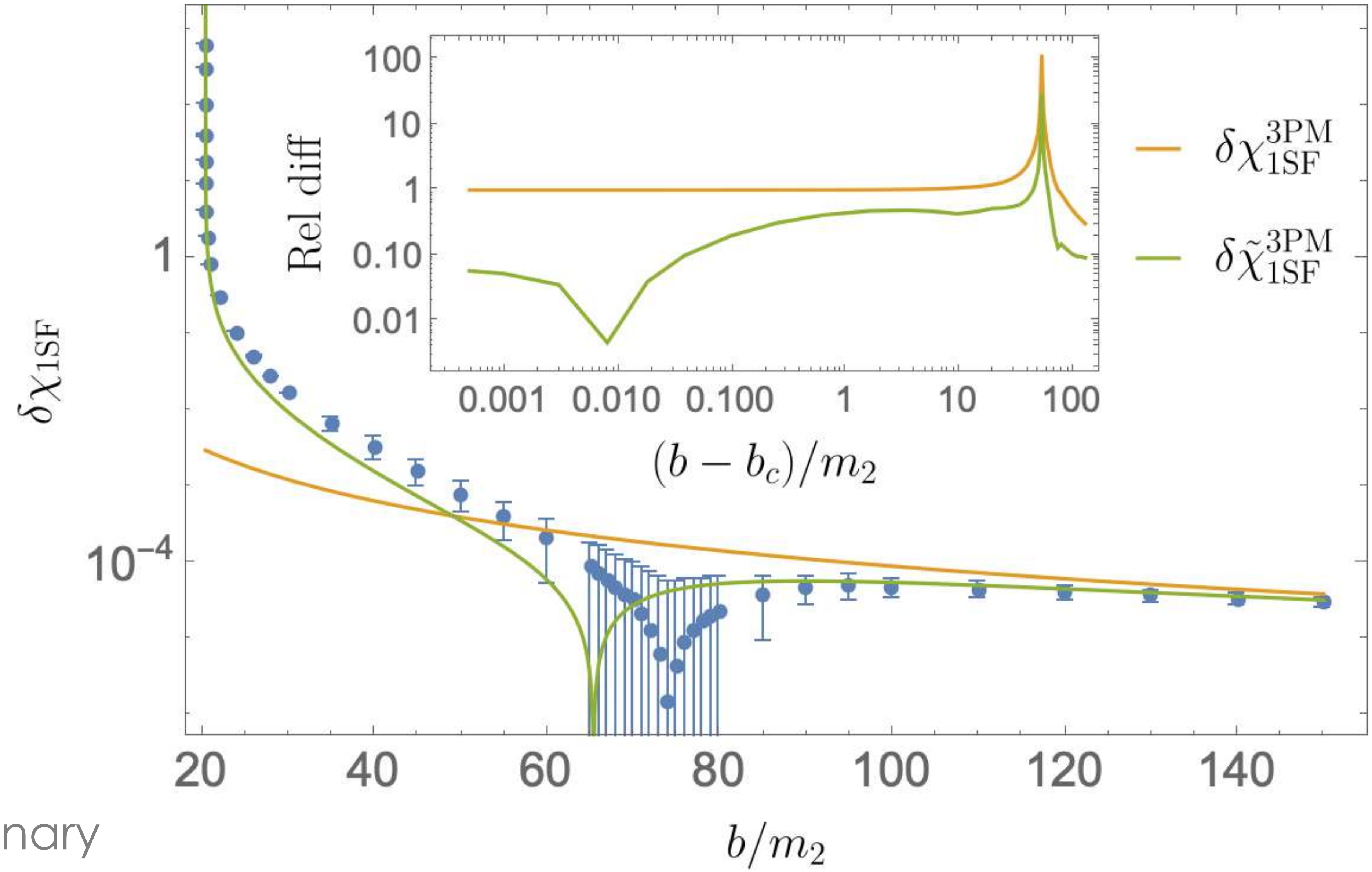
Use the numerical data to extract the coefficient $B(v)$.

# Resummed 1SF scattering angle



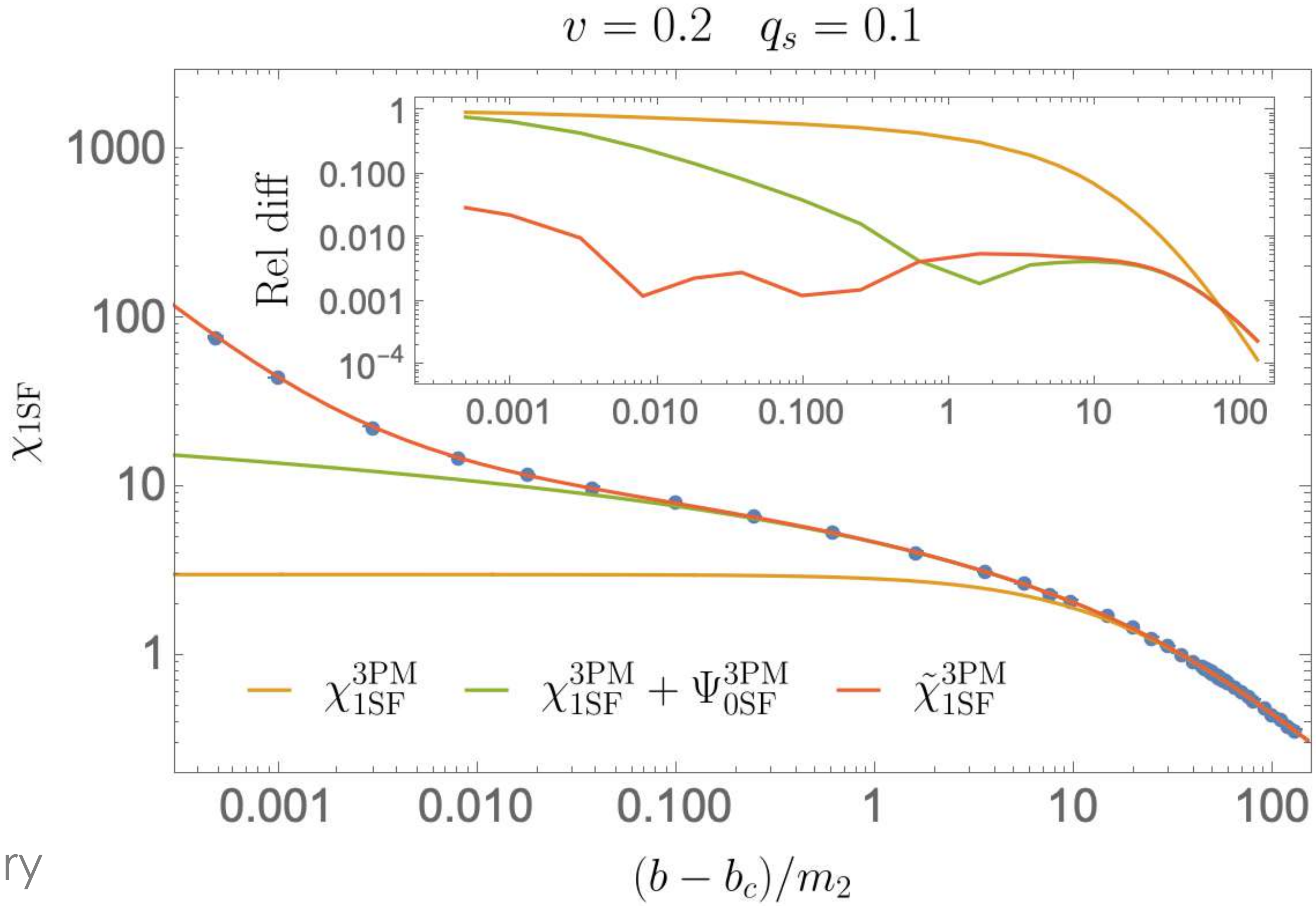

Preliminary

# Resummed 1SF scattering angle



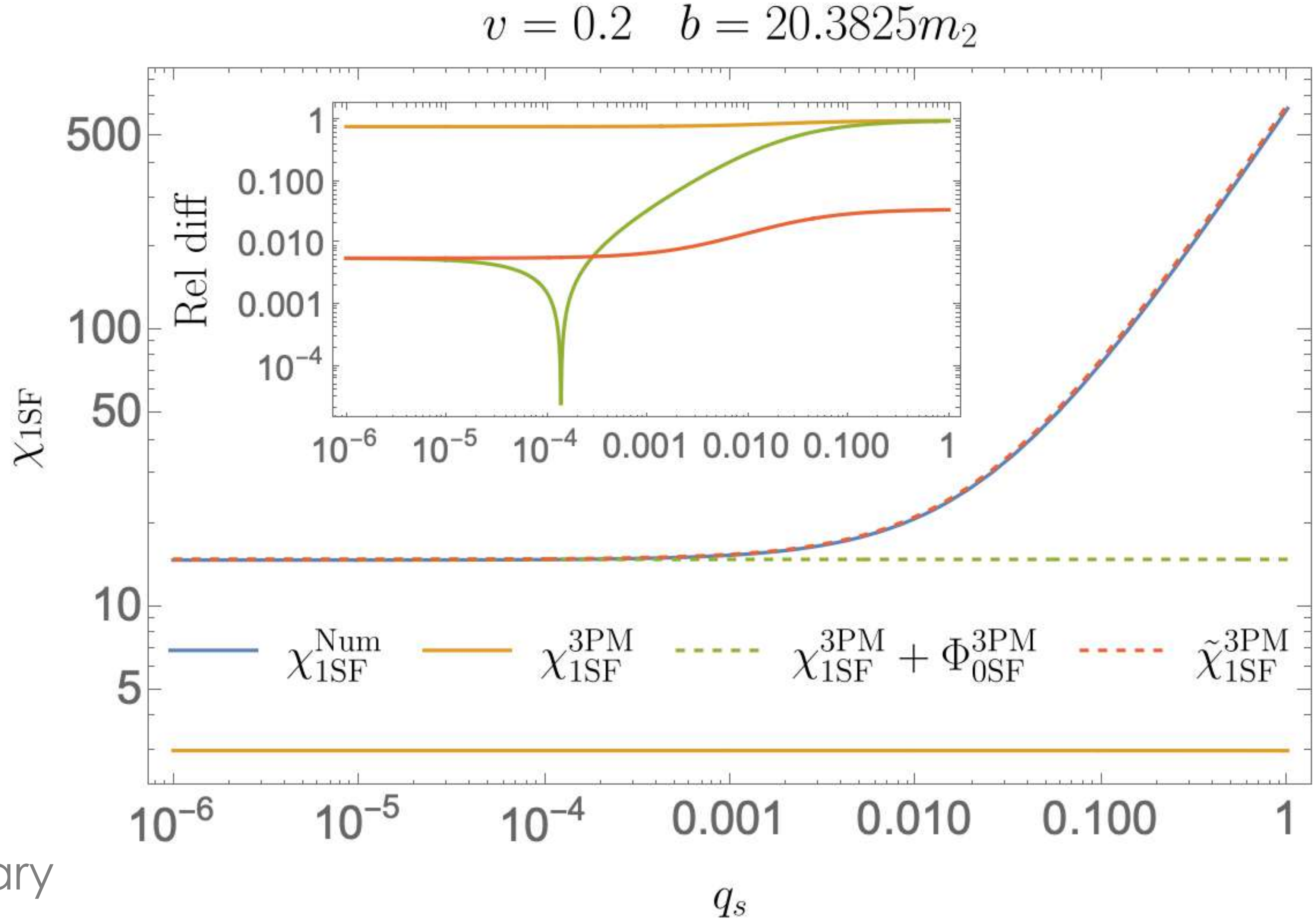

Preliminary

# Resummed scattering angle results



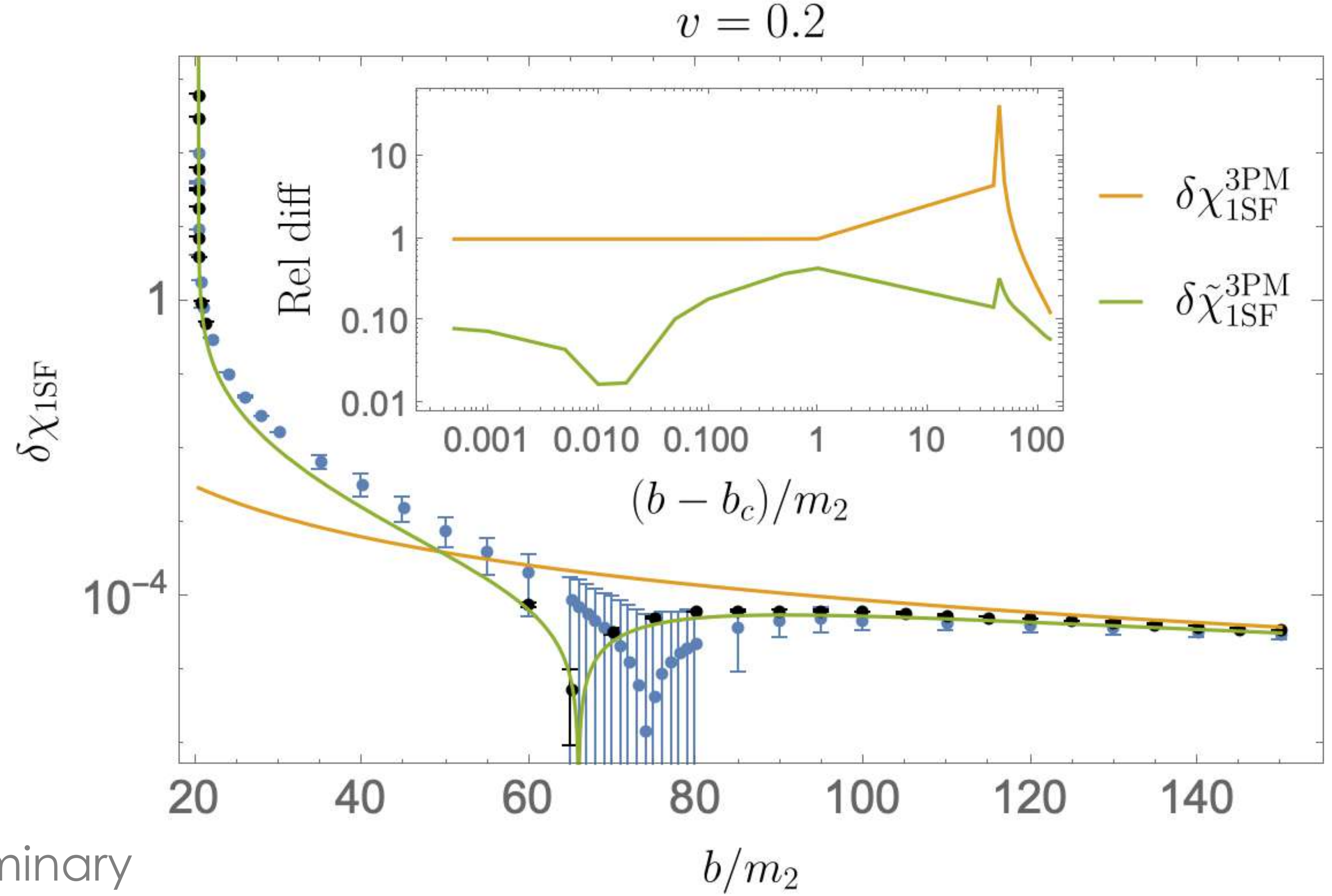

Preliminary

# Resummation as a function of $q_s$



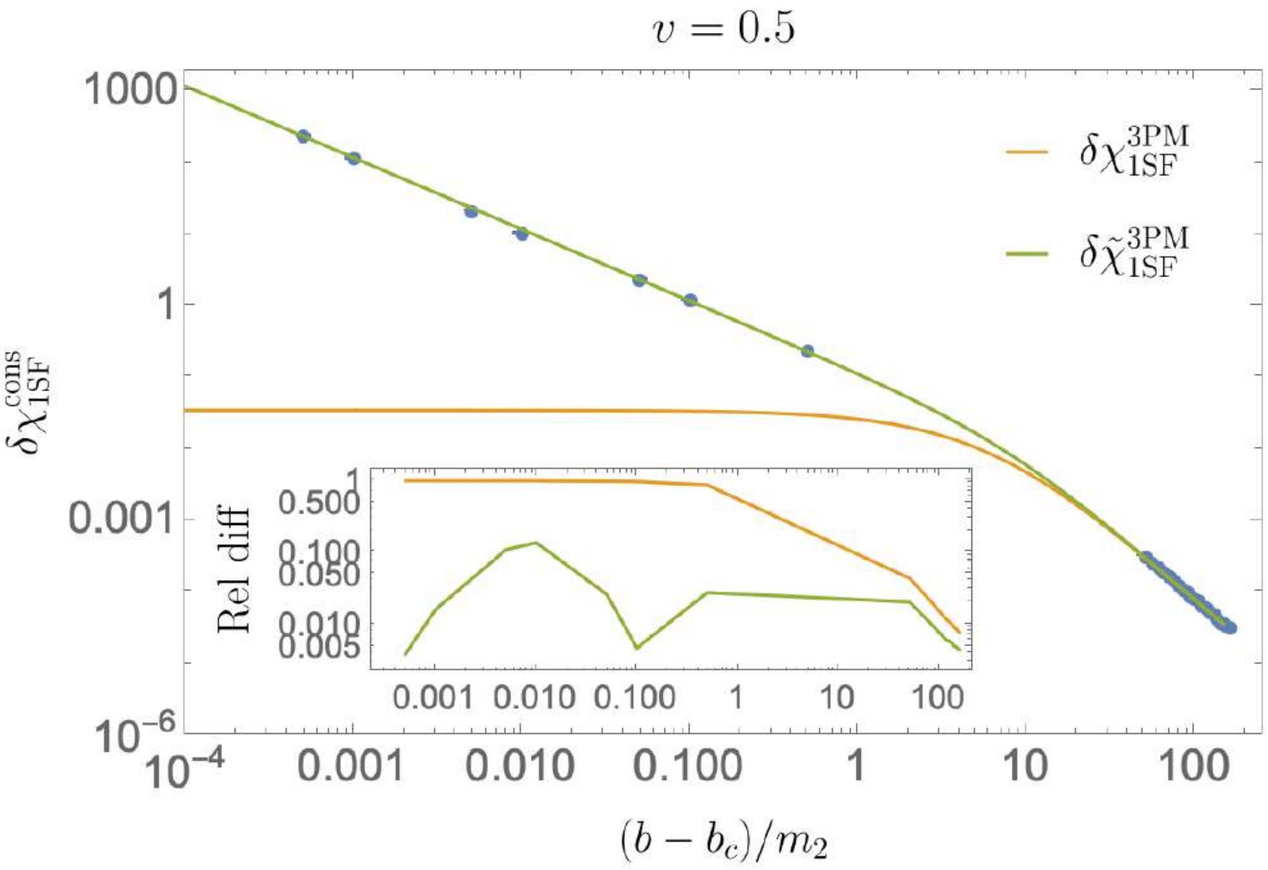

Preliminary

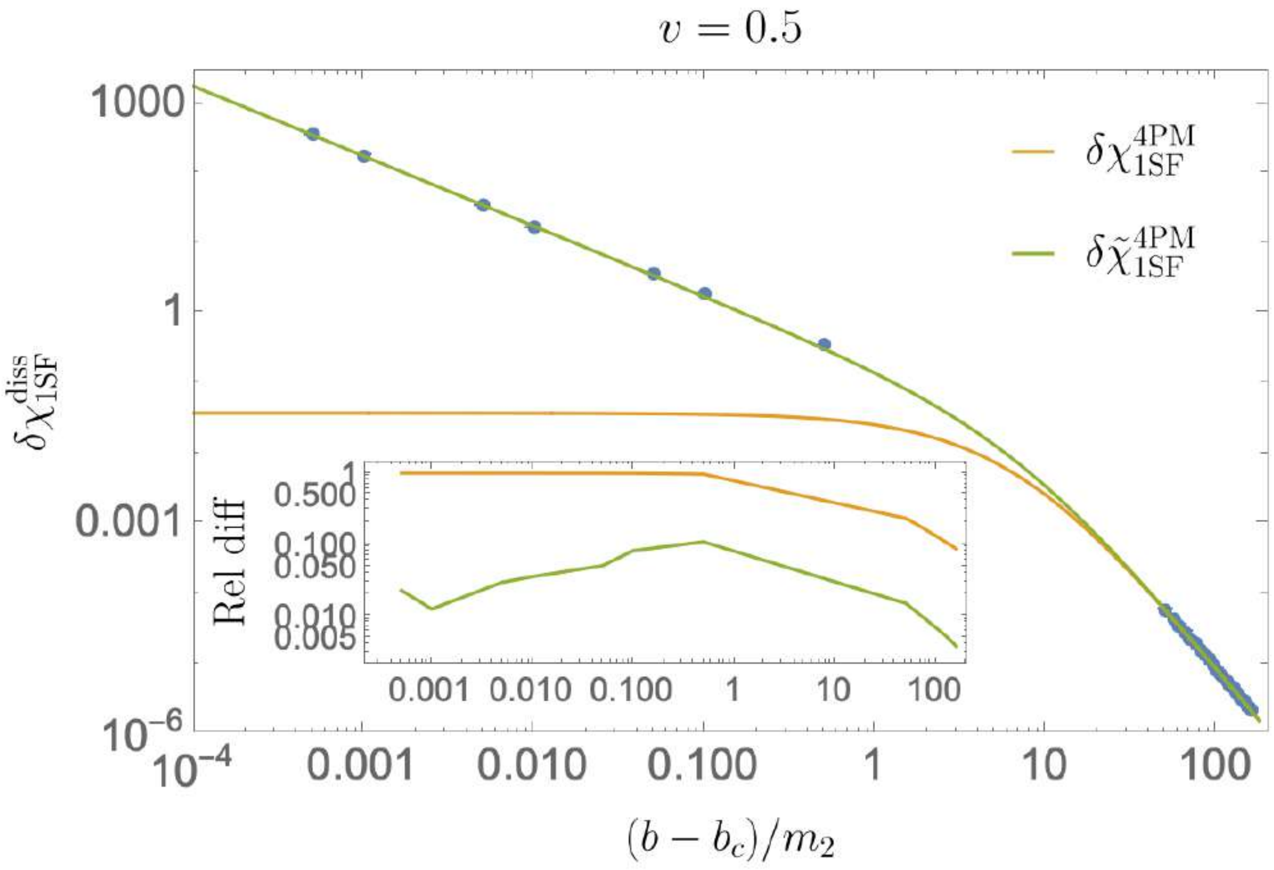
# Resummed 1SF scattering angle II




**Very** preliminary

# Resummed 1SF scattering angle III




**Very** preliminary

# Future work

Scalar self-force correction to the scattering angle:

- Combine weak-field calibration with strong-field resummation.
- Larger range of high accuracy data.
- Numerically extract resummation coefficient $B(v)$.

Gravitational self-force correction to the scattering angle:

- Calculation of gravitational self-force: spin ±2 Teukolsky equation.
- Scattering angle in COM frame: current formulation in the rest frame of the large BH.

Extension to Kerr.

Extension to 2nd order gravitational self-force.