Fakultät für Physik

UNIVERSITÄT DUISBURG ESSEN

Random Matrix Theory: from Single– to Many–Body Quantum Chaos

Lecture III — Stochastic Scattering

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Outline — Stochastic Scattering

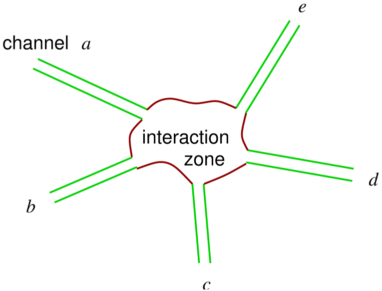
- some background: scattering theory
- (quantum) chaotic or stochastic scattering
- supersymmetry for distributions
- exact results for scattering matrix elements
- exact results for cross sections
- comparison with microwave experiments
- comparison with nuclear data

Introduction to Scattering Theory

Scattering Process

waves propagate in (fictitious) channels, scattered at target

scattering matrix S connects ingoing and outgoing waves



M channels,

S is $M \times M$

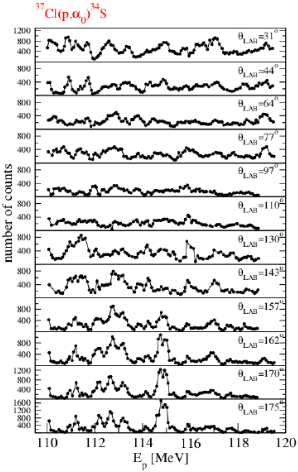
flux conservation

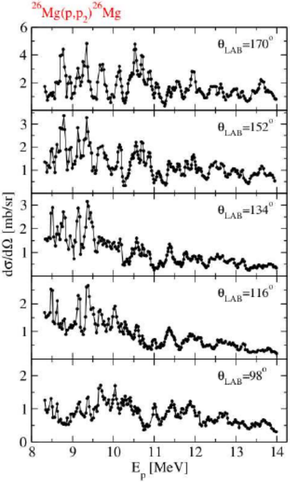
$SS^\dagger = \mathbb{1}_M = S^\dagger S$

no direct reactions ($a \neq b$) $\longrightarrow$ energy average $\overline{S}$ diagonal

transmission coefficients $T_a = 1 - |\overline{S_{aa}}|^2$

Scattering Experiments in Nuclear Physics

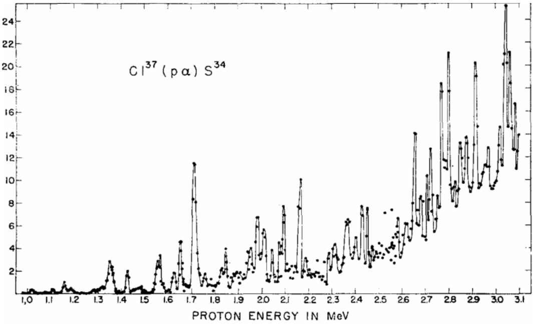




differential cross sections, squares of scattering matrix elements

this example: Richter et al. (1960's)

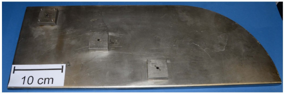
Different Regimes in Nuclear Scattering

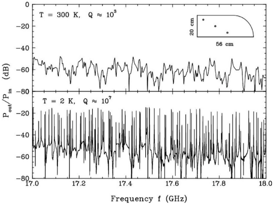


from isolated resonances towards Ericson regime of strongly overlapping resonances

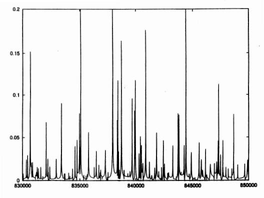
Clarke, Almqvist, Paul (1960's)

Scattering Experiments with Classical Waves





microwaves



elastic
reveberations

direct measurement of the scattering matrix

Weaver, Ellegaard, Stöckmann, Richter, Shridar groups (90's...10's)

(Quantum) Stochastic/Chaotic Scattering

Mexico Approach to Stochastic Scattering

to study statistics, S itself modeled as a stochastic quantity
minimum information principle yields probability measure

$$P(S)d\mu(S) \sim \frac{d\mu(S)}{|\det^{\beta(M-1)+2}(\mathbb{1}_M - S\langle S\rangle^\dagger)|}$$

- no invariance under time–reversal: S unitary, $\beta = 2$
- invariance under time–reversal:
  - spin–rotation symmetry: S unitary symmetric, $\beta = 1$
  - no spin–rotation symmetry: S unitary self–dual, $\beta = 4$

input: ensemble average $\langle S\rangle$, assume $\langle S\rangle = \overline{S}$

problem: energy and parameter dependence not clear !

Mello, Pereyra, Seligman (1980's)

Microscopic Description of Scattering Process ...

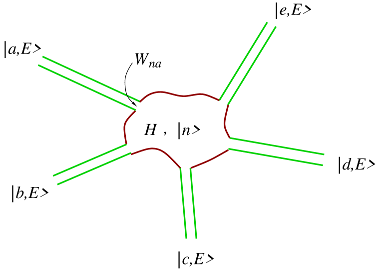
$$\mathcal{H} = \sum_{n,m=1}^{N} |n\rangle\, H_{nm}\, \langle m| + \sum_{a=1}^{M} \int dE\, |a,E\rangle\, E\, \langle a,E| + \sum_{n,a} \left(|n\rangle \int dE\, W_{na}\, \langle a,E| + \mathrm{c.c.} \right)$$

bound states
Hamiltonian H

$N \gg 1$ bound states $|n\rangle$

M channel states $|a, E\rangle$

coupling W_{na}



... Yields Scattering Matrix

$$S_{ab}(E) \ = \ \delta_{ab} - i2\pi W_a^\dagger G(E) W_b$$

with matrix resolvent containing bound states Hamiltonian H

$$G(E) \ = \ \frac{\mathbb{1}_N}{E\mathbb{1}_N - H + i\pi \sum_{c=1}^{M} W_c W_c^\dagger}$$

absence of direct reactions consistent with orthogonality

$$W_a^\dagger W_b \ = \ \frac{\gamma_a}{\pi}\delta_{ab}$$

Mahaux, Weidenmüller (1969)

Heidelberg Approach to Stochastic Scattering

Hamiltonian H modeled as a Gaussian random matrix

$$P(H) \sim \exp\left(-\frac{N\beta}{4v^2}\mathrm{tr}\, H^2\right)$$

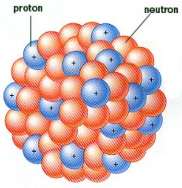
form of $P(H)$ irrelevant on local scale of mean level spacing

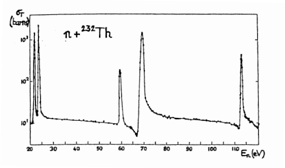
$\longrightarrow$ two universalities, experimental and mathematical

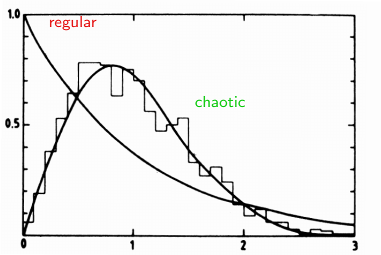
- no invariance under time–reversal: H Hermitean, $\beta = 2$
- invariance under time–reversal:
  - spin–rotation symmetry: H real symmetric, $\beta = 1$
  - no spin–rotation symmetry: H Hermitean self–dual, $\beta = 4$

Weidenmüller (1960's)

Chaotic Statistics, Example: Compound Nucleus







spacing distribution $p(s)$

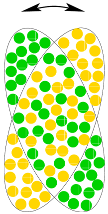
probability density to find two adjacent levels in distance s

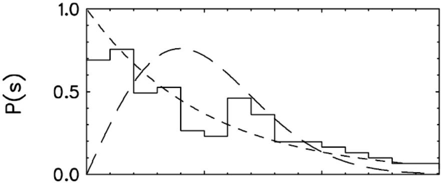
Bohigas, Haq, Pandey (1983)

Counter Example: Collective Excitations in Nuclei

single particle versus collective excitations

scissors mode oscillations, all neutrons $\leftrightarrow$ all protons





$\longrightarrow$ chaotic versus regular statistics

$\longrightarrow$ crossover transitions are frequent !

Enders, TG, Huxel, von Neumann–Cosel, Rangacharyulu, Richter (2000)

Supersymmetry for Correlations

Correlation Functions in RMT

Gaussian ensemble ($\beta = 1, 2, 4$) of $N \times N$ random matrices H

k–level correlations are probability density to find a level in each interval $[x_p, x_p + dx_p]$, $p = 1, \ldots, k$

can be expressed with resolvent

$$R_k^{(\beta)}(x_1, \ldots, x_k) \;=\; \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \mathrm{tr}\, \frac{\mathbb{1}_N}{H - x_p \mathbb{1}_N}$$

(notation is a bit simplified)

Generating Function for Correlations

introduce scalar source variables J_p

$$R_k^{(\beta)}(x_1, \ldots, x_k) \;=\; \frac{\partial^k}{\prod_{p=1}^k \partial J_p} Z_k^{(\beta)}(x+J)\bigg|_{J=0}$$

and generating function

$$Z_k^{(\beta)}(x+J) \;=\; \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)}$$

Supersymmetric Representation

vectors z_p, ζ_p with commuting and anticommuting entries

$$\frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)} = \int d[z_p] \exp\left(i z_p^\dagger (H - x_p + J_p) z_p\right) \int d[\zeta_p] \exp\left(i \zeta_p^\dagger (H - x_p - J_p) \zeta_p\right)$$

average over H just Gaussian!

intermediate result is integral over $z_p, \zeta_p,\ p = 1, \ldots, k$

but depends only on scalar products

$z_p^\dagger z_q,\ \zeta_p^\dagger \zeta_q$ commuting

$z_p^\dagger \zeta_q,\ \zeta_p^\dagger z_q$ anticommuting

Supermatrix Integral

Hubbard–Stratonovitch transformation: "use scalar products as integration variables, remaining ones are trivial"

identity (yes, this is exact) for generating function

$$Z_k^{(\beta)}(x+J) \ = \ \int d[\sigma]\exp(-\mathrm{str}\,\sigma^2)\mathrm{sdet}^{\,-N}(\sigma-x-J)$$

where σ is a $2k \times 2k$ or $4k \times 4k$ supermatrix

$\longrightarrow$ drastic reduction of dimensions

Efetov (1983), Verbaarschot, Zirnbauer (1985), TG (1991,2006)

Scattering Matrix

$$S_{ab}(E) \; = \; \delta_{ab} \; - \; i2\pi W_a^\dagger G(E) W_b$$

does not depend on an invariant, but on resolvent matrix

$$G(E) \; = \; \frac{\mathbb{1}_N}{E\mathbb{1}_N - H + i\pi \sum_{c=1}^{M} W_c W_c^\dagger}$$

introduce $N \times N$ matrix source variable J

$$G_{nm}(E) \; = \; \frac{\partial}{\partial J_{nm}} \frac{\det(G^{-1}(E) - J)}{\det(G^{-1}(E) + J)} \bigg|_{J=0}$$

determinants linear in H $\longrightarrow$ supersymmetry method

Verbaarschot, Weidenmüller, Zirnbauer (1985)

Many Results Obtained in this Way, for Example

two–point correlation functions $\langle S_{ab}(E_1) S_{cd}(E_2) \rangle$

$\beta = 1$ Verbaarschot, Weidenmüller, Zirnbauer (1985)
$\beta = 2$ Savin, Fyodorov, Sommers (2006)

higher order correlations, perturbative time–invariance breaking

Davis, Boosé (1988, 1989), Davis, Hartmann (1990)

distribution of diagonal elements $P(S_{aa}(E))$

Fyodorov, Savin, Sommers (2005)

... but: does not work for distribution $P(S_{ab}(E)),\ a \neq b$

$\longrightarrow$ new method needed

Supersymmetry for Distributions

Distribution of Scattering Matrix Elements

$$S_{ab}(E) \;=\; \delta_{ab} \,-\, i2\pi W_a^\dagger G(E) W_b$$

wish to calculate distribution of real and imaginary part

$$\wp_s(S_{ab}) \;=\; \pi\big((-i)^s W_a^\dagger G W_b + i^s W_b^\dagger G^\dagger W_a\big)$$

such that

$$x_1 = \wp_1(S_{ab}) = \mathrm{Re}\, S_{ab}(E) \quad \text{and} \quad x_2 = \wp_2(S_{ab}) = \mathrm{Im}\, S_{ab}(E)$$

distribution given by

$$P_s(x_s) \;=\; \int d[H] \exp(-\mathrm{tr}\, H^2) \delta(x_s - \wp_s(S_{ab})) \;, \quad s = 1, 2$$

Characteristic Function

obtain distribution by Fourier backtransform of

$$R_s(k) \;=\; \int d[H]\exp(-\mathrm{tr}\,H^2)\exp(-ik\wp_s(S_{ab}))$$

insert definition of scattering matrix

$$R_s(k) \;=\; \int d[H]\exp(-\mathrm{tr}\,H^2)\exp(-ik\pi W^\dagger A_s W)$$

with $\quad W \;=\; \begin{bmatrix} W_a \\ W_b \end{bmatrix} \quad$ and $\quad A_s \;=\; \begin{bmatrix} 0 & (-i)^s G \\ i^s G^\dagger & 0 \end{bmatrix}$

where A_s Hermitean, but contains H inverse

problem: have to invert A_s to perform H average !

Crucial Trick

Fourier transform in W space ! — Yields

$$\exp(-ik\pi W^\dagger A_s W) \\ \sim \int d[z] \exp\left(\frac{i}{2}(W^\dagger z + z^\dagger W)\right) \det{}^{\beta/2} A_s^{-1} \exp\left(\frac{i}{4\pi k} z^\dagger A_s^{-1} z\right)$$

now use anticommuting variables

$$\det{}^{\beta/2} A_s^{-1} \;\sim\; \int d[\zeta] \exp\left(\frac{i}{4\pi k} \zeta^\dagger A_s^{-1} \zeta\right)$$

now H linear in exponent $\longrightarrow$ supersymmetry applicable !

different rôle of commuting and anticommuting variables

Supermatrix Model

Hubbard–Stratonovitch transformation gives

$$R_s(k) \;=\; \int d[\varrho] \exp\Big(- r \mathrm{str}\, \varrho^2 - \frac{\beta}{2} \mathrm{str}\, \ln \boldsymbol{\Xi} - \frac{i}{4} F_s \Big)$$

with $8/\beta \times 8/\beta$ supermatrix ϱ and $r = 4\beta\pi^2 k^2 N/v^2$

$$\boldsymbol{\Xi} \;=\; \varrho_E \otimes \mathbb{1}_N + \frac{i}{4k} L \otimes \sum_{c=1}^{M} W_c W_c^\dagger \;, \quad \varrho_E = \varrho - \frac{E}{4\pi k} \mathbb{1}_{8/\beta}$$

matrix L is some superspace metrik

$F_s \;\sim\; [W^\dagger \; 0^\dagger] \, \boldsymbol{\Xi}^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix}$, projects onto boson–boson space

$\longrightarrow$ symmetry breaking differs from the one for correlations

Supersymmetric Non–Linear sigma Model

limit $N \longrightarrow \infty$, unfolding by saddlepoint approximation
integrate out "massive" modes

left with integral over "Goldstone" modes Q,
free rotations, coset manifold in superspace

$$R_s(k) \;=\; \int d\mu(Q) \exp\left(-\frac{i}{4}F_s\right) \prod_{c=1}^{M} \mathrm{sdet}^{\,-\beta/2}\left(1\!\!1_{8/\beta} + \frac{i\gamma_c}{4\pi k} Q_E^{-1} L\right)$$

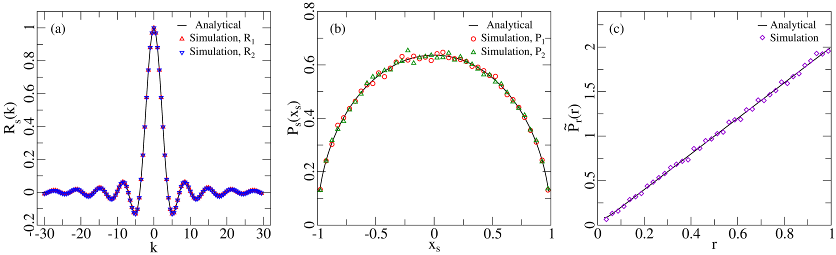
integrate out all remaining anticommuting variables

left with ordinary integrals, two for $\beta = 2$, four for $\beta = 1$

$\longrightarrow$ drastically reduced number of integration variables

Analytical Results versus Numerics

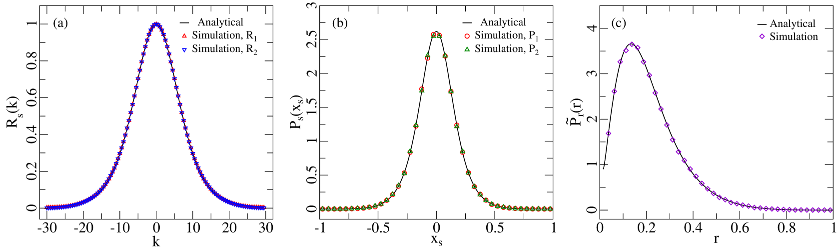
Reproducing the Circular Ensemble for $\beta = 2$



number of channels $M = 2$, energy $E = 0$,
width parameters $\gamma_1/D = 1,\ \gamma_2/D = 1$

real and imaginary parts always equally distributed for $\beta = 2$

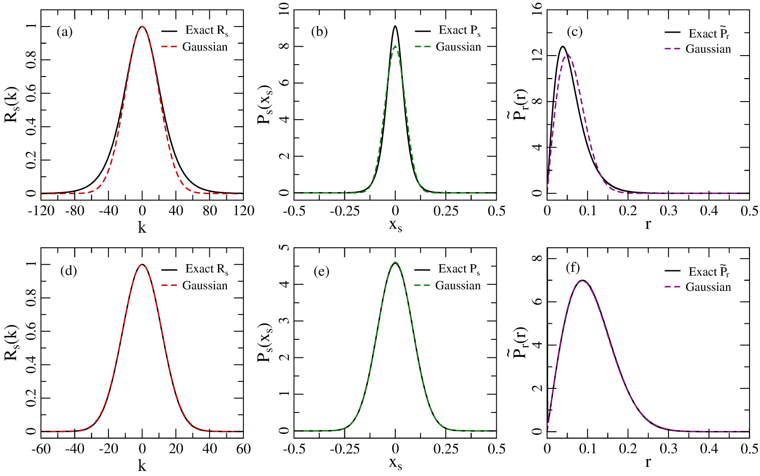
Far Away from the Circular Ensemble for $\beta = 2$



number of channels $M = 5$, energy $E/D = 1.2$,
width parameters γ_j/D between 0.08...0.72

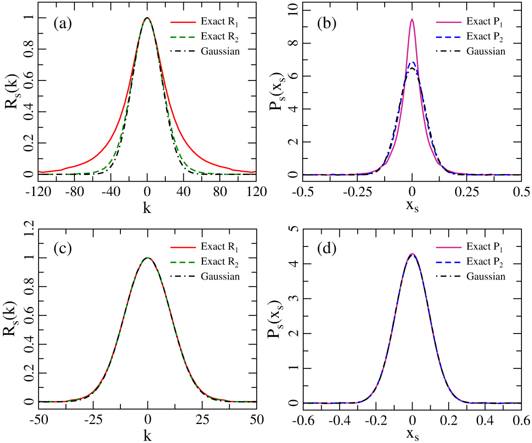
real and imaginary parts always equally distributed for $\beta = 2$

Towards Ericsson Regime for $\beta = 2$



average resonance width / mean level spacing $\Gamma/D = 0.716$ (top) and $\Gamma/D = 8.594$ (bottom)

Towards Ericsson Regime for $\beta = 1$

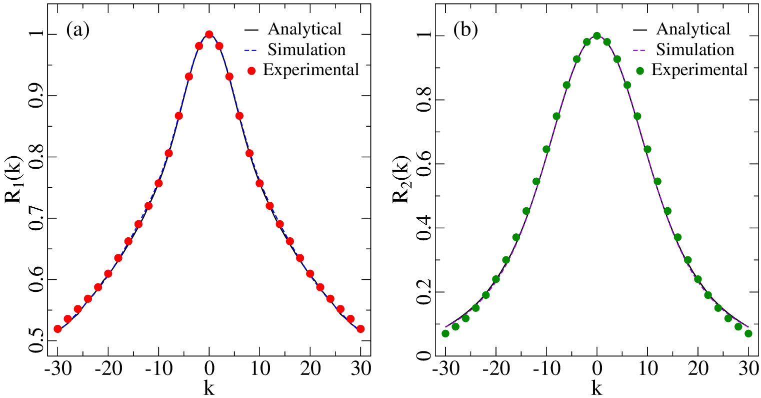


average resonance width / mean level spacing $\Gamma/D = 1.273$ (top) and $\Gamma/D = 7.162$ (bottom)

real and imaginary parts not equally distributed for $\beta = 1$

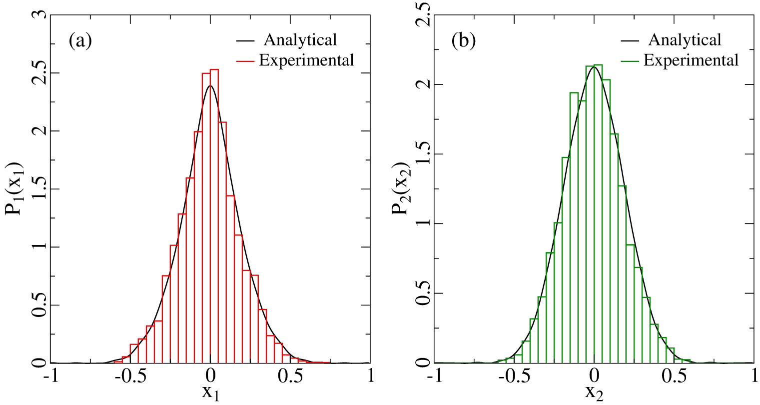
Comparison with Microwave Experiments

... vs Numerics and Experiment for $\beta = 1$



frequency range $10 \ldots 11\mathrm{GHz}$,
average resonance width / mean level spacing $\Gamma/D = 0.234$

Analytical Result vs Experiment for $\beta = 1$



frequency range $24\ldots25\mathrm{GHz}$,
average resonance width / mean level spacing $\Gamma/D = 1.21$

Distribution of Cross Sections

No Way Around the Joint Probability Density

cross section $\sigma_{ab}(E) \;=\; |S_{ab}(E)|^2 \;=\; \mathrm{Re}^2 S_{ab}(E) + \mathrm{Im}^2 S_{ab}(E)$

need joint pdf $P(\mathrm{Re}\, S_{ab}, \mathrm{Im}\, S_{ab}) = P(S_{ab}, S_{ab}^*)$

to calculate $p(\sigma_{ab}) \;=\; \int d^2 S_{ab}\, P(S_{ab}, S_{ab}^*)\, \delta(\sigma_{ab} - |S_{ab}|^2)$

good news: can extend previous calculation into complex plane

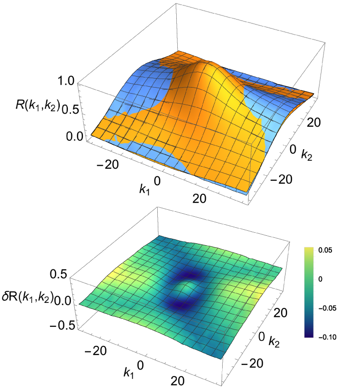
characteristic $R(k, k^*) \;=\; \int d[H] \exp(-\mathrm{tr}\, H^2) \exp(-i\mathrm{Re}\, k^* S_{ab})$

simply replace real k with complex $k = k_1 + ik_2$ everywhere

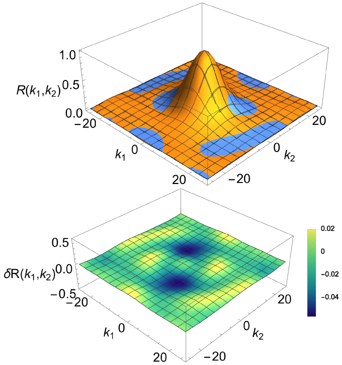
distribution $p(\sigma_{ab}(E)) \;=\; \int d^2 k\, R(k, k^*)\, J_0(\sqrt{\sigma_{ab}(E)}|k|)$

Comparison with Microwave and Nuclear Data

Characteristic Functions for Microwave Data

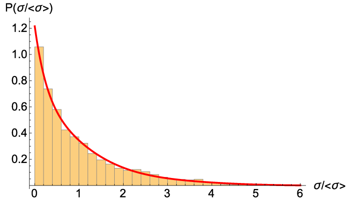


$\Gamma/D = 0.234$

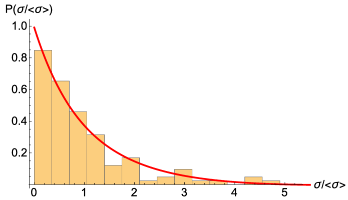


$\Gamma/D = 1.21$

Cross Section Distributions



microwaves $\Gamma/D = 1.21$

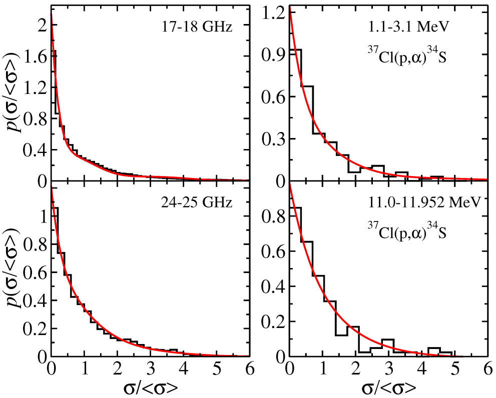


nuclear data ^{37}Cl(p,α)^{34}S

$p(0) \approx 1$ indicates Ericson regime

cross section distribution becomes exponential in Ericson regime

Analytical Results vs Cross Section Data



microwave data (left) $\Gamma/D \;=\; 0.7,\; 1.2$
nuclear data (right) $\Gamma/D \;\approx\; 1,\; 30$

$p(0) \approx 1$ indicates Ericson regime

What have we learned ?

Conclusions and Outlook

- Mexico and Heidelberg approaches to stochastic sacttering
- solved longstanding problem within Heidelberg approach
- now have supersymmetry for distributions
- distributions of scattering matrix elements and cross sections
- additional results: characteristic function generates moments, integral representations for all of them
- full analytical understanding of transition to Ericson regime
- Brouwer's equivalence proof Heidelberg–Mexico implies: now have explicit handle on Mexico approach for arbitrary channel number
- comparison with microwave and nuclear data
- also: condensed matter and wireless communication

Thank You for Your Attention !