NRGR for the two-body problem: an overview

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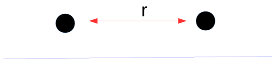
14/8/23 – Principia Institute – QCD meets Gravity in Brasil

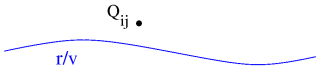
Fundamental aspects of the EFT formalism

NRGR & PN approximation to GR: Small expansion parameter v, related to metric perturbation $v^2 \sim \frac{GM}{r}$

Near zone, $D \sim r$



Far zone, $D \gtrsim \lambda = r/v$



Describe conservative dynamics

conservative + dissipative

EFT framework pioneered by W. Goldberger & I. Rothstein, hep-th/0409156
for spin R. Porto, gr-qc/0511061 (no spin in this talk)
Inspired by NRQCD, see e.g. Manohar & Stewart hep-ph/0605001

PN approximation for compact binary systems

	Near	Far
World-line	$-m_a \int dt + \int dx^\mu S^{ij}\omega_{\mu ij}$	$\int d^4x \left(Eh_{00} + \frac{1}{2}S^{ij}h_{0i,j}\right.$
	$\int dt \left(Q_{ij}(S)R_{0i0j} + R^2_{\mu\nu\rho\sigma} + \ldots\right)$	$\left.+Q_{ij}\underbrace{E^{ij} + O_{ijk}}_{v}E^{ij,k} + J_{ij}B_{ij}\ldots\right)$
Bulk	$\frac{1}{16\pi G}\int d^4x \left[R - \frac{1}{2}\left(g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\right)^2\right]$	
5PN	Blumlein, Maier, Marquard, Schäfer	Foffa, RS : 1907.02869, 2103.03190
	2110.13822	+Almeida 2008.06195, 2107.02634
	2208.04552	2110.14146, 2209.11594
		+Müller 2307.05327

Summary: 2 body dynamics expansions (spin-less)

Post-Minkowskian expansion parameter is $G_N M/r$, vs PN expansion

$$\mathcal{L} = -Mc^2 + \frac{\mu v^2}{2} + \frac{GM\mu}{r} + \frac{1}{c^2}[\dots] + \frac{1}{c^4}[\dots]$$

		N	1PN	2PN	3PN	4PN	5PN*	6PN	...
0PM	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	v^{14}	...
1PM		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	v^{12}/r	...
2PM			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	v^{10}/r	...
3PM				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	v^8/r^3	...
4PM*					$1/r^4$	v^2/r^4	v^4/r^4	v^6/r^4	...
5PM						$1/r^5$	v^2/r^5	v^4/r^5	...
6PM							$1/r^6$	v^2/r^6	...
...								$1/r^7$	...

4PM (but for some conservative processes involving radiation) by Bern, Parra-Martinez, Roiban, Ruf, Shen 2101.07254, 2112.10750
Dlapa, Kälin, Liu, (Nef), Porto 2210.05541, 2106.08276

Still some issues at 5PN-4PM to be solved, see later in this talk

Fundamental GR: inspiral analytic model

Inspiral $h = A\cos(\phi(t))$ $\qquad \frac{\dot{A}}{A} \ll \dot{\phi}$

Virial relation:

$$v \equiv (G_N M\pi f_{GW})^{1/3} \qquad \eta = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$\begin{aligned} E(v) &= -\frac{1}{2}\eta M v^2 \left(1 + \#(\eta, S_i/m_i^2)v^2 + \#(\eta, S_i/m_i^2)v^4 + \ldots\right) \\ P(v) \equiv -\frac{dE}{dt} &= \frac{32}{5G_N}v^{10}\left(1 + \#(\eta, S_i/m_i^2)v^2 + \#(\eta, S_i/m_i^2)v^3 + \ldots\right) \end{aligned}$$

E(v)(P(v)) known up to 4(4.5)PN Blanchet, Faye, Henry, Larrouturou, Trestini 2304.11186, Marchand, Blanchet, and G. Faye 1607.07601

$$\begin{aligned} \frac{1}{2\pi}\phi(T) &= \frac{1}{2\pi}\int^T \omega(t)dt = -\int^{v(T)} \frac{\omega(v)dE/dv}{P(v)}dv \\ &\sim \int \left(1 + \#(\eta, S_i/m_i^2)v^2 + \ldots + \#(\eta, S_i/m_i^2)v^6 + \ldots\right)\frac{dv}{v^6} \end{aligned}$$

Fundamental GR: inspiral analytic model

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Virial relation:

$$v \equiv (G_N M \pi f_{GW})^{1/3} \qquad \eta = \frac{m_1 m_2}{(m_1+m_2)^2}$$

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$$P(v) \equiv -\frac{dE}{dt} = \frac{32}{5G_N}v^{10}\left(1 + \#(\eta, S_i/m_i^2)v^2 + \#(\eta, S_i/m_i^2)v^3 + \ldots\right)$$

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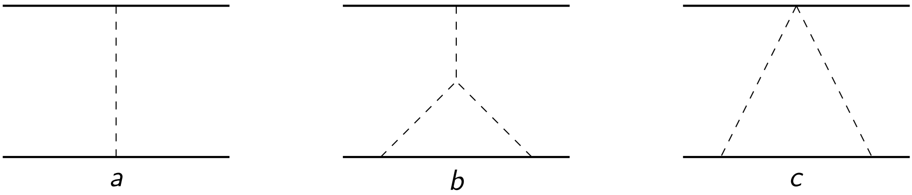
$$\sim \int \left(1 + \#(\eta, S_i/m_i^2)v^2 + \ldots + \#(\eta, S_i/m_i^2)v^6 + \ldots\right)\frac{dv}{v^6}$$

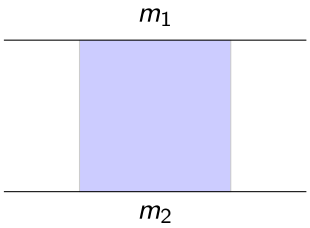
PN Coefficients (absorption $\sim v^8$, tidal $\sim v^{10}$)

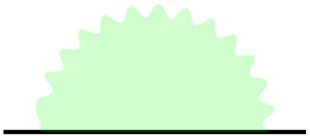
Near Zones examples

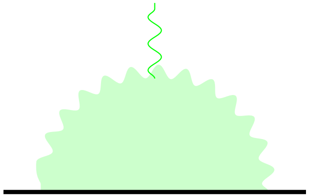


Examples of NZ processes at LO and NLO. From *Riemann*^{2} (diag c) in the world-line one has $(\partial^2)^2 1/r^2 \sim 1/r^6 \sim 5PN$ finite size effect

Near vs. Far zone graphs







And 1 pt diagrams $\rightarrow$ radiation
In this talk no radiation to ∞

Method of regions

Method of regions: Internal graviton momentum can be expanded following the scaling:

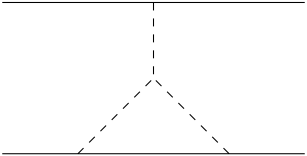
hard	(m, m)	quantum	×
soft	$(\|\vec{q}\|, \|\vec{q}\|)$	quantum	×
potential	$(v/r, 1/r)$	classical	✓
radiation	$(v/r, v/r)$	classical	✓

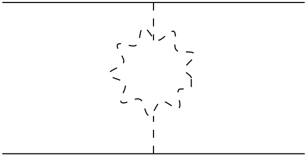
and then integrated over the full phase space

Only potential and radiation gravitons exchanged in classical processes: theory in terms of world lines selects diagrams that do not send source off-shell

Potential graviton $\rightarrow$ small change in energy wrt momentum, dominate classically

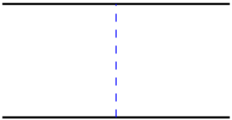
Ex. of classical/quantum connected diagrams





Near zone conservative dynamics

The potential V (via Feynman Green function):

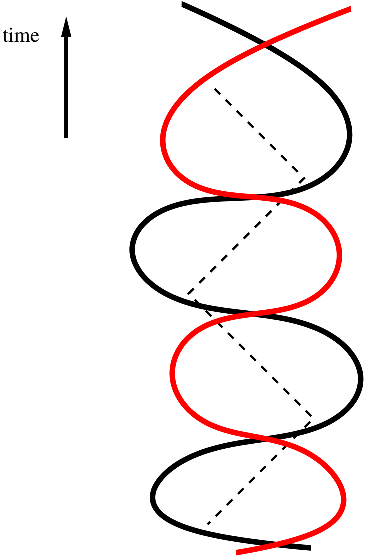


$$
\begin{aligned}
V & \propto \int d\omega d^3k \frac{e^{-i\omega t_{12}+i\vec{k}\cdot(\vec{x}_1(t_1)-\vec{x}_2(t_2))}}{k^2-\omega^2-i\epsilon} = \int d\omega d^3k \frac{e^{-i\omega t_{12}+i\vec{k}\cdot\vec{x}_{12}}}{k^2}\left(1+\frac{\omega^2}{k^2}+\ldots\right) \\
& = \delta(t_1-t_2)\int d^3k \frac{e^{i\vec{k}\cdot\vec{x}_{12}}}{k^2}\left(1+\frac{\partial_{t_1}\partial_{t_2}}{k^2}+\ldots\right) \\
& = \int d^3k \frac{e^{i\vec{k}\cdot\vec{x}_{12}}}{k^2}\left(1-\frac{\vec{k}\cdot\vec{v}_1\vec{k}\cdot\vec{v}_2}{k^2}+\ldots+\frac{\vec{k}\cdot\frac{d^{n-1}\vec{v}_1}{dt^{n-1}}\vec{k}\cdot\frac{d^{n-1}\vec{v}_2}{dt^{n-1}}}{k^{2n}}\right)
\end{aligned}
$$

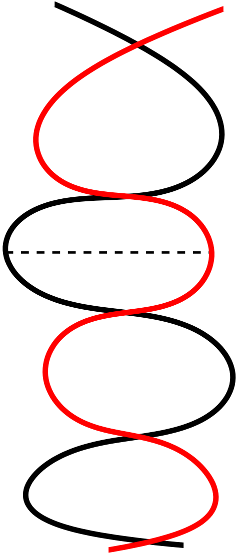
- Instantaneous interaction + trajectory derivatives
- Near zone amplitude integrands clearly bad behaved for $k \to 0$ (though no IR divs at $O(G)$). Integrated all the way to $k \to 0$!
  Fix: add the contribution of far-zone, for demonstration see e.g.

  Manohar & Stewart hep-ph/0605001, Jantzen 1111.2589, Blumlein+ 1902.11180, Foffa & RS 2103.03190

NR zone trades knowledge over the full trajectory with knowledge of all derivatives of the trajectory at equal time (PN approximation)



$\rightarrow$



$$+\dot{v} \sim \frac{v^2}{r}$$
$$+\ddot{v} \sim \frac{v^3}{r^2}$$
$$+\ldots$$

Scale separation → Near + Far

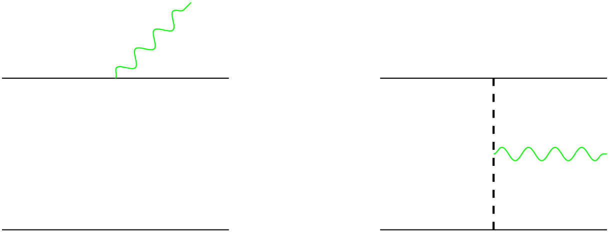
The full (I), Near, Far zone integral are

$$
\begin{aligned}
I &\equiv e^{-i\omega t}e^{i\mathrm{k}\cdot\mathrm{x}}\frac{1}{\mathrm{k}^2-\omega^2}\,,\\
N &\equiv e^{-i\omega t}e^{i\mathrm{k}\cdot\mathrm{x}}\frac{1}{\mathrm{k}^2}\sum_{m\geq 0}\left(\frac{\omega^2}{\mathrm{k}^2}\right)^m\,,\\
F &\equiv e^{-i\omega t}\sum_{n\geq 0}\frac{(i\mathrm{k}\cdot\mathrm{x})^n}{n!}\frac{1}{\mathrm{k}^2-\omega^2}\,,
\end{aligned}
$$

Decomposing

$$
\begin{aligned}
\int_{\mathrm{k}} I &= \int_{\mathrm{k}} F + \int_{\mathrm{k}} N + \int_{\mathrm{k}<\bar{\mathrm{k}}}\Big(\underbrace{I-F}_{=0}-N\Big) + \int_{\mathrm{k}>\bar{\mathrm{k}}}\Big(\underbrace{I-N}_{=0}-F\Big)\\
&= \int_{\mathrm{k}} F + \int_{\mathrm{k}} N + \int_{\mathrm{k}<\bar{\mathrm{k}}} N + \int_{\mathrm{k}>\bar{\mathrm{k}}} F\\
&= \int_{\mathrm{k}} F + \int_{\mathrm{k}} N + \underbrace{\int_{\mathrm{k}}\sum_{n,m}\frac{(i\mathrm{k}\cdot\mathrm{x})^n}{n!}\left(\frac{\omega^2}{\mathrm{k}^2}\right)^m}_{=0\ in\ dim.\ reg.}
\end{aligned}
$$

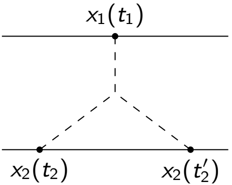
Matching: Multipoles in terms of two-body variables



Expanding in powers of the external momentum one obtains the coupling between long wavelength and multipole moments

(IR) Divergences in the Near zone

IR divergences due to splitting into Near and Far



In the full theory:

$$
\begin{aligned}
V \quad &\supset \quad \int dt_{1,2,2'}\, d^4k\, e^{ik_\mu(x_1^\mu(t_1)-x_2^\mu(t_2))}\frac{p^\alpha p^\beta}{p^2}\int d^4q\,\frac{e^{iq^\mu(x_2(t_2)-x_2(t_2'))}}{(k-q)^2q^2} \\
&\simeq \quad \int dt_{1,2,2'}\, d^4k e^{ik_\mu(x_1^\mu(t_1)-x_2^\mu(t_2))}\frac{k^\alpha k^\beta}{k^2}\Delta(k^\mu(x_2(t_2)-x_2(t_2')))
\end{aligned}
$$

after near/far breaking:

$$
\int dt\, d\omega d^3k\, e^{-i\omega t_{12}}e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}\frac{k^ik^j}{|\mathrm{k}^2|}\int d^3q\frac{1}{|\mathrm{q}|^2|\mathrm{k}-\mathrm{q}|^2}\left(1+\ldots+\frac{\omega^4}{|\mathrm{k}|^4}+\frac{\omega^6}{|\mathrm{k}|^6}+\ldots\right)
$$

$$
=\int dt\, d^3ke^{i\vec{k}\cdot\vec{x}_{12}}\frac{k^ik^j}{|\mathrm{k}|^3}\left\{1+\ldots+\frac{1}{|\mathrm{k}|^4}\left[\ldots\underbrace{\vec{k}\cdot\vec{a}_1\vec{k}\cdot\vec{a}_2}_{0\times IR\ \ div}\right]+\frac{1}{|\mathrm{k}^6|}\left[\cdots+\underbrace{\vec{k}\cdot\dot{\vec{a}}_1\vec{k}\cdot\dot{\vec{a}}_2}_{IR\ \ div}\right]\right\}
$$

Compensating divergences Near - Far

- Near IR/Far UV

$$V \supset GM\dddot{Q}_{ij}^2 \left(\frac{1}{\epsilon_{UV}} + \log(\mu\omega)\right) + GMm_1 m_2 r^2 \dot{a}_1^i \dot{a}_{2i} \left(\frac{1}{\epsilon_{IR}} + \log(\mu r)\right)$$

Theory at short and large distances have compensating spurious divergences, finite terms derived straightforwardly (Manohar & Stewart, Jantzen)

- Near zone UV divergences canceled by local counterterms:

$$G^2 m_a^3 \int d\tau \, (a^\mu \dot{v}_\mu + R_{\mu\nu} v^\mu v^\nu)$$

Foffa, RS, Porto, Rothstein 1903.05118

$a^\mu = 0 = R_{\mu\nu}$ on the equations of motion

- No far zone IR divergences
- From far zone alone → leading UV logs in the Energy function at all orders via Renormalization group flow (see later in this talk)

Goldberger, Ross, Rothstein 0912.4254, 1211.6095, Blanchet, Foffa, Larrouturou, RS 1912.12359

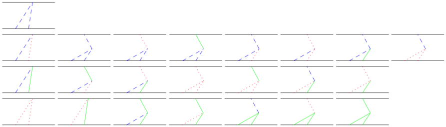
Diagrams proliferations

E.g. at 5PN: G 3, G^2 72, G^3 2702, G^4 41676, G^5 116498, G^6 27582
at 6PN: G 3, G^2 24, G^3 485, G^4 5553 (symmetrised)

Blumlein, Maier, Marquard, Schäfer 2010.13672, 2101.08630

For PN expansion it is useful to distinguish gravity polarizations, e.g. G_N^2

2 topologies



23 Feynman diagrams at 4PN ($G^2 v^3$)

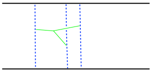
- - - - - $\sim h_{00} \to v^0$; .... $\sim h_{0i} \to v^i$; ——— $\sim h_{ij} \to v^i v^j$;

However only 1 of the two topologies is intrinsically G_N^2

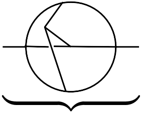
EFT and amplitude: tale of a happy marriage

The other obstruction to scalability of the NRGR PN calculation program (commonn problem with other framework) is the computation of master integrals

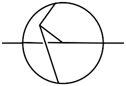
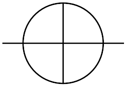
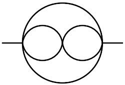
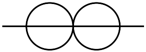
E.g. in the static 4PN sector (i.e. G_N^5) one meets

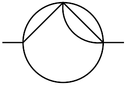
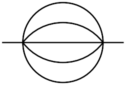


$$= -\mathrm{i}\,(8\pi G_N)^5 \left(\frac{(d-2)}{(d-1)}\; m_1 m_2\right)^3$$



$$\int_{k_{1,2,3,4}} \frac{N_{50}}{k_1^2\;k_2^2\;k_3^2\;k_4^2\;k_{12}^2\;k_{34}^2\;\hat{k}_{24}^2\;p_{13}^2\;\hat{p}_{14}^2}$$

 $= c_1$  $+ c_2$  $+ c_3$ 

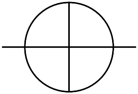
$+ c_4$  $+ c_5$ 

in terms of 4-loop self-energy diagrams in gauge theory

Reduction in terms of master integrals

No new master integrals at 5PN, 4PN ones did it all

Foffa, Mastrolia, RS, Sturm '17



$$= \frac{e^{2\varepsilon\gamma_E}}{s^{2-2\varepsilon}\left(4\pi\right)^{4+2\varepsilon}}\left\{\frac{1}{2\varepsilon^2}-\frac{1}{2\varepsilon}-4+\frac{\pi^2}{24}\right.$$

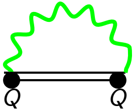
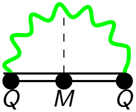
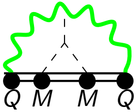
$$\left.-\varepsilon\left[9-\pi^2\left(\frac{13}{8}-\log 2\right)-\frac{77}{6}\zeta_3\right]+\mathcal{O}\left(\varepsilon^2\right)\right\}$$

Numerical result obtained via Summertime by Lee& Mingulov
analytic result via PSLQ algorithm, fitting trascendentals to numerical result

Confirmed up to $O(\varepsilon^0)$ by Damour, Jaranowski '18

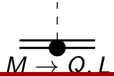
Including radiative modes

In Far zone self-energy diagrams, careful is needed when using radiative modes
Two oriented propagators treatable with standard Feynman Green functions (in-out)

Real part E:	Self-E$\sim$ $(\ddot{Q})(\dddot{Q}) \sim 0$	$\text{tail}^1 \sim$ $GM(\dddot{Q})^2 \log$ $x^4 \log x$	$\text{tail}^2 \sim$ $(GM)^2(\dddot{Q})(\ddddot{Q})$ $x^{11/2}$
			
Imaginary $\frac{dE}{dt}$:	LO flux	$\text{tail}^1 \sim \pi x^{3/2}$	$\text{tail}^2 \sim x^3 log(x)$

$$
\begin{aligned}
G_F &= -i(\theta_+\Delta_+ + \theta_-\Delta_-) \\
G_{R/A} &= \mp i\theta_\pm (\Delta_+ - \Delta_-) \\
\tilde{G}_F^2 &= \frac{1}{2}(\tilde{G}_A^2 + \tilde{G}_R^2) - \tilde{\Delta}_+\tilde{\Delta}_- + i\left(\tilde{G}_R + \tilde{G}_A\right)\left(\tilde{\Delta}_+ + \tilde{\Delta}_-\right)
\end{aligned}
$$

Other insertions possible:



When more than two rad. props in-in needed
(see talks in the next weeks by Almeida and Foffa)

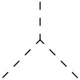
Summary

- NRGR is an efficient method for computations from first-principle, automatically kills quantum contributions
- Divergences (both UV and IR) well understood (indeed highly constrained hence helpful for sanity checks)
- Higher order $\rightarrow$ new master integrals, same problem for any perturbative method (PN, PM. . .)
- Diagram proliferation: necessary smart ideas and/or efficient codes
- Conservative+Averaged flux from in-out formalism for up to 2 radiative modes, for more radiative modes WIP

Double copy for EFT

Master integrals have to do with denominators, however numerators can be simplified too by writing $A_{GR} = A_{YM}^2$

Bern, Carrasco, Johansson PRD '08

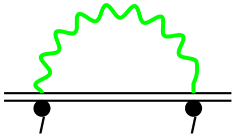
On-shell three vertices can be mapped: 

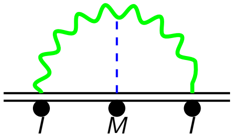
$$
\begin{array}{ccc}
\text{gauge theory} & \to & \text{gravity theory} \\
gf^{abc}\left(\eta_{\mu\nu}\left(k_1 - k_2\right)_\rho + \text{cyclic}\right) & \to & \sqrt{G_N}\left(\eta_{\mu\nu}\left(k_1 - k_2\right)_\rho + \text{cyclic}\right) \\
 & & \times\left(\eta_{\mu\nu}\left(k_1 - k_2\right)_\rho + \text{cyclic}\right) \\
\text{color factors} & \to & \text{kinematic numerator}
\end{array}
$$

Interactions at high loop $\to$ lengthy expressions, double copy?

See e.g. Goldberger+ '18, Chen '18, Almeida+ '20

Double Copy of Far amplitudes





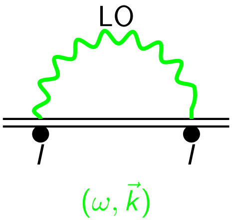
also doubling of world-line vertex  $d_i F_{0i} \to I_{ij} E_{ij}$, $\mu_i F_{ij} \to J_{ij} B_{ij}$

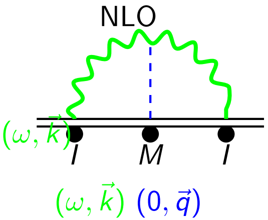
Verified for both electric and magnetic multipoles at NLO in G_N

Goldberger, Ridgway '18, Shen '18, Almeida+ '20

Yet to be verified derivation how YM$\to$ GR mapping propagates from "microscopic physics" to multipoles

Double Copy Far continued

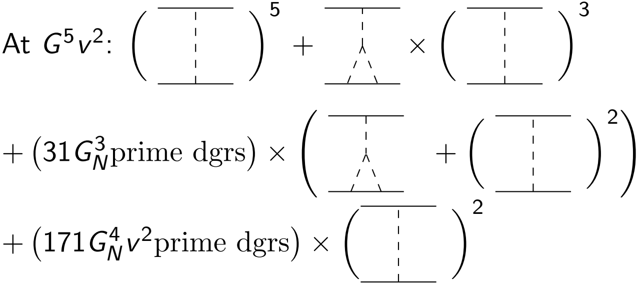




- Electric Self-Energy
  1. LO: $I^{iji_1\ldots i_r}(\omega) I^{iji_1\ldots i_r}(-\omega) \frac{k_{i_1}\ldots k_{i_r} k_{k_1}\ldots k_{k_r}}{k^2-\omega^2} \left(\omega^2\delta_{ik} - k_i k_k\right) \left(\omega^2 \delta jl - k_j k_l\right)$
  2. NLO: $I^{iji_1\ldots i_r}(\omega) I^{iji_1\ldots i_r}(-\omega) \frac{k_{i_1}\ldots k_{i_r} k_{k_1}\ldots k_{k_r}}{(k^2-\omega^2)((k+q)^2-\omega^2)q^2} \omega^2$
     $\times \left(\omega^2\delta_{ik} - (k+q)_i k_k + q_i q_k\right) \left(\omega^2\delta_{jl} - (k+q)_j q_l + q_j q_l\right)$
- analogously for the magnetic self-Energy at LO and NLO

Gravity+dilaton+anti-symmetric tensor amplitude matches gauge^2

Factorizable diagrams at G^5 5PN



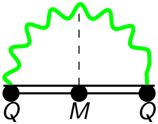
They amount to 1232 out of 1907 $G^5 v^2$ diagrams

Foffa, RS. W. J. Torres Bobadilla, 2010.14550

All leading logs in $E_{circ}(x)$

$$E_{circ} = -\frac{M\nu}{2}x\left(1+\frac{16\nu x^2}{15\beta_I}\left[\left(1+24\beta_I x^3\log x\right)x^{4\beta_I x^3}-1\right]\right) \qquad \beta_I = -\frac{214}{105}$$

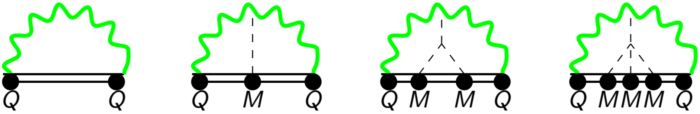
- In PN approximation Log terms arise from tail processes at 4PN order, non-local (but causal) effective term in conservative dynamics ($x \sim v^2 \sim Gm/r \equiv \gamma$):



$$\mathcal{L} = \frac{M\nu}{2}v^2 + \ldots + \frac{2G^2M}{5}\dddot{Q}_{ij}(t)\int d\tau \log(\tau)\ddddot{Q}_{ij}(t-\tau)\ldots$$
$$\to E_{circ} = -\frac{M\nu x}{2}\left(1+\ldots+\frac{448}{15}\nu x^5\log x+\ldots\right)$$

which turns local on circular orbits

- Expansion in $GM\omega = \frac{GM}{r}\times r\omega \sim v^3$:



LO tail: Blanchet, Damour PRD ('88)

Leading Logs at all orders

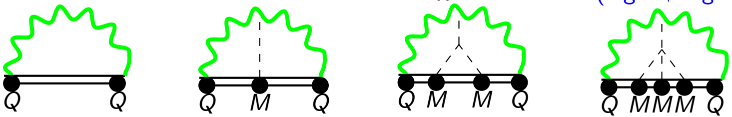
Real part E:

Self-E$\sim$ $(\ddot{Q})(\dddot{Q}) \sim 0$

tail^1 $\sim$ $GM(\dddot{Q})^2 \log t$ $x^4 \log x$

tail^2 $\sim$ $(GM)^2(\dddot{Q})(\ddddot{Q}) \log$ $\sim (GM)^2(\dddot{Q})(\ddddot{Q})$ $x^{11/2}$

tail^3 $\sim$ $(GM)^3 \left(\log + \log^2\right) \times$ $(\ddddot{Q})(\ddddot{Q})$ $x^7(\log x + \log^2 x)$



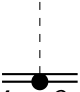
Imaginary $\frac{dE}{dt}$:

LO flux

tail^1 $\sim \pi x^{3/2}$

tail^2 $\sim x^3 log(x)$

tail^3 $\sim ? \times x^{9/2}$

Other insertions possible:  $M \to Q, L$

but do not contribute to leading E-logs: $\nu^2 x^{3n+1} \log^n x$ from tail^{2n-1}

Renormalization group enables to compute all leading logs: E-logs formula extends logs in $E(x)$ at $O(\nu)$ from self-force expanded up to 22 PN in Kavanagh-Ottewill-Wardell PRD 92 (2015)

Far UV divergences

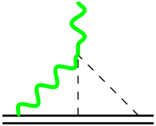
Suppose one had the Far zone theory only: the UV divergence is not compensated by the NZ but it can be renormalized:

- drop the divergence (absorb it with a local counterterm)
- impose $\mu-$independence Goldberger, Ross, Rothstein PRD '14

$$\frac{d\mathcal{L}_{tail}}{d\log\mu} = 0 \implies \frac{dM}{d\log\mu} = -\frac{2G^2M}{5}\left(2Q_{ij}^{(1)}Q^{(5)} - 2Q_{ij}^{(2)}Q_{ij}^{(4)} + \left(Q_{ij}^{(3)}\right)^2\right)$$

which can be solved by short-circuiting with analog equation for

$$\frac{dQ_{ij}}{d\log\mu} = \frac{214}{105}(GM)^2\,\ddot{Q}_{ij}(t,\mu)$$



Goldberger, Ross PRD '09
(see also Anderson+ '82!)
Adding analogous formula for J (Bernard, Blanchet, Faye, Marchand, Phys. Rev. D97 (2018)) and taking orbital average:

$$M(\mu) = M(\mu_0) - MG^2\sum_{n\geq 1}\frac{(2\log(\mu/\mu_0))^n}{n!}\left(\beta_I G^2M^2\right)^{n-1}\langle Q_{ij}^{(n+2)}Q_{ij}^{(n+2)}\rangle$$

$$L(\mu) = L(\mu_0) - \frac{12MG^2}{5}\sum_{n\geq 1}\frac{(2\log(\mu/\mu_0))^n}{n!}\left(\beta_I G^2M^2\right)^{n-1}\langle Q_{ij}^{(n+1)}Q_{ij}^{(n+2)}\rangle$$

Not quite there for E_{circ}: need for $dE = \omega dL$

Using $Q_{ij}(M, \mu)$ one has (leading log part of) $M(M_0, v, \gamma)$, $L(L_0, v, \gamma)$, adding $dE = \omega dL$ one can compute $r(v)$ on circular orbits: $Energy(r, v) \to E_{circ}(x)$ $(x \equiv (GM\omega)^{2/3})$

$$\gamma \equiv \frac{GM}{r} = x\left[1 + \frac{32\nu}{15}\sum_{n\geq 1}\frac{3n-7}{n!}\left(4\beta_I\right)^{n-1}x^{3n+1}\left(\log x\right)^n\right]$$

$$\begin{aligned} E &= -\frac{m\nu x}{2}\left[1 + \frac{64\nu}{15}\sum_{n\geq 1}\frac{6n+1}{n!}\left(4\beta_I\right)^{n-1}x^{3n+1}\left(\log x\right)^n\right] \\ J &= \frac{m^2\nu}{\sqrt{x}}\left[1 - \frac{64\nu}{15}\sum_{n\geq 1}\frac{3n+2}{n!}\left(4\beta_I\right)^{n-1}x^{3n+1}\left(\log x\right)^n\right] \end{aligned}$$

Remarkably $E(x)$ agrees 22PN order $x^{3n+1}(\log x)^n$ (up to $n = 7$), expanded self-force result by Kavanagh, Ottewill, Wardell, PRD (2015)