Conservative Binary Dynamics from Gravitational Tail Emission Processes

[& Anomalies in Classical Amplitudes]

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[GLA, A Müller, S Foffa, R Sturani - arXiv:2307.05327]

Gravitational Waves meet Amplitudes in the Southern Hemisphere

São Paulo, 21 Aug 2023

Hierarchy of scales and the **method of regions** for bound binary systems
[Goldberger and Rothstein, Phys. Rev. D **73**, 104029 (2006)]

$$\text{Orbital scale:} \qquad v^2 \sim \frac{G_N m}{r} \quad \Rightarrow \quad r_s \sim 2G_N m \sim r v^2$$

$$\text{GW scale:} \qquad \lambda \sim \frac{r}{v}$$

$$\Rightarrow \qquad r_s \;\sim\; r v^2 \;\sim\; \lambda v^3$$

In the nonrelativistic regime, $v \ll 1$, hierarchy of scales:

$$r_s \;\ll\; r \;\ll\; \lambda$$

Method of regions: $$h_{\mu\nu} = \underbrace{H_{\mu\nu}}_{\text{potential modes}} + \underbrace{\bar{h}_{\mu\nu}}_{\text{radiative modes}}$$

$$H_{\mu\nu} \;:\; \text{off-shell modes scaling as } (k^0, \mathbf{k}) \sim (v/r, 1/r)$$

$$\bar{h}_{\mu\nu} \;:\; \text{on-shell modes scaling as } (k^0, \mathbf{k}) \sim (v/r, v/r)$$

Integrating out the potential modes:

$$e^{iS_{\rm eff}[x_a,\bar{h}_{\mu\nu}]} = \int \mathcal{D}H_{\mu\nu}\exp\{iS_{\rm EH+GF}[H_{\mu\nu}+\bar{h}_{\mu\nu}] + iS_{\rm pp}[x_a(t), H_{\mu\nu}+\bar{h}_{\mu\nu}]\}$$

$$\Rightarrow S_{\rm eff} = \frac{1}{2}\int d^4x\, T^{\mu\nu}\bar{h}_{\mu\nu}$$

Multipole expansion, $\lambda \gg r$, makes $S_{\rm eff} \to S_{\rm mult}$:
[Goldberger and Ross, Phys. Rev. D **81**, 124015 (2010)]

$$S_{\rm mult} = -E\int d\tau - \frac{1}{2}\int dx^\mu\, L_{ab}\omega_\mu^{ab} + \frac{1}{2}\sum_{n=0}^{\infty}\int d\tau c_n^{(I)} I^{aba_1\cdot a_n}(\tau)\nabla_{a_1}\cdots\nabla_{a_n}E_{ab}(x)$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\int d\tau c_n^{(J)} J^{aba_1\ldots a_n}(\tau)\nabla_{a_1}\cdots\nabla_{a_n}B_{ab}(\tau)$$

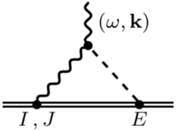
GW observables can be computed, e.g.:

$$P = \frac{1}{2T}\sum_{\rm pol}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,|\mathcal{A}(\omega,\mathbf{k})|^2$$

Nonlinear Effects: Emission and Radiation-Reaction

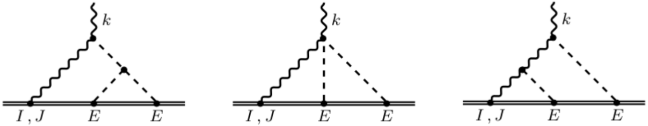
Emission diagrams

$i\mathcal{A}_{\rm tail}(\omega, \mathbf{k}) =$

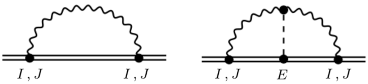


- IR and UV divergences
- Renormalization group evolution

[GLA, Foffa, Sturani, PRD **104**, 084095 (2021)]



Self-energy diagrams



- $\mathrm{Im}(S_{\rm self}) \Rightarrow$ Energy flux
- $\mathrm{Re}(S_{\rm self}) \Rightarrow$ Conservative contributions

Computed for arbitrary multipole moments

[GLA, Foffa, Sturani, Phys. Rev. D **104**, 124075 (2021)]

Consider an arbitrary source of size r emitting GWs with wavelength λ.

Assumption: compact source $\quad\Rightarrow\quad \lambda \gg r\,.$

In this *long wavelength* regime, the interaction of the system with gravity is given by a multipolar coupling through the following effective action:

$$S_0 = \int dt \left[\frac{1}{2} E h_{00} - \frac{1}{2} J^{b|a} h_{0b,a} - \sum_{r\geq 0} \left(c_r^{(I)} I^{ijR} \partial_R R_{0i0j} + \frac{c_r^{(J)}}{2} J^{b|iRa} \partial_R R_{0iab} \right) \right] ,$$

$$\text{with} \qquad c_r^{(I)} = \frac{1}{(r+2)!}\,, \qquad c_r^{(J)} = \frac{2(r+2)}{(r+3)!}\,.$$

$\Rightarrow$ Radiation is sourced by the multipole moments I^{ijR} and $J^{b|iRa}$. $[R = i_1 \dots i_r]$

We work with standard GR in the harmonic gauge $(\Gamma^\mu \equiv g^{\rho\sigma}\Gamma^\mu_{\rho\sigma}.)$

$$S_{\text{bulk}} = 2\Lambda^2 \int \mathrm{d}^{d+1}x \sqrt{-g} \left[R(g) - \frac{1}{2}\Gamma_\mu \Gamma^\mu \right] ,$$

where $\Lambda^{-2} \equiv 32\pi G_N$.

The **Classical Gravitational Field** at a spacetime position x is given by

$$\langle h_{\mu\nu}(x)\rangle = \int \mathcal{D}h\, e^{iS[h]} h_{\mu\nu}(x)\,.$$

The most relevant role is played by the **trace-reversed** quantity $\bar{h}_{\mu\nu}$, defined by

$$\bar{h}_{\mu\nu} = P_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta}\,, \qquad \text{with} \qquad P_{\mu\nu}{}^{\alpha\beta} = \frac{1}{2}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} - \eta_{\mu\nu}\eta^{\alpha\beta}\right)\,.$$

When interactions are considered, the field $h_{\mu\nu}$ will have the generic form

$$\langle h_{\mu\nu}(x)\rangle = \int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 - (\omega + i\mathsf{a})^2} \times i\mathcal{A}_{\mu\nu}(\omega, \mathbf{k})\,.$$

This equation defines the **Gravitational Scattering Amplitude** $i\mathcal{A}_{\mu\nu}$.

In particular, in direct space, this takes the form

$$\langle \bar{h}_{\mu\nu}(x)\rangle = -16\pi G_N \int d^{d+1}x' G_R(t-t', \mathbf{x}-\mathbf{x}') T_{\mu\nu}(x')\,.$$

Hence, we have the identification

$$T_{\mu\nu}(x) \qquad \sim \qquad i\bar{\mathcal{A}}_{\mu\nu}(\omega, \mathbf{k})\,.$$

It follows directly from the trace-reversed version of $\langle h_{\mu\nu}(x)\rangle$ that

$$\partial^\mu \langle \bar{h}_{\mu\nu}(x)\rangle = -\int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{e^{ik\cdot x}}{\mathbf{k}^2 - (\omega + i\mathtt{a})^2} \times k^\mu \bar{\mathcal{A}}_{\mu\nu}(\omega, \mathbf{k})\,.$$

Hence, we immediately see that, if the condition $k^\mu \bar{\mathcal{A}}_{\mu\nu} = 0$ is satisfied, we have

$$k^\mu \bar{\mathcal{A}}_{\mu\nu}(\omega, \mathbf{k}) = 0 \quad \Longrightarrow \quad \partial^\mu \langle \bar{h}_{\mu\nu}(x)\rangle = 0 \quad \text{and} \quad \partial^\mu T_{\mu\nu}(x) = 0\,.$$

The harmonic gauge condition: $\Gamma^\mu = 0$.

- Pertubatively in G_N, $\mathcal{O}(G_N^n)$: $\quad \partial^\mu \bar{h}^{(n)}_{\mu\nu} = \lambda[h^{(n-1)}, h^{(n-2)}, \ldots, h^{(1)}]$.

From this, it is easy to derive the important result: $k^\mu \bar{\mathcal{A}}_{\mu\nu}(\omega, \mathbf{k}) \propto (\omega^2 - \mathbf{k}^2)$.

Thus: Physically relevant amplitudes $i\mathcal{A}_{\mu\nu}$ are such that, on-shell ($\mathbf{k} \equiv \omega\hat{\mathbf{n}}$)

$k^\mu \bar{\mathcal{A}}_{\mu\nu}(\omega, \omega\hat{\mathbf{n}}) = 0 \quad \Rightarrow \quad$ This is the statement of the **Ward identity**.

On-shell amplitudes: Useful to build h^{TT}_{ij} in the far field approximation, $D \gg r$:

$$h^{TT}_{ij}(x) \equiv \langle \bar{h}^{TT}_{ij}(x)\rangle = -\frac{1}{4\pi D}\Lambda_{ijkl} \int \frac{d\omega}{2\pi}\, i\bar{\mathcal{A}}_{kl}(\omega, \omega\mathbf{n}) e^{-i\omega t_{\rm ret}}\,.$$

The computation of the gravitational amplitudes for the Energy E and angular momentum $J^{b|a}$, which in $d=3$ can be represented by $L_i = \frac{1}{2}\epsilon_{ijk}J^{j|k}$, yields

$$i\bar{\mathcal{A}}_{00}^{(LO)} = 16\pi G_N E(\omega)\,, \quad i\bar{\mathcal{A}}_{0k}^{(LO)} = 64\pi G_N i k_i \epsilon_{ijk} L_j(\omega)\,, \quad i\bar{\mathcal{A}}_{kl}^{(LO)} = 0\,.$$

The gauge condition is verified in this case, following from

$$\omega E(\omega) = 0\,, \qquad \omega L_i(\omega) = 0\,.$$

$\Rightarrow$ Satisfied at this perturbative order by admitting that E, L_i are conserved.

The **leading-order electric and magnetic multipole** amplitudes read

$$i\bar{\mathcal{A}}_{\mu\nu}^{(I)} = -16\pi G_N (-i)^r c_r^{(I)} k_R I^{ijR}(\omega) a_{\mu\nu,ij}\,,$$
$$i\bar{\mathcal{A}}_{\mu\nu}^{(J)} = -8\pi G_N (-i)^r c_r^{(J)} k_R k_a J^{b|iRa}(\omega) b_{\mu\nu,ib}\,.$$

$$a_{00,ij} = k_i k_j\,, \qquad a_{0k,ij} = -\omega k_j \delta_{ik}\,, \qquad a_{kl,ij} = \omega^2 \delta_{ik}\delta_{jl}\,.$$
$$b_{00,ib} = 0\,, \qquad b_{0k,ib} = k_i \delta_{bk}\,, \qquad b_{kl,ib} = -\omega(\delta_{ik}\delta_{bl} + \delta_{il}\delta_{bk})\,.$$

$\Rightarrow$ **In this case, the Ward identities are trivially satisfied.**

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The M-tail amplitude is divergent for $d \to 3$ and its radiative, TT, on-shell part is

$$\mathcal{A}_{ij}^{(e,\text{tail})TT} = 32\pi(-i)^r \omega^3 c_r^{(I)} G_N^2 E \Lambda_{ij\,,kl}^{TT} k_R I^{klR}(\omega) \times \left(\frac{1}{\epsilon} - \kappa_{r+2} + \frac{\log x}{2} \right) ,$$
$$\mathcal{A}_{ij}^{(m,\text{tail})TT} = 32\pi(-i)^r \omega^2 c_r^{(J)} G_N^2 E \Lambda_{ij\,,kl}^{TT} k_R k_n J^{n|kRl}(\omega) \times \left(\frac{1}{\epsilon} - \pi_{r+2} + \frac{\log x}{2} \right) ,$$

with

$$\kappa_l = \frac{2l^2 + 5l + 4}{l(l+1)(l+2)} + \sum_{i=1}^{l-2} \frac{1}{i} \,, \qquad \pi_l = \frac{l-1}{l(l+1)} + \sum_{i=1}^{l-1} \frac{1}{i} \,,$$

where $\epsilon \equiv d - 3$, $x \equiv -e^{\gamma}\omega^2/\mu\pi$, and length scale μ^{-1} defined by $G_d = G_N \mu^{-\epsilon}$.

In particular, as expected, for the full amplitudes, we find

$$k^{\mu} \mathcal{A}_{\mu\nu}^{(e,m;\text{tail})}(\omega, \omega\mathbf{n}) = 0 \,.$$

For the angular momentum failed tail, the radiative, TT, on-shell part is given by

$$
i\mathcal{A}_{ij}^{(e,J-\text{ftail})TT} = \frac{32\pi(-i)^r c_r^{(I)}\omega^2}{(r+1)(r+2)(r+3)(r+4)} G_N^2 \Lambda_{ij,(kl)}^{TT} J^{m|n} k_{R-1} I^{pRk}(\omega)
\times \left\{ k_n \left[2(r^2+4r+6)\delta_{lm}k_p k_{i_1} - r(r^2+5r+10)\delta_{lp}\delta_{i_1 m}\omega^2\right] + 24\delta_{0r}\delta_{lm}\delta_{np}k_{i_1}\omega^2 \right\}.
$$

From the full amplitude, we have

$$
k^\mu \bar{\mathcal{A}}_{\mu 0}(\omega, \omega\mathbf{n}) = 0 \,.
$$

$$
k^\mu \bar{\mathcal{A}}_{\mu l}(\omega, \omega\mathbf{n}) = (-i)^{r+1}\frac{c_r^{(I)}}{2\Lambda^4}\left(\frac{i\omega}{4}\right)\left[k_a\omega^2 J^{i|a} I^{iRl}(\omega)\int_{\mathbf{q}} \frac{q_R}{(\mathbf{q}^2-\omega^2)}\right].
$$

Hence, we notice that, since the integral in $\mathbf{q}$ is proportional to δ_R, this result vanishes on account of the tracelessness of I^{iRl}, unless $r=0$, in which

$$
k^\mu \bar{\mathcal{A}}_{\mu l}\Big|_{r=0} = 16\pi i G_N^2 k_a \omega^4 J^{i|a} I^{il}(\omega)\,.
$$

The presence of this “anomaly”can be linked to the term in blue above.

Variation of the Einstein-Hilbert plus gauge-fixing action, with metric expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, yields

$$S_{EH+GF} \sim \Lambda^2 \int d^4x \, (h\partial^2 h + h^2 \partial^2 h + \dots)$$
$$\Rightarrow \qquad \Box \bar{h}_{\mu\nu} = N_{\mu\nu}[h,h] + M_{\mu\nu}[h,h,h] + \dots .$$

Perturbative expansion in G_N, with $h^{(n)}$ denoting contributions of $\mathcal{O}(G_N^n)$, gives

$$\begin{aligned} \Box \bar{h}^{(1)}_{\mu\nu} &= 0 \,, \\ \Box \bar{h}^{(2)}_{\mu\nu} &= N_{\mu\nu}[h^{(1)}, h^{(1)}] \,, \\ \Box \bar{h}^{(3)}_{\mu\nu} &= N_{\mu\nu}[h^{(1)}, h^{(2)}] + M_{\mu\nu}[h^{(1)}, h^{(1)}, h^{(1)}] \,, \\ & \dots . \end{aligned}$$

$\Rightarrow$ Explicit check for $h^{(\mathrm{J-ftail})}_{\mu\nu}$, for the electric quadrupole, shows that it is indeed a solution of the perturbed Einstein's equation $\Box \bar{h}^{(2)}_{\mu\nu} = N_{\mu\nu}[h^{(1)}, h^{(1)}]$.

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The problem of solving perturbatively the Einstein field equations is translated into solving simultaneously the two equations

$$\Box \bar{h}_{\mu\nu} = \Lambda_{\mu\nu} \qquad \text{and} \qquad \partial^\mu \bar{h}_{\mu\nu} = 0 \,.$$

Once we obtain a particular solution $h^p_{\mu\nu}$ of $\Box \bar{h}_{\mu\nu} = \Lambda_{\mu\nu}$, we can always find a homogeneous solution $h^h_{\mu\nu}$ that precisely cancels the divergence in $\partial^\mu \bar{h}_{\mu\nu}$.

A general solution to the homogeneous equation, based on $\Box(\partial^\mu \bar{h}_{\mu\nu}) = 0$, can be always obtained in terms of four SFT tensors, say N_L, P_L, Q_L, R_L, such that

$$\Box \bar{h}^h_{\mu\nu} = 0 \qquad \text{and} \qquad \partial^\mu \bar{h}^h_{\mu\nu} = -\partial^\mu \bar{h}^p_{\mu\nu} \,.$$

$\Rightarrow$ General solution to the problem is $h_{\mu\nu} = h^p_{\mu\nu} + h^h_{\mu\nu}$.

In terms of amplitudes, we have ($\bar{\mathcal{A}}_\nu \equiv i k^\mu \bar{\mathcal{A}}_{\mu\nu}$)

$$\bar{\mathcal{A}}^0 = \sum_{l=0}^{\infty} i^l k_L N_L(\omega) \,,$$

$$\bar{\mathcal{A}}^i = \sum_{l=0}^{\infty} i^{l+1} k_i k_L P_L(\omega) + \sum_{l=1}^{\infty} \left[i^{l-1} k_{L-1} Q_{iL-1}(\omega) + \epsilon_{iab} i^l k_a k_{L-1} R_{bL-1}(\omega) \right] \,.$$

A solution $i\mathbf{a}_{\mu\nu}$ to this, corresponding to the homogeneous solution and being such that $k^\mu \bar{\mathbf{a}}_{\mu\nu} = -k^\mu \bar{A}_{\mu\nu}$, reads

$$\bar{\mathbf{a}}_{00} = -\frac{i}{\omega}N(\omega) + ik_a\left[-\frac{i}{\omega}N_a(\omega) - \frac{1}{\omega^2}Q_a(\omega) - 3P_a(\omega)\right].$$

$$\bar{\mathbf{a}}_{0i} = \frac{i}{\omega}Q_i(\omega) + 3i\omega P_i(\omega) - \epsilon_{iab}\frac{k_a}{\omega}R_b(\omega) + \sum_{l=2}^{\infty} i^{l-1}k_{L-1}N_{iL-1}(\omega),$$

$$\bar{\mathbf{a}}_{ij} = -\delta_{ij}P(\omega) + \sum_{l=2}^{\infty} i^{l-1}\Big\{2\delta_{ij}k_{L-1}P_{L-1}(\omega) - 6k_{L-2}k_{(i}P_{j)L-2}(\omega)$$
$$- ik_{L-2}\left[-i\omega N_{ijL-2}(\omega) - 3\omega^2 P_{ijL-2}(\omega) - Q_{ijL-2}(\omega)\right] - 2k_{aL-2}\epsilon_{ab(i}R_{j)bL-2}(\omega)\Big\}.$$

For the electric J-failed tail, we derive

$$Q_{al} = 16\pi i G_N^2\omega^4 J^{i|(a}I^{l)i} \qquad \text{and} \qquad R_b = -8\pi i G_N^2\omega^4\epsilon_{bcd}J^{i|c}I^{id}.$$

From which the following results are obtained

$$\bar{\mathbf{a}}_{00} = 0$$
$$\bar{\mathbf{a}}_{0i} = -8\pi i G_N^2\omega^3 J^{b|k}(k_j\delta_{ib} - k_b\delta_{ij})I^{jk},$$
$$\bar{\mathbf{a}}_{ij} = -16\pi i G_N^2\omega^4 J^{m|(i}I^{j)m}.$$

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By performing cuts in self-energy diagrams, we can see how this type of diagram is related to the emission amplitudes present in the subdiagrams, e.g., notice that:
Self-energy for quadrupole-quadruple interaction from EFT methods

$$iS_{\text{eff}} = -\frac{1}{16\Lambda^2}\int \frac{d\omega}{2\pi}\omega^4 I_{ij}(\omega)I^*_{kl}(\omega)\int_{\mathbf{k}} \frac{1}{\mathbf{k}^2-\omega^2}\left(-\frac{i}{2}\right)$$
$$\times \frac{1}{2}\Big[\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{(d-1)}\delta_{ij}\delta_{kl}+\frac{2}{(d-1)\omega^2}(k_ik_j\delta_{kl}+k_kk_l\delta_{ij})$$
$$-\frac{1}{\omega^2}(k_ik_k\delta_{jl}+k_ik_l\delta_{jk}+k_jk_k\delta_{il}+k_jk_l\delta_{ik})+\frac{4}{c_d\omega^4}k_ik_jk_kk_l\Big]\,.$$

The content of the last two lines is precisely the d-dimensional sum of the physical polarizations $\epsilon_{ij}(\mathbf{k},h)$ over $h=+,\times$, computed on the mass-shell $|\mathbf{k}|^2=\omega^2$, so:

$$iS_{\text{eff}} = -\frac{1}{16\Lambda^2}\int \frac{d\omega}{2\pi}\omega^4 I_{ij}(\omega)I^*_{kl}(\omega)\int_{\mathbf{k}} \frac{1}{\mathbf{k}^2-\omega^2}\left(-\frac{i}{2}\right)\left[\sum_h \epsilon_{ij}(\mathbf{k},h)\epsilon^*_{kl}(\mathbf{k},h)\right]$$
$$=\frac{1}{2}\sum_h\int_{\mathbf{k}}\frac{d\omega}{2\pi}\left[i\mathcal{A}_0(\omega,\mathbf{k})\right]\left(\frac{-i}{\mathbf{k}^2-\omega^2}\right)\left[i\mathcal{A}_0(-\omega,-\mathbf{k})\right].$$

$\Rightarrow$ Product of two leading-order emission amplitudes.

More generally, for the gluing of two amplitudes $\mathcal{A}_{\mu\nu}$ and $\mathcal{B}_{\mu\nu}$, we have

$$\begin{aligned} iS_{\text{eff}} &= \frac{1}{2}\int_{\mathbf{k}} \frac{d\omega}{2\pi} \mathcal{A}_{\mu\nu}(\omega, \mathbf{k}) \mathcal{D}[h_{\mu\nu}, h_{\rho\sigma}] \mathcal{B}_{\rho\sigma}(-\omega, -\mathbf{k}) \\ &= \frac{1}{2}\int_{\mathbf{k}} \frac{d\omega}{2\pi} \mathcal{A}_{ij}^{TT}(\omega, \mathbf{k}) \mathcal{D}[h_{ij}, h_{kl}] \mathcal{B}_{kl}^{TT}(-\omega, -\mathbf{k}) \,. \end{aligned}$$

As a consequence of the Ward identity, the TT part alone of the emission amplitude is sufficient to reconstruct the self-energy diagram.

For the mass tail, gluing $\mathcal{A}_{\mu\nu}^{(\text{tail})}$ to $\mathcal{A}_{\mu\nu}^{(LO)}$, we obtain

$$\begin{aligned} S_{\text{eff}}^{(e,\text{tail})} = -G_N^2 E \frac{2^{r+2}(r+3)(r+4)}{(r+1)(r+2)(2r+5)!} \\ \times \int \frac{d\omega}{2\pi} (\omega^2)^{r+3} I^{ijR}(\omega) I^{ijR}(-\omega) \left(\frac{1}{\epsilon} - \gamma_r^{(e)} + \log x \right) \,, \end{aligned}$$

This allows one to derive explicit relations between $\gamma_r^{(e)}$ and κ_{r+2}:

$$\gamma_r^{(e)} = \kappa_{r+2} - \left(\frac{1}{2} + \frac{1}{r+3} + \frac{1}{r+4} - \frac{1}{2} H_{r+\frac{5}{2}} - \log 2 \right) \,.$$

And similar to the magnetic case, connecting $\gamma_r^{(m)}$ and π_{r+2}.

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Since the J-failed tail presents no anomaly for $r > 0$ in the electric case, the computation of the self-energy from standard EFT methods or by gluing of amplitudes should result in the same expression. Indeed, we have:

$$iS_{\text{eff}}^{(J-\text{tail})} = G_N^2 \frac{2^r(12+50r+35r^2+10r^3+r^4)}{(r+1)^3(r+2)^3(r+3)(1+2r)(3+2r)(5+2r)(2r)!} \times J^{b|a} \int \frac{d\omega}{2\pi}\, \omega^{7+2r} I^{aiR}(\omega) I^{biR}(-\omega)\,. \qquad (r>0)$$

For the quadrupole case, $r = 0$, we must glue the corrected amplitude $i\mathcal{M}_{\mu\nu} = i\mathcal{A}_{\mu\nu} + i\mathbf{a}_{\mu\nu}$ previously obtained. In this case, we get:

$$iS_{\text{eff}}^{(r=0,J-\text{tail})} = -\frac{1}{30} G_N^2 J^{i|k} \int \frac{d\omega}{2\pi} I^{ij}(\omega) I^{jk}(-\omega)\omega^7\,.$$

$\Rightarrow$ Standard self-energy computation gives the coefficient $8/15$. This is obtained by just gluing $i\mathcal{A}_{\mu\nu}$, and thus, does not correspond to the physically correct value.

Interestingly, by setting $r \to 0$ in the generic formula above (for $r > 0$) gives the correct value $1/30$.

Conclusions

- We have studied Gravitational Scattering Amplitudes for leading-order processes, the simple tail, and the angular momentum failed tail;
- We have identified, for the first time, a classical anomaly in the quadrupole cases of the angular momentum failed tail;
- A fixing at the level of the amplitudes could be implemented by the introduction of counter-terms, within a consistent framework.

Besides this,

- We have learned how emission amplitudes could be used to compute self-energy diagrams;
- In this case, we were able to correct previous results for the conservative dynamics stemming from the angular momentum failed tail.

$\Rightarrow$ The work presented here is important to correctly account for the far-zone effects of back-scattering in the conservative dynamics of compact binary systems.

$\Rightarrow$ Particularly important in the completion of the 5PN dynamics.

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Thank you