# Overview: scattering amplitudes and the gravitational two-body problem

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Talk at *Gravitational Waves meet Amplitudes in the Southern Hemisphere,*
ICTP-SAIFR, Brazil, 15 Aug 2023

# Outline

(1) Background – GW physics & emergence of amplitudes approach

(2) Connecting quantum to classical

(3) Gravitational amplitudes: Loop integrand

(4) Loop integration

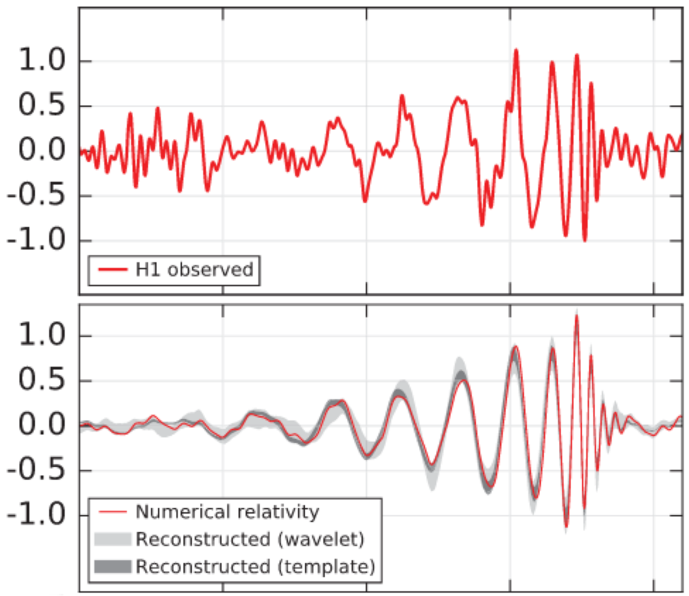
(5) Results – spinless case

(6) Other developments: spin, dissipation, higher orders, etc.

*An incomplete overview – the field exploded in the past few years!*

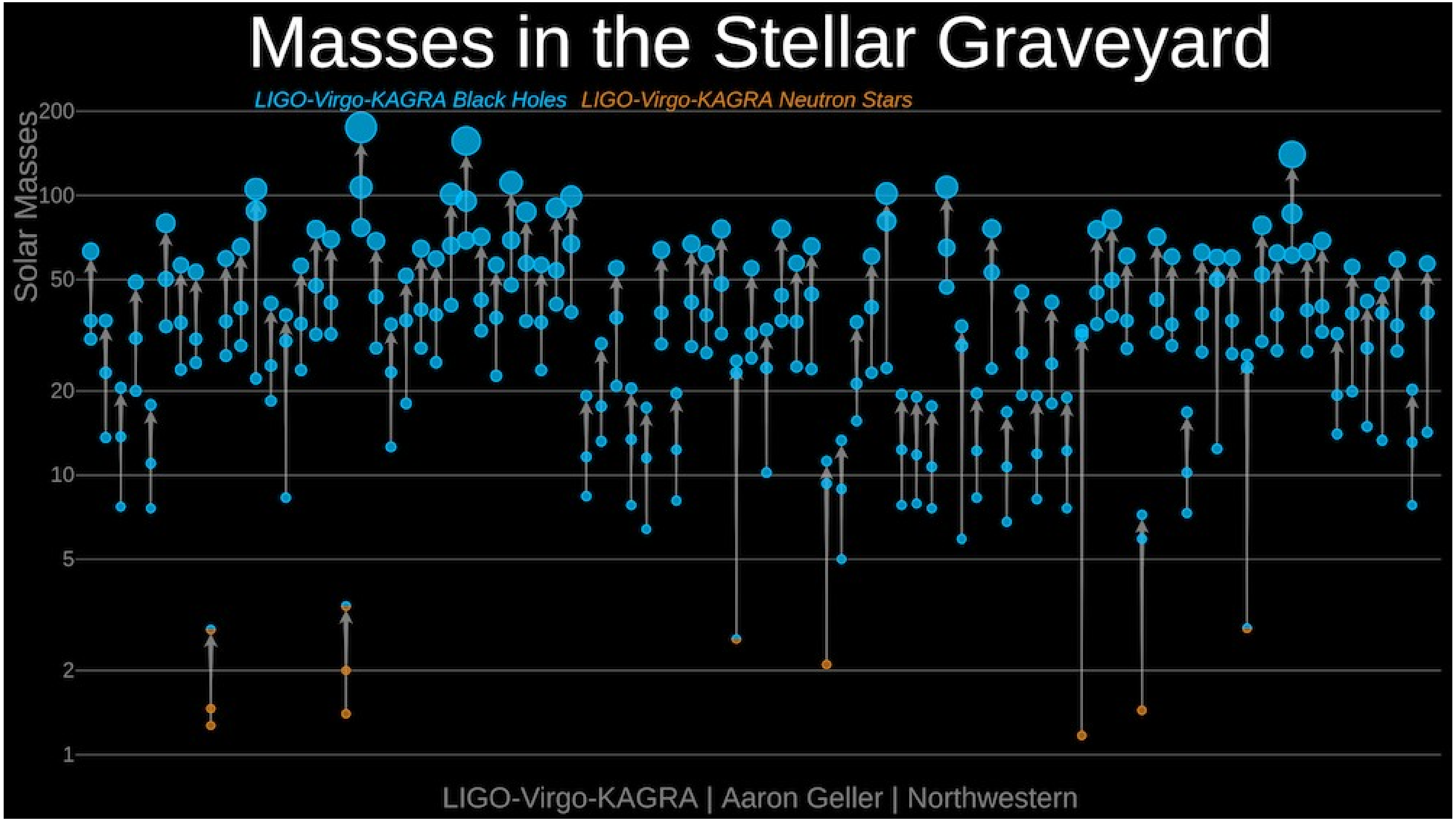
# Background – GW physics & emergence of amplitudes approach

# Beginning of new era: GW detection, 2015-



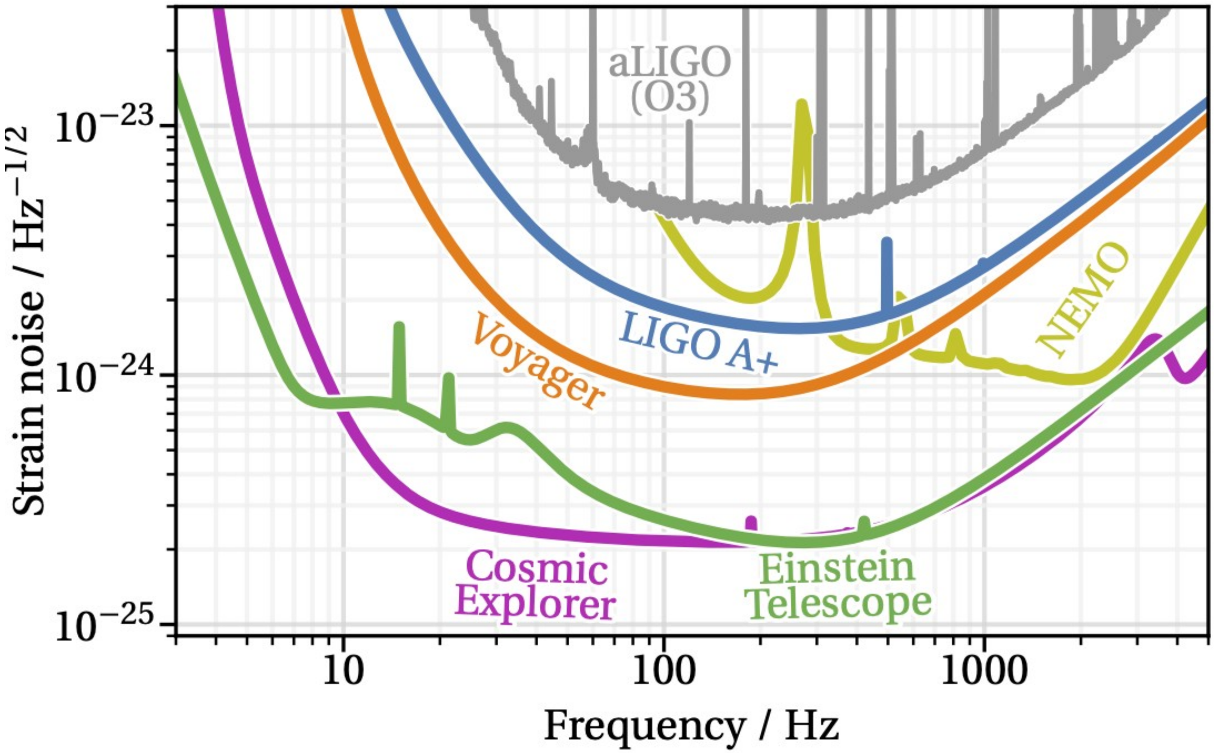

- General relativity in strong-field regime
- Dense nuclear matter in neutron stars
- BSM compact objects, dark matter
- Black hole formation mechanism, populations
- …

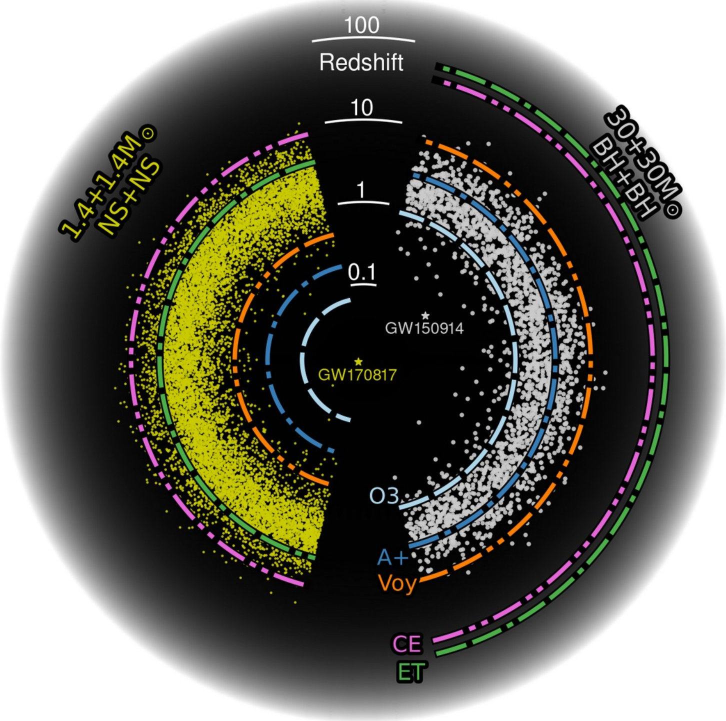
# LIGO-VIRGO-KAGRA O3B Catalog (late 2021)



# Future gravitational wave detectors

- Ground based: Advanced LIGO A+, Einstein Telescope, Cosmic Explorer
- ~100 times increase in strain sensitivity depending on frequency

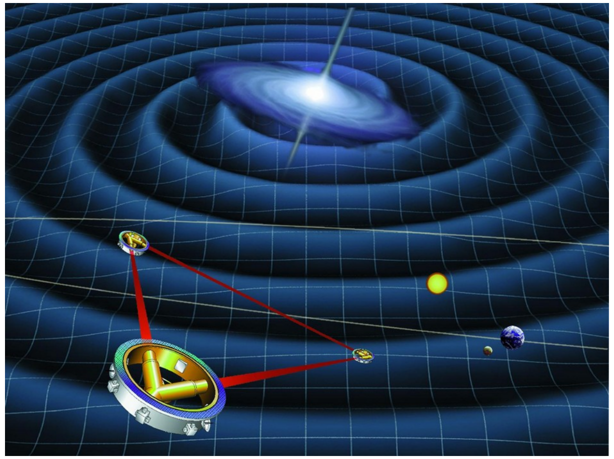


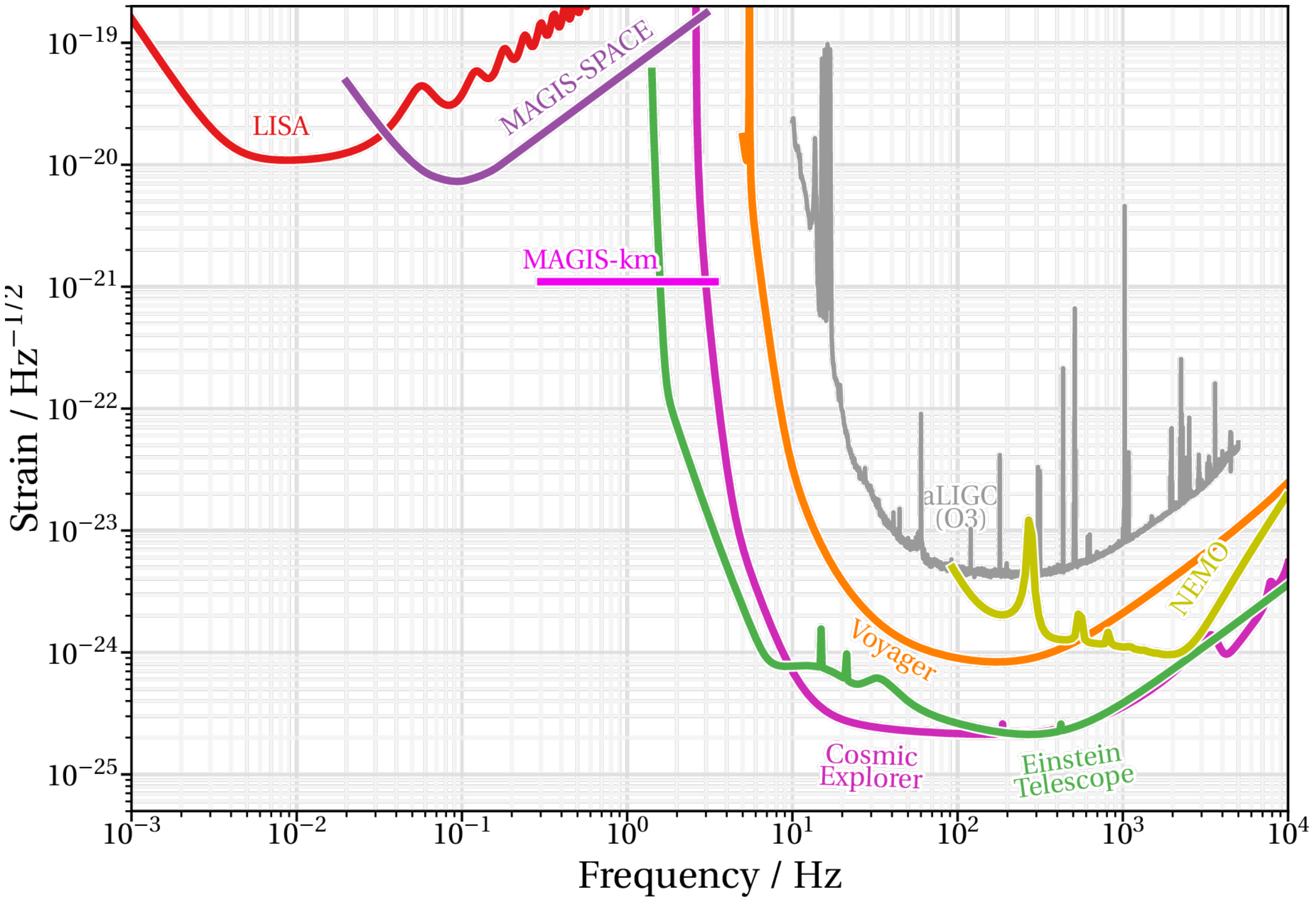




Source: https://cosmicexplorer.org/sensitivity.html

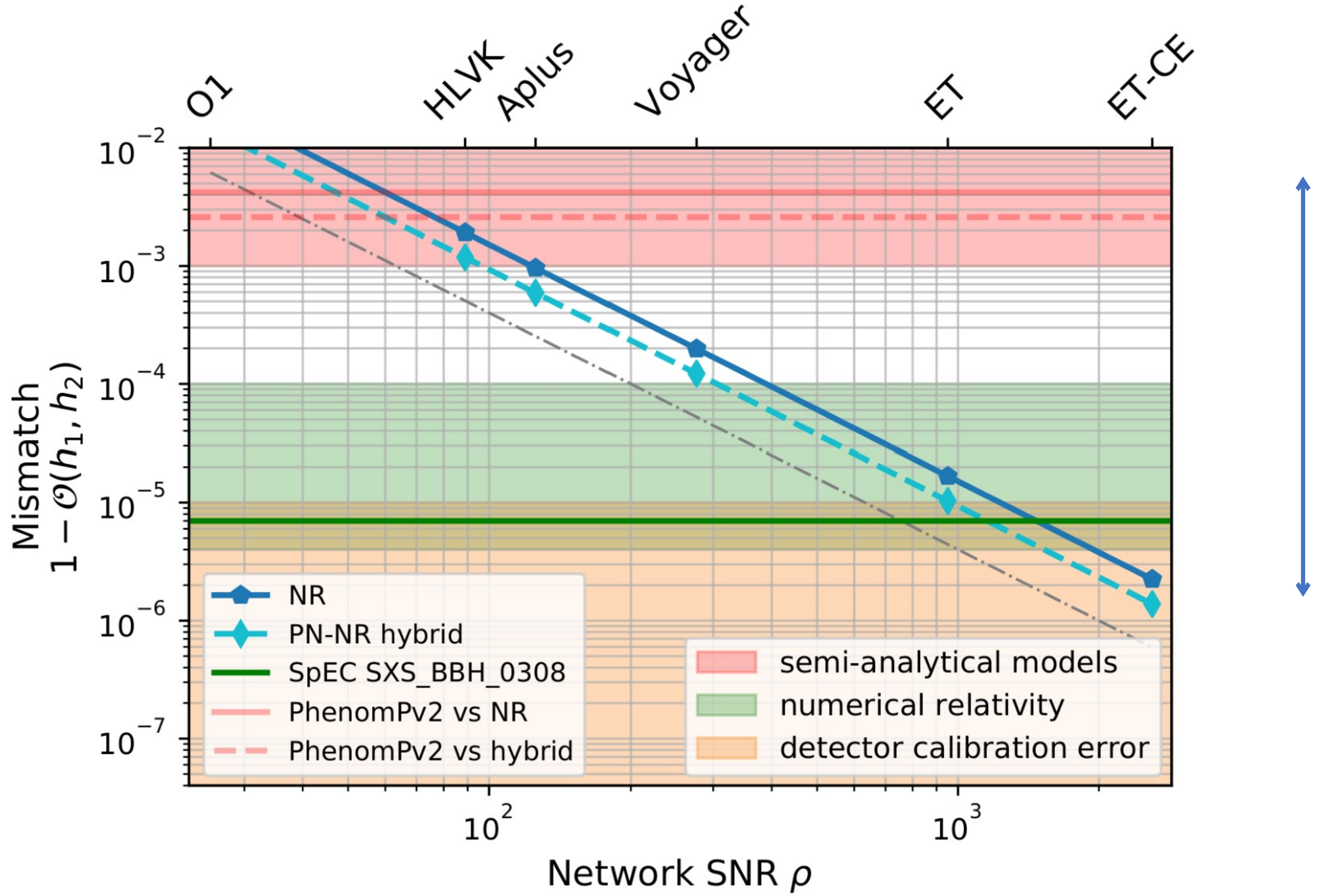
# Future gravitational wave detectors

- Space-based: LISA (2035+), TianQin (2035+).
- Atom interferometry: MAGIS km, MAGIS-SPACE



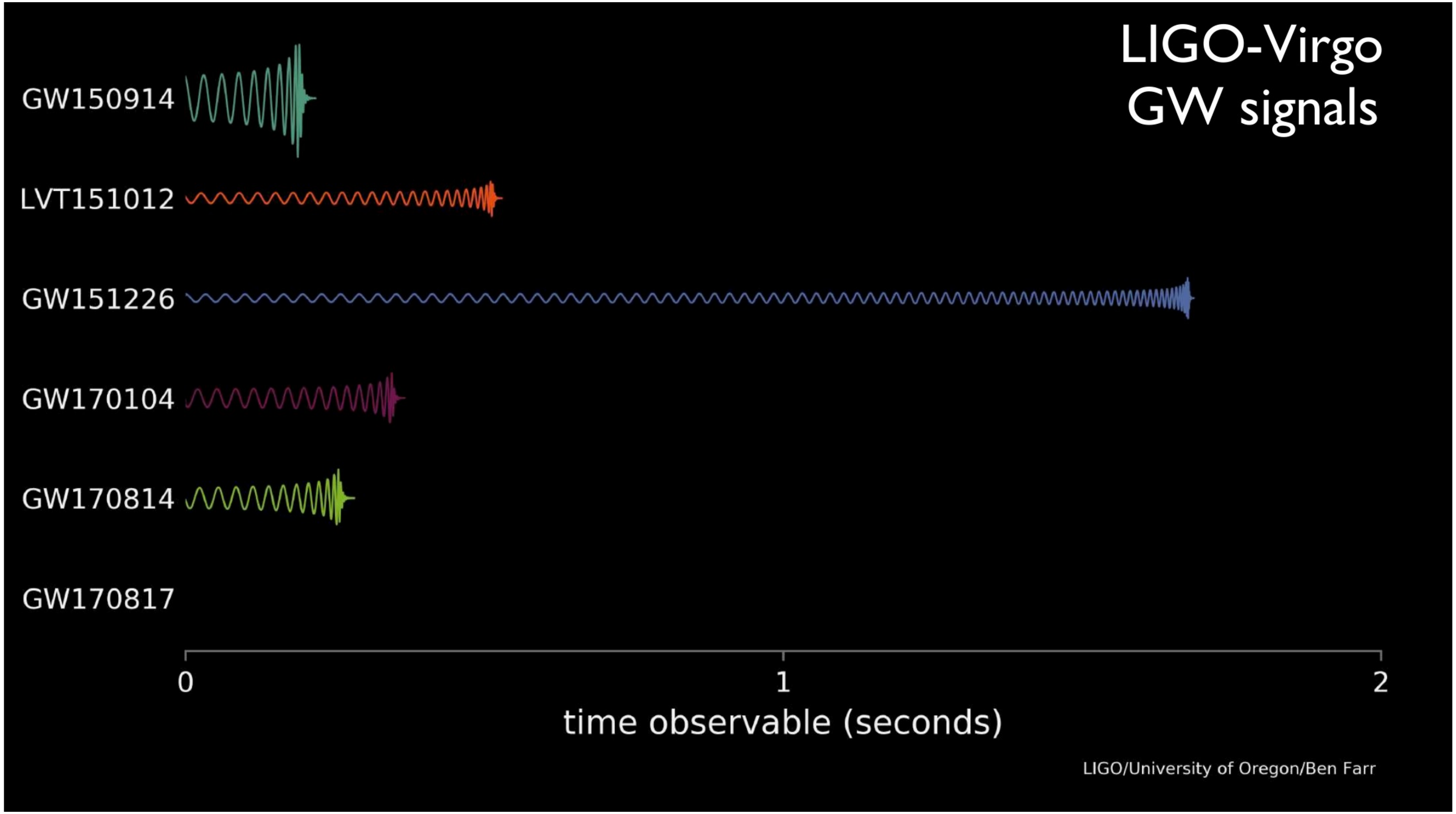


# Precision requirements for theory predictions

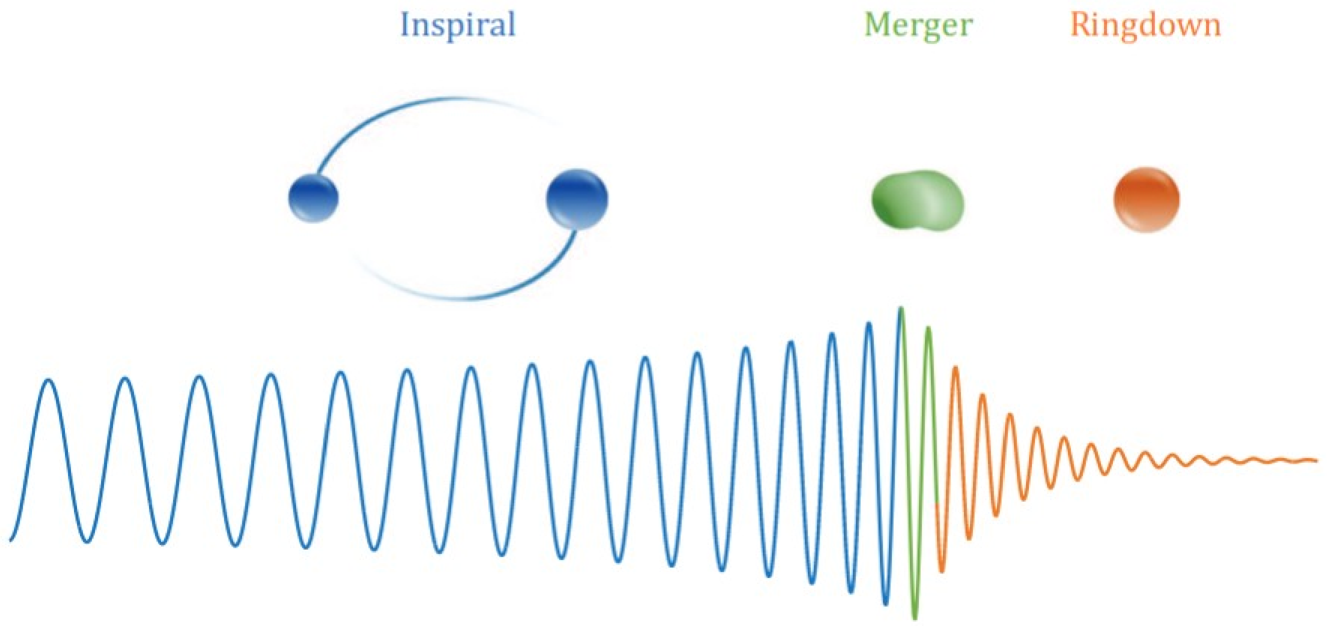



Need orders of magnitude improvement

# Waveforms from binary inspiral



# Waveforms from binary inspiral




[Picture: Antelis, Moreno, 1610.03567]

- Inspiral: **perturbative expansions,** e.g. post-Newtonian (PN), post-Minkowskian (PM), self force (SF), resummations such as Effective one-body (EOB) [Buoanno, Damour, ’98].
- Merger: numerical relativity, semi-analytic models
- Ringdown: quasi-normal modes

# Post-Newtonian (PN) expansion

- Joint expansion in $v^2$ and $GM/R$, locked together by Virial's theorem.

$m = m_A + m_B, \quad \nu = \mu/m$

$\mu = m_A m_B/m$

$\mathcal{O}(v^2)$ $\quad \mathcal{O}(G)$

$$\frac{H}{\mu} = \boxed{\frac{v^2}{2} - \frac{Gm}{R}} + H_{1\mathrm{PN}} + H_{2\mathrm{PN}} + H_{3\mathrm{PN}} + H_{4\mathrm{PN}} \ldots$$

0PN, Newton

$H_{2\mathrm{PN}}$ (1980) $\quad H_{3\mathrm{PN}}$ (2000) $\quad H_{4\mathrm{PN}}$ (2014)

$H_{1\mathrm{PN}}$:

$$\mathcal{O}(v^4) + \mathcal{O}(Gv^2) + \mathcal{O}(G^2)$$

$$+\frac{1}{c^2}\left\{-\frac{v^4}{8} + \frac{3\nu v^4}{8} + \frac{Gm}{R}\left(-\frac{v_R^2 \nu}{2} - \frac{3v^2}{2} - \frac{\nu v^2}{2}\right) + \frac{G^2 m^2}{2R^2}\right\}$$

1PN, Lorentz, Droste, 1917; Einstein, Infeld, Hoffman, 1938

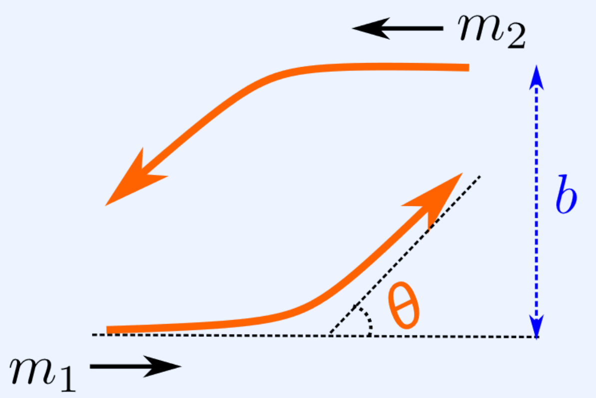
# Post-Minkowskian (PM) expansion

- Expansion in coupling constant *G*, **exact velocity dependence.** [Bertotti, Kerr, Plebanski, Portilla, Westpfahl, Gollder, Bel, Damour, Derulle, Ibanez, Martin, Ledvinka, Schaefer, Bicak…]

- Most accurate PM scattering angle until 2019 [Westpfahl, '85]

**Scattering angle** of two black holes, as function of $m_1, m_2, b, \sigma \equiv \hat{p}_1 \cdot \hat{p}_2$ .

$$\theta = \frac{4G(m_1+m_2)}{b}\frac{2\sigma^2-1}{2(\sigma^2-1)} + \frac{3\pi}{4}\frac{G^2(m_1+m_2)^2}{b^2}\frac{5\sigma^2-1}{\sigma^2-1} + \mathcal{O}(G^3)$$



# Scattering amplitudes approach: “pre-history”

**The Two-body Problem in the Theory of the Quantized Gravitational Field †**

By E. CORINALDESI ‡
Dublin Institute for Advanced Studies

*Communicated by L. Rosenfeld ; MS. received 7th June 1955 and in amended form 17th November 1955*

*Abstract.* The equations of the two-body problem of general relativity are derived by a Hamiltonian method based on an expansion of the general covariant Lagrangian in powers of the gravitational constant and by employing the techniques and the viewpoint of quantum field theory. It is found that, within the approximation in which they have so far been calculated, the equations could have been obtained identically from a linear theory of gravitation.

**Fourth-Order Gravitational Potential Based on Quantum Field Theory.**

Y. Iwasaki
*Research Institute for Fundamental Physics, Kyoto University - Kyoto*

(ricevuto l’1 Marzo 1971)

There have been many attempts (1) to understand the gravitational interaction in terms of quantum field theory in flat Minkowskian space-time in analogy to the electromagnetic interaction. Since in the case of the electromagnetic interaction there is excellent agreement between the quantized theory and experiment (2), we also believe that the gravitational interaction can be and should be understood by means of quantum field theory. This is the starting point of our discussions.

*Courtesy of Michael Ruf’s screenshots*

- Iwasaki ’71: reproduced 1PN potential from QFT viewpoint.
- Worldline QFT (NRGR) applied to PN expansion: [Goldberger, Rothstein, ’04 …]
- Early study of modern amplitude methods (generalized unitarity, double copy) for *PN expansion:* [Neil, Rothstein, ’13. Vaidya, ’14]

# Scattering orbits of interest to GR community

**High-energy gravitational scattering and the general relativistic two-body problem**

Thibault Damour*

*Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France*

(Received 29 October 2017; published 26 February 2018)

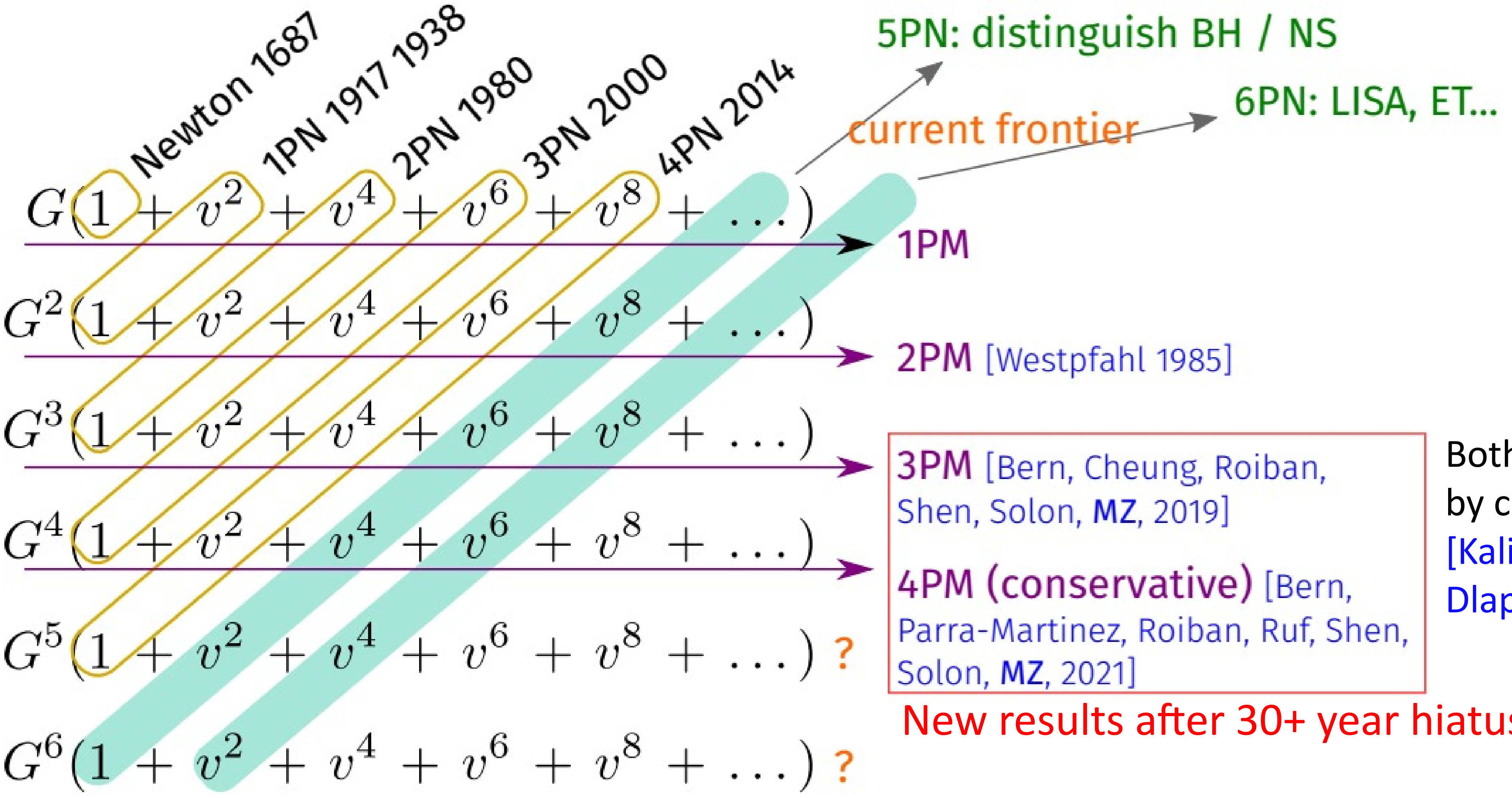
A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's

- Hyperbolic encounters. Relates to transient signals in high-eccentrity binaries.
- Scattering observables can be matched to an EOB Hamiltonian, as alternative to first EOB approach based on bound state energy levels.
- Interest in self-force calculations for scattering orbits [Barack, Long, '21-, multiple papers]

# Scattering amplitudes approach: fruition

- **Post-Minkowskian classical scattering observables** from quantum scattering amplitudes.
  - Many methods: *eikonal [Amati, Ciafaloni, Veneziano, '90. Di Vecchia, Heissenberg, Russo, Veneziano, '20 ...], non-relativistic potential EFT [Cheung, Rothstein, Solon, '18 ...], heavy-mass EFT [Damgaard, Helset '19; Aoude, Helset, '20], observable formalism [Kosower, Maybee, O'Connell, '18]* ...
- **Loop integrand** tamed by modern methods: *generalized unitarity, double copy [Bern, Carrasco, Johanson, '08, '10], massive spinor-helicity formalism [Arkani, Huang, Huang, '17], nonlinear gauge fixing [Cheung, Remmen '16, '17...]*
- **Loop integration techniques** developed for particle physics – adapted to cutting edge GR calculations. *Integration by parts [Chetyrkin, Tkachov '81. Laporta, '01], Differential equations [Kotikov, Bern, Dixon, Kosower, Gehrmann, Remmidi, Henn …], method of regions [Beneke, Smirnov, '98], reverse unitarity [Anastasiou, Melnikov, '04]*
- **Bound-state dynamics** can also be obtained, through matching to *effective-one-body formalism* [Buonanno, Damour, '99; Damour '16], *EFT potential*, or by *analytic continuation (B2B)* [Kalin, Porto, '20...].

# Scattering Amplitudes PM Results – spinless case




[adapted from Mikhail Solon's slide]

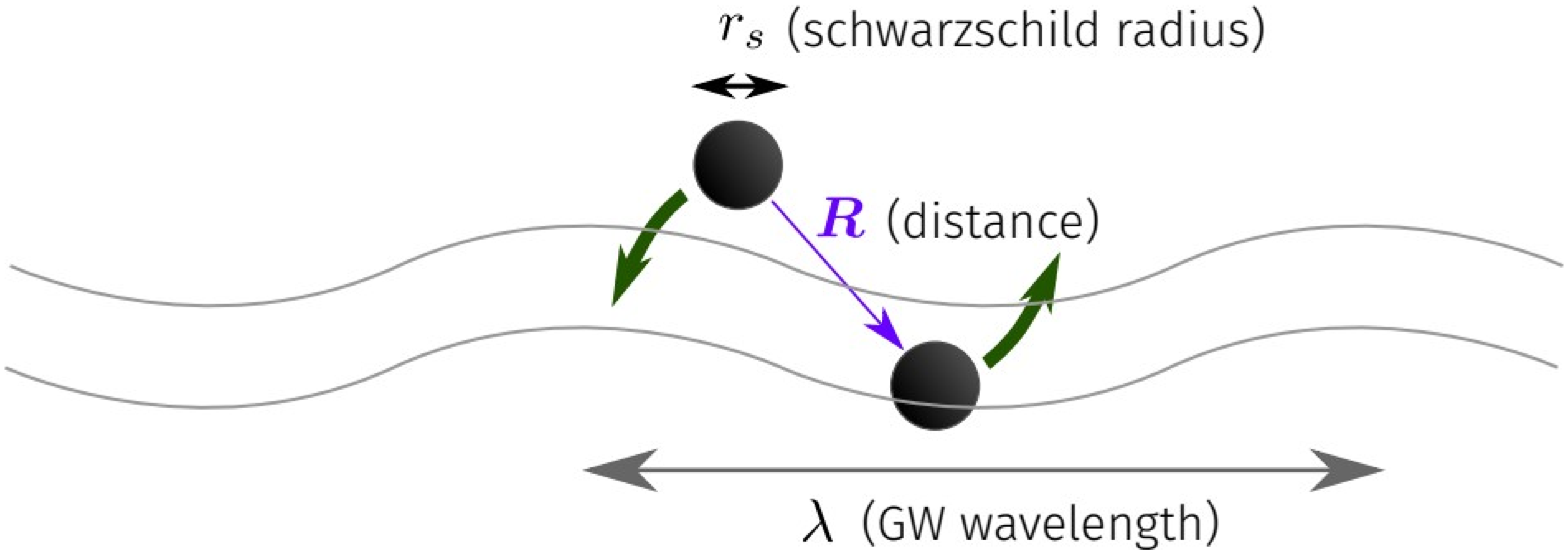
Both results reproduced by classical worldline EFT [Kalin, Liu, Porto, ’20; Dlapa, Kalin, Liu, Porto, ‘21]

New results after 30+ year hiatus.

# Connecting quantum to classical

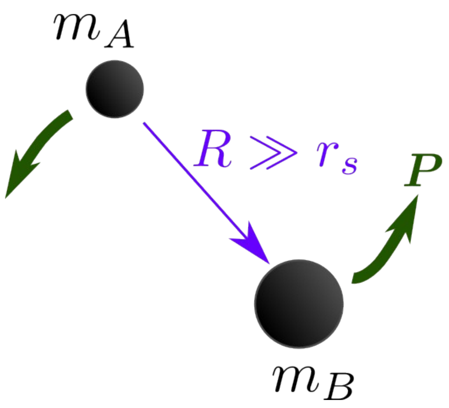
# Scale hierarchy in binary problem

- During inspiral phase, $R_s \leq r \leq \lambda \implies$
- Use effectuve field theory [Goldberger, Rothstein, '04; Cheung, Rothstein, Solon, '18; Damgaard, Haddad, Helset, '19; Kalin, Porto, '20]

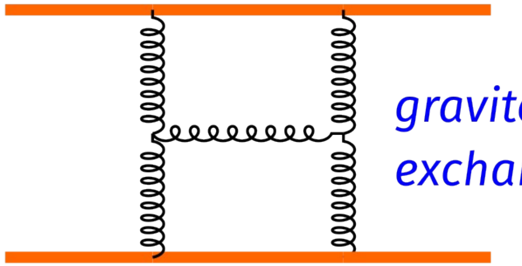



- Multipole expansion / EFT operator expansion around ***point-particle*** limit.

# Point-particle effective field theory



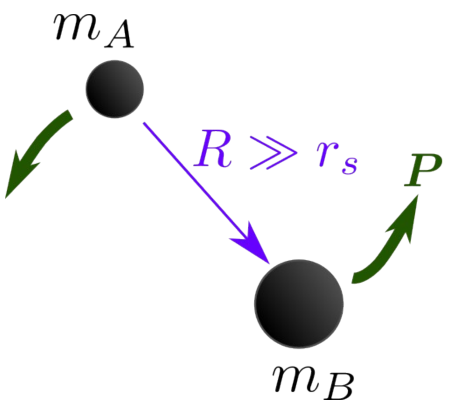





- Spinless gravitating objects replaced by massive scalar particles coupled to gravity.
- Lagrangian:

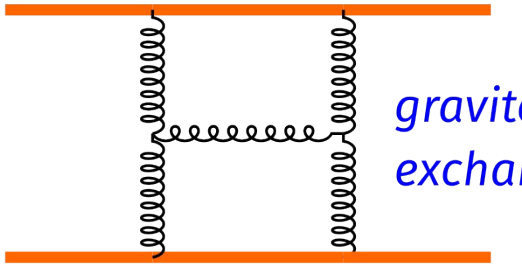
$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{2} \sum_i \left( \nabla^\mu \phi_i \nabla_\mu \phi_i + m^2 \phi_i^2 \right) \right] + S_{\text{finite-size}}$$

# Point-particle effective field theory




point particle EFT

[Goldberger, Rothstein, '04]



*graviton exchange*

- Beyond minimal coupling: [Cheung, Solon, '20]

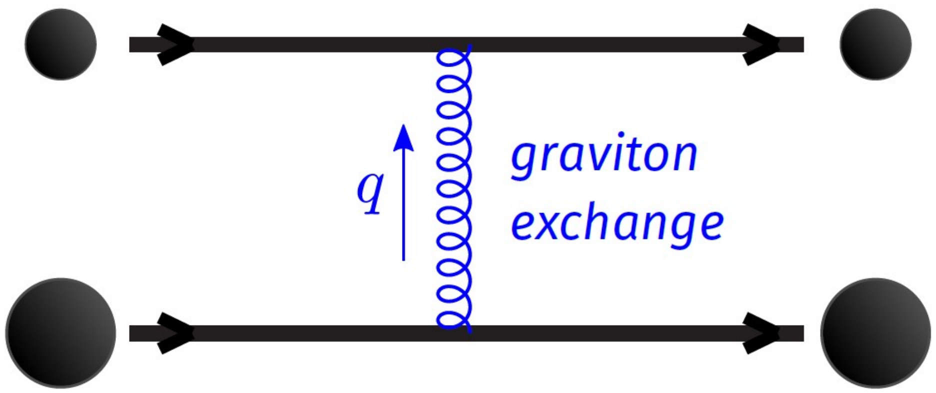
$$S_{\text{finite-size}} = \int d^4x \sqrt{-g}\frac{1}{4} C_{\mu\alpha\nu\beta} C^{\rho\alpha\sigma\beta} \sum_i \left( \lambda_i \phi_i^2 \delta_\rho^\mu + \frac{\eta_i}{m_i^4} \nabla^\mu \nabla^\nu \phi_i \nabla_\rho \nabla_\sigma \phi_i \right)$$

"gravito-magnetic coupling"

Mixes with "gravito-electric coupling"

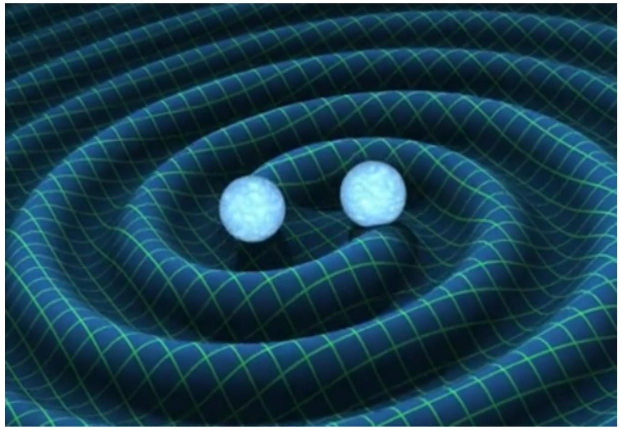
- ***Highly suppressed*** for generic compact objects / black holes: $\sim \mathcal{O}(G^5)$ / $\mathcal{O}(G^6)$

# Classical scaling of amplitudes




momentum transfer $q \sim \hbar/R \ll m_1, m_2$

$\hbar \to 0?$



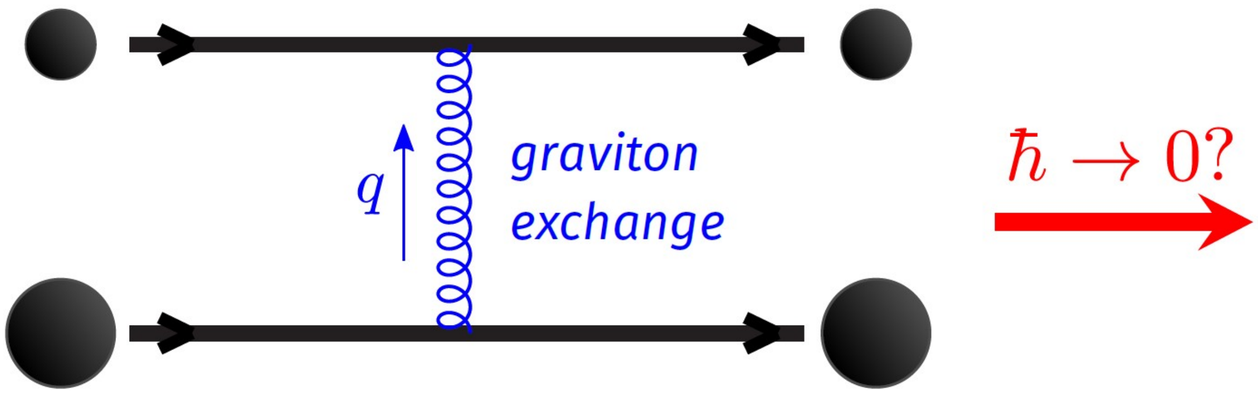
[picture: LIGO]

- Classical: large quantum numbers, e.g. large angular momentum

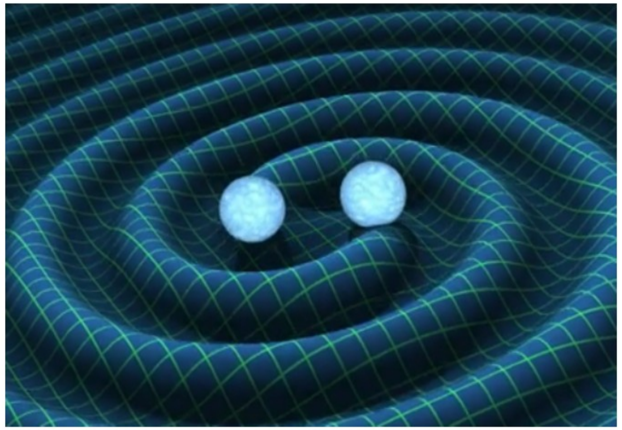
$$J/\hbar \sim |p|R \sim MR \sim M/|q| \gg 1$$

- If considering masses as O(1), classical limit ~ small $|q|$ limit.

# Classical limit diverges naively!




momentum transfer $q \sim \hbar/R \ll m_1, m_2$



[picture: LIGO]

- QFT textbooks: count $\hbar$ powers of integrand (propagators & vertices) $\Rightarrow$ $\hbar^L$ at $L$ loops.
- What's different here: $m_1, m_2 \gg |q| \sim \hbar/R$ . NOT true elementary particles. Loop integration produces $m_i/|q| \sim 1/\hbar$ divergences.

# Classical limit diverges naively!

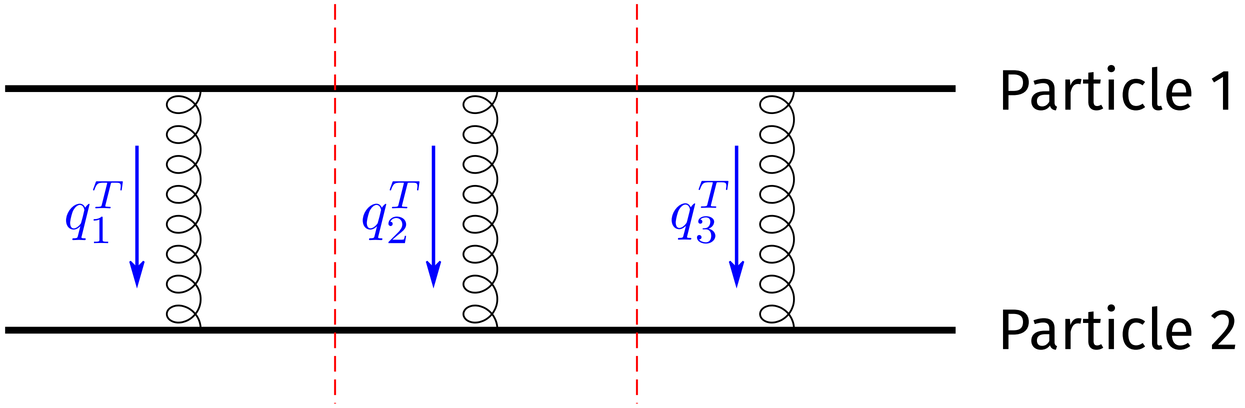
- Intuition from JWKB: $\mathcal{M} \sim \exp\left(\frac{i}{\hbar}\int V(x)dx\right)$

$\implies 1/\hbar,\ 1/\hbar^2,\ 1/\hbar^3 \ldots$ divergences in perturbative expansion.

- ***Super-classical*** or ***classically divergent*** terms.

- How do we extract ***finite classical quantities?*** A few methods below.

# Method 1: Eikonal exponentiation



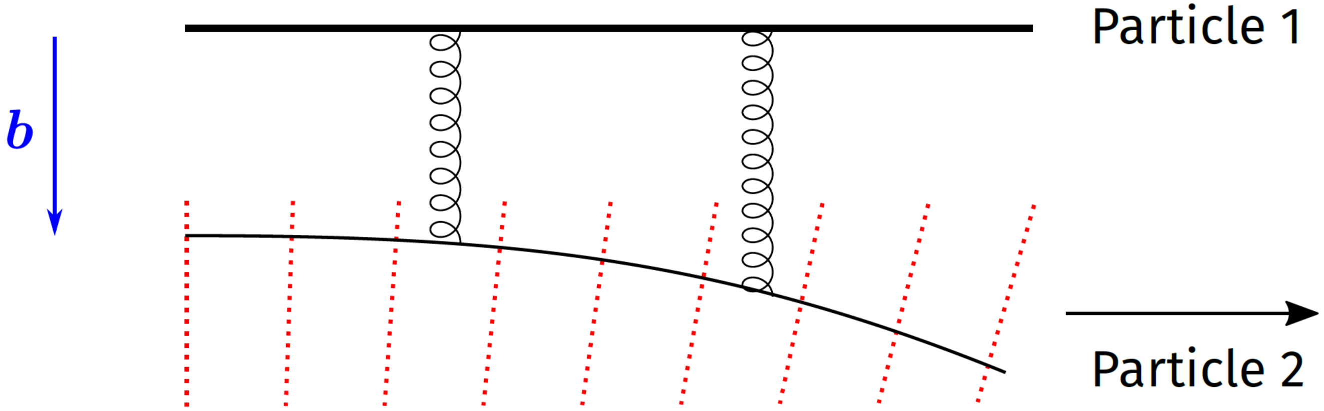

$$q^T = q_1^T + q_2^T + q_3^T + \dots$$

- ***Iterated*** soft graviton exchange with nearly on-shell intermediate matter lines. This sets $q_i$ to be purely transverse, in leading approximation.
- Convolution in momentum space ⇒ product in transverse impact parameter space:

$$\mathcal{M}(b) \propto e^{i\delta(b)}$$

[Glauber, '59; Levy, Sucher, '69; Soldate, '87; 't Hooft, '87; Amati, Ciafoloni, Veneziano, '87, '88, '90, '07; Muzinich, Soldate, '88; Kobat, Ortiz, '92; Akhoury, Saotome, Sterman ’13; Bjerrum-Bohr, Damgaard, Festuccia, Plante, Vanhove, ’18 ...]

# Scattering angle from eikonal




$$\mathcal{M}(b) \propto e^{i\delta(b)} \, .$$

*Huygens principle:* Scattering angle ∝ phase gradient $\chi \propto \frac{\partial \delta}{\partial b}$

Made rigrous by *steepest descent / stationary phase* approximation. E.g. modern derivation in [Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White, '21]

Generaling to spin is an active frontier. E.g. proposals in above reference, and [Bern, Luna, Roiban, Shen, MZ, '20]

# Method 2: non-relativistic EFT

[Cheung, Rothstein, Solon '19]

Lagrangian: two scalars, no antiparticles, no particle creation

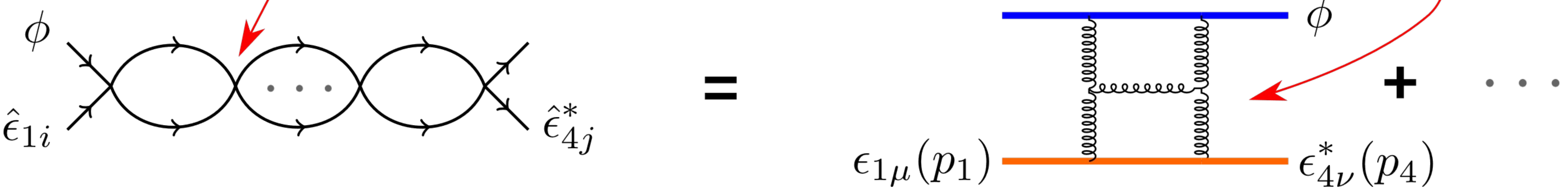
$$\mathcal{L} = \sum_{i=1,2} \int_{\boldsymbol{k}} \phi_i^\dagger(-\boldsymbol{k}) \left( i\partial_t - \sqrt{\boldsymbol{k}^2 + m_i^2} \right) \phi_i(\boldsymbol{k}) + \int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}') \phi_1^\dagger(-\boldsymbol{k}) \phi_2^\dagger(-\boldsymbol{k}') \phi_1(\boldsymbol{k}) \phi_2(-\boldsymbol{k}^\dagger)$$

Kinematic term

Instantaneous potential

*EFT instantaneous potential*

*full theory w/ dynamical gravitons*

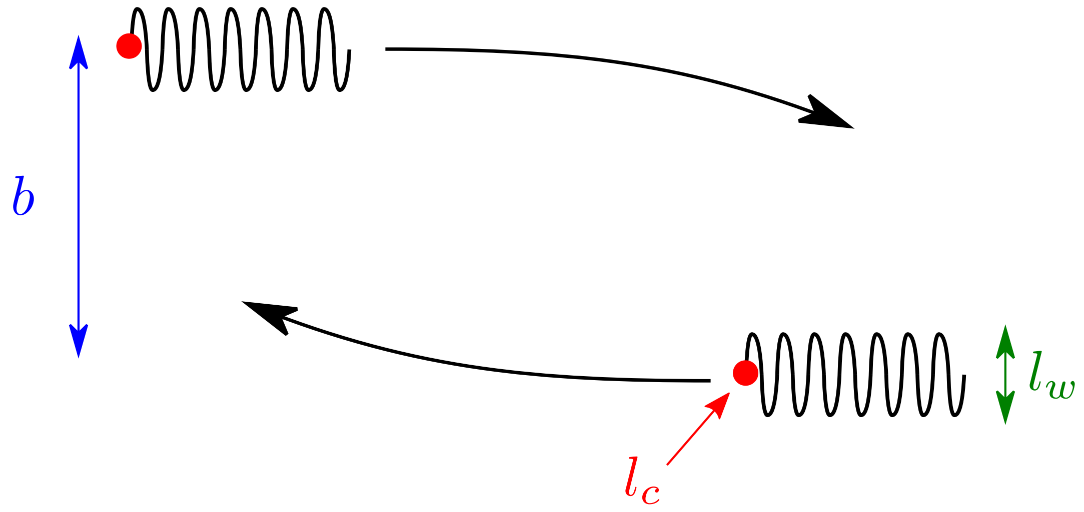



***EFT matching,*** i.e. equating EFT amplitude and full-theory amplitude, fixes potential.

# Method 3: wavepackets and S-matrix

[Kosower, Maybee, O'Connel, '18]

Wavepacket states with semi-classical localization in both position and momentum. "Goldilock condition": compton length $l_c \ll$ wavepacket spread $l_w \ll$ impact parameter




$$|\text{in}\rangle \approx \int_{\boldsymbol{q}_T} e^{i\boldsymbol{b}\cdot\boldsymbol{q}_T} |\boldsymbol{p}+\boldsymbol{q}_T\rangle$$

Then compute expectation values of $\langle\text{out}|\mathcal{O}|\text{out}\rangle = \langle\text{in}|\mathcal{S}^\dagger\mathcal{O}\mathcal{S}|\text{in}\rangle$ .

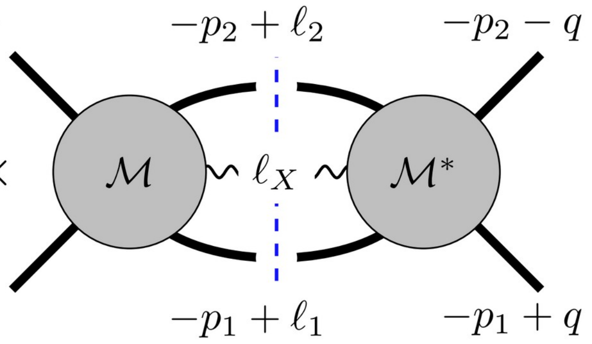
Order-by-order evaluation of ordinary momentum-space amplitudes, then integrate against plane wave profile – simplest case is basically a transverse Fourier tranform, as in eikonal.

# Method 3: wavepackets and S-matrix

[Kosower, Maybee, O'Connel, '18]

- Insert observable operator into final state, to do the "measurement".

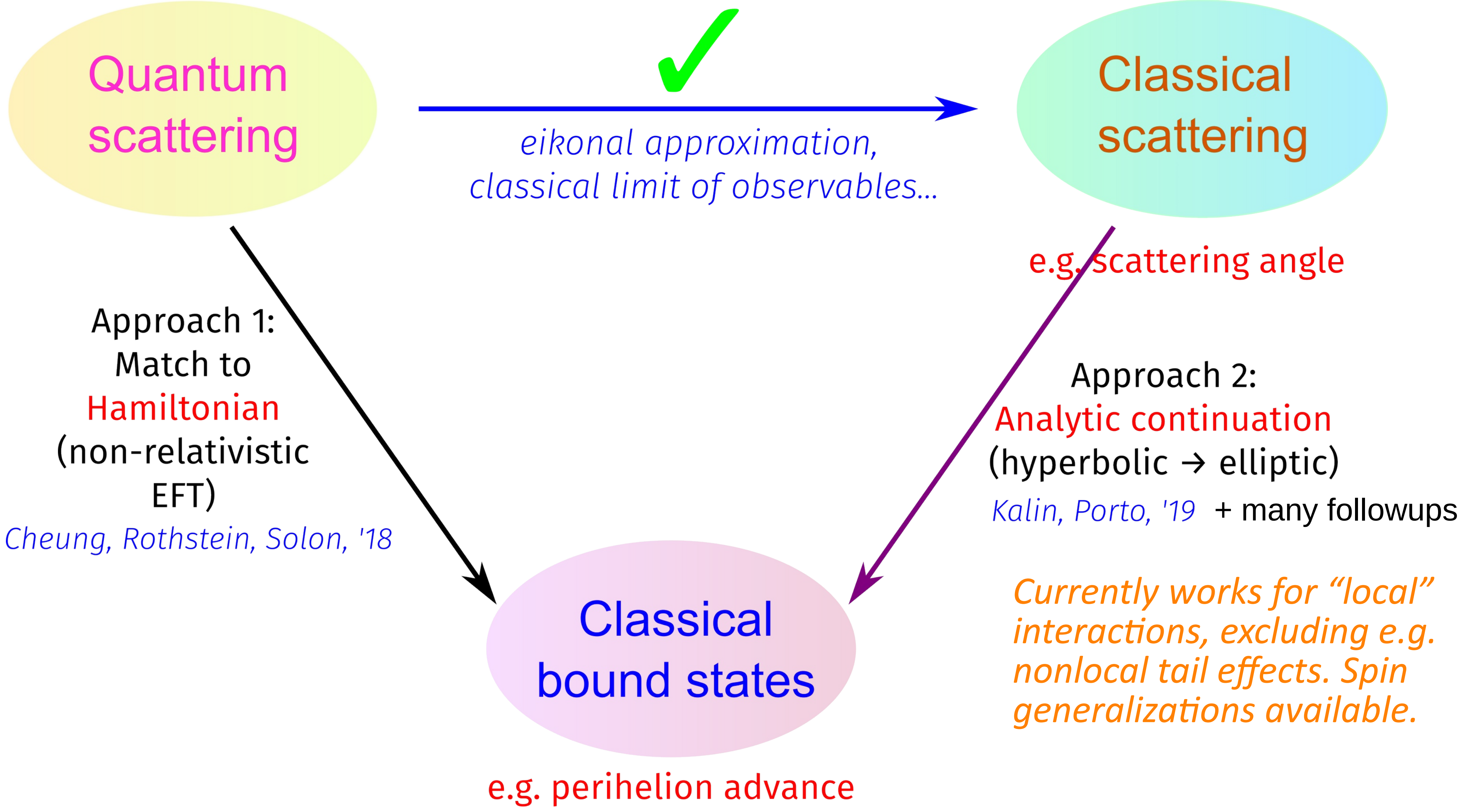
$$\langle \text{out}|\mathcal{O}|\text{out}\rangle = \langle \text{in}|\mathcal{S}^\dagger \mathcal{O}\mathcal{S}|\text{in}\rangle \,. \qquad \sum_X \int \mathrm{d}\Phi_{2+X}\; \ell_X^\mu \times$$




- Literally considers *O(ħ/R)* momentum transfer, in contrary to resummed viewpoint of eikonal
  - Assuming exponentiated S matrix, re-derives eikonal result [Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White, '21]
  - High-order calculations with KMOC, e.g. complete conservative + dissipative dynamics at 3PM [Herrmann, Parra-Martinez, Ruf, MZ, '21] confirms that KMOC agrees with other methods
  - Ehrenfest theorem in QM: simultaneous localization of *x* and *p* is *enough* to exhibit classical e.o.m.

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}, \quad \frac{d}{dt}\langle p\rangle = -\langle V'(x)\rangle$$

# From scattering to bound binaries



# Gravitational amplitudes: Loop integrand

# Perturbative gravity as a field theory

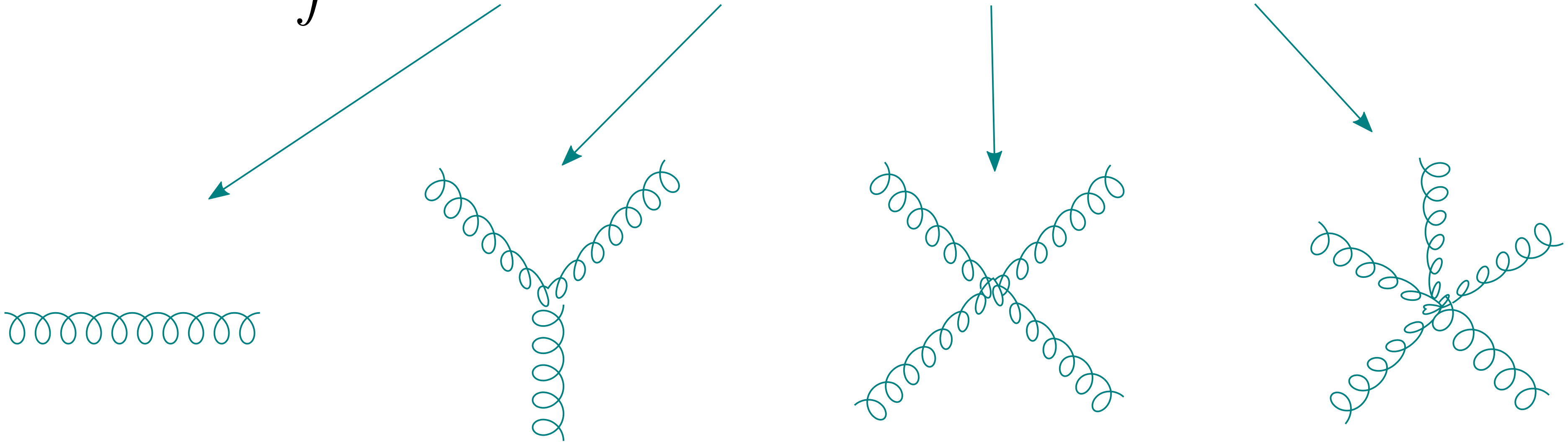
Einstein-Hilbert action

spacetime volume    curvature scalar

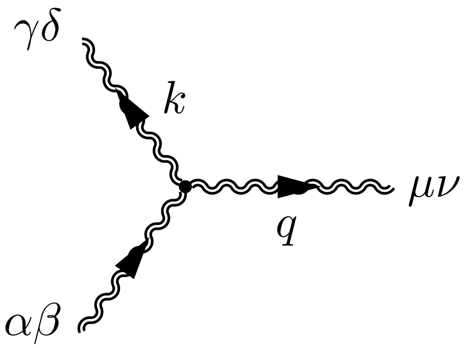
$$S_{\rm EH} = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}R, \qquad \kappa \equiv \sqrt{8\pi G}$$

Expand around flat Minkowski space: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

$$S_{\rm EH} = \int d^dx[h\partial^2 h + \kappa h^2\partial^2 h + \kappa^2 h^3\partial^2 h + \kappa^3 h^4 \partial^2 h + \ldots]$$



# Feynman vertices for gravity



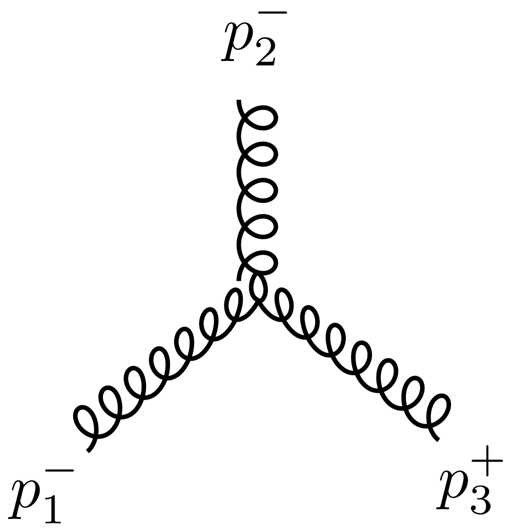

$$
\begin{aligned}
\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) = -\frac{i\kappa}{2}\Big\{ & P_{\alpha\beta,\gamma\delta}\Big[k^\mu k^\nu + (k+q)^\mu (k+q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\Big] \\
& +2q_\lambda q_\sigma\Big[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta} \\
& \qquad -I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta} - I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}\Big] \\
& +\Big[q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\nu,}{}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\mu,}{}_{\alpha\beta}) \\
& \quad -q^2(\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})\Big] \\
& +\Big[2q^\lambda\Big(I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu + I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu \\
& \qquad -I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k+q)^\mu - I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k+q)^\nu\Big) \\
& \quad +q^2(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu,}{}_{\gamma\delta}) \\
& \quad +\eta^{\mu\nu}q^\lambda q_\sigma(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta})\Big] \\
& +\Big[(k^2+(k+q)^2)\Big(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\Big) \\
& \quad -((k+q)^2\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + k^2\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta})\Big]\Big\}
\end{aligned}
$$

*Background field gauge vertex [Holstein,Ross, 0802.0716]*

- Leading non-linear interaction: ~100 terms in 3-graviton vertex!
- Modern methods to the rescue:
  - ***Double copy construction:*** "Gravity = $YM^2$" [Bern, Carrasco, Johansson, '08]
  - ***Generalized unitarity cuts:*** loop integrand from on-shell trees [Bern, Dixon, Dubar, Kosower, '94. Britto, Cachazo, Feng, '04]
- Alternative: exploit gauge fixing freedom to "shrink" the size of vertex expressions. [Cheung, Remmen, '16, '17. Porto, Kalin, '20]

# Gravity from Yang-Mills

- Gravity = (Yang-Mills)$^2$. 3-point amplitude example:




*Yang-Mills* — *Gravity*

$$\mathcal{A}_3(1^-2^-3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \quad \mathcal{M}_3(1^-2^-3^+) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}$$

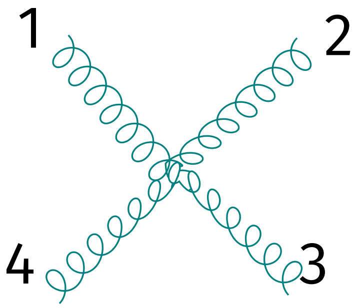
- factored out color factor $f^{abc}$

*square!*

- 4 points and above: generally a sum of squares.
  - *Kawai-Tye-Lewellen relations* from string theory, then take field theory limit
  - Modern incarnation: *double copy* from color-kinematic duality
    [Bern, Carrasco, Johansson, ’08]

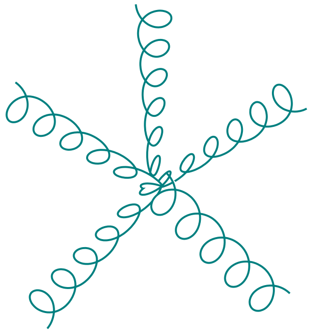
# Kawai-Lewellen-Tye (KLT) relations

- Gravity amplitude as a sum over quadratics in Yang-Mill partial amplitudes




$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}\, A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)$$

$$\mathcal{A}_4^{\text{tree}} \equiv g^2 \sum_{\sigma \in S_4/Z_4} A_4^{\text{tree}}(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\, \mathrm{tr}\left(T^{a_{\sigma_1}} T^{a_{\sigma_2}} T^{a_{\sigma_3}} T^{a_{\sigma_4}}\right)$$



$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}\, A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24}\, A_5^{\text{tree}}(1,3,2,4,5)\, A_5^{\text{tree}}(3,1,4,2,5)$$

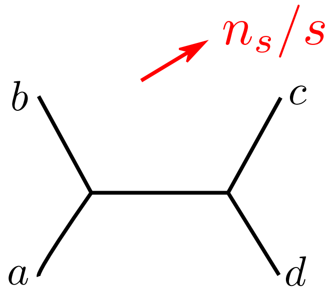
$$\vdots$$

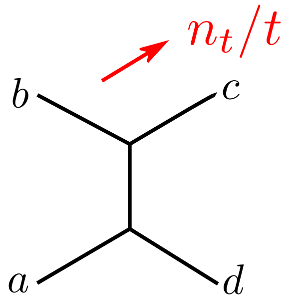
- Challenges in applications: cleaning up spurious poles, e.g. doubled propagators

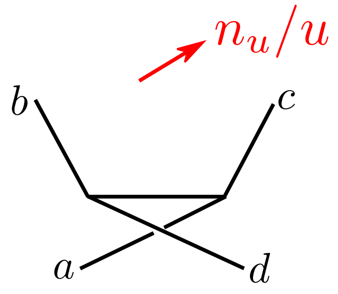
# Double copy / color-kinematic duality

[Bern, Carrasco, Johansson, '08]

- Simplest example: 4-gluon amplitude in a cubic representation:

$\mathcal{A}_4 \ = \ C_s$ 

$+\, C_t$ 

$+\, C_u$ 


$$C_s = f^{abf} f^{cdf}\,, \quad C_t = f^{bcf} f^{daf}\,, \quad C_u = f^{afc} f^{dbf}\,,$$

- Jacobi identity: $C_s + C_t + C_u = 0\,.$ Surprise: $n_s + n_t + n_u = 0\,.$

color ⟷ kinematics

- Gravity amplitude: $\mathcal{M}_4 = \mathcal{A}_4\big|_{C_i \to n_i} = n_s^2/s + n_t^2/t + n_u^2/u\,.$

# Many incarnations of double copy

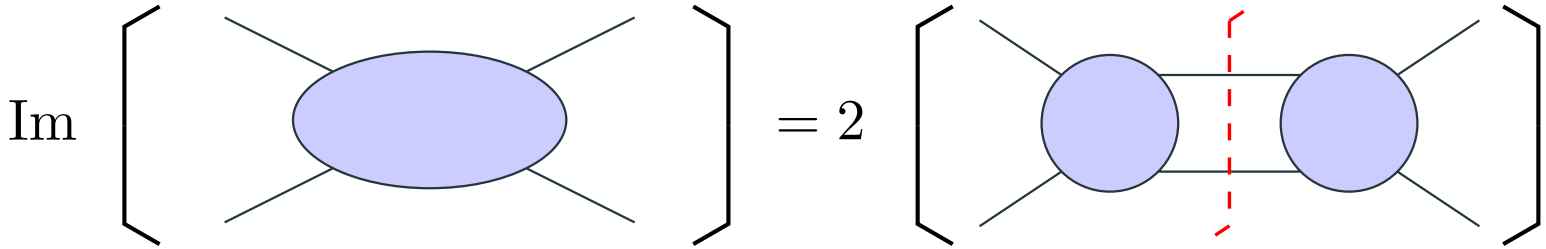
See e.g. [Snowmass white paper, Adamo et. al., arXiv:2204.06547]

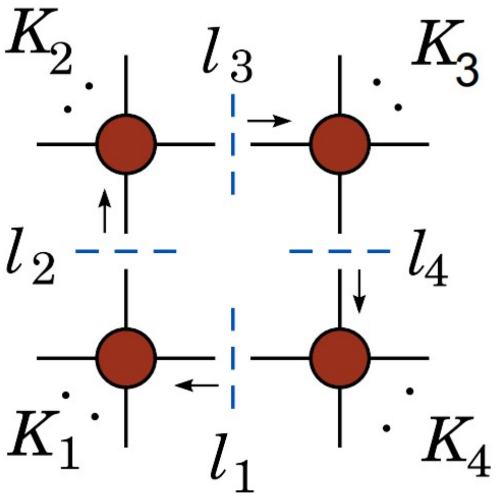
- **Kerr-Schild double copy:** relating a class of classical Yang-Mills solutions and gravity solutions [Monteiro, O'Connell, White, '14]. For example, Coulomb charge solution related to Schwarzschild black hole.
- **Classical worldline double copy** [Goldberger, Ridgway, '16]. Recent new work in "WQFT" setup [Comberiati, Shi, '22]
- Closer to amplitudes context, **heavy mass EFT double copy** for amplitudes of massive matter particles and gravitons [Branhuber, Chen, Travaglini, Wen, '21]. Applied to reproduce 3PM potential and calculate NLO waveform in BH scattering [Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini, '23]
- Strong-field double copy for massive particles in **plane wave background** [Adamo, Ilderton, '20]
- **…**

# Generalized unitarity

[Bern, Dixon, Dubar, Kosower, '94. Britto, Cachazo, Feng, '04]

- Optical theorem: imaginary part of forward amplitude ≅ decay cross section



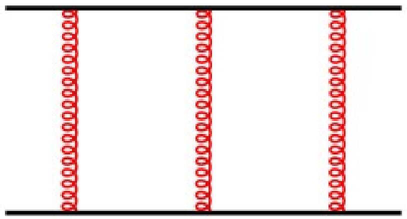




- Generalized unitarity: "cutting" with complex $l^\mu$ .
  Box cut examples: $l_1^2 = l_2^2 = l_3^2 = l_4^2$ .
- Loop integrand factorizes into gauge-invaraint on-shell tree amps. (e.g. from double copy). Fixes integrand.
- Works in 4 dimensions, or general dimensions for dimensional regularization.

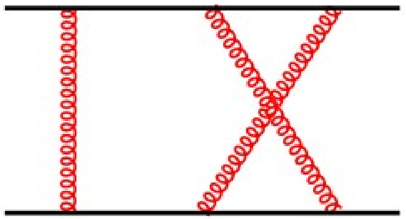
# Loop integration (how collider physics helps)

# Challenges in loop integration

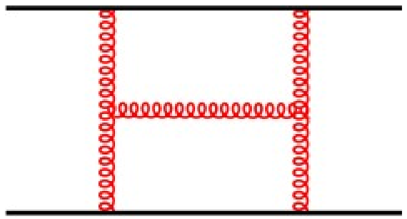
- 2→2 and 2→3 scattering of masses $m_1$, $m_2$ exchanging gravitons. Similar to integrals encounterd in Bhabha / $e\mu$ scattering, but need 3 loops and beyond!
- Essential to exploit classical limit to simplify integration, by *asymptotic expansions*



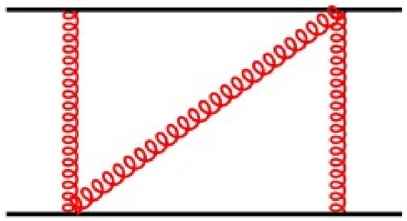
Smirnov, '01;
Henn, Smirnov, '13
Two-mass: Heller '21



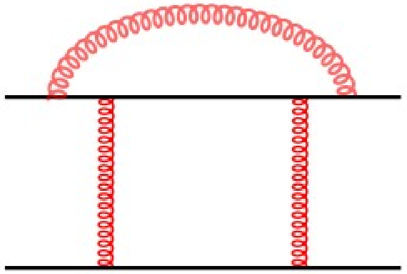
Heinrich, Smirnov, '04
*(only the $1/\epsilon^2$ pole)*



Leoni, Bianchi, '16;
Kreer, Weinzierl, '21



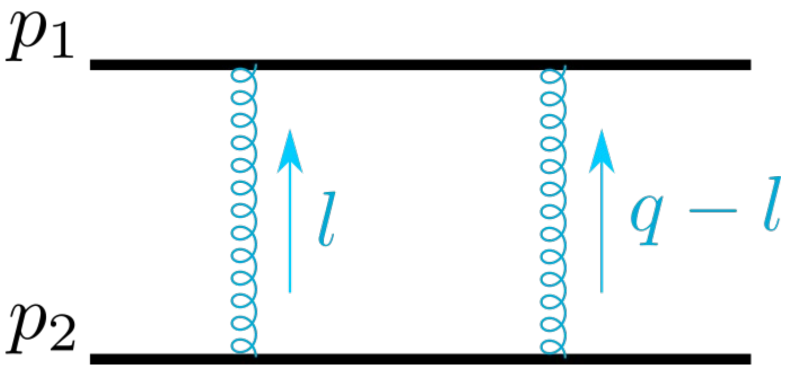
Heller, von Manteuffel, Schabinger, 19;
Broedel, Duhr, Dulat, Penante, Tancredi, '19



Duhr, Smirnov, Tancredi, '21

# Method of regions [Beneke, Smirnov '98]

Asymptotic series in $|q|/|p|$, to any order, is a sum of two integrand-level expansions, both integrated over full $\mathbb{R}^d$ :




1. **Soft region,** $|l| \sim |q| \ll |p|$ , expand in small $|l|/|p|,\ |q|/|p|$

2. Hard region, $|q| \ll |l| \sim |p|$ , **not needed for classical physics.**

Near static limit, soft-region sub-divides into *potential* region, *radiation* region, and *quantum soft* region. Integral values depend on $u_1 \cdot u_2 \equiv y = \sqrt{1+v^2}$ .

*3 dimensionless parameters → 1 dimensionless parameter. Vastly reduced complexity*

# Differential equations & velocity dependence

$$v=0 \quad \xrightarrow{\partial/\partial v} \quad v=1$$

**Post-Newtonian boundary condition.**

**Post-Minkowskian: exact function in velocity ✓**

**Ultra-relativistic limit ✓**

Integration by parts:
[Tkachov, ’81. Chetyrkin, ’81]

“Pfaffian system” of diff. eqs.

$$\mathcal{M} = \vec{c}\cdot\vec{I}, \qquad \frac{\partial}{\partial v}\vec{I} = A\cdot\vec{I}$$

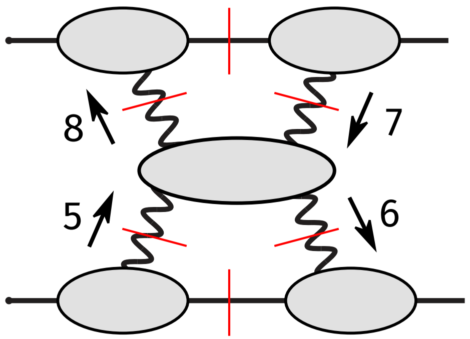
[Kotikov ’91. Bern, Dixon, Kosower, ’92. Gehrmann, Remiddi, ’99. Henn, ’13]

Introduced to post-Minkowskian gravity in [Parra-Martinez, Ruf, **MZ**, ’20]. Since then, widely adopted in cutting edge calculations in classical gravity.

# Results and comparisons

# $O(G^3)$ / 2-loop conservative dynamics

- First demonstration of power of scattering amplitudes approach, beyond the reach of direct classical methods. [Bern, Cheung, Roiban, Shen, Solon, **MZ**, '19]




Feynman diagrams from quantized Einstein-Hilbert action: **expect more than $10^5$ terms.**

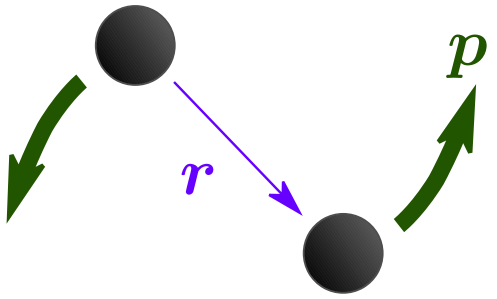
*Not truly prohibitive for computers yet, but would deteriorate quickly at higher orders.*

- Compact integrand from *unitarity cuts* obtained from spinor helicity + double copy

$$\mathrm{Cut} = -i\left\{2t^2 m_1^2 m_2^4 + \frac{1}{t^6}\left[\mathrm{Tr}[\not{7}\not{2}\not{8}\not{6}\not{1}\not{5})^4\right] + (7 \leftrightarrow 8)\right]\right\}\left(\frac{1}{(k_5-k_8)^2} + \frac{1}{(k_6+k_8)^2}\right)$$

# $O(G^3)$ / 2-loop conservative dynamics

[Bern, Cheung, Roiban, Shen, Solon, **MZ**, '19 (PRL)]




$$H^{3\text{PM}}(p,r) = \sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2} + V^{3\text{PM}}(p,r),$$

$$V^{3\text{PM}}(p,r) = \left(\frac{G}{|r|}\right) c_1(p^2) + \left(\frac{G}{|r|}\right)^2 c_2(p^2) + \left(\frac{G}{|r|}\right)^3 c_3(p^2)\,.$$

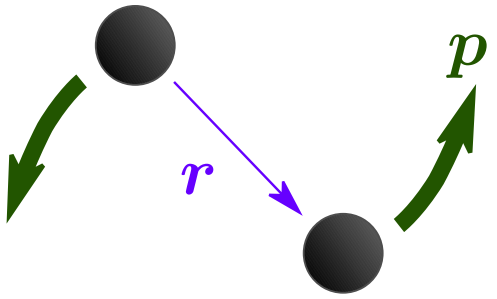
Westpfahl, '85 ($c_2$) Our new result ($c_3$)

$$m = m_1+m_2,\ \nu = \frac{m_1 m_2}{m^2},\ E = E_1+E_2,\ \xi = \frac{E_1E_2}{E^2},\ \gamma = \frac{E}{m},\ \sigma = \frac{p_1\cdot p_2}{m_1 m_2}$$

$$\mathbf{c_1} = \frac{\nu^2 m^2}{\gamma^2\xi}\left(1-2\sigma^2\right), \quad \mathbf{c_2} = \frac{\nu^2 m^3}{\gamma^2\xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$\mathbf{c_3} = \frac{\nu^2 m^4}{\gamma^2\xi}\left[\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right.$$
$$- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} + \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3}$$
$$\left. - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\right].$$

# $O(G^3)$ / 2-loop conservative dynamics

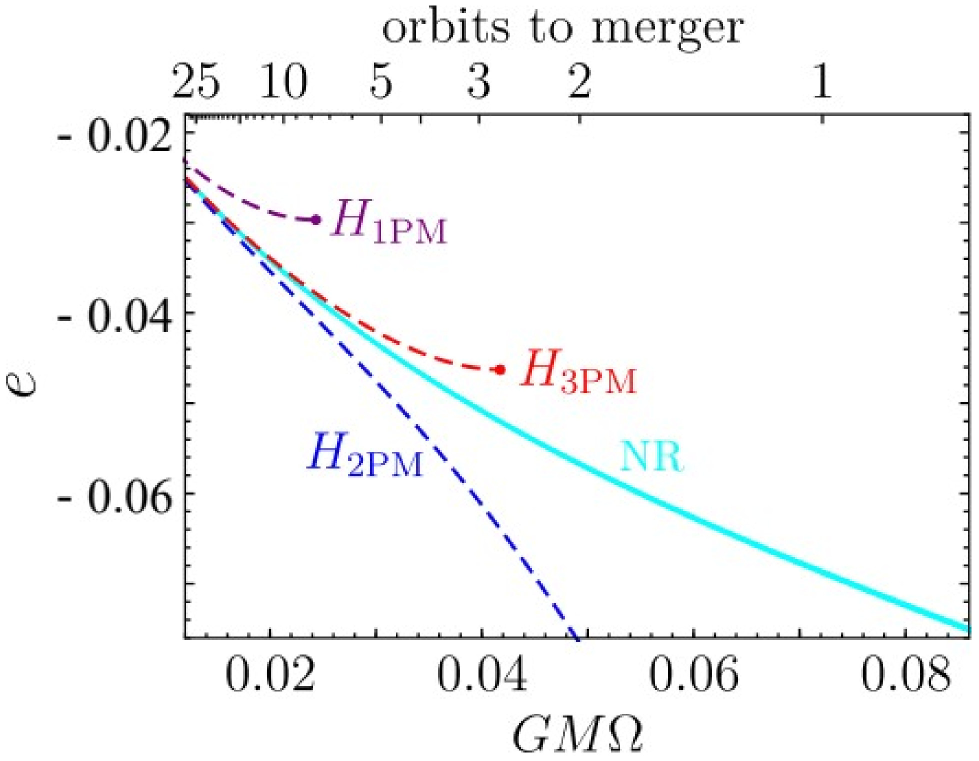



[Bern, Cheung, Roiban, Shen, Solon, **MZ**, '19 (PRL)]

$$H^{3\mathrm{PM}}(p,r) = \sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2} + V^{3\mathrm{PM}}(p,r),$$

$$V^{3\mathrm{PM}}(p,r) = \left(\frac{G}{|r|}\right) c_1(p^2) + \left(\frac{G}{|r|}\right)^2 c_2(p^2) + \left(\frac{G}{|r|}\right)^3 c_3(p^2)\,.$$

Westpfahl, ’85 Our new result



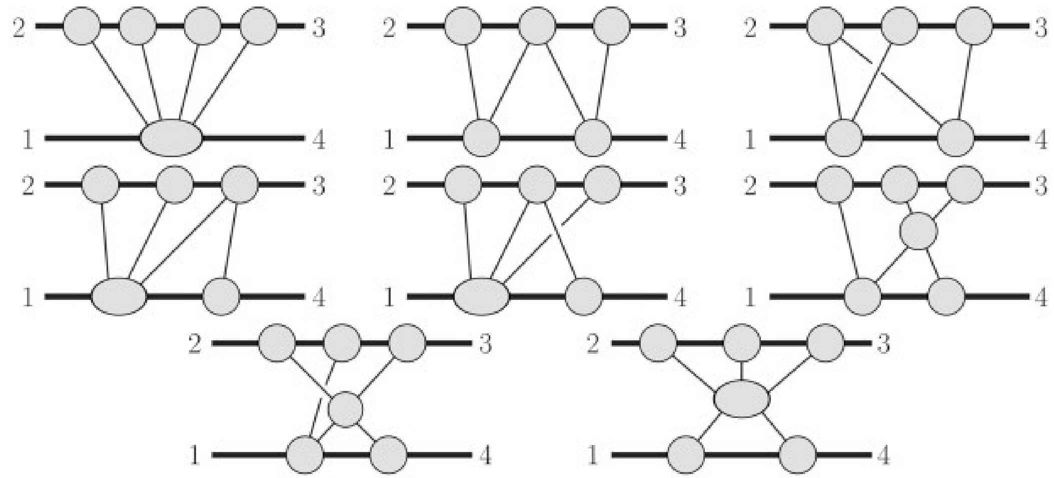

Binding energy: improved agreement with numerical relativity over lower PM orders. (See Figure) [Antonelli, Buonanno, Steinhoff, Vines, '19]

Result subsequently reproduced with several methods, e.g. Worldline PM EFT [Porto, Kalin, ’20], HEFT double copy [Brandhuber, Chen, Travaglini, Wen ’20], eikonal [Bjerrum-Bohr, Damgaard, Plante, Vanhove, ’21]

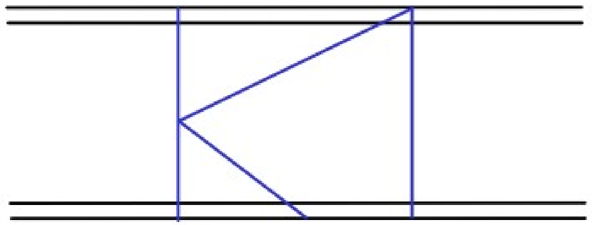
# $O(G^4)$ / 3-loop dynamics - potential region

[Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, 2101.07254]

- ***Loop integrand*** from 8 generalized unitarity cuts.



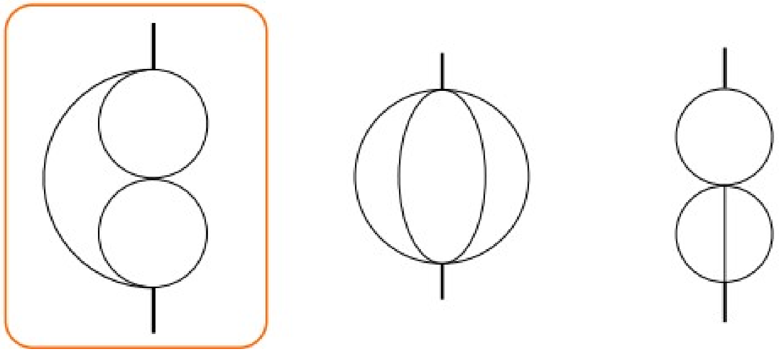
- ***Integration:***
  - IBP reduction with FIRE6 *[Smirnov]*.
  - Used `epsilon` *[Prauso, '17]* to find canonical form for DEs + one elliptic sector



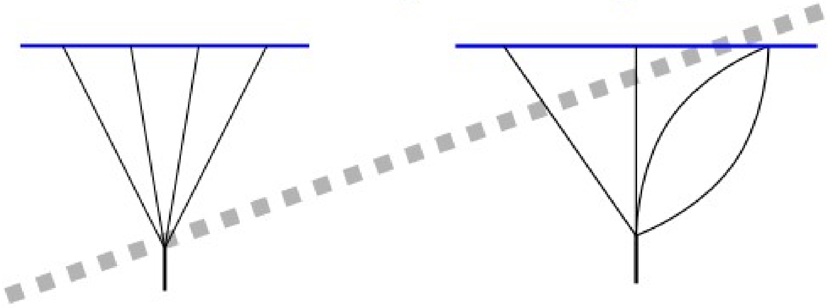
(3 master integrals)

- ***Boundary conditions*** in terms of 3D integrals

*Contribute to elliptic integrals after solving DEs*

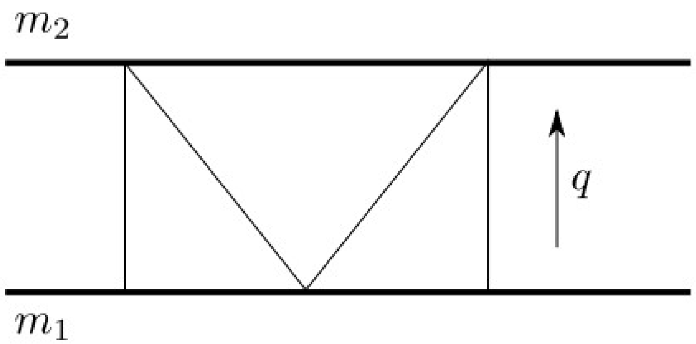


*Divergent 3D integrals set to zero*



# 3-loop amplitude - potential region

[Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, 2101.07254]




$+ \dots$

An infrared divergence!

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{1/3}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^p + \nu\left(\frac{\mathcal{M}_4^t}{\epsilon} + \mathcal{M}_4^f\right)\right] + \text{Iterations}$$

$M = m_1 + m_2$

$\nu = \frac{m_1 m_2}{M^2}$, **1SF** term vanishes in probe limit

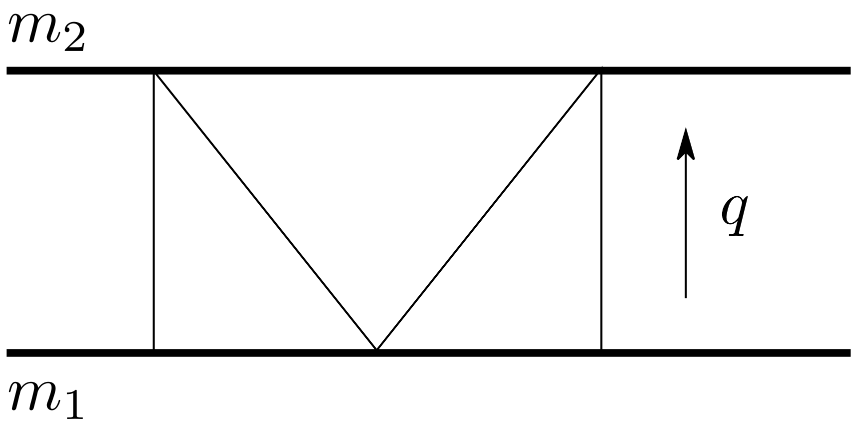
$\frac{1}{r^4}$ potential after 3D Fourier transform

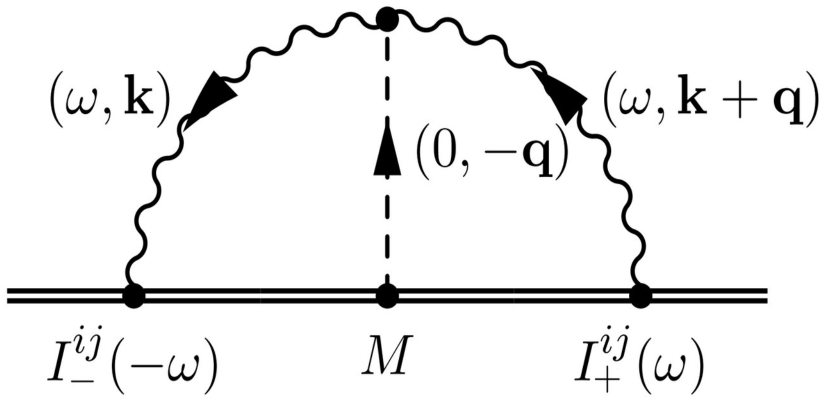
$$\mathcal{M}_4^{\mathrm{p}} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}, \quad \mathcal{M}_4^{\mathrm{t}} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}}.$$

$$\sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}$$

$h_i$ are rational functions, see next slide

# IR divergence in 2-body potential




*Potential region:* spatial momentum exchange

*Radiation region:* couples to quadruple moments of binary system, “bounces back”

$$\mathcal{M}_4^{\text{pot}} \propto \left[\frac{\mathcal{M}_4^t}{\epsilon} + \text{finite}\right] + \text{iterations}$$

$$\mathcal{M}_4^{\text{rad}} \propto \left[-\frac{\mathcal{M}_4^t}{\epsilon} + 2\log(v^2) + \text{finite}\right]$$

*Divergence cancels in the sum,* leaving log(v) term analogous to Lamb shift in QED with log(α) term.

# Finite 3-loop amplitude: potential + radiation

[Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, ’21].
Indep. verification by PM EFT [Dlapa, Kalin Liu, Porto, ’21]

$$\mathcal{M}_4^{\text{cons}} = G^4 M^7 \nu^2 |\boldsymbol{q}| \pi^2 \left[ \mathcal{M}_4^{\text{probe}} + \nu \left( 4\mathcal{M}_4^{\text{tail}} \log\left(\tfrac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right) \right] + \text{Iterations}$$

$p_\infty \equiv \sqrt{(u_1 \cdot u_2)^2 - 1}$

$$\mathcal{M}_4^{\text{probe}} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}, \quad \mathcal{M}_4^{\text{tail}} = r_1 + r_2 \log\left(\tfrac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}},$$

$$\mathcal{M}_4^{\pi^2} = r_4 \pi^2 + r_5\, K\left(\frac{\sigma - 1}{\sigma + 1}\right) E\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_6\, K^2\left(\frac{\sigma - 1}{\sigma + 1}\right) + r_7\, E^2\left(\frac{\sigma - 1}{\sigma + 1}\right),$$

$$\mathcal{M}_4^{\text{rem}} = r_8 + r_9 \log\left(\frac{\sigma + 1}{2}\right) + r_{10} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} + r_{11} \log(\sigma) + r_{12} \log^2\left(\frac{\sigma + 1}{2}\right) + r_{13} \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \log\left(\frac{\sigma + 1}{2}\right)$$
$$+ r_{14} \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2 - 1} + r_{15} \operatorname{Li}_2\left(\tfrac{1-\sigma}{2}\right) + r_{16} \operatorname{Li}_2\left(\frac{1 - \sigma}{1 + \sigma}\right) + r_{17} \frac{1}{\sqrt{\sigma^2 - 1}} \left[ \operatorname{Li}_2\left(-\sqrt{\frac{\sigma - 1}{\sigma + 1}}\right) - \operatorname{Li}_2\left(\sqrt{\frac{\sigma - 1}{\sigma + 1}}\right) \right]$$

complete elliptic integrals of the 1st & 2nd kind

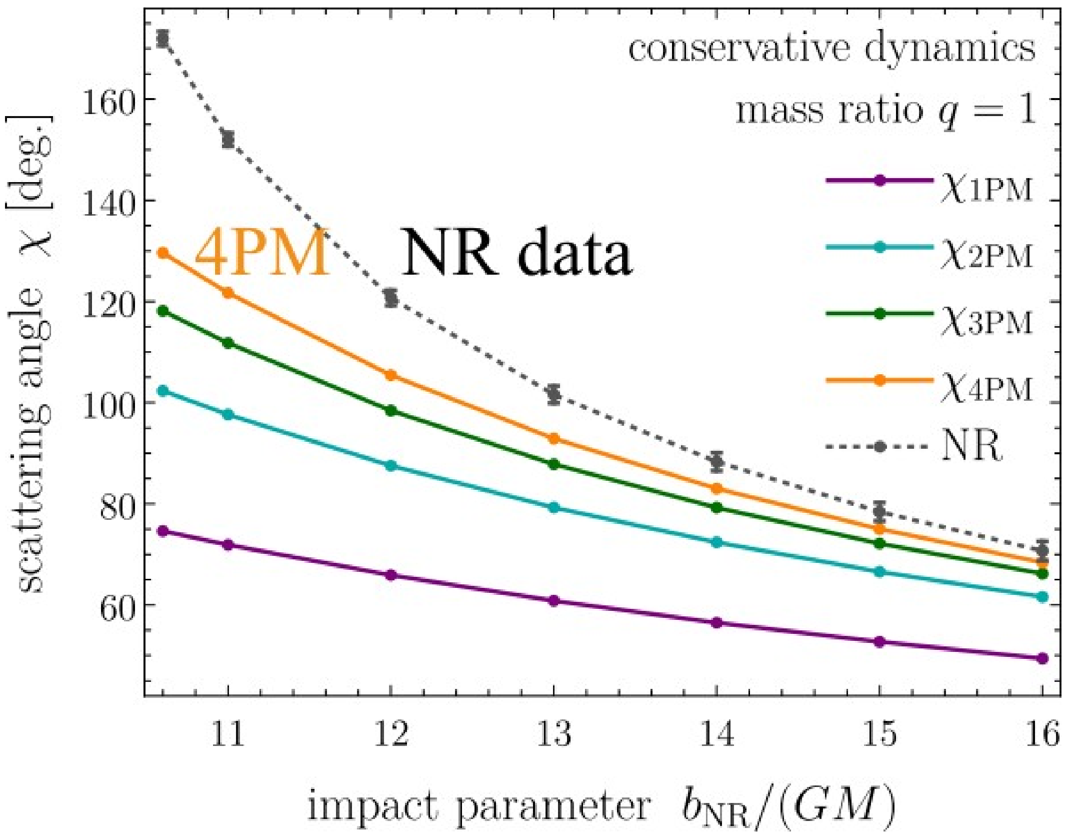
polylogarithms up to transcendental weight 2

Rational functions:

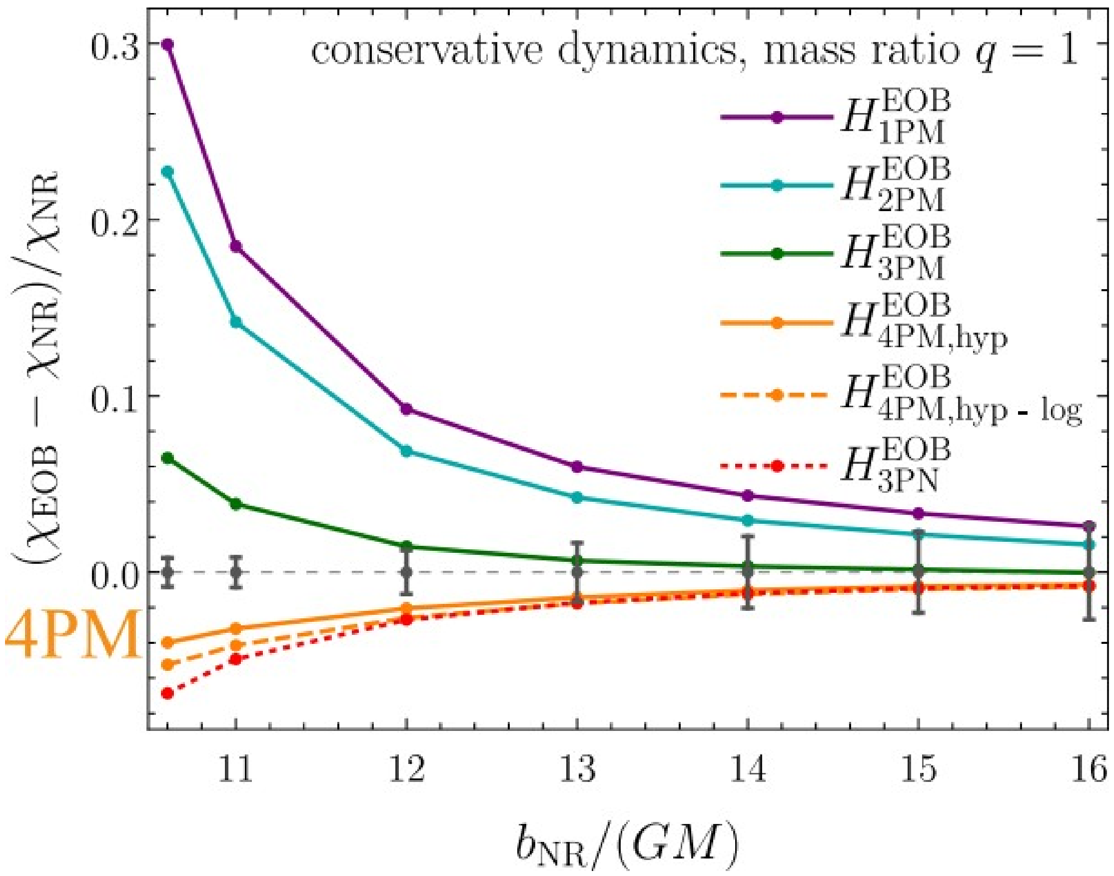
$$r_1 = \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2 - 1)}, \quad r_2 = \frac{1}{2}\left(5 - 76\sigma + 150\sigma^2 - 60\sigma^3 - 35\sigma^4\right), \quad \ldots$$

# $O(G^4)$ scattering angle vs. numerical relativity

Original angle in PM perturbation




Angle improved by EOB resummation




[Khalil, Buonanno, Steinhoff, Vines]

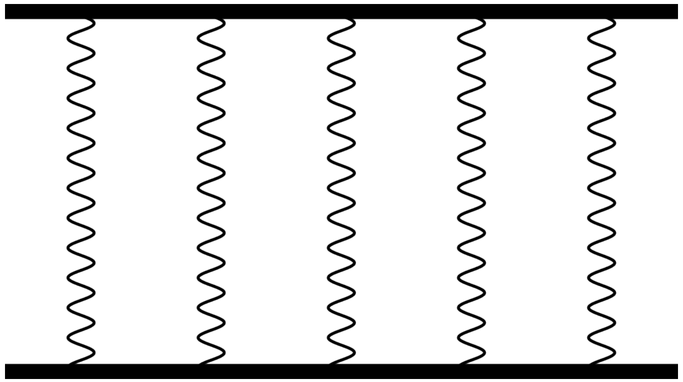
with numerical data from [Damour, Guercilena, Hinder, Hopper, Nagar, Rezzolla]

# Other developments

# *High order frontier* in post-Minkowskian expansion

[Bern, Herrmann, Roiban, Ruf, Smirnov, Smirnov, **MZ**, 2305.08981]

- 4 loops / 5PM. Start from toy theory: two charges in electrodynamics.



- "Potential region" only. integration-by-parts reduction to 1107 master integrals with FIRE6 [A.V. Smirnov]. Evaluate master integrals with differential equations.

- Scattering angle in terms of ***cyclotomic polylogarithms*** [Ablinger, Blumlein, Schneider, '11], i.e. iterated integrals over kernels built from cyclotomic polynomials:

$$x-1,\ x+1,\ x^2+x+1,\ x^2+1,\ x^4+x^3+x^2+x+1\ \ldots$$

# *Incorporating spin* – amplitudes approaches

- Finite-spin massive particles. [Holstein, Ross, 0802.0716. Vaidya, 1410.5348. Maybee, O'Connell, Vines, '19]
  - ✓ Friendly for loop integration techniques, limited to finite orders in spin. **3PM $O(S^2)$ calculation:** [Febres Cordero, Krauss, Lin, Ruf, MZ, '22]

- Arbitrary-spin massive spinor-helicity amplitudes. [Chung, Huang, Kim, Lee, 1812.08752. Arkani-Hamed, Huang, O'Connell, 1906.10100. Guevara, Ochirov, Vines, 1906.10071 …] **All order spin results obtained for 1PM, and 2PM** subject to undetermined matching coeffs **(conjectures put forward for Kerr BHs)** [Bern, Kosmopoulos, Luna, Roiban, Teng, '22. Aoude, Haddad, Helset, '22, '23]

- Higher-spin non-transverse fields permitting changes in $|S|$. [Bern, Luna, Roiban, Shen, Zeng, 2005.03071. Kosmopoulos, Luna, 2012.10137. Bern, Kosmopoulos, Luna, Roiban, Teng, 2203.06197…]

- *Worldline:* as static source [Kalin, Liu, Porto, 2007.04977] or promoted to quantum displacements ("WQFT") [Mogull, Plefka, Steinhoff, 2010.02865], including spin by supersymmetrization [Jakobsen, Mogull, Plefka, Steinhoff, 2201.07778], reached **4PM order for spin-orbit coupling** [Jakobsen, Mogull, Plefka, Sauer, Xu, '23].

# Massive vector particles

- Einstein-Hilbert + Scalar + Proca Lagrangian (minimally coupled massive spin-1)

$$\mathcal{L} = \sqrt{-g}\left[-\frac{2R}{\kappa^2} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} + \frac{1}{2}m_A^2 g^{\mu\nu}A_\mu A_\nu\right]$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad \kappa = \sqrt{32\pi G}$$

Used for PN expansion:
[Vaidya '14. Holstein, Ross '08]
Double copy:
[Bautista, Guevara, '19. Johansson, Ochirov '19]

- Finite-size operator for neutron stars: $R_{\mu\nu\alpha\beta}G^{\mu\nu}G^{\alpha\beta}$
  - Straightforward to add; included in wordline calculations [Jakobsen, Mogull, '22]

# Example 2-loop / 3PM potential: spin-orbit term

[Febres Cordero, Krauss, Lin, Ruf, **MZ**, 2205.07357]

$$c_i^{(2)}(\boldsymbol{k}^2) = c_{i,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) + c_{i,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) + \frac{c_i^{(1)}(\boldsymbol{k}^2)}{m_A^2(\gamma_1+1)}$$

$$c_{1,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = -\frac{2\sigma m_\phi}{E\xi}\,, \qquad c_{2,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = \frac{m_\phi(4m_A+3m_\phi)\sigma(5\sigma^2-3)}{4E\xi(\sigma^2-1)}\,,$$

$$c_{3,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = \frac{m_\phi}{E\xi(\sigma^2-1)^2}\Big[-2m_A^2\sigma(3-12\sigma^2+10\sigma^4)-\left(\tfrac{83}{6}+27\sigma^2-52\sigma^4+\tfrac{44}{3}\sigma^6\right)m_A m_\phi - m_\phi^2\sigma\left(\tfrac{7}{2}-14\sigma^2+12\sigma^4\right)$$
$$+\frac{(4m_A+3m_\phi)E}{4}\sigma(2\sigma^2-1)(5\sigma^2-3)+4m_A m_\phi\sigma(\sigma^2-6)(2\sigma^2+1)\sqrt{\sigma^2-1}\,\mathrm{arccosh}(\sigma)\Big]\,,$$

- Multiplies $(\boldsymbol{r}\wedge\boldsymbol{p})\cdot S$ in the two-body Hamiltonian
- Hamiltonian confirmed and extended by worldline-based calculation [Jakobsen, Mogull, 2210.06451]

# *Radiation & Dissipation*

Adapted from Radu Roiban's slide from QCD Meets Gravity 2022

- Radiation reaction & disspation effects known through $O(G^4)$.
- 3PM radiation reaction effect to scattering angle [Di Vecchia, Heissenberg, Russo, Veneziano; Damour] - cures high-energy divergence of conservative scattering angle.
- Linear reponse formula for radiation reaction correction to scattering angle (exact at 3PM) [Bini, Damour, '20]

$$2\chi^{\rm rr} = \frac{\partial \chi^{\rm cons}}{\partial E} \Delta E + \frac{\partial \chi^{\rm cons}}{\partial J} \Delta J$$

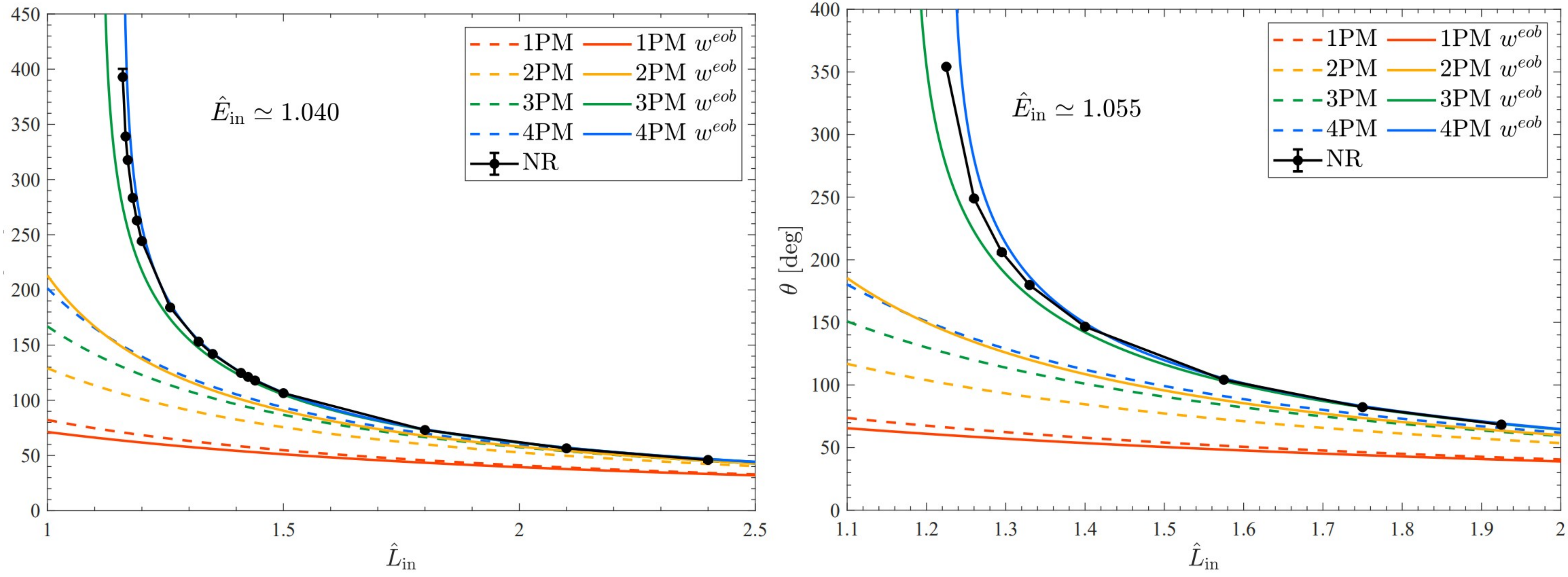
- 4PM radiation reacted impulse / angle

$$\Delta p = \Delta p^{\rm cons} + \Delta p^{\rm rr,even} + \Delta p^{\rm rr,odd}$$

- Dissipative / odd part: 3PM energy loss [Herrmann, Parra-Martinez, Ruf, MZ, '21]. 3PM angular momentum loss [Manohar, Ridgway, Shen, '22]. Complete (even + odd) 4PM impulse: [Dlapa, Kalin, Liu, Neef, Porto, '22]

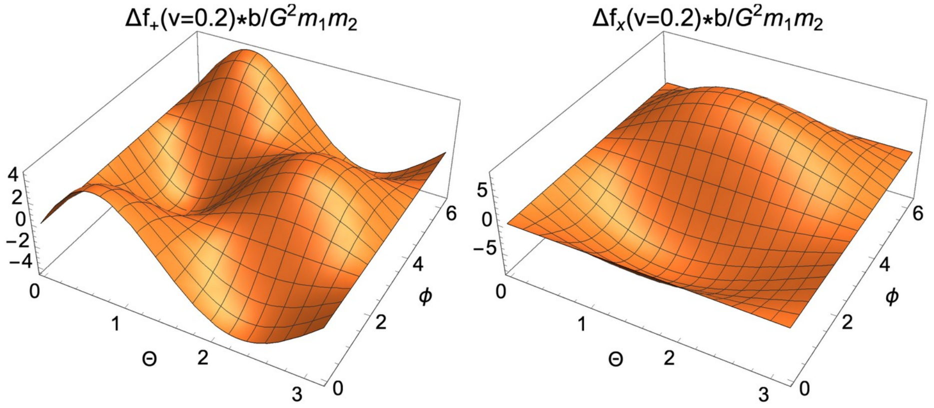
# *PM vs NR:* Conservative + Disspative Angle

Rettegno, Pratten, Thomas, Schmidt, Damour, '23



# *Waveforms (in scattering)*

- From 5-point amplitude involving 1 graviton emission, and worldline approaches.
- Classical limit of massless radiation captured by coherent states; otherwise straightforward extension of KMOC [Cristofoli, Gonzo, Kosower, O'Connell, 22]
- Leading order waveform, radiated energy & angular momentum per solid angle [Jakobsen, Mogull, Plefka, Steinhoff, '21. Mougiakakos, Riva, Vernizzi, '21]:




- NLO waveform from 1-loop 5-point amplitude [Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini, '23. Herderschee, Teng, Roiban, '23. Elkhidir, O'Connell, Sergola, Vazquez-Holm, '23. Georgoudis, Heissenberg, Vazquez-Holm, '23. Caron-Huot, Giroux, Hennesdottir, Mizera, '23]

# Conclusions

- QFT scattering amplitudes have been used to obtain *new results for post-Minkowskian binary dynamics*
- Complements classical and NRGR approaches to post-Newtonian expansion, and offers new analytic insights (special functions, high energy behavior…)
- Relies on synthesis of many modern developments: *on-shell methods for scattering amplitudes, non-relativistic effective field theories, loop integration methods…*
- Wide range of phenomena addressed: spin effects in orbital dynamics, tidal effects and finite-size (non-minimal coupling), dissipation / radiation (5-point amplitudes…)
- Hope to contribute to *coming decades of GW physics!*

# Thank you!