UNIVERSITÄT DUISBURG ESSEN

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Thomas Guhr:

Random Matrix Theory: from Single– to Many–Body Quantum Chaos

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1. Phenomenological Model for the Spacing Distribution

The nearest–neighbor–spacing distribution $p(s)$ is one of the most important spectral observables. It is easily accessible in experiments and numerical simulations, in contrast to the energy correlation functions $X_k(\xi_1 \ldots, \xi_k)$ of higher order k.

(a) Compare $p(s)$ und $X_2(r)$, $r = \xi_1 - \xi_2$, we recall that the unfolded energy correlation functions only depend on the differences $\xi_i - \xi_j$. Which $X_k(\xi_1 \ldots, \xi_k)$ are needed to calculate $p(s)$, i.e. to which order k do they enter $p(s)$?

(b) In a phenomenological model, the spacing distribution satisfies the integral equation

$$p(s) \ = \ \mu(s) \int\limits_s^\infty p(s')ds' \ .$$

Solve this equation for arbitrary $\mu(s)$, use $p(0) = \mu(0)$. Show that the normalization $\int_0^\infty p(s)ds = 1$ holds, if $\int_0^\infty \mu(s)ds \to \infty$.

(c) Which are the nearest–neighbor–spacing distributions that you get for the choices $\mu(s) = 1$ and $\mu(s) = \pi s/2$? Make qualitative drawings. What is the phenomenological meaning of the function $\mu(s)$?

2. Heuristic Derivation of Wigner's Surmise

Wigner put forward the surmise that the nearest–neighbor–spacing distribution $p(s)$ for Gaussian random matrices is well approximated by the formula

$$p(s) \ = \ p^{(W)}(s) \ = \ \frac{\pi}{2}\, s \, \exp\left(-\frac{\pi}{4}s^2\right) \ .$$

Show that this is the spacing distribution for real symmetric 2×2 matrices

$$H \ = \ \left[\begin{array}{cc} H_{11} & H_{12} \\ H_{12} & H_{22} \end{array} \right] \ .$$

The matrix elements are Gaussian distributed,

$$P(H) \;=\; \frac{\sqrt{2}}{\sqrt{2\pi v^2}^3}\,\exp\left(-\frac{1}{2v^2}\mathrm{tr}\,H^2\right)$$

with variance v^2. The eigenvalues of H are E_1 und E_2. The dimensionless distance to the next neighbor is in this case just $s = |E_1 - E_2|/D$, where D formally represents the mean level spacing. The nearest–neighbor–spacing distribution is then given as

$$p(s) \;=\; \int\limits_{-\infty}^{+\infty} dH_{11} \int\limits_{-\infty}^{+\infty} dH_{22} \int\limits_{-\infty}^{+\infty} dH_{12}\, P(H)\, \delta\left(s - |E_1 - E_2|/D\right) \ .$$

Compute $p(s)$ by making the following steps:

(a) The matrix H is diagonalized by an orthogonal 2×2 matrix $\mathrm{U}(\varphi)$ which depends on an angle φ. We have $H = U^T E U$ with $E = \mathrm{diag}\,(E_1, E_2)$. Carry out a change of variables ino eigenvalue–angle coordinates. Hint: the volume element transforms as

$$dH_{11} dH_{22} dH_{12} \;=\; \frac{1}{4}|E_1 - E_2| dE_1 dE_2 d\varphi \ ,$$

where E_1 und E_2 take values in $\mathbb{R}$. Do the angular integration.

(b) Calculate the eigenvalue integrals, use $\widetilde{E} = (E_1 + E_2)/2$ and $\varepsilon = E_2 - E_1$ as new coordinates.

(c) Determine v^2 and D from the two normalization conditions

$$\int\limits_0^\infty p(s)\, ds \;=\; 1 \qquad \text{and} \qquad \int\limits_0^\infty s\, p(s)\, ds \;=\; 1 \ .$$

What is the meaning of the second condition?

(d) Why is this derivation heuristic? Why is this not a full derivation of the spacing distribution in Random Matrix Theory?

3. Level Number Variance for Regular Systems

The level number variance $\Sigma^2(L)$ was introduced in the lecture by dividing the unfolded spectrum into M windows. The window $m = 1, \ldots, M$ contains $\nu_m(L)$ levels. The level number variance is then

$$\Sigma^2(L) \;=\; \langle \nu^2(L) \rangle \;-\; \langle \nu(L) \rangle^2 \ ,$$

where $\langle \ldots \rangle$ is the average over all windows. On the unfolded scale, there are $L \pm \sqrt{\Sigma^2(L)}$ levels in an interval of length L. Thus, in contrast to the nearest–neighbor–spacing distribution, the level number variance gives information on long–range properties of the level statistics. For chaotic systems, $\Sigma^2(L)$ behaves logarithmically for larger L. For regular systems, one often finds $\Sigma^2(L) \sim L$. The case of the harmonic oscillator in one dimension is very special.

(a) Calculate $\Sigma^2(L)$ for the unfolded spectrum of the harmonic oscillator in one dimension. Begin with considering integer $L \in \mathbb{N}$. What is then $\Sigma^2(L)$? Show $\Sigma^2(L) = \Sigma^2(L+1)$. In the general case, decompose $L = [L] + x$ where $0 \leq x < 1$ with $[L]$ being the largest integer smaller L.

(b) Show $\Sigma^2(L) = L$ for a system with Poissonian spectrum.

4. **Spectral Rigidity**

The spectral rigidity $\Delta_3(L)$ is closely related to the level number variance. In the lecture, it was introduced for window m as the minimum

$$\Delta_{3m}(L) = \frac{1}{L}\min_{A,B} \int_0^L d\xi \, (\nu_m(\xi) - A\xi - B)^2 \ ,$$

where $\nu_m(\xi)$ is the number of levels in window m.

(a) Express A und B in terms of $\nu_m(\xi)$ by minimizing the above integral.

(b) Re–insert A and B in the above expression to show that $\Delta_{3m}(L)$ can be written as

$$\Delta_{3m}(L) = \frac{1}{L}\int_0^L \nu_m^2(\xi) d\xi - \frac{1}{L}\vec{I}^T C \vec{I} \ ,$$

where the vector $\vec{I}$ is defined as

$$\vec{I} = \begin{bmatrix} \int_0^L \xi \nu_m(\xi) d\xi \\ \int_0^L \nu_m(\xi) d\xi \end{bmatrix} .$$

What is the meaning of these two integrals? Determine the matrix C.

5. **Spherical Functions — Itzykson–Zuber Integral**

To compute spectral transitions, group integrals are needed. Unfortunately, only the case of integrals over the unitary group is explicitly doable for arbitary matrix dimension N. Consider the simplest orthogonal case, the integral over SO(2),

$$\Phi_2^{(1)}(x,k) = \int_{\mathrm{SO}(2)} \exp\left(i \,\mathrm{tr}\, x U^T k U\right) d\mu(U) \ ,$$

here we have $x = \mathrm{diag}\,(x_1, x_2)$, $k = \mathrm{diag}\,(k_1, k_2)$. To calculate it, proceed as follows:

(a) Write $U \in \mathrm{SO}(2)$ as a rotation matrix with angle φ in two real dimensions. Hint: $d\mu(U) = d\varphi$.

(b) Calculate $\mathrm{tr}\, xU^TkU$, write it in terms of only $\cos 2\varphi$. Hint: differences and sums $x_1 \pm x_2$, $k_1 \pm k_2$ appear.

(c) Which function results from the remaining integral over φ ?

6. **Spectrum of Chiral (Random) Matrices**

In chiral Random Matrix Theory, one starts from the Dirac operator in the chiral basis and replaces it by matrices of the form

$$D = \begin{bmatrix} 0 & M \\ M^\dagger & 0 \end{bmatrix} ,$$

such that $D = D^\dagger$. Here, D should not be confused with the mean level spacing introduced previously. We notice that the matrix M has no further symmetries. In particular, M can be a rectangular matrix of dimension $N \times L$. Without loss of generality, we may assume that $L \geq N$. Moreover, we restrict ourselves to the case that M and hence D are real. Chiral symmetry reflects itself in a pointwise symmetry of the spectrum. Show the following:

(a) In the case $L > N$, there are $L - N$ eigenvalues which are zero. They are called zero modes.

(b) All non–zero eigenvalues λ of D come in pairs. What does that imply for the level density?

(c) The spectrum of chiral random matrices is closely related to that of Wishart correlation matrices.