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Random Matrix Theory: from Single– to Many–Body Quantum Chaos

Lecture II — Mathematical Physics Aspects

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Outline — Mathematical Physics Aspects

- more complicated random matrix models: transitions
- connection to harmonic analysis and spherical functions: Harish–Chandra's and Gelfand's
- ... and to exactly solvable systems
- supersymmetry as a new mathematical concept
- why is supersymmetry so powerful for random matrices and disordered systems ?

Random Matrix Models

Matrix Spaces

- no invariance under time–reversal: H Hermitean, $\beta = 2$
- invariance under time–reversal:
  - integer spin or half–odd spin with rotation symmetry:
    H real symmetric, $\beta = 1$
  - half–odd spin without rotation symmetry (Kramers):
    H Hermitean selfdual, $\beta = 4$

entries of H are real, complex, quaternionic for $\beta = 1, 2, 4$

for mathematicians: symmetric spaces

$\mathrm{U}(N)/\mathrm{O}(N)$	real symmetric	$\beta = 1$
$\mathrm{U}(N)/1$	Hermitean	$\beta = 2$
$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	Hermitean selfdual	$\beta = 4$

Wigner, von Neumann (1929), Hua (50's)

Dyson's Threefold Way

probability density and flat integration measure

$$P_N^{(\beta)}(H)d[H] \sim \exp\left(-\frac{\beta}{4v^2}\mathrm{tr}\, H^2\right) d[H]$$

Gaussian Ensembles: GOE, GUE, GSE $(\beta = 1, 2, 4)$

rotation invariance in matrix space $\longrightarrow$ eigenvalues, angles

$$H = U^{-1}EU \ , \quad U \in \mathrm{O}(N), \mathrm{U}(N), \mathrm{USp}(2N) \quad (\beta = 1, 2, 4)$$

where $E = \mathrm{diag}\,(E_1, \ldots, E_N)$ and doubled for $\beta = 4$

$$P_N^{(\beta)}(H)d[H] \sim \exp\left(-\frac{\beta}{4v^2}\mathrm{tr}\, E^2\right) |\Delta_N(E)|^\beta d[E]d\mu(U)$$

with Vandermonde determinant $\quad \Delta_N(E) = \prod_{n<m}(E_n - E_m)$

Dyson (60's, 70's), Altland, Zirnbauer (90's)

Correlation Functions of k Levels

pdf of finding a level in each interval $[E_p, E_p + dE_p], \; p = 1, ..., k$

$$R_k^{(\beta)}(E_1, \ldots, E_k) \sim \int\limits_{-\infty}^{+\infty} dE_{k+1} \cdots \int\limits_{-\infty}^{+\infty} dE_N |\Delta_N(E)|^\beta \exp\left(-\frac{\beta}{4v^2} \operatorname{tr} E^2\right)$$

amazing: can be done in closed form

$$R_k^{(2)}(E_1, \ldots, E_k) = \det\left[\sum_{n=0}^{N-1} \varphi_n(E_p)\varphi_n(E_q)\right]_{p,q=1,\ldots,k}$$

φ_n oscillator wave function

for unitary case $\beta = 2$, similar but more complicated for $\beta = 1, 4$

Unfolded Correlation Functions

unfolding and limit of infinitely many levels yield for level density

$$X_1^{(\beta)}(\xi_1) = 1$$

and for the k–level correlation functions

$$X_k^{(2)}(\xi_1, \ldots, \xi_k) = \det\left[\frac{\sin \pi(\xi_p - \xi_q)}{\pi(\xi_p - \xi_q)}\right]_{p,q=1,\ldots,k}$$

for $\beta = 2$, similar but more complicated for $\beta = 1, 4$

only differences, translation invariance

Mehta, Gaudin (60's), Dyson (70's)

Transition Ensembles and Harmonic Analysis

Transitions Imply Non–Trivial Eigenvalues

ensemble of matrices $H(\alpha) = H^{(0)} + \alpha H^{(\beta)}$

arbitrary $H^{(0)}$, but symmetric space of $H^{(\beta)}$ includes that of $H^{(0)}$

eigenvalues of $H(\alpha)$ highly non–trivial

Gaussian probability density

$$P^{(\beta)}(H^{(\beta)}, v) = \frac{1}{2^{N/2}} \left(\frac{\beta}{\pi v^2}\right)^{\beta N(N-1)+N/2} \exp\left(-\frac{\beta}{4v^2} \operatorname{tr} H^{(\beta)2}\right)$$

express $H^{(\beta)}$ as $(H(\alpha) - H^{(0)})/\alpha$ use $H = H(\alpha)$ as integration variables, new Gaussian probability density $P^{(\beta)}(H - H^{(0)}, \alpha v)$

correct limit: $\lim_{\alpha \to 0} P^{(\beta)}(H - H^{(0)}, \alpha v) = \delta(H - H^{(0)})$

Group Integrals Inevitable

need joint pdf of eigenvalues of H, requires diagonalizations $H = U^{-1}EU$ and $H^{(0)} = V^{-1}E^{(0)}V$ and integration over diagonalizing matrices U and V,

$$\int d\mu(U) \int d\mu(V) P^{(\beta)}(H - H^{(0)}, \alpha v)$$

$$\sim \exp\left(-\frac{\beta}{4\alpha^2 v^2}\mathrm{tr}\,(E^2 + E^{(0)2})\right)$$

$$\int d\mu(U) \int d\mu(V) \exp\left(\frac{\beta}{2\alpha^2 v^2}\mathrm{tr}\, U^{-1}EUV^{-1}E^{(0)}V\right)$$

invariance of the Haar measure $d\mu(UV^{-1}) = d\mu(U)$ implies that double group integral is equal to a single group integral

$$\int d\mu(U) \exp\left(\frac{\beta}{2\alpha^2 v^2}\mathrm{tr}\, U^{-1}EUE^{(0)}\right)$$

Group Integrals in Standard Notation

non–trivial group integrals have to be done

$$\Phi_N^{(\beta)}(x, k) = \int d\mu(U) \exp(i \mathrm{tr}\, U^{-1} x U k)$$

notation: x, k diagonal matrices of eigenvalues

more precisely: x, k are radial coordinates on symmetric spaces

also useful to include imaginary unit

Matrix Plane Waves

H and K are $N \times N$ matrices

matrix plane wave $\exp(i\operatorname{tr} HK)$ satisfies

$$\Delta_H \exp(i\operatorname{tr} HK) = -\operatorname{tr} K^2 \exp(i\operatorname{tr} HK)$$

with Cartesian Laplacean over matrices

$$\Delta_H = \operatorname{tr} \frac{\partial^2}{\partial H^2} = \sum_{n,m} c_{nm} \frac{\partial^2}{\partial {H_{nm}}^2} \ , \quad c_{nm} = \text{const}$$

matrix gradient $\partial/\partial H$ follows from H by replacing H_{nm} with $\partial/\partial H_{nm}$ and inserting some factors $1/2$.

Integrating over Diagonalizing Groups

diagonalize $H = U^{-1} x U$ and $K = V^{-1} k V$

$U, V \in \mathrm{O}(N)$	if H, K real symmetric	$\beta = 1$
$U, V \in \mathrm{U}(N)$	if H, K Hermitean	$\beta = 2$
$U, V \in \mathrm{USp}(2N)$	if H, K Hermitean selfdual	$\beta = 4$

integrate plane wave equation

$$\Delta_H \exp(i \operatorname{tr} HK) = -\operatorname{tr} K^2 \exp(i \operatorname{tr} HK)$$

over diagonalizing matrix V of K, use $\operatorname{tr} K^2 = \operatorname{tr} k^2$

$$\Delta_H \int d\mu(V) \exp(i \operatorname{tr} HK) = -\operatorname{tr} k^2 \int d\mu(V) \exp(i \operatorname{tr} HK)$$

Gelfand's Spherical Functions

invariance of the Haar measure $d\mu(VU^{-1}) = d\mu(V)$ implies

$$\int d\mu(V) \exp(i \mathrm{tr}\, HK) = \int d\mu(V) \exp(i \mathrm{tr}\, U^{-1} x U V^{-1} k V)$$
$$= \int d\mu(V) \exp(i \mathrm{tr}\, x V^{-1} k V)$$

Gelfand's spherical function

$$\Phi_N^{(\beta)}(x, k) = \int d\mu(V) \exp(i \mathrm{tr}\, x V^{-1} k V)$$

depends only on radial variables (eigenvalues) x, k
$\longrightarrow$ may replace Δ_H by its radial part Δ_x

Matrix Bessel Functions

radial equation $\Delta_x \Phi_N^{(\beta)}(x,k) = -\mathrm{tr}\, k^2 \Phi_N^{(\beta)}(x,k)$

radial Laplacean $\Delta_x = \sum_{n=1}^{N} \frac{1}{|\Delta_N(x)|^\beta} \frac{\partial}{\partial x_n} |\Delta_N(x)|^\beta \frac{\partial}{\partial x_n}$

with the Vandermonde determinant $\Delta_N(x) = \prod_{n<m}(x_n - x_m)$

$$\longrightarrow \quad \Delta_x = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \sum_{n<m} \frac{\beta}{x_n - x_m} \left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m} \right)$$

$\longrightarrow$ matrix generalization of Bessel operator

symmetry $\Phi_N^{(\beta)}(x,k) = \int d\mu(U) \exp(i\mathrm{tr}\, U^{-1} x U k) = \Phi_N^{(\beta)}(k,x)$

Vector and Matrix Bessel Functions

usual Bessel functions in two and three dimensional vector spaces

$$J_0(z) = \int_0^{2\pi} \exp\left(iz\cos\varphi\right) d\varphi \sim \sum_{\kappa=0}^{\infty} (-1)^\kappa \frac{(z/2)^{2\kappa}}{\kappa!\nu!}$$

$$j_0(z) = \int_0^{2\pi} \int_0^{\pi} \exp\left(iz\cos\vartheta\right) \sin\vartheta d\vartheta d\varphi \sim \frac{\sin z}{z}$$

Bessel functions in $N \times N$ matrix spaces

$$\Phi_N^{(\beta)}(x,k) = \int d\mu(U) \exp(i \operatorname{tr} U^{-1} x U k)$$

difficult objects, complexity depends strongly on β

Link to Exactly Solvable Systems

Hamiltonian Dynamics and Matrix Bessel Fuctions

ansatz $\quad \Phi_N^{(\beta)}(x,k) = \dfrac{\Theta_N^{(\beta)}(x,k)}{(\Delta_N(x)\Delta_N(k))^{\beta/2}}$

eigenvalue equation $\quad L_x \Theta_N^{(\beta)}(x,k) = \operatorname{tr} k^2 \Theta_N^{(\beta)}(x,k)$

Hamilton or Schrödinger operator

$$L_x = -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \frac{\beta(\beta-2)}{2} \sum_{n<m} \frac{1}{(x_n - x_m)^2}$$

view β as continuous parameter and the x_n as positions of N particles in one dimension $\quad \longrightarrow \quad$ Calogero–Sutherland

Jack polynomials, not symmetric under $x \leftrightarrow k$

no $n \neq m$ interaction for $\beta = 2$

Harish-Chandra and Itzykson–Zuber Integrals

Itzykson–Zuber Integral

in the unitary case $\beta = 2$ the matrix Bessel function can be calculated in closed form

$$\Phi_N^{(2)}(x,k) = \int\limits_{\mathrm{U}(N)} d\mu(U) \exp(i \mathrm{tr}\, U^{-1} x U k)$$

$$= \frac{\det[\exp(i x_n k_m)]_{n,m=1,\ldots,N}}{\Delta_N(x)\Delta_N(k)}$$

with Vandermonde determinant $\Delta_N(x) = \prod_{n<m}(x_n - x_m)$

reason is a separability of the radial Laplacean !

or, equivalently, absence of interaction in associated Hamilton (Schrödinger) operator

Itzykson, Zuber (1980)

Harish-Chandra Integral

$\mathcal{G}$ compact semi–simple Lie group, a, b fixed elements in Cartan subalgebra $\mathcal{H}_0$ of $\mathcal{G}$

$$\int_{\mathcal{G}} \exp\left(\operatorname{tr} U^{-1} a U b\right) d\mu(U) = \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \frac{\exp\left(\operatorname{tr} w(a) b\right)}{\Pi(a)\Pi(w(b))}$$

$\Pi(a)$ product of all positive roots of $\mathcal{H}_0$, $\mathcal{W}$ Weyl reflection group

everything stays in the space of the Lie group and its algebra !

$\longrightarrow$ Gelfand's and Harish–Chandra's spherical functions are very different objects

coincide only in the unitary case, because $\mathrm{U}(N)/1 \equiv \mathrm{U}(N)$

Harish-Chandra (1957)

Application to Time–Reversal Invariance Breaking

GOE ($\beta = 1$) and GSE ($\beta = 4$) preserve time–reversal invariance, thus there are two models for time–reversal invariance breaking

GOE–GUE transition

$$H(\alpha) = H^{(1)} + \alpha H^{(2)}$$

GSE–GUE transition

$$H(\alpha) = H^{(4)} + \alpha H^{(2)}$$

using Itzykson–Zuber integral, all diagonalizing groups can be integrated out, yielding joint pdf of eigenvalues of $H(\alpha)$

furthermore: eigenvalue integrals can be done in closed form, all k–level correlations written in terms of Pfaffian determinants

Mehta, Pandey (1983)

Supermathematics

Variables

k_1 complex commuting variables $\quad z_p,\ p = 1, \dots, k_1$

k_2 complex anticommuting variables $\quad \zeta_p,\ p = 1, \dots, k_2$

$\zeta_p \, \zeta_q \ = \ -\zeta_q \, \zeta_p \quad ,$ in particular $\quad \zeta_p^2 \ = \ 0$

every function is a finite polynomial, for example for $k_2 = 2$

$$f(\zeta_1, \zeta_2) \ = \ c_0 + c_{11}\zeta_1 + c_{12}\zeta_2 + c_2\zeta_1\zeta_2$$

complex conjugation $\quad \zeta_p \ \longrightarrow \ \zeta_p^* \ \longrightarrow \ \zeta_p^{**} = -\zeta_p$

$$\zeta_p \, \zeta_q^* \ = \ -\zeta_q^* \, \zeta_p$$

commuting and anticommuting variables commute

$$z_p \, \zeta_q \ = \ \zeta_q \, z_p \quad \text{and} \quad z_p \, \zeta_q^* \ = \ \zeta_q^* \, z_p$$

Example: Strange Identities for Functions

functions such as exp or cos etc.. involving anticommuting variables can only be interpreted as power series

but, as the square of an anticommuting variable is zero, these power series must terminate

every function of anticommuting variables is a finite polynomial

for example

$$\exp(\zeta_p^* \zeta_p) \;=\; 1 + \zeta_p^* \zeta_p \;=\; \frac{1}{1 - \zeta_p^* \zeta_p} \;=\; \sqrt{1 + 2\zeta_p^* \zeta_p}$$

$$= \; 1 + \sin(\zeta_p^* \zeta_p) \;=\; 1 + \ln(1 + \zeta_p^* \zeta_p)$$

these are identities !

Linear Algebra

supervectors $\psi = \begin{bmatrix} z \\ \zeta \end{bmatrix}$ and supermatrices $\sigma = \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix}$

matrices a, b have commuting entries
matrices μ, ν have anticommuting entries

$$\sigma\psi = \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix} \begin{bmatrix} z \\ \zeta \end{bmatrix} = \begin{bmatrix} az + \mu\zeta \\ \nu z + b\zeta \end{bmatrix} = \begin{bmatrix} z' \\ \zeta' \end{bmatrix} = \psi'$$

supertrace $\quad \operatorname{str}\sigma = \operatorname{tr} a - \operatorname{tr} b \quad \longrightarrow \quad \operatorname{str}\sigma_1\sigma_2 = \operatorname{str}\sigma_2\sigma_1$

superdeterminant $\quad \operatorname{sdet}\sigma = \dfrac{\det(a - \mu b^{-1}\nu)}{\det b}$

$$\longrightarrow \quad \operatorname{sdet}\sigma_1\sigma_2 = \operatorname{sdet}\sigma_1 \operatorname{sdet}\sigma_2$$

Analysis

derivative $\quad \dfrac{\partial \zeta_p}{\partial \zeta_q} = \delta_{pq} \quad$ and $\quad \dfrac{\partial \zeta_p^*}{\partial \zeta_q} = 0$

Berezin integral $\quad \int d\zeta_p = 0 \quad$ and $\quad \int \zeta_p d\zeta_p = \dfrac{1}{\sqrt{2\pi}}$

for example

$$\int \exp(-a\zeta_p^*\zeta_p) d\zeta_p^* d\zeta_p = \int \left(1 - a\zeta_p^*\zeta_p\right) d\zeta_p^* d\zeta_p = \frac{a}{2\pi}$$

apart from factors, derivative and integral are the same !

change of variables $\quad \psi \ \rightarrow \ \chi = \chi(\psi) \quad$ requires

Jacobian or Berezinian $\quad \int f(\psi) d[\psi] = \int f(\psi(\chi)) \mathrm{sdet} \dfrac{\partial \psi}{\partial \chi} d[\chi]$

Gaussian Integrals over Bosons and Fermions

matrix A has commuting entries
vector z has commuting entries ("Bosons")
vector ζ has anticommuting entries ("Fermions")

$$\int \exp(-iz^\dagger Az)d[z] \;=\; \det{}^{-1}\frac{A}{2\pi}$$

$$\int \exp(-i\zeta^\dagger A\zeta)d[\zeta] \;=\; \det\frac{A}{2\pi}$$

σ is a supermatrix and ψ a supervector

$$\int \exp(-\psi^\dagger\sigma\psi)d[\psi] = \mathrm{sdet}^{-1}\frac{\sigma}{2\pi}$$

$\longrightarrow$ divergencies removed, renormalization (field theory)

Supersymmetry in Random Matrix Theory

Preparing for Supersymmetry: Generating Function

Gaussian ensembles ($\beta = 1, 2, 4$) of $N \times N$ random matrices H,
(formulae a bit simplified, apply in this form to $\beta = 2$)

k–level correlation functions

$$R_k^{(\beta)}(x_1, \ldots, x_k) \ = \ \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \mathrm{tr}\, \frac{\mathbb{1}_N}{x_p^{\pm} - H}$$

can be written as derivatives

$$R_k^{(\beta)}(x_1, \ldots, x_k) \ = \ \frac{\partial^k}{\prod_{p=1}^{k} \partial J_p} Z_k^{(\beta)}(x + J) \bigg|_{J=0}$$

of generating function

$$Z_k^{(\beta)}(x + J) \ = \ \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)}$$

Ensemble Average – FT in Ordinary Space

$$\frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)} = \int d[z_p] \int \exp(-iz_p^\dagger(H - x_p + J_p)z_p)$$
$$\int d[\zeta_p] \exp(-i\zeta_p^\dagger(H - x_p - J_p)\zeta_p)$$

collect total dependence on random matrix H

ensemble average becomes Fourier transform in matrix space !

$$\int d[H] \exp(-\mathrm{tr}\, H^2) \exp\left(-i\mathrm{tr}\, H \sum_{p=1}^{k}(z_p z_p^\dagger - \zeta_p\zeta_p^\dagger)\right)$$
$$= \exp\left(-\mathrm{tr}\left(\sum_{p=1}^{k}(z_p z_p^\dagger - \zeta_p\zeta_p^\dagger)\right)^2\right) = \exp(-\mathrm{str}\, B^2)$$

Ensemble Average – FT in Superspace

B is a $2k \times 2k$ ($\beta = 2$) or $4k \times 4k$ ($\beta = 1, 4$) supermatrix, entries are all scalar products ordered in blocks

$$B_{pq} = \begin{bmatrix} z_p^\dagger z_q & z_p^\dagger \zeta_q \\ \zeta_p^\dagger z_q & \zeta_p^\dagger \zeta_q \end{bmatrix}$$

introduce supermatrix σ with the same symmetries and do another Fourier transform, now in superspace

$$\begin{aligned} \exp(-\mathrm{str}\, B^2) &= \int d[\sigma] \exp(-\mathrm{str}\, \sigma^2) \exp(-i\mathrm{str}\, \sigma B) \\ &= \int d[\sigma] \exp(-\mathrm{str}\, \sigma^2) \exp\left(-i \sum_{p=1}^{k} \psi_p^\dagger (\mathbb{1}_N \otimes \sigma) \psi_p\right) \end{aligned}$$

all integrals over $\psi_p = \begin{bmatrix} z_p \\ \zeta_p \end{bmatrix}$ doable, yield superdeterminants !

Supersymmetric Representation of RMT

Gaussian ensemble ($\beta = 1, 2, 4$) of $N \times N$ random matrices H

k–level correlations

$$R_k^{(\beta)}(x_1, \ldots, x_k) \; = \; \frac{\partial^k}{\prod_{p=1}^k \partial J_p} Z_k^{(\beta)}(x+J)\bigg|_{J=0}$$

generating function obeys the identity (yes, this is exact!)

$$Z_k^{(\beta)}(x+J) \; = \; \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)}$$

$$= \; \int d[\sigma] \exp(-\mathrm{str}\, \sigma^2) \mathrm{sdet}^{\,-N}(\sigma - x - J)$$

where σ is a $2k \times 2k$ or $4k \times 4k$ supermatrix

$\longrightarrow$ drastic reduction of dimensions, N explicit parameter

Efetov (1983), Verbaarschot, Zirnbauer (1985)

Relevant Supergroups and Symmetric Superspaces

Random Matrices and Corresponding Supermatrices

as we have seen, ensemble average over random matrices H in ordinary space becomes an ensemble average over supermatrices σ

diagonalization in the form

$$\sigma = u^{-1} s u \quad , \quad s = \begin{bmatrix} s_1 & 0 \\ 0 & is_2 \end{bmatrix} \quad \begin{matrix} \text{“Bosonic”} \\ \text{“Fermionic”} \end{matrix} \quad \text{eigenvalues}$$

inspite of this jargon, eigenvalues are always commuting !

the diagonalizing supermatrices u have commuting and anticommuting entries, they are elements of supergroups

Relevant Supergroups

for the GUE ($\beta = 2$)

unitary supergroup $\mathrm{U}(k_1|k_2)$ with $u^\dagger u = \mathbb{1}\, k_1 + k_2$

$$\longrightarrow \quad \mathrm{U}(k_1|k_2) \supset \mathrm{U}(k_1) \otimes \mathrm{U}(k_2)$$

for the GOE and GSE ($\beta = 1, 4$)

unitary ortho–symplectic supergroup $\mathrm{UOSp}(k_1|2k_2)$ with

$$u^\dagger u = \mathbb{1}_{k_1+2k_2} \quad \text{and} \quad u^T M u = M \ ,$$

$$M = \begin{bmatrix} 1_{k_1} & 0 & 0 \\ 0 & 0 & 1_{k_2} \\ 0 & -1_{k_2} & 0 \end{bmatrix}$$

$$\longrightarrow \quad \mathrm{UOSp}(k_1|k_2) \supset \mathrm{O}(k_1) \otimes \mathrm{USp}(2k_2)$$

Kac (1977), Berezin (1983)

Symmetric Ordinary and Super Spaces

random matrix H lives in symmetric ordinary spaces

$\mathrm{U}(N)/\mathrm{O}(N)$	real symmetric	$\beta = 1$
$\mathrm{U}(N)/1$	Hermitean	$\beta = 2$
$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	Hermitean selfdual	$\beta = 4$

corresponding supermatrix σ lives in symmetric superspaces

$$\mathrm{U}(k_1|k_2)/1 \quad \text{for} \quad \beta = 2$$
$$\mathrm{Gl}(k_1|2k_2)/\mathrm{OSp}(k_1|2k_2) \;\; \text{(two forms)} \quad \text{for} \quad \beta = 1, 4$$

symmetric spaces have an angular or group part
and a radial or eigenvalue part

Integrals over Unitary Supergroup $\mathrm{U}(k_1|k_2)$

supersymmetric generalization of Itzykson-Zuber integral

$$\int d\mu(u)\exp(i\,\mathrm{str}\,u^{-1}sur) \;=\; \frac{\det[\exp(is_{p1}r_{q1})]\det[\exp(is_{p2}r_{q2})]}{B_{k_1k_2}(s)B_{k_1k_2}(r)}$$

with $\quad B_{k_1k_2}(s) \;=\; \dfrac{\Delta_{k_1}(s_1)\Delta_{k_2}(is_2)}{\prod_{p,q}(s_{p1}-is_{q2})}$

for $\;k_1=k_2=k\;$ get $\;B_{kk}(s)=B_k(s)=\det\left[\dfrac{1}{s_{p1}-is_{q2}}\right]_{p,q=1,\ldots,k}$

for $\;k_1,k_2\;$ equal to $\;N,0\;$ or $\;0,N\;$ recover ordinary integral

$$\int d\mu(U)\exp(i\,\mathrm{tr}\,U^{-1}xUk)=\frac{\det[\exp(ix_nk_m)]}{\Delta_N(x)\Delta_N(k)}$$

TG (1991, 1996)

Application to Random Matrix Problems

Exact Solutions

Exact Solutions for Transitions to GUE

ensemble of matrices $H(\alpha) = H^{(0)} + \alpha H^{(2)}$

arbitrary $H^{(0)}$, space included in Hermitean space of $H^{(2)}$

generating function acquires the form

$$Z_k^{(\beta)}(x+J,\alpha) = \int d[\sigma] \exp\Big(-\frac{1}{\alpha^2}\mathrm{str}\,(\sigma - x - J)^2\Big) \\ \mathrm{sdet}^{-1}(\mathbb{1}_N \otimes \sigma + H^{(0)} \otimes \mathbb{1}_{2k})$$

supersymmetric Itzykson-Zuber integral yields immediately

$$Z_k^{(\beta)}(x+J,\alpha) = \frac{1}{B_k(x+J)} \int d[s] B_k(s) \exp\Big(-\frac{1}{\alpha^2}\mathrm{str}\,(s - x - J)^2\Big) \\ \mathrm{sdet}^{-1}(\mathbb{1}_N \otimes s + E^{(0)} \otimes \mathbb{1}_{2k}) \\ = \frac{1}{B_k(x+J)} \det\left[K_N((x+J)_{p1}, (x+J)_{q2}, \alpha)\right]_{p,q=1,\ldots,k}$$

... with Kernel

$2k$–fold integral collapses to determinant of twofold integrals

they are given by the kernel

$$K_N(r_{p1}, ir_{q2}, \alpha) \;=\; \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \frac{ds_1 ds_s}{s_1 - is_2} \exp\Big(-\frac{1}{\alpha^2}\left((s_1 - r_1)^2 + (s_2 - r_2)^2\right)\Big)$$

$$\prod_{n=1}^{N} \frac{is_2 + E_n^{(0)}}{s_1^{\pm} + E_n^{(0)}}$$

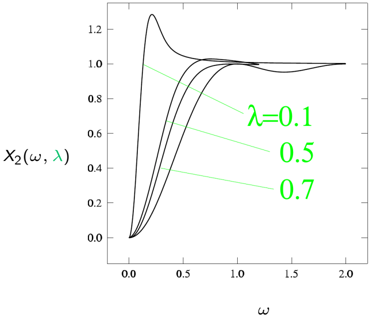
derivatives with respect to the J_p are easy,
get all k–level correlation functions exactly

TG (1996)

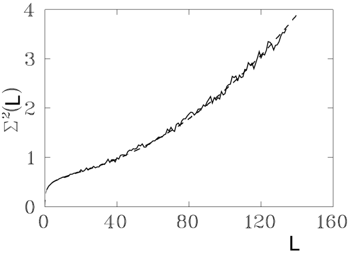
Regularity–Chaos Transition

unitary case: $H(\alpha) = H^{(\text{regular})} + \alpha H^{(2)}$

two–level correlation function $X_2(\omega, \lambda)$ with $\lambda = \alpha / D$



$\longrightarrow$



transition towards GUE spectral correlations very fast !

Built–in Structures of Supersymmetry

consider $\beta = 2$

Hermitean $N \times N$ ordinary matrix $H = U^\dagger x U$

$$d[H] = \Delta_N^2(x) d[x] d\mu(U) \quad \text{where} \quad \Delta_N(x) = \prod_{n<m} (x_n - x_m)$$

$\longrightarrow$ level repulsion

Hermitean $2k \times 2k$ supermatrix $\sigma = u^\dagger s u$

$$d[\sigma] = B_k^2(s) d[s] d\mu(u) \quad \text{where}$$

$$B_k(s) = \det \left[\frac{1}{s_{p1} - i s_{q2}} \right]_{p,q=1,\ldots,k}$$

$\longrightarrow$ k–level correlations, determinantal processes

Application to Random Matrix Problems

Supersymmetric Nonlinear sigma Model

Saddle Point Approximation for $k = 2$ and Large N

asymptotic approach for two-level ($k = 2$) correlation functions
for all $\beta = 1, 2, 4$, write

$$Z_k^{(\beta)}(x + J) = \int d[\sigma] \exp(-\mathrm{str}\, \sigma^2) \mathrm{sdet}^{-N}(\sigma - x - J)$$
$$= \int d[\sigma] \exp(-L(\sigma))$$

with "free energy" $\quad L(\sigma) = \mathrm{str}\,(\sigma^2 + N \ln(\sigma - x - J))$

at $J = 0$, $dL = 0$ determines saddle points, $\quad 2s_0 + \dfrac{N}{s_0 - x} = 0$

solution $\quad s_0 = \dfrac{x \pm i\sqrt{2N - x^2}}{2}$

imaginary part is Wigner's semicircle !

Efetov (1983)

Coset Supermanifolds and Nonlinear sigma Models

“massive” modes corresponding to the level densities are simple Gaussian integrals

remaining integrals explore surrounding space, correlation functions result from corresponding integrals over “massless” Goldstone modes, parametrized by supermatrices Q which are elements of the coset supermanifolds

$\mathrm{U}(1,1|2)/(\mathrm{U}(1|1) \times \mathrm{U}(1|1))$ for $\beta = 2$

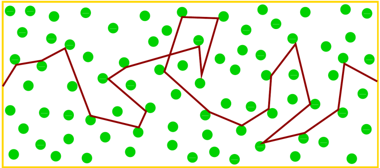
$\mathrm{UOSp}(2,2|4)/(\mathrm{UOSp}(2|2) \times \mathrm{UOSp}(2|2))$ for $\beta = 1, 4$

widely applicable, particularly in d–dimensional field theories

Efetov (1983)

Supersymmetry and Disordered Systems

electron moves diffusively in a probe with scatterers (impurities)



d dimensions, localization ?

random disorder potential

$$\langle V(\vec{r})V(\vec{r}\,')\rangle \sim \delta^{(d)}(\vec{r}-\vec{r}\,')$$

field theory: supersymmetric non–linear σ model with action

$$S[Q] = \mathrm{str} \int d^d r \left(\mathcal{D}\partial_i Q(\vec{r})\partial_i Q(\vec{r}) - i\omega\Lambda Q(\vec{r}) \right)$$

where $Q = Q(\vec{r})$ is supermatrix field in coset space

Efetov (1983)

Harish–Chandra and Supersymmetry

supersymmetric Itzykson-Zuber integral (TG (1991,1996))

$\longrightarrow$ most interesting remaining case is UOSp($k_1/2k_2$)

conjecture: Serganova (1992) and Zirnbauer (1996)

proof: Guhr, Kohler (2002)

Laplacean Δ_A over Lie superalgebra uosp($k_1/2k_2$)

construct radial part Δ_a over Cartan subalgebra

identify Harish–Chandra integrals as eigenfunctions of Δ_a

realize that Δ_a is separable

$\longrightarrow$ solution of eigenequation is trivial

proof also comprises Lie groups in ordinary space

What have we learned ?

Conclusions

- group integrals are needed to treat transitions
- integrals of Itzykson–Zuber, Gelfand and Harish-Chandra
- direct connection to exactly solvable systems
- supermathematics and the supersymmetry method
- exact solutions in the unitary case, in general asymptotic supersymmetric nonlinear sigma models

Thank You for Your Attention !