From classical scattering amplitudes to bound state observables

Riccardo Gonzo
based on work with with C.Shi (2304.06066), T.Adamo (2212.13269)
and work in progress with T.Adamo and A.Ilderton

THE UNIVERSITY of EDINBURGH

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Content

Motivation and introduction (I)

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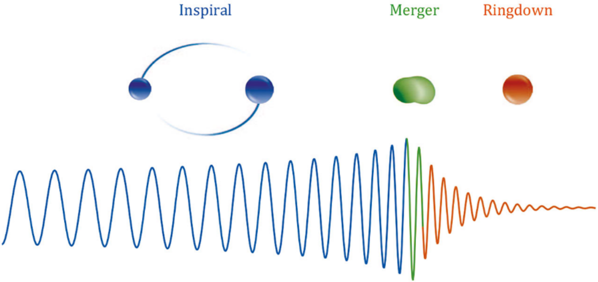

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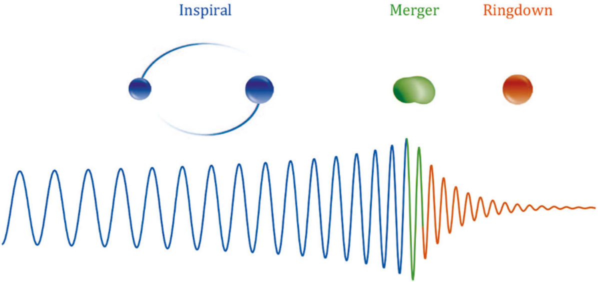

- Today: focus on the inspiral phase, where we can model compact objects as point particles in the spirit of effective field theory [Goldberger,Rothstein]

Motivation and introduction (II)

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but why amplitudes?

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Amplitudes are gauge-invariant, universal objects which encode in a compa
way the perturbative scattering dynamics for point particles in a QFT and can be e ciently computed in an analytic form. New perspective on GR!

Motivation and introduction (III)

Question 1: How can we study classical bound states from QFT amplitude techniques?

This seems an hard question ...
'It must be said that the theory of relativistic e ects and radiative corrections in bound states is not yet in an entirely satisfactory shape.'
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E ective one body method, ...

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We are familiar to the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle\,; \qquad H = \frac{p^2}{2m} + V\,; \qquad V(r) \propto -\frac{g}{r}\,;$$

which can be solved exactly (at all orders in the coupling g)

- $E > 0 \Rightarrow$ scattering plane wave $\propto e^{i\vec{k}\cdot\vec{x}} \Rightarrow$ continuous spectrum $E_{\vec{k}}$
- $E < 0 \Rightarrow$ normalizable wavefunction $\propto e^{-\kappa r} \Rightarrow$ discrete spectrum E_n

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Using perturbation theory this would correspond to an infinite sum $V + VGV + \dots + V(GV)^n$: bound states are intrinsically non-perturbative!

The bound state equation in quantum mechanics (II)

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which is actually used to study QED bound systems like positronium e^+e^- !

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We can write it as an amplitude recursion relation

which is nothing else that the (quantum) Bethe-Salpeter equation!

The bound state equation in quantum eld theory

The Bethe-Salpeter equation is a non-perturbative recursion relation for 4-
amplitudes, which generate the bound state energy poles via the iteration
a two-massive particle irreducible kernel K

equation

Bethe- Salpeter

$$M_4(p_1; p_1^0; P) = K(p_1; p_1^0; P) + \int \hat{d}^4 l \, K(p_1; l; P) G(l; P) M_4(l; p_1^0; P) ;$$

where $G(l; P)$ is the two-body propagator.

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How can we take the classical limit?

What is the classical expansion?

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Consider the conservative 4-pt amplitude: the classical expansion $\hbar \to 0$ is equivalent to considering the Heavy Particle Effective Theory (HEFT) scheme [Damgaard,Aoude,Haddad,Helset;Brandhuber,Chen,Travaglini,Wen]

$$p_1 := p_A + \hbar\frac{\bar q}{2}; \qquad (p_1') := p_A - \hbar\frac{\bar q}{2}; \qquad s = (p_A + p_B)^2;$$

$$p_2 := p_B - \hbar\frac{\bar q}{2}; \qquad (p_2') := p_B + \hbar\frac{\bar q}{2}; \qquad t = -\hbar^2 |\bar q|^2;$$

where p_A, p_B are the classical momenta and $\bar q$ is the momentum transfer.

The classical Bethe-Salpeter equation

We define the classical Bethe-Salpeter equation from quotienting diagrams symmetrization over internal graviton exchanges: [Adamo, RG]

$$
\mathcal{M}^{cl}_{4,(n+1)}(p_A, p_B, q) = \begin{cases} K_{cl}(p_A, p_B, q) & \text{if } n = 0 \\ \frac{1}{n+1}\int \hat{d}^4 l \, K_{cl}(p_A, p_B, l) G_{cl}(p_A, p_B, l) \mathcal{M}^{cl}_{4,(n)}(p_A, p_B, q - l) & \text{if } n \geq 1 \end{cases} ;
$$

where the two-body propagator is replaced by its on-shell version

$$
G_{cl}(p_A, p_B, l) = \hat{\delta}(2l \cdot p_A)\hat{\delta}(2l \cdot p_B) ;
$$

and $\Gamma(n)$ is the number of classical two-massive particle irreducible diagrams

Going to impact parameter space

$$\tilde{f}(x_q) \equiv \int \hat{d}^4 q\, \hat{\delta}(2p_A \cdot q)\, \hat{\delta}(2p_B \cdot q)\, e^{i(q\cdot x_q)/\hbar} f(q)\,,$$

the classical BSE becomes

$$\tilde{\mathcal{M}}^{cl}_{4,(n+1)}(p_A, p_B, x_\perp) = \begin{cases} \tilde{\mathcal{K}}_{cl}(p_A, p_B, x_\perp) & \text{if } n = 0 \\ \frac{1}{n+1}\tilde{\mathcal{K}}_{cl}(p_A, p_B, x_\perp)\tilde{\mathcal{M}}^{cl}_{4,(n)}(p_A, p_B, x_\perp) & \text{if } n \geq 1 \end{cases},$$

which means that the final solution exponentiates exactly

$$\tilde{\mathcal{M}}^{cl}_{4}(p_A, p_B, x_\perp) = e^{\tilde{\mathcal{K}}_{cl}(p_A, p_B, x_\perp)}\,.$$

Natural generalization for spinning particles! [Adamo,RG; Haddad]

Exponentiation of the classical kernel: an exact solutio

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the classical BSE becomes

$$\tilde{\mathcal{M}}^{\rm cl}_{4;(n+1)}(p_A; p_B; x_\perp) = \begin{cases} \tilde{K}_{\rm cl}(p_A; p_B; x_\perp) & \text{if } n = 0 \\ \frac{1}{n+1} \tilde{K}_{\rm cl}(p_A; p_B; x_\perp)\, \tilde{\mathcal{M}}^{\rm cl}_{4;(n)}(p_A; p_B; x_\perp) & \text{if } n \geq 1 \end{cases};$$

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The analytic structure (poles, etc.) in momentum space arise completely fro

$$i\mathcal{M}^{\rm cl}_4(p_A; p_B; q_\perp) = \frac{4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}}{\hbar^2} \int d^2 x_\perp\, e^{-iq_\perp \cdot x_\perp/\hbar} \left(e^{\tilde{K}_{\rm cl}(p_A; p_B; x_\perp)} - 1 \right);$$

An example: classical kernel for spinless particles at 2PM

We can consider for example the classical kernel up to 2 PM

$$\tilde{K}^{\mathrm{cl};>}(p_A, p_B, x_\perp) = \frac{i}{\hbar}\Bigg[-2G_N \log(\mu_{IR}|x_\perp|)\, m_A m_B \frac{2y^2-1}{\sqrt{y^2-1}} + \frac{3}{4} G_N^2 m_A m_B (m_A+m_B) \frac{5y^2-1}{\sqrt{y^2-1}} \frac{1}{|x_\perp|} \Bigg],$$

which encodes the conservative dynamics of two spinless particles.

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which encodes the conservative dynamics of two spinless particles.

Note that the motion is restricted to a plane and completely determined by the conserved quantities $(\mathcal{E}; L)$!

$$\mathcal{E} := \frac{E - m_A - m_B}{\mu}; \qquad L = p_1(E; m_A; m_B)|x_\perp|; \qquad y = \frac{E^2 - m_A^2 - m_B^2}{2m_A m_B};$$

The Hamilton-Jacobi action from amplitudes (I)

Since $E > 0$ for scattering orbits and $E < 0$ for bound orbits, we introduce the superscript $>$ (resp. $<$) to denote an expression valid for scattering orbits (resp. bound orbits). How do we perform an analytic continuation?

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Natural connection of the kernel with the scattering Hamilton-Jacobi action

$$\widetilde{K}^{>}_{cl}(p_A; p_B; x_?) = \frac{i}{\hbar} I^{>}(E; L) ; \quad I_r^{>}(E; L) = \oint_{C^>} dr\, p_r(r; E; L) + L ;$$

where p_r is the radial momentum and $C^>$ is the contour of integration for scattering orbits. This is the "amplitude-action" relation! [Bern et al.; Damgaard,Plante,Vanhove; Kol,O'Connell,Telem; Adamo,RG]

The Hamilton-Jacobi action from amplitudes (II)

There is a remarkable analytic continuation between the boundary conditions for scattering and bound planar orbits [Kälin,Porto]

$$\int_{C_r^>} = 2\int_{r_m(E;L)}^{\infty} \; ; \qquad \int_{C_r^<} = 2\int_{r_-(E;L)}^{r_+(E;L)} \; ;$$

$$r_-(E;L) \overset{E\leq 0}{=} r_m(E;L) \; ; \qquad r_+(E;L) \overset{E\leq 0}{=} r_m(E;-L) \; ;$$

thanks to which p_r is invariant under $L \to -L$

$$I_r^<(E<0;L) = I_r^>(E<0;L) - I_r^>(E<0;-L) \; .$$

Alternatively, analytically continue in the rapidity at fixed L [Adamo, RG; Di Vecchia, Heissenberg, Russo, Veneziano]

$$I_r^<\left(\sqrt{y^2-1};L\right) = I_r^>\left(i\sqrt{1-y^2};L\right) + I_r^>\left(-i\sqrt{1-y^2};-L\right) \; .$$

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thanks to which (p_r is invariant under $L \to\ L$)

$$I_r^<(E<0;L) = I_r^>(E<0;L)\quad I_r^>(E<0;\ L) :$$

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This picture generalize to the case of aligned-spin particles $a_1;a_2$, since the motion still remains on the equatorial plane. [Kalin,Porto]

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Today: What happens when $\vec{L}$ and $\vec{a}$ are oriented in generic directions?

Hamiltonian for geodesics in Kerr

Let's consider the generic orbit in a Kerr black hole of spin a for a massive point particle with orbital angular momentum $\vec{L} \cdot \vec{a} \neq |\vec{L}||\vec{a}|$

which are described by the geodesic Hamiltonian $H(x; p) = 1/2\, g^{\mu\nu}_{\mathrm{Kerr}} p_\mu p_\nu$ in some convenient Boyer-Lindquist coordinates (t, r, θ, ϕ).

The Hamilton-Jacobi action for generic Kerr orbits

Using the integrability of Kerr, we can generalize the H-J action for generic orbits of a massive spinless probe in a Kerr black hole [Carter; RG,Shi]

$$\boxed{I := I_r + I} \; ; \qquad I_r = \int_{C_r} p_r \, dr \; ; \qquad I = \int_{C} p \; d \; ;$$

with the radial (resp.polar) momentum p_r (resp. p) and contour C_r (resp. C)

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The momenta are expressed in terms of radial $R(r)$ and polar potential $(\ )$

$$p_r \propto \frac{\sqrt{R(r)}}{(\ r)} ; \qquad p\ \propto \sqrt{(\ )} ;$$

which are functions of the conserved charges $(E; L; Q)$, with the Carter constant being a measure of the motion of the particle o the equatorial plan

$$Q = |\tilde{L}|^2 \quad L^2 \quad |p|^2 (a \quad \hat{r})^2 :$$

Radial action for generic Kerr orbits (I)

The radial action for unbound and bound orbits is, respectively,

$$I_r^> = 2\int_{r_m(\hat{E},L,a,L_Q)}^{+\infty} p_r(\hat{E},L,a,L_Q)\,dr\,; \qquad I_r^< = 2\int_{r_-(\hat{E},L,a,L_Q)}^{r_+(\hat{E},L,a,L_Q)} p_r(\hat{E},L,a,L_Q)\,dr\,;$$

which are expressed in terms of $L_Q = \sqrt{Q + L^2}$ and $\hat{E} = (E^2 - m^2)/m^2$.

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which are expressed in terms of $L_Q = \sqrt{Q + L^2}$ and $\mathcal{E} = (E^2 - m^2)/m^2$.

With our radial potential we find the analytic continuation [RG,Shi]

$$r_-(E;L;a;L_Q) \overset{\mathcal{E}\le 0}{=} r_m(E;L;a;L_Q)\,; \qquad r_+(E;L;a;L_Q) \overset{\mathcal{E}\le 0}{=} r_m(E;-L;-a;-L_Q)\,;$$

which implies p_r is invariant under $(a;L;L_Q) \to (-a;-L;-L_Q)$

$$\boxed{I_r^<(E;L;a;L_Q) \overset{\mathcal{E}\le 0}{=} I_r^>(E;L;a;L_Q) - I_r^>(E;-L;-a;-L_Q)}\,:$$

Radial action for generic Kerr orbits (II)

The radial action for scattering orbits is

$$I_r^> = \frac{\sqrt{E}}{\sqrt{M^2 - a^2}} \int_0^{u_m} \frac{du}{u^2} \prod_{j=1}^{4} \left(1 - \frac{u}{u_j}\right)^{-\frac{1}{2}} \left(\frac{1}{u_B - u} - \frac{1}{u_A - u}\right);$$

where we have defined the (inverse of) radial roots $\{u_j\}_{j=1,\dots,4}$ and

$$R(u) = -\frac{a^2 Q}{u^4} \prod_{j=1}^{4} (u - u_j) ; \quad u_A = \frac{M + \sqrt{M^2 - a^2}}{a^2} ; \quad u_B = \frac{M - \sqrt{M^2 - a^2}}{a^2} :$$

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The exact all-order expression is (ϵ IR-regulator >0) [RG, Shi]

$$I_r^{>;\epsilon} = \frac{m\sqrt{E}}{u_m^{1-\epsilon}\sqrt{M^2-a^2}} \frac{\Gamma(3/2)\,\Gamma(-1+\epsilon)}{\Gamma(1/2+\epsilon)} \left\{ \frac{1}{u_B} F_D^{(4)}\left(-1; 1; -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}+\epsilon; \frac{u_m}{u_B}, \frac{u_m}{u_1}, \frac{u_m}{u_2}, \frac{u_m}{u_3}\right) - (u_B \leftrightarrow u_A) \right\}.$$

Polar action for generic Kerr orbits (I)

The most general polar action is [Kapec,Lupsasca; RG, Shi]

$$\int_{C} = 2n \int_{=2} + {}_{\text{in}} \int_{=2}^{\text{in}} \quad {}_{\text{out}} \int_{=2}^{\text{out}} ;$$

where $_{\text{in}}$ (resp. $_{\text{out}}$) is the initial (resp. nal) polar angle, n is the number of turning points, are the roots of the polar potential and $_{\text{in}}$; $_{\text{out}}$ = 1.

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For generic Kerr orbits of interest for the amplitude community,

$$I = 2n \int_{=2} p \quad {}_{out} \int_{=2}^{out} p ;$$

where we set $_{in}$ = $= 2$ and $(n^{>} ; {}^{>}_{out}) = (1 ; \; 1)$, $(n^{<} ; {}^{<}_{out}) = (2; +1)$ are xed by the PM expansion of the equations of motion.

Polar action for generic Kerr orbits (II)

We obtain the exact all order polar action

$$I^{>} = \frac{}{2}{}^{p}\,\overline{QU_+}\,F_D^{(2)}\ \frac{1}{2};{}^{n}\ 1;\ \frac{1}{2}{}^{o};2;U_+;\frac{U_+}{U}$$

$$+\ {}^{q}\,\overline{QU^{>}_{out}}\,F_D^{(3)}\ \frac{1}{2};{}^{n}\ 1;\ \frac{1}{2};\ \frac{1}{2}{}^{o};\frac{3}{2};U^{>}_{out};\frac{U^{>}_{out}}{U_+};\frac{U^{>}_{out}}{U}\quad;$$

where $U = \cos^2(\ )$ and of the outgoing angle $U^{>}_{out} = \cos^2(\ _{out})$ is

$$U^{>}_{out} = U_+\ sn^2\ \ X^{>}_0\ \frac{U}{U_+}\quad;$$

$$X^{>}_0 = \ 4u_m{}^{p}\,\overline{U\ \ a^2}\,F_D^{(3)}\ \ 1;{}^{n}\frac{1}{2};\frac{1}{2};\frac{1}{2}{}^{o};\frac{3}{2};\frac{u_m}{u_1};\frac{u_m}{u_2};\frac{u_m}{u_3}\quad:$$

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$$+\ q\ \overline{QU^{>}_{out}}\, F_D^{(3)} \left(\frac{1}{2};\ n\ 1;\ \frac{1}{2};\ \frac{1}{2}\ o\ ; \frac{3}{2}; U^{>}_{out}; \frac{U^{>}_{out}}{U_+}; \frac{U^{>}_{out}}{U} \right) ;$$

where $U = \cos^2(\ )$ and of the outgoing angle $U^{>}_{out} = \cos^2(\ _{out})$ is

$$U^{>}_{out} = U_+ \, sn^2 \ X^{>}_0 \ \frac{U}{U_+} ;$$

$$X^{>}_0 = 4u_m\ p\ \overline{U\ a^2}\, F_D^{(3)} \left(1;\ n\ \frac{1}{2}; \frac{1}{2}; \frac{1}{2}\ o\ ; \frac{3}{2}; \frac{u_m}{u_1}; \frac{u_m}{u_2}; \frac{u_m}{u_3} \right) :$$

The polar action obeys the analytic continuation [RG, Shi]

$$\boxed{I^{<}(E; L; a; L_Q; n^{>};\ ^{>}_{out}) \overset{E \leq 0}{=} I^{>}(E; L; a; L_Q; n^{<};\ ^{<}_{out})} :$$

Polar action for generic Kerr orbits (II)

We obtain the exact all order polar action

$$I^{>} = \frac{}{2}\sqrt{QU_+}\,F_D^{(2)}\left(\frac{1}{2};\left\{1;\ \frac{1}{2}\right\};2;U_+;\frac{U_+}{U}\right)$$
$$+\sqrt{QU^{>}_{out}}\,F_D^{(3)}\left(\frac{1}{2};\left\{1;\ \frac{1}{2};\ \frac{1}{2}\right\};\frac{3}{2};U^{>}_{out};\frac{U^{>}_{out}}{U_+};\frac{U^{>}_{out}}{U}\right);$$

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Now it's time to have fun and compute scattering and bound observables!

Azimuthal de ection angle for generic Kerr orbits

Once we have the H-J action, we can obtain the azimuthal de ection angle by di erentiation over L

$$+ \quad = \frac{\partial}{\partial L} = \frac{\partial_r}{\partial L} \quad \frac{\partial}{\partial L}$$

which gives in the perturbative expansion in $G_N M$ $(a \quad G_N M)$

$$= \frac{2G_N MmL(2E+1)}{\sqrt{E} L_Q^2} + \frac{3 \quad G_N^2 M^2 m^2 L(5E+4)}{4L_Q^3}$$
$$+ \, a\sqrt{E+1} \quad \left(\frac{4G_N Mm^2\sqrt{E}(L_Q^2 \quad 2L^2)}{L_Q^4} \right) + \cdots$$

Azimuthal deflection angle for generic Kerr orbits

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Matches known result in the equatorial limit $Q \to 0$ [Damgaard, Hoogeveen, Luna, Vines], but the expansion is new for generic orbits! [RG,Shi]

Polar de ection angle for generic Kerr orbits

From the r equations of motion we get the polar de ection angle in the perturbative expansion

$$\frac{}{\sqrt{Q}} = \frac{2G_N M m(2E+1)}{\sqrt{E} L_Q^2} \quad \frac{3 \; G_N^2 M^2 m^2 (5E+4)}{4 L_Q^3} + a \sqrt{E+1} \left(\frac{8 G_N M m^2 L \sqrt{E}}{L_Q^4} \right) + \cdots$$

which vanish identically in the equatorial limit.

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which vanish identically in the equatorial limit.

Matches expectations that when $a \to 0$ we get $\xrightarrow{a \to 0}$ $(L/\sqrt{Q})$ at the lowest order (the motion is happening on an inclined plane)!

Time delay for generic null Kerr orbits

The time delay is obtained by di erentiating over the energy E in the HJ action. For generic null geodesics with xed b relative to an observer with $b' \gg b$ but at the same energy $E' = E$ [Camanho, Edelstein, Maldacena, Zhiboedov; Accettulli Huber, Brandhuber, De Angelis, Travaglini]

$$
\begin{aligned}
T &= \left.\frac{\partial I}{\partial E}\right|_{L_Q;E} - \left.\frac{\partial I}{\partial E}\right|_{L_Q^0 \gg L_Q;E^0=E} \\
&= 4G_N M \log\left(\frac{L_Q^0}{L_Q}\right) + \frac{15\pi G_N^2 M^2 E}{2L_Q} + \frac{64 G_N^3 M^3 E^2}{L_Q^2} \\
&\quad - \frac{aLE}{L_Q^4}\left(8G_N M L_Q^2 + 15\pi G_N^2 M^2 E L_Q + 256 G_N^3 M^3 E^2\right) + \dots
\end{aligned}
$$

which in the equatorial limit matches [Bautista, Guevara, Kavanagh, Vines]

Fundamental frequencies for Kerr orbits

The H-J action provides an intrinsic definition of the fundamental frequencies for the bound motion $\omega_r; \omega_\theta; \omega_\phi$ via the action-angle representation [Schmidt]

$$\omega_r = \frac{1}{\Upsilon}\frac{\partial J_\theta}{\partial Q}; \quad \omega_\theta = \frac{1}{\Upsilon}\frac{\partial J_r}{\partial Q}; \quad \omega_\phi = \frac{1}{\Upsilon}\left(\frac{\partial J_r}{\partial L}\frac{\partial J_\theta}{\partial Q} - \frac{\partial J_r}{\partial Q}\frac{\partial J_\theta}{\partial L}\right);$$

with $\Upsilon := \frac{\partial J_r}{\partial H}\frac{\partial J_\theta}{\partial Q} - \frac{\partial J_r}{\partial Q}\frac{\partial J_\theta}{\partial H}$, $J_\phi = L$, $J_r = \oint p_r \, dr = I_r^<$, $J_\theta = \oint p_\theta \, d\theta = I_\theta^{<(1)}$.

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From the amplitude perspective we want bound observables invariant under the choice of the time coordinate, i.e. the frequency ratios [RG, Shi]

$$K^{\phi r} := \frac{\omega_\phi}{\omega_r} = \frac{\partial J_r/\partial Q}{\partial J_\theta/\partial Q}\frac{\partial J_\theta}{\partial L} - \frac{\partial J_r}{\partial L}; \qquad K^{\theta r} := \frac{\omega_\theta}{\omega_r} = -\frac{\partial J_r/\partial Q}{\partial J_\theta/\partial Q}.$$

corresponding to the precession of the periastron and of the orbital plane.

Frequency ratios for Kerr orbits

The periastron advance is [RG,Shi]

$$K^{r} = 1 + \frac{3G_N^2 M^2 m^2 (5E + 4)}{4L_Q^2} + \frac{a G_N^2 M^2 m^3 \sqrt{E + 1}\,(-L_Q - 3L)(5E + 2)}{L_Q^4}$$
$$+ \frac{3a^2 G_N^2 M^2 m^4}{32 L_Q^6}\Big[L^2 (445E^2 + 416E + 40) - L_Q (L_Q + 2L)(85E^2 + 80E + 8) \Big]$$

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The precession of the orbital plane is

$$K^{r} = 1 + \frac{3G_N^2 M^2 m^2 (5E+4)}{4L_Q^2} \quad \frac{3aG_N^2 M^2 m^3 L \sqrt{E+1}\,(5E+2)}{L_Q^4} + \frac{3a^2 G_N^2 M^2 m^4}{32 L_Q^6}\left[L^2(445E^2+416E+40) \quad L_Q^2(85E^2+80E+8)\right] :$$

As expected frequencies become degenerate in the spinless case $a \to 0$!

Radiative observables in the S-matrix formalism (I)

How can the previous picture be generalized in the presence of radiation?
Consider the 5-pt recursion with the emission of a positive energy graviton

and apply the symmetrization procedure [Adamo, RG, Ilderton]

A similar recursion holds for the emission of n gravitons.

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A similar recursion holds for the emission of n gravitons.
Can we find an exact solution from the resummation?

Radiative observables in the S-matrix formalism (II)

The classical 5 pt recursion relation is of the form

$$M^{cl;\ 1\ 1}_{5;(1)}(p_A;p_B;q_1;q_2)\Big|_{E_{k_1^0}>0} = K_R^{cl;\ 1\ 1}(p_A;p_B;q_1;q_2)\,;$$

$$M^{cl;\ 1\ 1}_{5;(n+1)}(p_A;p_B;q_1;q_2)\Big|_{E_{k_1^0}>0}$$

$$= \frac{1}{n+1}\Big[\int \hat{d}^4 l\, K^{cl}(p_A;p_B;l)G^{cl}(p_A;p_B;l)M^{cl;\ 1\ 1}_{5;(n)}(p_A;p_B;q_1\quad l;q_2+l)$$

$$+ \int \hat{d}^4 l\, K_R^{cl;\ 1\ 1}(p_A;p_B;q_1\quad l;q_2+l)G^{cl}(p_A;p_B;l)M^{cl}_{4;(n)}(p_A;p_B;l)\Big]\,;$$

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We can then solve the recursion by working in impact parameter space

$$\tilde{f}(x_{q_1};x_{q_2}) \quad \int \hat{d}^4 q_1 \hat{d}^4 q_2\, \hat{}(2p_A\ \ q_1)\, \hat{}(2p_B\ \ q_2)$$

$$\hat{}(q_1+q_2\ \ k)\, e^{i(q_1\ x_{q_1}+q_2\ x_{q_2})/\hbar} f(q_1;q_2)\,;$$

with momentum transfers $(q_1;q_2)$ of initial and nal pair of massive states.

Radiative observables in the S-matrix formalism (III)

The classical S-matrix is [Cristofoli, RG, Moynihan, O'Connell, Ross, Sergo
White; Britto, RG, Jehu; Di Vecchia, Heissenberg, Russo, Veneziano]

$$\tilde{S}^{cl} \qquad {}_{E_{k_1};\ldots;E_{k_N}>0} \qquad e^{i\tilde{\chi}^{cl}(p_A;p_B;x_{q_1};x_{q_2})}\, e^{\sum \int d(k)\,\tilde{\alpha}^{cl}_{R}(p_A;p_B;x_{q_1};x_{q_2};k)\,a^{y}(k)+h.c.} ;$$

which includes a coherent state of gravitons representing the classical gravitational wave (see D.Kosower's talk).

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Open problem: can we understand the analytic continuation of the waveform

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- Future directions: understand the amplitude-action relation for generic spin orientations, generalized Carter constant for spinning particles, analytic continuation for radiative observables, self-force from amplitudes, . . .

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