## From classical scattering amplitudes to bound state observables

Riccardo Gonzo<br>based on work with with C.Shi (2304.06066), T.Adamo (2212.13269) and work in progress with T.Adamo and A.Ilderton



# THE UNIVERSITY of EDINBURGH 

ICTP-SAIFR

GWs meet Amplitudes, 22 August 2023

## Content

(1) Motivation and introduction
(2) Amplitude-action relation and classical bound states
(3) Boundary to bound dictionary for generic Kerr orbits

4 All-order scattering and bound observables for Kerr
(5) The classical S-matrix and radiative observables

6 Conclusion

## Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.


## Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.
- We need waveform templates to extract the signal: the effective one-body (EOB) [Buonanno, Damour] allows to combine analytical and numerical techniques valid for different stages of the evolution of compact binaries



## Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study the two-body problem.
- We need waveform templates to extract the signal: the effective one-body (EOB) [Buonanno, Damour] allows to combine analytical and numerical techniques valid for different stages of the evolution of compact binaries

- Today: focus on the inspiral phase, where we can model compact objects as point particles in the spirit of effective field theory [Goldberger,Rothstein]


## Motivation and introduction (II)

- The program is about scattering amplitudes meeting gravitational waves

but why amplitudes?


## Motivation and introduction (II)

- The program is about scattering amplitudes meeting gravitational waves

but why amplitudes?
- Amplitudes are gauge-invariant, universal objects which encode in a compact way the perturbative scattering dynamics for point particles in a QFT and can be efficiently computed in an analytic form. New perspective on GR!


## Motivation and introduction (III)

- Question 1: How can we study classical bound states from QFT amplitudes techniques?

This seems an hard question... 'It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in an entirely satisfactory shape.' (Weinberg, QFT I, page 560)


## Motivation and introduction (III)

- Question 1: How can we study classical bound states from QFT amplitudes techniques?

This seems an hard question... 'It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in an entirely satisfactory shape.' (Weinberg, QFT I, page 560)


Amplitude-action relation, ...

## Motivation and introduction (III)

- Question 1: How can we study classical bound states from QFT amplitudes techniques?

This seems an hard question... 'It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in an entirely satisfactory shape.' (Weinberg, QFT I, page 560)


Amplitude-action relation, ...

- Question 2: How far can we push our analytic tools to understand the bound dynamics from the scattering one?


## Motivation and introduction (III)

- Question 1: How can we study classical bound states from QFT amplitudes techniques?

This seems an hard question... 'It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in an entirely satisfactory shape.' (Weinberg, QFT I, page 560)


Amplitude-action relation, ...

- Question 2: How far can we push our analytic tools to understand the bound dynamics from the scattering one?

Boundary to bound dictionary, ...

## Motivation and introduction (III)

- Question 1: How can we study classical bound states from QFT amplitudes techniques?

This seems an hard question... 'It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in an entirely satisfactory shape.' (Weinberg, QFT I, page 560)


## Amplitude-action relation, ...

- Question 2: How far can we push our analytic tools to understand the bound dynamics from the scattering one?

Boundary to bound dictionary, ...

- Question 3: How important is the all-order resummation for scattering and bound state observables?


## Motivation and introduction (III)

- Question 1: How can we study classical bound states from QFT amplitudes techniques?

This seems an hard question... 'It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in an entirely satisfactory shape.' (Weinberg, QFT I, page 560)


Amplitude-action relation, ...

- Question 2: How far can we push our analytic tools to understand the bound dynamics from the scattering one?

Boundary to bound dictionary, ...

- Question 3: How important is the all-order resummation for scattering and bound state observables?

Effective one body method, ...

## The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the (non-relativistic) hydrogen atom problem with a fixed potential.


## The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the (non-relativistic) hydrogen atom problem with a fixed potential.
- We are familiar to the eigenvalue problem

$$
H|\psi\rangle=E|\psi\rangle, \quad H=\frac{p^{2}}{2 m}+V, \quad V(r) \propto-\frac{g}{r},
$$

which can be solved exactly (at all orders in the coupling $g$ )
$E>0 \leftrightarrow$ scattering plane wave $\psi \propto e^{i \vec{k} \cdot \cdot \vec{x}} \leftrightarrow$ continuous spectrum $E_{\vec{k}}$
$E<0 \leftrightarrow$ normalizable wavefunction $\psi \propto e^{-\vec{k}^{<} \cdot \vec{x}} \leftrightarrow$ discrete spectrum $E_{n}$

## The bound state equation in quantum mechanics (I)

- How can we describe bound states of point particles? Start with the (non-relativistic) hydrogen atom problem with a fixed potential.
- We are familiar to the eigenvalue problem

$$
H|\psi\rangle=E|\psi\rangle, \quad H=\frac{p^{2}}{2 m}+V, \quad V(r) \propto-\frac{g}{r},
$$

which can be solved exactly (at all orders in the coupling $g$ )
$E>0 \leftrightarrow$ scattering plane wave $\psi \propto e^{i \vec{k} \cdot \vec{x}} \leftrightarrow$ continuous spectrum $E_{\vec{k}}$
$E<0 \leftrightarrow$ normalizable wavefunction $\psi \propto e^{-\vec{k}^{<} \cdot \vec{x}} \leftrightarrow$ discrete spectrum $E_{n}$

- Using perturbation theory this would correspond to an infinite sum $V+V G V+\cdots+V(G V)^{n}$ : bound states are intrinsically non-perturbative!



## The bound state equation in quantum mechanics (II)

- The natural generalization of the previous picture to the non-relativistic two-body problem is given by the "ladder approximation"

which is actually used to study QED bound systems like positronium $e^{+} e^{-}$!


## The bound state equation in quantum mechanics (II)

- The natural generalization of the previous picture to the non-relativistic two-body problem is given by the "ladder approximation"

which is actually used to study QED bound systems like positronium $e^{+} e^{-}$!
- We can write it as an amplitude recursion relation

which is nothing else that the (quantum) Bethe-Salpeter equation!


## The bound state equation in quantum field theory

- The Bethe-Salpeter equation is a non-perturbative recursion relation for 4-pt amplitudes, which generate the bound state energy poles via the iteration of a two-massive particle irreducible kernel $\mathcal{K}$


Bethe-

equation

$$
\mathcal{M}_{4}\left(p_{1}, p_{1}^{\prime} ; P\right)=\mathcal{K}\left(p_{1}, p_{1}^{\prime} ; P\right)+\int \hat{\mathrm{d}}^{4} / \mathcal{K}\left(p_{1}, I ; P\right) G(I, P) \mathcal{M}_{4}\left(I, p_{1}^{\prime} ; P\right)
$$

where $G(I, P)$ is the two-body propagator.

## The bound state equation in quantum field theory

- The Bethe-Salpeter equation is a non-perturbative recursion relation for 4-pt amplitudes, which generate the bound state energy poles via the iteration of a two-massive particle irreducible kernel $\mathcal{K}$


Bethe-

equation

$$
\mathcal{M}_{4}\left(p_{1}, p_{1}^{\prime} ; P\right)=\mathcal{K}\left(p_{1}, p_{1}^{\prime} ; P\right)+\int \hat{\mathrm{d}}^{4} / \mathcal{K}\left(p_{1}, I ; P\right) G(I, P) \mathcal{M}_{4}\left(I, p_{1}^{\prime} ; P\right)
$$

where $G(I, P)$ is the two-body propagator.

- How can we take the classical limit?


## What is the classical expansion?

- Framework: QFT scattering amplitudes techniques for the classical gravitational interaction of two massive (spinless or spinning) point particles


## What is the classical expansion?

- Framework: QFT scattering amplitudes techniques for the classical gravitational interaction of two massive (spinless or spinning) point particles
- Consider the conservative 4-pt amplitude: the classical expansion $\hbar \rightarrow 0$ is equivalent to considering the Heavy Particle Effective Theory (HEFT) scheme [Damgaard,Aoude,Haddad,Helset;Brandhuber, Chen, Travaglini,Wen]

$$
\begin{aligned}
& p_{1}^{\mu}:=p_{A}^{\mu}+\hbar \frac{\bar{q}^{\mu}}{2}, \quad\left(p_{1}^{\prime}\right)^{\mu}:=p_{A}^{\mu}-\hbar \frac{\bar{q}^{\mu}}{2}, \quad s=\left(p_{A}+p_{B}\right)^{2}, \\
& p_{2}^{\mu}:=p_{B}^{\mu}-\hbar \frac{\bar{q}^{\mu}}{2}, \quad\left(p_{2}^{\prime}\right)^{\mu}:=p_{B}^{\mu}+\hbar \frac{\bar{q}^{\mu}}{2}, \quad t=-\hbar^{2}|\overrightarrow{\vec{q}}|^{2},
\end{aligned}
$$

where $p_{A}, p_{B}$ are the classical momenta and $q$ is the momentum transfer.


## The classical Bethe-Salpeter equation

- We define the classical Bethe-Salpeter equation from quotienting diagrams by symmetrization over internal graviton exchanges: [Adamo, RG]

$$
\begin{aligned}
& \mathcal{M}_{4,(n+1)}^{\mathrm{cl}}\left(p_{A}, p_{B}, q\right) \\
& =\left\{\begin{array}{ll}
\mathcal{K}_{\mathrm{cl}}\left(p_{A}, p_{B}, q\right) & \text { if } n=0 \\
\frac{1}{n+1} \int \hat{\mathrm{~d}}^{4} / \mathcal{K}_{\mathrm{cl}}\left(p_{A}, p_{B}, l\right) G_{\mathrm{cl}}\left(p_{A}, p_{B}, /\right) \mathcal{M}_{4,(n)}^{\mathrm{cl}}\left(p_{A}, p_{B}, q-l\right) & \text { if } n \geq 1
\end{array} .\right.
\end{aligned}
$$

where the two-body propagator is replaced by its on-shell version

$$
G_{\mathrm{cl}}\left(p_{A}, p_{B}, l\right)=\hat{\delta}\left(2 l \cdot p_{A}\right) \hat{\delta}\left(2 l \cdot p_{B}\right)
$$

and $(n)$ is the number of classical two-massive particle irreducible diagrams.


## Exponentiation of the classical kernel: an exact solution

- Going to impact parameter space

$$
\widetilde{f}\left(x_{q}\right) \equiv \int \hat{\mathrm{d}}^{4} q \hat{\delta}\left(2 p_{A} \cdot q\right) \hat{\delta}\left(2 p_{B} \cdot q\right) e^{i\left(q \cdot x_{q}\right) / \hbar} f(q)
$$

the classical BSE becomes

$$
\widetilde{\mathcal{M}}_{4,(n+1)}^{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right)=\left\{\begin{array}{ll}
\widetilde{\mathcal{K}}_{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right) & \text { if } n=0 \\
\frac{1}{n+1} \widetilde{\mathcal{K}}_{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right) \widetilde{\mathcal{M}}_{4,(n)}^{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right) & \text { if } n \geq 1
\end{array},\right.
$$

which means that the final solution exponentiates exactly

$$
\widetilde{\mathcal{M}}_{4}^{c \mid}\left(p_{A}, p_{B}, x_{\perp}\right)=e^{\tilde{\mathcal{K}}_{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right)}
$$

Natural generalization for spinning particles! [Adamo,RG; Haddad]

## Exponentiation of the classical kernel: an exact solution

- Going to impact parameter space

$$
\widetilde{f}\left(x_{q}\right) \equiv \int \hat{\mathrm{d}}^{4} q \hat{\delta}\left(2 p_{A} \cdot q\right) \hat{\delta}\left(2 p_{B} \cdot q\right) e^{i\left(q \cdot x_{q}\right) / \hbar} f(q)
$$

the classical BSE becomes

$$
\widetilde{\mathcal{M}}_{4,(n+1)}^{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right)= \begin{cases}\widetilde{\mathcal{K}}_{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right) & \text { if } n=0 \\ \frac{1}{n+1} \widetilde{\mathcal{K}}_{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right) \widetilde{\mathcal{M}}_{4,(n)}^{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right) & \text { if } n \geq 1\end{cases}
$$

which means that the final solution exponentiates exactly

$$
\widetilde{\mathcal{M}}_{4}^{c \mid}\left(p_{A}, p_{B}, x_{\perp}\right)=e^{\widetilde{\mathcal{K}}_{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right)} .
$$

Natural generalization for spinning particles! [Adamo,RG; Haddad]

- The analytic structure (poles, etc.) in momentum space arise completely from
$\mathrm{i} \mathcal{M}_{4}^{\mathrm{cl}}\left(p_{A}, p_{B} ; q_{\perp}\right)=\frac{4 \sqrt{\left(p_{A} \cdot p_{B}\right)^{2}-m_{A}^{2} m_{B}^{2}}}{\hbar^{2}} \int \mathrm{~d}^{2} x_{\perp} \mathrm{e}^{-\mathrm{i} \overline{q_{\perp}} \cdot x_{\perp}}\left(\mathrm{e}^{\tilde{\mathcal{K}}_{\mathrm{cl}}\left(p_{A}, p_{B}, x_{\perp}\right)}-1\right)$


## An example: classical kernel for spinless particles at 2PM

- We can consider for example the classical kernel up to 2 PM

$$
\begin{aligned}
\widetilde{\mathcal{K}}^{\mathrm{cl},>}\left(p_{A}, p_{B}, x_{\perp}\right)= & \frac{i}{\hbar}\left[-2 G_{N} \log \left(\mu_{\mathrm{IR}}\left|x_{\perp}\right|\right) m_{A} m_{B} \frac{2 y^{2}-1}{\sqrt{y^{2}-1}}\right. \\
& \left.+\frac{3 \pi}{4} G_{N}^{2} m_{A} m_{B}\left(m_{A}+m_{B}\right) \frac{5 y^{2}-1}{\sqrt{y^{2}-1}} \frac{1}{\left|x_{\perp}\right|}\right]
\end{aligned}
$$

which encodes the conservative dynamics of two spinless particles.


## An example: classical kernel for spinless particles at 2PM

- We can consider for example the classical kernel up to 2 PM

$$
\begin{aligned}
\tilde{\mathcal{K}}^{\mathrm{cl},>}\left(p_{A}, p_{B}, x_{\perp}\right)= & \frac{i}{\hbar}\left[-2 G_{N} \log \left(\mu_{\mathrm{IR}}\left|x_{\perp}\right|\right) m_{A} m_{B} \frac{2 y^{2}-1}{\sqrt{y^{2}-1}}\right. \\
& \left.+\frac{3 \pi}{4} G_{N}^{2} m_{A} m_{B}\left(m_{A}+m_{B}\right) \frac{5 y^{2}-1}{\sqrt{y^{2}-1}} \frac{1}{\left|x_{\perp}\right|}\right]
\end{aligned}
$$

which encodes the conservative dynamics of two spinless particles.


- Note that the motion is restricted to a plane and completely determined by the conserved quantities $(\mathcal{E}, L)$ !

$$
\mathcal{E}:=\frac{E-m_{A}-m_{B}}{\mu}, \quad L=p_{\infty}\left(E, m_{A}, m_{B}\right)\left|x_{\perp}\right|, \quad y=\frac{E^{2}-m_{A}^{2}-m_{B}^{2}}{2 m_{A} m_{B}},
$$

## The Hamilton-Jacobi action from amplitudes (I)

- Since $\mathcal{E}>0$ for scattering orbits and $\mathcal{E}<0$ for bound orbits, we introduce the superscript $>($ resp. $<)$ to denote an expression valid for scattering orbits (resp. bound orbits). How do we perform an analytic continuation?



## The Hamilton-Jacobi action from amplitudes (I)

- Since $\mathcal{E}>0$ for scattering orbits and $\mathcal{E}<0$ for bound orbits, we introduce the superscript $>($ resp. $<)$ to denote an expression valid for scattering orbits (resp. bound orbits). How do we perform an analytic continuation?

- Natural connection of the kernel with the scattering Hamilton-Jacobi action

$$
\widetilde{\mathcal{K}}_{c l}^{>}\left(p_{A}, p_{B} ; x_{\perp}\right)=\frac{i}{\hbar} I^{>}(\mathcal{E}, L), \quad l_{r}^{>}(\mathcal{E}, L)=\oint_{\mathcal{C}>} d r p_{r}(r, \mathcal{E}, L)+L \pi,
$$

where $p_{r}$ is the radial momentum and $\mathcal{C}^{>}$is the contour of integration for scattering orbits. This is the "amplitude-action" relation! [Bern et al.; Damgaard,Plante, Vanhove; Kol,O'Connell,Telem; Adamo,RG]

## The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between the boundary conditions for scattering and bound planar orbits [Kälin,Porto]

$$
\begin{gathered}
\int_{\mathcal{C}_{r}^{>}}=2 \int_{r_{m}(\mathcal{E}, L)}^{\infty}, \quad \int_{\mathcal{C}_{r}^{<}}=2 \int_{r_{-}(\mathcal{E}, L)}^{r_{+}(\mathcal{E}, L)}, \\
r_{-}(\mathcal{E}, L) \stackrel{\mathcal{E} \leq 0}{=} r_{m}(\mathcal{E}, L), \quad r_{+}(\mathcal{E}, L) \stackrel{\mathcal{E} \leq 0}{=} r_{m}(\mathcal{E},-L),
\end{gathered}
$$

thanks to which ( $p_{r}$ is invariant under $L \rightarrow-L$ )

$$
I_{r}^{<}(\mathcal{E}<0, L)=I_{r}^{>}(\mathcal{E}<0, L)-I_{r}^{>}(\mathcal{E}<0,-L) .
$$

Alternatively, analytically continue in the rapidity $y$ at fixed $L$ [Adamo, RG; Di Vecchia, Heissenberg, Russo, Veneziano]

$$
I_{r}^{<}\left(\sqrt{y^{2}-1}, L\right)=I_{r}^{>}\left(i \sqrt{1-y^{2}}, L\right)+I_{r}^{>}\left(-i \sqrt{1-y^{2}}, L\right) .
$$

## The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between the boundary conditions for scattering and bound planar orbits [Kälin,Porto]

$$
\begin{gathered}
\int_{\mathcal{C}_{r}^{>}}=2 \int_{r_{m}(\mathcal{E}, L)}^{\infty}, \quad \int_{\mathcal{C}_{r}^{<}}=2 \int_{r_{-}(\mathcal{E}, L)}^{r_{+}(\mathcal{E}, L)}, \\
r_{-}(\mathcal{E}, L) \stackrel{\mathcal{E} \leq 0}{=} r_{m}(\mathcal{E}, L), \quad r_{+}(\mathcal{E}, L) \stackrel{\mathcal{E} \leq 0}{=} r_{m}(\mathcal{E},-L),
\end{gathered}
$$

thanks to which ( $p_{r}$ is invariant under $L \rightarrow-L$ )

$$
I_{r}^{<}(\mathcal{E}<0, L)=I_{r}^{>}(\mathcal{E}<0, L)-I_{r}^{>}(\mathcal{E}<0,-L) .
$$

Alternatively, analytically continue in the rapidity $y$ at fixed $L$ [Adamo, RG; Di Vecchia, Heissenberg, Russo, Veneziano]

$$
I_{r}^{<}\left(\sqrt{y^{2}-1}, L\right)=I_{r}^{>}\left(i \sqrt{1-y^{2}}, L\right)+I_{r}^{>}\left(-i \sqrt{1-y^{2}}, L\right) .
$$

- This picture generalize to the case of aligned-spin particles $\vec{L} / / \vec{a}_{1}, \vec{a}_{2}$, since the motion still remains on the equatorial plane. [Kälin,Porto]


## The Hamilton-Jacobi action from amplitudes (II)

- There is a remarkable analytic continuation between the boundary conditions for scattering and bound planar orbits [Kälin,Porto]

$$
\begin{gathered}
\int_{\mathcal{C}_{r}^{>}}=2 \int_{r_{m}(\mathcal{E}, L)}^{\infty}, \quad \int_{\mathcal{C}_{r}^{<}}=2 \int_{r_{-}(\mathcal{E}, L)}^{r_{+}(\mathcal{E}, L)}, \\
r_{-}(\mathcal{E}, L) \stackrel{\mathcal{E} \leq 0}{=} r_{m}(\mathcal{E}, L), \quad r_{+}(\mathcal{E}, L) \stackrel{\mathcal{E} \leq 0}{=} r_{m}(\mathcal{E},-L),
\end{gathered}
$$

thanks to which ( $p_{r}$ is invariant under $L \rightarrow-L$ )

$$
I_{r}^{<}(\mathcal{E}<0, L)=I_{r}^{>}(\mathcal{E}<0, L)-I_{r}^{>}(\mathcal{E}<0,-L) .
$$

Alternatively, analytically continue in the rapidity $y$ at fixed $L$ [Adamo, RG; Di Vecchia, Heissenberg, Russo, Veneziano]

$$
I_{r}^{<}\left(\sqrt{y^{2}-1}, L\right)=I_{r}^{>}\left(i \sqrt{1-y^{2}}, L\right)+I_{r}^{>}\left(-i \sqrt{1-y^{2}}, L\right) .
$$

- This picture generalize to the case of aligned-spin particles $\vec{L} / / \vec{a}_{1}, \vec{a}_{2}$, since the motion still remains on the equatorial plane. [Kälin,Porto]
- Today: What happens when $\vec{L}$ and $\vec{a}$ are oriented in generic directions?


## Hamiltonian for geodesics in Kerr

- Let's consider the generic orbit in a Kerr black hole of spin $\vec{a}$ for a massive point particle with orbital angular momentum $\vec{L} \cdot \vec{a} \neq|\vec{L}||\vec{a}|$

which are described by the geodesic Hamiltonian $H(x, p)=1 / 2 g_{\text {Kerr }}^{\mu \nu} p_{\mu} p_{\nu}$ in some convenient Boyer-Lindquist coordinates ( $t, r, \theta, \phi$ ).


## The Hamilton-Jacobi action for generic Kerr orbits

- Using the integrability of Kerr, we can generalize the H-J action for generic orbits of a massive spinless probe in a Kerr black hole [Carter; RG,Shi]

$$
I:=I_{r}+I_{\theta}, \quad I_{r}=\int_{\mathcal{C}_{r}} p_{r} \mathrm{~d} r, \quad I_{\theta}=\int_{\mathcal{C}_{\theta}} p_{\theta} \mathrm{d} \theta
$$

with the radial (resp.polar) momentum $p_{r}$ (resp. $p_{\theta}$ ) and contour $\mathcal{C}_{r}\left(\right.$ resp. $\left.\mathcal{C}_{\theta}\right)$

## The Hamilton-Jacobi action for generic Kerr orbits

- Using the integrability of Kerr, we can generalize the H-J action for generic orbits of a massive spinless probe in a Kerr black hole [Carter; RG,Shi]

$$
I:=I_{r}+I_{\theta}, \quad I_{r}=\int_{\mathcal{C}_{r}} p_{r} \mathrm{~d} r, \quad I_{\theta}=\int_{\mathcal{C}_{\theta}} p_{\theta} \mathrm{d} \theta
$$

with the radial (resp.polar) momentum $p_{r}$ (resp. $p_{\theta}$ ) and contour $\mathcal{C}_{r}\left(\right.$ resp. $\left.\mathcal{C}_{\theta}\right)$

- The momenta are expressed in terms of radial $R(r)$ and polar potential $\Theta(\theta)$

$$
p_{r} \propto \frac{\sqrt{R(r)}}{\Delta(r)}, \quad p_{\theta} \propto \sqrt{\Theta(\theta)}
$$

which are functions of the conserved charges ( $m, E, L, Q$ ), with the Carter constant being a measure of the motion of the particle off the equatorial plane

$$
Q=|\vec{L}|^{2}-L^{2}-|\vec{p}|^{2}(a \cdot \hat{r})^{2} .
$$

## Radial action for generic Kerr orbits (I)

- The radial action for unbound and bound orbits is, respectively,

$$
I_{r}^{>}=2 \int_{r_{m}\left(\mathcal{E}, L, a, L_{Q}\right)}^{+\infty} p_{r}\left(\mathcal{E}, L, a, L_{Q}\right), \quad I_{r}^{<}=2 \int_{r_{-}\left(\mathcal{E}, L, a, L_{Q}\right)}^{r_{+}\left(\mathcal{E}, L, a, L_{Q}\right)} p_{r}\left(\mathcal{E}, L, a, L_{Q}\right),
$$

which are expressed in terms of $L_{Q}=\sqrt{Q+L^{2}}$ and $\mathcal{E}=\left(E^{2}-m^{2}\right) / m^{2}$.

## Radial action for generic Kerr orbits (I)

- The radial action for unbound and bound orbits is, respectively,

$$
I_{r}^{>}=2 \int_{r_{m}\left(\mathcal{E}, L, a, L_{Q}\right)}^{+\infty} p_{r}\left(\mathcal{E}, L, a, L_{Q}\right), \quad I_{r}^{<}=2 \int_{r_{-}\left(\mathcal{E}, L, a, L_{Q}\right)}^{r_{+}\left(\mathcal{E}, L, a, L_{Q}\right)} p_{r}\left(\mathcal{E}, L, a, L_{Q}\right),
$$

which are expressed in terms of $L_{Q}=\sqrt{Q+L^{2}}$ and $\mathcal{E}=\left(E^{2}-m^{2}\right) / m^{2}$.

- With our radial potential we find the analytic continuation [RG,Shi]

$$
r_{-}\left(\mathcal{E}, L, a, L_{Q}\right) \stackrel{\mathcal{E} \leq 0}{=} r_{m}\left(\mathcal{E}, L, a, L_{Q}\right), \quad r_{+}\left(\mathcal{E}, L, a, L_{Q}\right) \stackrel{\mathcal{E} \leq 0}{=} r_{m}\left(\mathcal{E},-L,-a,-L_{Q}\right),
$$

which implies $\left(p_{r}\right.$ is invariant under $\left.\left(a, L, L_{Q}\right) \rightarrow\left(-a,-L,-L_{Q}\right)\right)$

$$
I_{r}^{<}\left(\mathcal{E}, L, a, L_{Q}\right) \stackrel{\mathcal{E} \leq 0}{=} I_{r}^{>}\left(\mathcal{E}, L, a, L_{Q}\right)-I_{r}^{>}\left(\mathcal{E},-L,-a,-L_{Q}\right) .
$$

## Radial action for generic Kerr orbits (II)

- The radial action for scattering orbits is

$$
I_{r}^{>}=\frac{\sqrt{\mathcal{E}}}{\sqrt{M^{2}-a^{2}}} \int_{0}^{u_{m}} \frac{\mathrm{~d} u}{u^{2}} \prod_{j=1}^{4}\left(1-\frac{u}{u_{j}}\right)^{\frac{1}{2}}\left(\frac{1}{u_{B}-u}-\frac{1}{u_{A}-u}\right)
$$

where we have defined the (inverse of) radial roots $\left\{u_{j}\right\}_{j=1, \ldots, 4}$ and

$$
R(u)=-\frac{a^{2} Q}{u^{4}} \prod_{j=1}^{4}\left(u-u_{j}\right), \quad u_{A}=\frac{M+\sqrt{M^{2}-a^{2}}}{a^{2}}, \quad u_{B}=\frac{M-\sqrt{M^{2}-a^{2}}}{a^{2}} .
$$

## Radial action for generic Kerr orbits (II)

- The radial action for scattering orbits is

$$
I_{r}^{>}=\frac{\sqrt{\mathcal{E}}}{\sqrt{M^{2}-a^{2}}} \int_{0}^{u_{m}} \frac{\mathrm{~d} u}{u^{2}} \prod_{j=1}^{4}\left(1-\frac{u}{u_{j}}\right)^{\frac{1}{2}}\left(\frac{1}{u_{B}-u}-\frac{1}{u_{A}-u}\right)
$$

where we have defined the (inverse of) radial roots $\left\{u_{j}\right\}_{j=1, \ldots, 4}$ and

$$
R(u)=-\frac{a^{2} Q}{u^{4}} \prod_{j=1}^{4}\left(u-u_{j}\right), \quad u_{A}=\frac{M+\sqrt{M^{2}-a^{2}}}{a^{2}}, \quad u_{B}=\frac{M-\sqrt{M^{2}-a^{2}}}{a^{2}} .
$$

- The exact all-order expression is $(\epsilon=\mathrm{IR}$-regulator $>0)[\mathrm{RG}$, Shi]

$$
\begin{aligned}
& I_{r}^{>, \epsilon}=\frac{m \sqrt{\mathcal{E}}}{u_{m}^{1-\epsilon} \sqrt{M^{2}-a^{2}}} \frac{\Gamma(3 / 2) \Gamma(-1+\epsilon)}{\Gamma(1 / 2+\epsilon)} \\
& \times\left[\frac{1}{u_{B}} F_{D}^{(4)}\left(\epsilon-1,\left\{1,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\}, \frac{1}{2}+\epsilon ; \frac{u_{m}}{u_{B}}, \frac{u_{m}}{u_{1}}, \frac{u_{m}}{u_{2}}, \frac{u_{m}}{u_{3}}\right)-\left(u_{B} \leftrightarrow u_{A}\right)\right] .
\end{aligned}
$$

## Polar action for generic Kerr orbits (I)

- The most general polar action is [Kapec,Lupsasca; RG, Shi]

$$
\int_{\mathcal{C}_{\theta}}=2 n\left|\int_{\pi / 2}^{\theta_{ \pm}}\right|+\eta_{\text {in }}\left|\int_{\pi / 2}^{\theta_{\text {in }}}\right|-\eta_{\text {out }}\left|\int_{\pi / 2}^{\theta_{\text {out }}}\right|
$$

where $\theta_{\text {in }}$ (resp. $\theta_{\text {out }}$ ) is the initial (resp. final) polar angle, $n$ is the number of turning points, $\theta_{ \pm}$are the roots of the polar potential and $\eta_{\text {in }}, \eta_{\text {out }}= \pm 1$.

## Polar action for generic Kerr orbits (I)

- The most general polar action is [Kapec,Lupsasca; RG, Shi]

$$
\int_{\mathcal{C}_{\theta}}=2 n\left|\int_{\pi / 2}^{\theta_{ \pm}}\right|+\eta_{\text {in }}\left|\int_{\pi / 2}^{\theta_{\text {in }}}\right|-\eta_{\text {out }}\left|\int_{\pi / 2}^{\theta_{\text {out }}}\right|
$$

where $\theta_{\text {in }}$ (resp. $\theta_{\text {out }}$ ) is the initial (resp. final) polar angle, $n$ is the number of turning points, $\theta_{ \pm}$are the roots of the polar potential and $\eta_{\text {in }}, \eta_{\text {out }}= \pm 1$.

## Polar action for generic Kerr orbits (I)

- The most general polar action is [Kapec,Lupsasca; RG, Shi]

$$
\int_{\mathcal{C}_{\theta}}=2 n\left|\int_{\pi / 2}^{\theta_{ \pm}}\right|+\eta_{\text {in }}\left|\int_{\pi / 2}^{\theta_{\text {in }}}\right|-\eta_{\text {out }}\left|\int_{\pi / 2}^{\theta_{\text {out }}}\right|
$$

where $\theta_{\text {in }}\left(\right.$ resp. $\left.\theta_{\text {out }}\right)$ is the initial (resp. final) polar angle, $n$ is the number of turning points, $\theta_{ \pm}$are the roots of the polar potential and $\eta_{\text {in }}, \eta_{\text {out }}= \pm 1$.

- For generic Kerr orbits of interest for the amplitude community,
$I_{\theta}=2 n\left|\int_{\pi / 2}^{\theta_{ \pm}}\right| p_{\theta}-\eta_{\text {out }}\left|\int_{\pi / 2}^{\theta_{\text {out }}}\right| p_{\theta}$,
where we set $\theta_{\text {in }}=\pi / 2$ and $\left(n^{>}, \eta_{\text {out }}^{>}\right)=(1,-1),\left(n^{<}, \eta_{\text {out }}^{<}\right)=$ $(2,+1)$ are fixed by the PM expan-
 sion of the equations of motion.


## Polar action for generic Kerr orbits (II)

- We obtain the exact all order polar action

$$
\begin{aligned}
I_{\theta}^{>}=\frac{\pi}{2} \sqrt{Q U_{+}} & F_{D}^{(2)}\left(\frac{1}{2},\left\{1,-\frac{1}{2}\right\}, 2 ; U_{+}, \frac{U_{+}}{U_{-}}\right) \\
& +\sqrt{Q U_{\text {out }}^{>}} F_{D}^{(3)}\left(\frac{1}{2},\left\{1,-\frac{1}{2},-\frac{1}{2}\right\}, \frac{3}{2} ; U_{\text {out }}^{>}, \frac{U_{\text {out }}^{>}}{U_{+}}, \frac{U_{\text {out }}^{>}}{U_{-}}\right),
\end{aligned}
$$

where $U_{ \pm}=\cos ^{2}\left(\theta_{ \pm}\right)$and of the outgoing angle $U_{\text {out }}^{>}=\cos ^{2}\left(\theta_{\text {out }}\right)$ is

$$
\begin{gathered}
U_{\text {out }}^{>}=U_{+} \operatorname{sn}^{2}\left(X_{0}^{>} \left\lvert\, \frac{U_{-}}{U_{+}}\right.\right), \\
X_{0}^{>}=-4 u_{m} \sqrt{-U_{-} a^{2}} F_{D}^{(3)}\left(1,\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \frac{3}{2} ; \frac{u_{m}}{u_{1}}, \frac{u_{m}}{u_{2}}, \frac{u_{m}}{u_{3}}\right) .
\end{gathered}
$$

## Polar action for generic Kerr orbits (II)

- We obtain the exact all order polar action

$$
\begin{aligned}
I_{\theta}^{>}=\frac{\pi}{2} \sqrt{Q U_{+}} & F_{D}^{(2)}\left(\frac{1}{2},\left\{1,-\frac{1}{2}\right\}, 2 ; U_{+}, \frac{U_{+}}{U_{-}}\right) \\
& +\sqrt{Q U_{\text {out }}^{>}} F_{D}^{(3)}\left(\frac{1}{2},\left\{1,-\frac{1}{2},-\frac{1}{2}\right\}, \frac{3}{2} ; U_{\text {out }}^{>}, \frac{U_{\text {out }}^{>}}{U_{+}}, \frac{U_{\text {out }}^{>}}{U_{-}}\right),
\end{aligned}
$$

where $U_{ \pm}=\cos ^{2}\left(\theta_{ \pm}\right)$and of the outgoing angle $U_{\text {out }}^{>}=\cos ^{2}\left(\theta_{\text {out }}\right)$ is

$$
\begin{gathered}
U_{\text {out }}^{>}=U_{+} \operatorname{sn}^{2}\left(X_{0}^{>} \left\lvert\, \frac{U_{-}}{U_{+}}\right.\right), \\
X_{0}^{>}=-4 u_{m} \sqrt{-U_{-} a^{2}} F_{D}^{(3)}\left(1,\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \frac{3}{2} ; \frac{u_{m}}{u_{1}}, \frac{u_{m}}{u_{2}}, \frac{u_{m}}{u_{3}}\right) .
\end{gathered}
$$

- The polar action obeys the analytic continuation [RG, Shi]

$$
I_{\theta}^{<}\left(\mathcal{E}, L, a, L_{Q} ; n^{>}, \eta_{\text {out }}^{>}\right)^{\mathcal{E} \leq 0} I_{\theta}^{>}\left(\mathcal{E}, L, a, L_{Q} ; n^{<}, \eta_{\text {out }}^{<}\right) .
$$

## Polar action for generic Kerr orbits (II)

- We obtain the exact all order polar action

$$
\begin{aligned}
I_{\theta}^{>}=\frac{\pi}{2} \sqrt{Q U_{+}} & F_{D}^{(2)}\left(\frac{1}{2},\left\{1,-\frac{1}{2}\right\}, 2 ; U_{+}, \frac{U_{+}}{U_{-}}\right) \\
& +\sqrt{Q U_{\text {out }}^{>}} F_{D}^{(3)}\left(\frac{1}{2},\left\{1,-\frac{1}{2},-\frac{1}{2}\right\}, \frac{3}{2} ; U_{\text {out }}^{>}, \frac{U_{\text {out }}^{>}}{U_{+}}, \frac{U_{\text {out }}^{>}}{U_{-}}\right),
\end{aligned}
$$

where $U_{ \pm}=\cos ^{2}\left(\theta_{ \pm}\right)$and of the outgoing angle $U_{\text {out }}^{>}=\cos ^{2}\left(\theta_{\text {out }}\right)$ is

$$
\begin{gathered}
U_{\text {out }}^{>}=U_{+} \operatorname{sn}^{2}\left(X_{0}^{>} \left\lvert\, \frac{U_{-}}{U_{+}}\right.\right), \\
X_{0}^{>}=-4 u_{m} \sqrt{-U_{-} a^{2}} F_{D}^{(3)}\left(1,\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \frac{3}{2} ; \frac{u_{m}}{u_{1}}, \frac{u_{m}}{u_{2}}, \frac{u_{m}}{u_{3}}\right) .
\end{gathered}
$$

- The polar action obeys the analytic continuation [RG, Shi]

$$
I_{\theta}^{<}\left(\mathcal{E}, L, a, L_{Q} ; n^{>}, \eta_{\text {out }}^{>}\right) \stackrel{\mathcal{E} \leq 0}{=} I_{\theta}^{>}\left(\mathcal{E}, L, a, L_{Q} ; n^{<}, \eta_{\text {out }}^{<}\right) .
$$

- Now it's time to have fun and compute scattering and bound observables!


## Azimuthal deflection angle for generic Kerr orbits

- Once we have the $\mathrm{H}-\mathrm{J}$ action, we can obtain the azimuthal deflection angle by differentiation over L

$$
\Delta \phi+\pi=-\frac{\partial I}{\partial L}=-\frac{\partial I_{r}}{\partial L}-\frac{\partial I_{\theta}}{\partial L}
$$

which gives in the perturbative expansion in $G_{N} M\left(a \ll G_{N} M\right)$

$$
\begin{aligned}
\Delta \phi= & \frac{2 G_{N} M m L(2 \mathcal{E}+1)}{\sqrt{\mathcal{E}} L_{Q}^{2}}+\frac{3 \pi G_{N}^{2} M^{2} m^{2} L(5 \mathcal{E}+4)}{4 L_{Q}^{3}} \\
& +a \sqrt{\mathcal{E}+1}\left(\frac{4 G_{N} M m^{2} \sqrt{\mathcal{E}}\left(L_{Q}^{2}-2 L^{2}\right)}{L_{Q}^{4}}\right)+\ldots
\end{aligned}
$$

## Azimuthal deflection angle for generic Kerr orbits

- Once we have the $\mathrm{H}-\mathrm{J}$ action, we can obtain the azimuthal deflection angle by differentiation over L

$$
\Delta \phi+\pi=-\frac{\partial I}{\partial L}=-\frac{\partial I_{r}}{\partial L}-\frac{\partial I_{\theta}}{\partial L}
$$

which gives in the perturbative expansion in $G_{N} M\left(a \ll G_{N} M\right)$

$$
\begin{aligned}
\Delta \phi= & \frac{2 G_{N} M m L(2 \mathcal{E}+1)}{\sqrt{\mathcal{E}} L_{Q}^{2}}+\frac{3 \pi G_{N}^{2} M^{2} m^{2} L(5 \mathcal{E}+4)}{4 L_{Q}^{3}} \\
& +a \sqrt{\mathcal{E}+1}\left(\frac{4 G_{N} M m^{2} \sqrt{\mathcal{E}}\left(L_{Q}^{2}-2 L^{2}\right)}{L_{Q}^{4}}\right)+\ldots
\end{aligned}
$$

- Matches known result in the equatorial limit $Q \rightarrow 0$ [Damgaard, Hoogeveen, Luna, Vines], but the expansion is new for generic orbits! [RG,Shi]


## Polar deflection angle for generic Kerr orbits

- From the $r-\theta$ equations of motion we get the polar deflection angle in the perturbative expansion

$$
\begin{aligned}
\frac{\Delta \theta}{\sqrt{Q}}=- & \frac{2 G_{N} M m(2 \mathcal{E}+1)}{\sqrt{\mathcal{E}} L_{Q}^{2}}-\frac{3 \pi G_{N}^{2} M^{2} m^{2}(5 \mathcal{E}+4)}{4 L_{Q}^{3}} \\
& +a \sqrt{\mathcal{E}+1}\left(\frac{8 G_{N} M m^{2} L \sqrt{\mathcal{E}}}{L_{Q}^{4}}\right)+\ldots
\end{aligned}
$$

which vanish identically in the equatorial limit.

## Polar deflection angle for generic Kerr orbits

- From the $r-\theta$ equations of motion we get the polar deflection angle in the perturbative expansion

$$
\begin{aligned}
\frac{\Delta \theta}{\sqrt{Q}}= & -\frac{2 G_{N} M m(2 \mathcal{E}+1)}{\sqrt{\mathcal{E}} L_{Q}^{2}}-\frac{3 \pi G_{N}^{2} M^{2} m^{2}(5 \mathcal{E}+4)}{4 L_{Q}^{3}} \\
& +a \sqrt{\mathcal{E}+1}\left(\frac{8 G_{N} M m^{2} L \sqrt{\mathcal{E}}}{L_{Q}^{4}}\right)+\ldots
\end{aligned}
$$

which vanish identically in the equatorial limit.

- Matches expectations that when $a \rightarrow 0$ we get $\Delta \phi \stackrel{a \rightarrow 0}{\sim}-(L / \sqrt{Q}) \Delta \theta$ at the lowest order (the motion is happening on an inclined plane)!


## Time delay for generic null Kerr orbits

- The time delay is obtained by differentiating over the energy $E$ in the HJ action. For generic null geodesics with fixed $b$ relative to an observer with $b^{\prime} \gg b$ but at the same energy $E^{\prime}=E$ [Camanho, Edelstein, Maldacena, Zhiboedov; Accettulli Huber, Brandhuber, De Angelis, Travaglini]

$$
\begin{aligned}
\Delta T= & \left.\frac{\partial I}{\partial E}\right|_{L_{Q}, E}-\left.\frac{\partial I}{\partial E}\right|_{L_{Q}^{\prime} \gg L_{Q}, E^{\prime}=E} \\
= & 4 G_{N} M \log \left(\frac{L_{Q}^{\prime}}{L_{Q}}\right)+\frac{15 \pi G_{N}^{2} M^{2} E}{2 L_{Q}}+\frac{64 G_{N}^{3} M^{3} E^{2}}{L_{Q}^{2}} \\
& -\frac{a L E}{L_{Q}^{4}}\left(8 G_{N} M L_{Q}^{2}+15 \pi G_{N}^{2} M^{2} E L_{Q}+256 G_{N}^{3} M^{3} E^{2}\right)+\ldots
\end{aligned}
$$

which in the equatorial limit matches [Bautista, Guevara, Kavanagh, Vines].

## Fundamental frequencies for Kerr orbits

- The H-J action provides an intrinsic definition of the fundamental frequencies for the bound motion $\omega_{r}, \omega_{\phi}, \omega_{\theta}$ via the action-angle representation [Schmidt]:

$$
\omega_{r}=-\frac{1}{\Omega} \frac{\partial J_{\theta}}{\partial Q}, \quad \omega_{\theta}=\frac{1}{\Omega} \frac{\partial J_{r}}{\partial Q}, \quad \omega_{\phi}=\frac{1}{\Omega}\left(\frac{\partial J_{r}}{\partial L} \frac{\partial J_{\theta}}{\partial Q}-\frac{\partial J_{r}}{\partial Q} \frac{\partial J_{\theta}}{\partial L}\right)
$$

with $\Omega:=\frac{\partial J_{r}}{\partial H} \frac{J_{\theta}}{\partial Q}-\frac{\partial J_{r}}{\partial Q} \frac{\partial J_{\theta}}{\partial H}, J_{\phi}=L, J_{r}=\oint p_{r} \mathrm{~d} r=I_{r}^{<}, J_{\theta}=\oint p_{\theta} \mathrm{d} \theta=I_{\theta}^{<(1)}$.

## Fundamental frequencies for Kerr orbits

- The $\mathrm{H}-\mathrm{J}$ action provides an intrinsic definition of the fundamental frequencies for the bound motion $\omega_{r}, \omega_{\phi}, \omega_{\theta}$ via the action-angle representation [Schmidt]:

$$
\omega_{r}=-\frac{1}{\Omega} \frac{\partial J_{\theta}}{\partial Q}, \quad \omega_{\theta}=\frac{1}{\Omega} \frac{\partial J_{r}}{\partial Q}, \quad \omega_{\phi}=\frac{1}{\Omega}\left(\frac{\partial J_{r}}{\partial L} \frac{\partial J_{\theta}}{\partial Q}-\frac{\partial J_{r}}{\partial Q} \frac{\partial J_{\theta}}{\partial L}\right),
$$

with $\Omega:=\frac{\partial J_{r}}{\partial H} \frac{J_{\theta}}{\partial Q}-\frac{\partial J_{r}}{\partial Q} \frac{\partial J_{\theta}}{\partial H}, J_{\phi}=L, J_{r}=\oint p_{r} \mathrm{~d} r=I_{r}^{<}, J_{\theta}=\oint p_{\theta} \mathrm{d} \theta=I_{\theta}^{<(1)}$.

- From the amplitude perspective we want bound observables invariant under the choice of the time coordinate, i.e. the frequency ratios [RG, Shi]

$$
K^{\phi r}:=\frac{\omega_{\phi}}{\omega_{r}}=\frac{\partial J_{r} / \partial Q}{\partial J_{\theta} / \partial Q} \frac{\partial J_{\theta}}{\partial L}-\frac{\partial J_{r}}{\partial L}, \quad K^{\theta r}:=\frac{\omega_{\theta}}{\omega_{r}}=-\frac{\partial J_{r} / \partial Q}{\partial J_{\theta} / \partial Q} .
$$

corresponding to the precession of the periastron and of the orbital plane.

## Frequency ratios for Kerr orbits

- The periastron advance is [RG,Shi]

$$
\begin{aligned}
K^{\phi r} & =1+\frac{3 G_{N}^{2} M^{2} m^{2}(5 \mathcal{E}+4)}{4 L_{Q}^{2}}+\frac{a G_{N}^{2} M^{2} m^{3} \sqrt{\mathcal{E}+1}\left(L_{Q}-3 L\right)(5 \mathcal{E}+2)}{L_{Q}^{4}} \\
& +\frac{3 a^{2} G_{N}^{2} M^{2} m^{4}}{32 L_{Q}^{6}}\left[L^{2}\left(445 \mathcal{E}^{2}+416 \mathcal{E}+40\right)-L_{Q}\left(L_{Q}+2 L\right)\left(85 \mathcal{E}^{2}+80 \mathcal{E}+8\right)\right]
\end{aligned}
$$

which matches [Kälin,Porto] in the equatorial limit.

## Frequency ratios for Kerr orbits

- The periastron advance is [RG,Shi]

$$
\begin{aligned}
K^{\phi r} & =1+\frac{3 G_{N}^{2} M^{2} m^{2}(5 \mathcal{E}+4)}{4 L_{Q}^{2}}+\frac{a G_{N}^{2} M^{2} m^{3} \sqrt{\mathcal{E}+1}\left(L_{Q}-3 L\right)(5 \mathcal{E}+2)}{L_{Q}^{4}} \\
& +\frac{3 a^{2} G_{N}^{2} M^{2} m^{4}}{32 L_{Q}^{6}}\left[L^{2}\left(445 \mathcal{E}^{2}+416 \mathcal{E}+40\right)-L_{Q}\left(L_{Q}+2 L\right)\left(85 \mathcal{E}^{2}+80 \mathcal{E}+8\right)\right]
\end{aligned}
$$

which matches [Kälin,Porto] in the equatorial limit.

- The precession of the orbital plane is

$$
\begin{aligned}
K^{\theta r} & =1+\frac{3 G_{N}^{2} M^{2} m^{2}(5 \mathcal{E}+4)}{4 L_{Q}^{2}}-\frac{3 a G_{N}^{2} M^{2} m^{3} L \sqrt{\mathcal{E}+1}(5 \mathcal{E}+2)}{L_{Q}^{4}} \\
& +\frac{3 a^{2} G_{N}^{2} M^{2} m^{4}}{32 L_{Q}^{6}}\left[L^{2}\left(445 \mathcal{E}^{2}+416 \mathcal{E}+40\right)-L_{Q}^{2}\left(85 \mathcal{E}^{2}+80 \mathcal{E}+8\right)\right] .
\end{aligned}
$$

As expected frequencies become degenerate in the spinless case $\omega_{\theta} \xrightarrow{a \rightarrow 0} \omega_{\phi}$

## Radiative observables in the S-matrix formalism (I)

- How can the previous picture be generalized in the presence of radiation?

Consider the 5 -pt recursion with the emission of a positive energy graviton

and apply the symmetrization procedure [Adamo, RG, Ilderton]


A similar recursion holds for the emission of $n$ gravitons.

## Radiative observables in the S-matrix formalism (I)

- How can the previous picture be generalized in the presence of radiation?

Consider the 5 -pt recursion with the emission of a positive energy graviton

and apply the symmetrization procedure [Adamo, RG, Ilderton]


A similar recursion holds for the emission of $n$ gravitons.

- Can we find an exact solution from the resummation?


## Radiative observables in the S-matrix formalism (II)

- The classical 5-pt recursion relation is of the form

$$
\begin{aligned}
& \left.\mathcal{M}_{5,(1)}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}, q_{2}\right)\right|_{E_{k_{1}}>0}=\mathcal{K}_{\mathcal{R}}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}, q_{2}\right), \\
& \left.\mathcal{M}_{5,(n+1)}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}, q_{2}\right)\right|_{E_{k_{1}^{\prime}}>0} \\
& \quad=\frac{1}{n+1}\left[\int \hat{\mathrm{~d}}^{4} / \mathcal{K}^{\mathrm{cl}}\left(p_{A}, p_{B} ; I\right) G^{\mathrm{cl}}\left(p_{A}, p_{B} ; l\right) \mathcal{M}_{5,(n)}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}-l, q_{2}+l\right)\right. \\
& \left.\quad+\int \hat{\mathrm{d}}^{4} / \mathcal{K}_{\mathcal{R}}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}-l, q_{2}+I\right) G^{\mathrm{cl}}\left(p_{A}, p_{B} ; l\right) \mathcal{M}_{4,(n)}^{\mathrm{cl}}\left(p_{A}, p_{B} ; l\right)\right],
\end{aligned}
$$

## Radiative observables in the S-matrix formalism (II)

- The classical 5-pt recursion relation is of the form

$$
\begin{aligned}
& \left.\mathcal{M}_{5,(1)}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}, q_{2}\right)\right|_{E_{k_{1}}>0}=\mathcal{K}_{\mathcal{R}}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}, q_{2}\right), \\
& \left.\mathcal{M}_{5,(n+1)}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}, q_{2}\right)\right|_{E_{k_{1}^{\prime}}>0} \\
& \quad=\frac{1}{n+1}\left[\int \hat{\mathrm{~d}}^{4} / \mathcal{K}^{\mathrm{cl}}\left(p_{A}, p_{B} ; I\right) G^{\mathrm{cl}}\left(p_{A}, p_{B} ; l\right) \mathcal{M}_{5,(n)}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}-l, q_{2}+l\right)\right. \\
& \left.\quad+\int \hat{\mathrm{d}}^{4} / \mathcal{K}_{\mathcal{R}}^{\mathrm{cl}, \mu_{1} \nu_{1}}\left(p_{A}, p_{B} ; q_{1}-l, q_{2}+I\right) G^{\mathrm{cl}}\left(p_{A}, p_{B} ; l\right) \mathcal{M}_{4,(n)}^{\mathrm{cl}}\left(p_{A}, p_{B} ; I\right)\right],
\end{aligned}
$$

- We can then solve the recursion by working in impact parameter space

$$
\begin{aligned}
& \tilde{f}\left(x_{q_{1}}, x_{q_{2}}\right) \equiv \int \hat{\mathrm{d}}^{4} q_{1} \hat{\mathrm{~d}}^{4} q_{2} \hat{\delta}\left(2 p_{A} \cdot q_{1}\right) \hat{\delta}\left(2 p_{B} \cdot q_{2}\right) \\
& \quad \times \hat{\delta}\left(q_{1}+q_{2}-k\right) e^{i\left(q_{1} \cdot x_{q_{1}}+q_{2} \cdot x_{q_{2}}\right) / \hbar} f\left(q_{1}, q_{2}\right)
\end{aligned}
$$

with momentum transfers ( $q_{1}, q_{2}$ ) of initial and final pair of massive states.

## Radiative observables in the S-matrix formalism (III)

- The classical S-matrix is [Cristofoli, RG, Moynihan, O'Connell, Ross, Sergola, White; Britto, RG, Jehu; Di Vecchia, Heissenberg, Russo, Veneziano]

$$
\left.\widetilde{\mathcal{S}}^{\mathrm{cl}}\right|_{E_{k_{1}}, \ldots, E_{k_{N}}>0} \sim e^{\widetilde{\mathcal{K}}^{c l}\left(p_{A}, p_{B} ; x_{q_{1}}, x_{q_{2}}\right)} e^{\sum_{\sigma} \int \mathrm{d} \Phi(k) \widetilde{\mathcal{K}}_{\mathcal{R}}^{c l}\left(p_{A}, p_{B} ; x_{q_{1}}, x_{q_{2}}, k^{\sigma}\right) a_{\sigma}^{\dagger}(k)+\text { h.c. }},
$$

which includes a coherent state of gravitons representing the classical gravitational wave (see D.Kosower's talk).


## Radiative observables in the S-matrix formalism (III)

- The classical S-matrix is [Cristofoli, RG, Moynihan, O'Connell, Ross, Sergola, White; Britto, RG, Jehu; Di Vecchia, Heissenberg, Russo, Veneziano]

$$
\left.\widetilde{\mathcal{S}}^{c l}\right|_{E_{k_{1}}, \ldots, E_{k_{N}}>0} \sim e^{\widetilde{\mathcal{K}}^{\mathrm{cl}}\left(p_{A}, p_{B} ; x_{q_{1}}, x_{q_{2}}\right)} e^{\sum_{\sigma} \int \mathrm{d} \Phi(k) \widetilde{\mathcal{K}}_{\mathcal{R}}^{\mathrm{c}}\left(p_{A}, p_{B} ; x_{q_{1}}, x_{q_{2}}, k^{\sigma}\right) a_{\sigma}^{\dagger}(k)+\text { h.c. }},
$$

which includes a coherent state of gravitons representing the classical gravitational wave (see D.Kosower's talk).


- Comments: 1) All amplitude observables for the two-body problem can derived from such gauge-invariant representation; 2) Compact expression which unifies the treatment of potential and radiative modes.


## Radiative observables in the S-matrix formalism (III)

- The classical S-matrix is [Cristofoli, RG, Moynihan, O'Connell, Ross, Sergola, White; Britto, RG, Jehu; Di Vecchia, Heissenberg, Russo, Veneziano]

$$
\left.\widetilde{\mathcal{S}}^{c l}\right|_{E_{k_{1}}, \ldots, E_{k_{N}}>0} \sim e^{\widetilde{\mathcal{K}}^{c l}\left(p_{A}, p_{B} ; x_{q_{1}}, x_{q_{2}}\right)} e^{\sum_{\sigma} \int \mathrm{d} \Phi(k) \widetilde{\mathcal{K}}_{\mathcal{R}}^{d}\left(p_{A}, p_{B} ; x_{q_{1}}, \alpha_{q_{2}}, k^{\sigma}\right) a_{\sigma}^{\dagger}(k)+\text { h.c. }},
$$

which includes a coherent state of gravitons representing the classical gravitational wave (see D.Kosower's talk).


- Comments: 1) All amplitude observables for the two-body problem can derived from such gauge-invariant representation; 2) Compact expression which unifies the treatment of potential and radiative modes.
- Open problem: can we understand the analytic continuation of the waveform?


## Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes


## Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes
- We solved the BS equation in impact parameter space, making a connection with the Hamilton-Jacobi action via the amplitude-action relation


## Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes
- We solved the BS equation in impact parameter space, making a connection with the Hamilton-Jacobi action via the amplitude-action relation
- We discussed a generalization of the boundary to bound dictionary for generic angular momentum orientations (i.e. including both radial and polar action), focusing on a massive probe particle moving in a Kerr background


## Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes
- We solved the BS equation in impact parameter space, making a connection with the Hamilton-Jacobi action via the amplitude-action relation
- We discussed a generalization of the boundary to bound dictionary for generic angular momentum orientations (i.e. including both radial and polar action), focusing on a massive probe particle moving in a Kerr background
- We provided all-order expressions for observables in the probe limit for the scattering (azimuthal and polar deflection angles, time delay) and bound (periastron advance, precession of the orbital plane) dynamics


## Summary and future directions

- We derived the classical Bethe-Salpeter equation which describes gravitational bound systems, both for particles with and without spin, using a new physical principles for classical amplitudes
- We solved the BS equation in impact parameter space, making a connection with the Hamilton-Jacobi action via the amplitude-action relation
- We discussed a generalization of the boundary to bound dictionary for generic angular momentum orientations (i.e. including both radial and polar action), focusing on a massive probe particle moving in a Kerr background
- We provided all-order expressions for observables in the probe limit for the scattering (azimuthal and polar deflection angles, time delay) and bound (periastron advance, precession of the orbital plane) dynamics
- Future directions: understand the amplitude-action relation for generic spin orientations, generalized Carter constant for spinning particles, analytic continuation for radiative observables, self-force from amplitudes, ...

Summary and future directions


