Random Matrix Theory: from Single– to Many–Body Quantum Chaos

Lecture III — Stochastic Scattering

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Outline — Stochastic Scattering

- some background: scattering theory
- (quantum) chaotic or stochastic scattering
- supersymmetry for distributions
- exact results for scattering matrix elements
- exact results for cross sections
- comparison with microwave experiments
- comparison with nuclear data
Introduction to Scattering Theory
Scattering Process

waves propagate in (fictitious) channels, scattered at target
scattering matrix $S$ connects ingoing and outgoing waves

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$M$ channels, $S$ is $M \times M$

flux conservation

$SS^\dagger = 1_M = S^\dagger S$

no direct reactions ($a \neq b$) $\rightarrow$ energy average $\overline{S}$ diagonal

transmission coefficients $T_a = 1 - |S_{aa}|^2$
Scattering Experiments in Nuclear Physics

\[ ^{37}\text{Cl}(p, \alpha)^{34}\text{S} \]

\[ ^{26}\text{Mg}(p, p')^{26}\text{Mg} \]

Differential cross sections, squares of scattering matrix elements

This example: Richter et al. (1960's)
Different Regimes in Nuclear Scattering

from isolated resonances towards Ericson regime of strongly overlapping resonances

Clarke, Almqvist, Paul (1960’s)
Scattering Experiments with Classical Waves

direct measurement of the scattering matrix

Weaver, Ellegaard, Stöckmann, Richter, Shridar groups (90’s...10’s)
(Quantum) Stochastic/Chaotic Scattering
to study statistics, $S$ itself modeled as a stochastic quantity
minimum information principle yields probability measure

$$P(S) d\mu(S) \sim \frac{d\mu(S)}{|\det^{\beta(M-1)+2}(1_M - S\langle S\rangle^\dagger)|}$$

- no invariance under time–reversal: $S$ unitary, $\beta = 2$
- invariance under time–reversal:
  - spin–rotation symmetry: $S$ unitary symmetric, $\beta = 1$
  - no spin–rotation symmetry: $S$ unitary self–dual, $\beta = 4$

input: ensemble average $\langle S \rangle$, assume $\langle S \rangle = \bar{S}$

problem: energy and parameter dependence not clear!

Mello, Pereyra, Seligman (1980's)
Microscopic Description of Scattering Process ...

\[
\mathcal{H} = \sum_{n,m=1}^{N} \langle n | H_{nm} | m \rangle + \sum_{a=1}^{M} \int dE | a, E \rangle E \langle a, E |
\]

\[
+ \sum_{n,a} \left( | n \rangle \int dE \ W_{na} \langle a, E | + c.c. \right)
\]

bound states
Hamiltonian \( H \)

\( N \gg 1 \) bound states \( | n \rangle \)

\( M \) channel states \( | a, E \rangle \)

coupling \( W_{na} \)
Yields Scattering Matrix

\[ S_{ab}(E) = \delta_{ab} - i2\pi W_a^\dagger G(E) W_b \]

with matrix resolvent containing bound states Hamiltonian \( H \)

\[ G(E) = \frac{1}{E1_N - H + i\pi \sum_{c=1}^{M} W_c W_c^\dagger} \]

absence of direct reactions consistent with orthogonality

\[ W_a^\dagger W_b = \frac{\gamma_a}{\pi} \delta_{ab} \]

Mahaux, Weidenmüller (1969)
Heidelberg Approach to Stochastic Scattering

Hamiltonian $H$ modeled as a Gaussian random matrix

$$P(H) \sim \exp \left( -\frac{N\beta}{4v^2} \text{tr} H^2 \right)$$

form of $P(H)$ irrelevant on local scale of mean level spacing

$\rightarrow$ two universalities, experimental and mathematical

- no invariance under time–reversal: $H$ Hermitean, $\beta = 2$
- invariance under time–reversal:
  - spin–rotation symmetry: $H$ real symmetric, $\beta = 1$
  - no spin–rotation symmetry: $H$ Hermitean self–dual, $\beta = 4$

Weidenmüller (1960's)
Chaotic Statistics, Example: Compound Nucleus

spacing distribution $p(s)$

probability density to find two adjacent levels in distance $s$

Bohigas, Haq, Pandey (1983)
Counter Example: Collective Excitations in Nuclei

single particle versus collective excitations

scissors mode oscillations, all neutrons ↔ all protons

→ chaotic versus regular statistics

→ crossover transitions are frequent!

Supersymmetry for Correlations
Correlation Functions in RMT

Gaussian ensemble ($\beta = 1, 2, 4$) of $N \times N$ random matrices $H$

$k$-level correlations are probability density to find a level in each interval $[x_p, x_p + dx_p], \ p = 1, \ldots, k$

can be expressed with resolvent

$$R_k^{(\beta)}(x_1, \ldots, x_k) = \int \mathrm{d}[H] \exp(-\text{tr} \ H^2) \prod_{p=1}^k \text{tr} \ \frac{1_N}{H - x_p 1_N}$$

(notation is a bit simplified)
introduce scalar source variables $J_p$

\[
R_k^{(\beta)}(x_1, \ldots, x_k) = \frac{\partial^k}{\prod_{p=1}^k \partial J_p} Z_k^{(\beta)}(x + J) \bigg|_{J=0}
\]

and generating function

\[
Z_k^{(\beta)}(x + J) = \int d[H] \exp(-\text{tr } H^2) \prod_{p=1}^k \frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)}
\]
Supersymmetric Representation

vectors $z_p, \zeta_p$ with commuting and anticommuting entries

$$\frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)} = \int d[z_p] \exp \left( iz_p^\dagger (H - x_p + J_p) z_p \right)$$

$$\int d[\zeta_p] \exp \left( i\zeta_p^\dagger (H - x_p - J_p) \zeta_p \right)$$

average over $H$ just Gaussian!

intermediate result is integral over $z_p, \zeta_p$, $p = 1, \ldots, k$

but depends only on scalar products

$z_p^\dagger z_q, \zeta_p^\dagger \zeta_q$ commuting

$z_p^\dagger \zeta_q, \zeta_p^\dagger z_q$ anticommuting
Supermatrix Integral

Hubbard–Stratonovitch transformation: “use scalar products as integration variables, remaining ones are trivial”

Identity (yes, this is exact) for generating function

\[ Z_k^{(\beta)}(x + J) = \int d[\sigma] \exp(-\text{str } \sigma^2) \text{sdet}^{-N}(\sigma - x - J) \]

where \( \sigma \) is a \( 2k \times 2k \) or \( 4k \times 4k \) supermatrix

\[ \rightarrow \text{ drastic reduction of dimensions} \]

Scattering Matrix

\[ S_{ab}(E) = \delta_{ab} - i2\pi W^\dagger_a G(E) W_b \]

does not depend on an invariant, but on resolvent matrix

\[ G(E) = \frac{1_N}{E1_N - H + i\pi \sum_{c=1}^{M} W_c W_c^\dagger} \]

introduce \( N \times N \) matrix source variable \( J \)

\[ G_{nm}(E) = \left. \frac{\partial}{\partial J_{nm}} \frac{\det(G^{-1}(E) - J)}{\det(G^{-1}(E) + J)} \right|_{J=0} \]

determinants linear in \( H \) \( \rightarrow \) supersymmetry method

Verbaarschot, Weidenmüller, Zirnbauer (1985)
Many Results Obtained in this Way, for Example

Two–point correlation functions $\langle S_{ab}(E_1)S_{cd}(E_2) \rangle$

$\beta = 1$ Verbaarschot, Weidenmüller, Zirnbauer (1985)

$\beta = 2$ Savin, Fyodorov, Sommers (2006)

Higher order correlations, perturbative time–invariance breaking


distribution of diagonal elements $P(S_{aa}(E))$

Fyodorov, Savin, Sommers (2005)

... but: does not work for distribution $P(S_{ab}(E)), \ a \neq b$

$\rightarrow$ new method needed
Supersymmetry for Distributions
\[
S_{ab}(E) = \delta_{ab} - i2\pi W_a^\dagger G(E) W_b
\]

Wish to calculate distribution of real and imaginary part

\[
\varphi_s(S_{ab}) = \pi \left( (-i)^s W_a^\dagger GW_b + i^s W_b^\dagger G^\dagger W a \right)
\]

such that

\[
x_1 = \varphi_1(S_{ab}) = \text{Re} \, S_{ab}(E) \quad \text{and} \quad x_2 = \varphi_2(S_{ab}) = \text{Im} \, S_{ab}(E)
\]

distribution given by

\[
P_s(x_s) = \int d[H] \exp(-\text{tr} \, H^2) \delta(x_s - \varphi_s(S_{ab})) , \quad s = 1, 2
\]
obtain distribution by Fourier backtransform of
\[ R_s(k) = \int d[H] \exp(-\text{tr} H^2) \exp(-ik\mathcal{O}_s(S_{ab})) \]

insert definition of scattering matrix
\[ R_s(k) = \int d[H] \exp(-\text{tr} H^2) \exp(-ik\pi W\dagger A_s W) \]

with \( W = \begin{bmatrix} W_a \\ W_b \end{bmatrix} \) and \( A_s = \begin{bmatrix} 0 & (-i)^s G \\ i^s G\dagger & 0 \end{bmatrix} \)

where \( A_s \) Hermitean, but contains \( H \) inverse

problem: have to invert \( A_s \) to perform \( H \) average!
Crucial Trick

Fourier transform in $W$ space — Yields

$$\exp(-ik\pi W^\dagger A_s W)$$

$$\sim \int d[z] \exp \left( \frac{i}{2} (W^\dagger z + z^\dagger W) \right) \det^{\beta/2} A_s^{-1} \exp \left( \frac{i}{4\pi k} z^\dagger A_s^{-1} z \right)$$

now use anticommuting variables

$$\det^{\beta/2} A_s^{-1} \sim \int d[\zeta] \exp \left( \frac{i}{4\pi k} \zeta^\dagger A_s^{-1} \zeta \right)$$

now $H$ linear in exponent $\rightarrow$ supersymmetry applicable!

different rôle of commuting and anticommuting variables
Hubbard–Stratonovitch transformation gives

\[ R_s(k) = \int d[\varrho] \exp \left( -r \text{str} \varrho^2 - \frac{\beta}{2} \text{str} \ln \Xi - \frac{i}{4} F_s \right) \]

with \( 8/\beta \times 8/\beta \) supermatrix \( \varrho \) and \( r = 4\beta \pi^2 k^2 N/v^2 \)

\[ \Xi = \varrho_E \otimes 1_N + \frac{i}{4k} L \otimes \sum_{c=1}^M W_c W_c^\dagger, \quad \varrho_E = \varrho - \frac{E}{4\pi k} 1_{8/\beta} \]

matrix \( L \) is some superspace metrik

\[ F_s \sim [W^\dagger 0^\dagger] \Xi^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix}, \quad \text{projects onto boson–boson space} \]

\[ \rightarrow \text{symmetry breaking differs from the one for correlations} \]
Supersymmetric Non–Linear sigma Model

limit $N \rightarrow \infty$, unfolding by saddlepoint approximation
integrate out “massive” modes

left with integral over “Goldstone” modes $Q$,
free rotations, coset manifold in superspace

$$R_s(k) = \int d\mu(Q) \exp \left(-\frac{i}{4} F_s \right) \prod_{c=1}^{M} \text{sdet}^{-\beta/2} \left(1_{8/\beta} + \frac{i\gamma_c}{4\pi k} Q_E^{-1} L \right)$$

integrate out all remaining anticommuting variables

left with ordinary integrals, two for $\beta = 2$, four for $\beta = 1$

$\rightarrow$ drastically reduced number of integration variables
Analytical Results versus Numerics
Reproducing the Circular Ensemble for $\beta = 2$

number of channels $M = 2$, energy $E = 0$, width parameters $\gamma_1/D = 1$, $\gamma_2/D = 1$

real and imaginary parts always equally distributed for $\beta = 2$
number of channels $M = 5$, energy $E/D = 1.2$, width parameters $\gamma_j/D$ between 0.08...0.72

real and imaginary parts always equally distributed for $\beta = 2$.
Towards Ericsson Regime for $\beta = 2$

average resonance width / mean level spacing $\Gamma / D = 0.716$ (top) and $\Gamma / D = 8.594$ (bottom)
Towards Ericsson Regime for $\beta = 1$

average resonance width / mean level spacing $\Gamma/D = 1.273$ (top) and $\Gamma/D = 7.162$ (bottom)

real and imaginary parts not equally distributed for $\beta = 1$
Comparison with Microwave Experiments
... vs Numerics and Experiment for $\beta = 1$

frequency range $10 \ldots 11\text{GHz}$,
average resonance width / mean level spacing $\Gamma/D = 0.234$
Analytical Result vs Experiment for $\beta = 1$

frequency range 24...25GHz,
average resonance width / mean level spacing $\Gamma/D = 1.21$
Distribution of Cross Sections
No Way Around the Joint Probability Density

cross section \( \sigma_{ab}(E) = |S_{ab}(E)|^2 = \text{Re}^2 S_{ab}(E) + \text{Im}^2 S_{ab}(E) \)

need joint pdf \( P(\text{Re} S_{ab}, \text{Im} S_{ab}) = P(S_{ab}, S_{ab}^*) \)

to calculate \( p(\sigma_{ab}) = \int d^2 S_{ab} P(S_{ab}, S_{ab}^*) \delta(\sigma_{ab} - |S_{ab}|^2) \)

good news: can extend previous calculation into complex plane

characteristic \( R(k, k^*) = \int d[H] \exp(-\text{tr} H^2) \exp(-i \text{Re} k^* S_{ab}) \)

simply replace real \( k \) with complex \( k = k_1 + ik_2 \) everywhere

distribution \( p(\sigma_{ab}(E)) = \int d^2 k R(k, k^*) J_0(\sqrt{\sigma_{ab}(E)}|k|) \)
Comparison with Microwave and Nuclear Data
Characteristic Functions for Microwave Data

\[ \Gamma/D = 0.234 \]

\[ \Gamma/D = 1.21 \]
Cross Section Distributions

$\Gamma / D = 1.21$

nuclear data $^{37}\text{Cl}(p,\alpha)^{34}\text{S}$

$p(0) \approx 1$ indicates Ericson regime

cross section distribution becomes exponential in Ericson regime
Analytical Results vs Cross Section Data

microwave data (left) $\Gamma/D = 0.7, 1.2$

nuclear data (right) $\Gamma/D \approx 1, 30$

$p(0) \approx 1$ indicates Ericson regime
What have we learned?
Conclusions and Outlook

- Mexico and Heidelberg approaches to stochastic scattering
- solved longstanding problem within Heidelberg approach
- now have supersymmetry for distributions
- distributions of scattering matrix elements and cross sections
- additional results: characteristic function generates moments, integral representations for all of them
- full analytical understanding of transition to Ericson regime
- Brouwer’s equivalence proof Heidelberg–Mexico implies: now have explicit handle on Mexico approach for arbitrary channel number
- comparison with microwave and nuclear data
- also: condensed matter and wireless communication
Thank You for Your Attention!