Semiclassical Foundations of Many-Body Quantum Chaos

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Regensburg

Deutsche Forschungsgemeinschaft DFG

3-burg cooperation & Reinhart-Koselleck-programme

Semiclassical foundations of MB quantum chaos

- **Lecture I: The semiclassical limit $\hbar \to 0$**
  - Quantum chaos and semiclassical methods: A brief history
  - Semiclassical quantization of single-particle dynamics
  - The mean density of states: from few to many particles
  - Brief outlook: N-particle Scattering

- **Lecture II: The semiclassical limit $\hbar_{\text{eff}} = 1/N \to 0$**
  - Concepts of MB semiclassics
  - MB Gutzwiller-van Vleck propagator
  - Applications: Many-Body Echoes
  - Brief outlook: Controlling MB quantum chaos

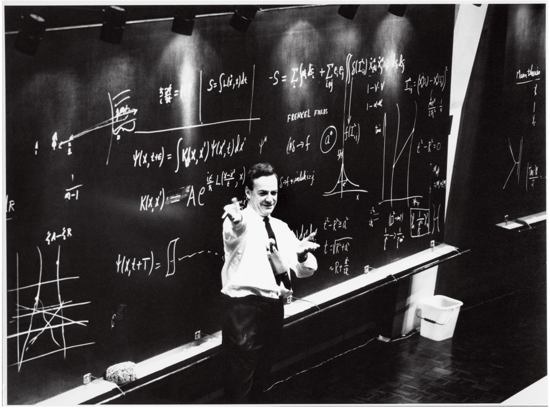
- **Lecture III: Semiclassical roots of universality**
  - Spectral statistics and encounters
  - Rewinding time: OTOCs
  - Brief outlook: A semiclassical way towards (quantum) gravity ?!

KR, J D Urbina, S Tomsovic, *Semiclassical roots of universality in MB quantum chaos*, J. Phys. A 2022

The semiclassical way to many-body complexity

- **approach based on the Feynman path integral:**

$$K\left(\mathbf{q}^{(f)}, \mathbf{q}^{(i)}; T\right) = \int \mathcal{D}\left[\mathbf{q}(t)\right] \mathrm{e}^{\frac{\mathrm{i}}{\hbar} R[\mathbf{q}(t)]}$$



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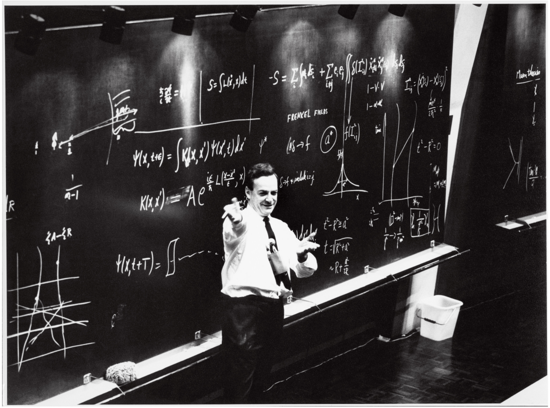

- **consider classical limit, but take into account interference effects !**

Single-particle wave propagation: semiclassics

J.H. van Vleck

M. Gutzwiller

- **semiclassical (stationary phase) approximation to propagator** (for $\hbar \ll R_\gamma$):

$$K^{(sc)}\left(\mathbf{q}^{(f)}, \mathbf{q}^{(i)}; T\right) = \frac{1}{(-2\pi\mathrm{i}\hbar)^{\frac{d}{2}}} \sum_{\gamma} \mathcal{D}_\gamma \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} R_\gamma}$$

- γ: **classical trajectories** $=$ solutions of classical equations of motion

$$\dot{q}_j = \frac{\partial H(\mathbf{p}, \mathbf{q})}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H(\mathbf{p}, \mathbf{q})}{\partial q_j}$$

with boundary conditions $\mathbf{q}(t=0) = \mathbf{q}^{(i)}$ $\qquad \mathbf{q}(t=T) = \mathbf{q}^{(f)}$

- **classical action:**

$$R_\gamma = \int_0^T \mathrm{d}t \left[\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{p}(t), \mathbf{q}(t))\right] \gg \hbar$$

Density of States: Weyl Term + Gutzwiller Term

- single-particle density of states $\varrho(E)$ in the semiclassical limit $\hbar \to 0$:

Hermann Weyl

$$\bar{\varrho}(E) + \sum_{\gamma:\,\mathrm{po}} A_\gamma \cos\left[S_\gamma(E)/\hbar\right]$$

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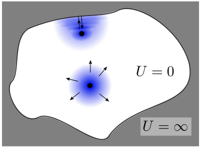
- The Weyl law (1915)

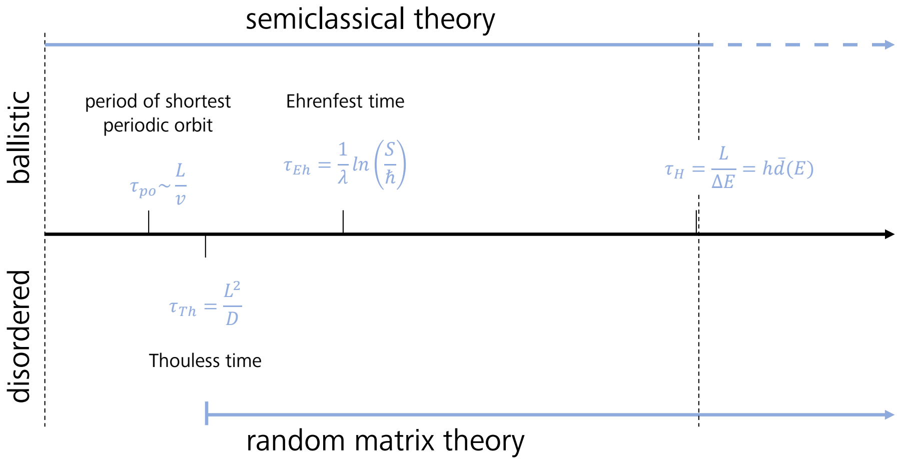
$$\bar{\varrho}(E) = \underbrace{\mathrm{const}\cdot V_D\, E^{\frac{D}{2}-1}}_{\text{locally free}} - \underbrace{\mathrm{const}\cdot S_{D-1}\, E^{\frac{D-1}{2}-1}}_{\text{reflection on flat boundary}} + \dots$$



- smooth DOS $\longleftrightarrow$ **short time propagation**
- smooth DOS dominates for large D, i.e. large particle number N !

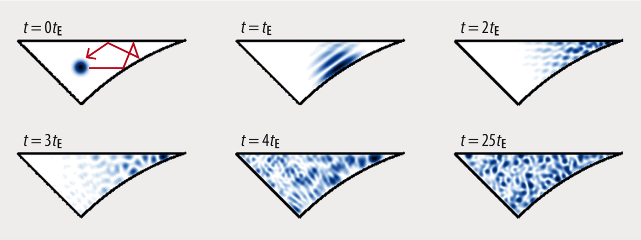
H. Weyl, *Rend. Circ. Matem. Palermo* (1915)

Relevant (single-particle) timescales



Ehrenfest (scrambling) time

- waves in the **semiclassical regime**: $\lambda_{\rm dB} \ll$ system size L



- **Ehrenfest time** $\tau_{\rm E}$: separates wavepacket evolution following classical dynamics from time scales dominated by wave interference

G P Berman & G M Zaslavsky, Physica A 1978

- $L = \lambda_{\rm dB} \exp\left(\lambda \tau_{\rm E}\right) \quad \Rightarrow \quad \tau_{\rm E} \sim \frac{1}{\lambda} \ln\left(L/\lambda_{\rm dB}\right) \sim \frac{1}{\lambda} \ln\left(S/\hbar\right)$

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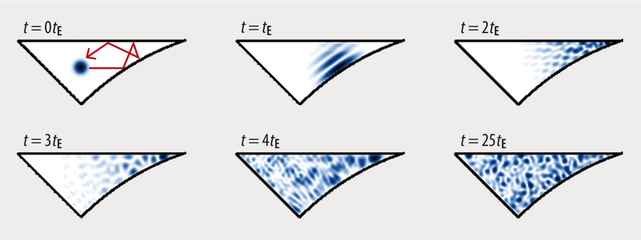

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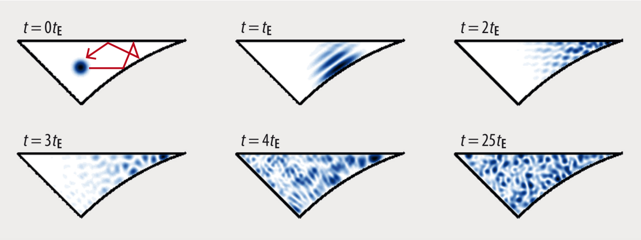

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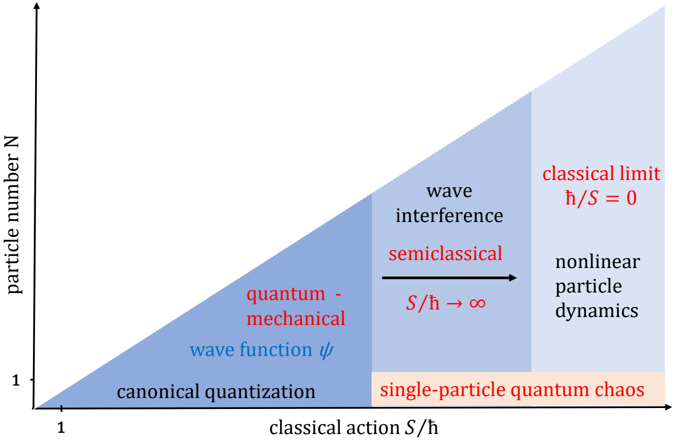
- prominent role in field of quantum chaos / mesoscopic physics

Casati, Aleiner, Larkin, Tian, Adagideli, Beenakker, Jacquod, Whitney, Brouwer, Waltner, KR, . . .

Concepts of MB semiclassics

Semiclassical limits in many-body systems



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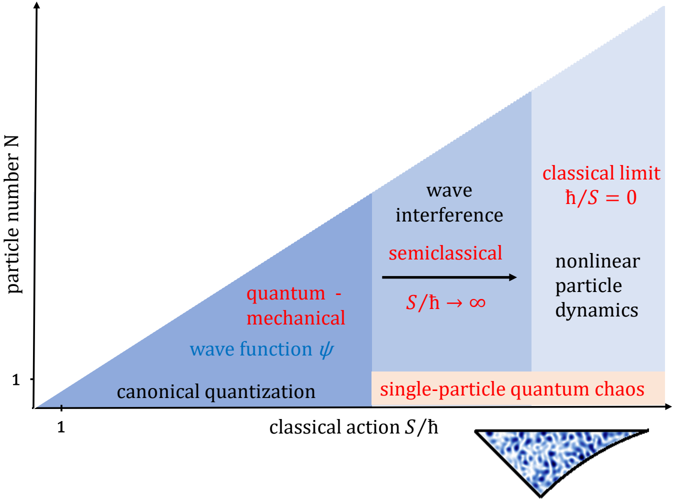

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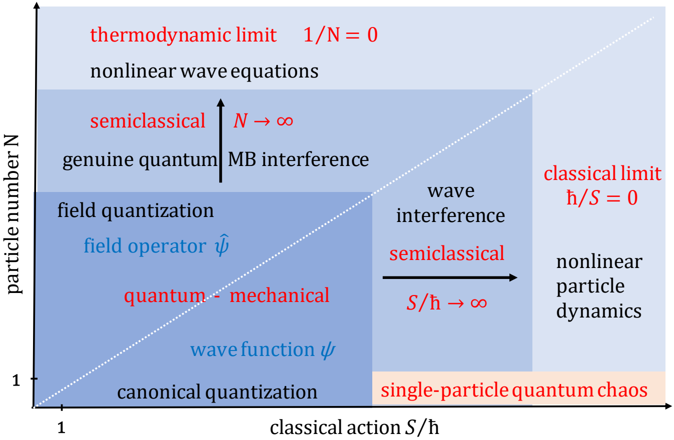
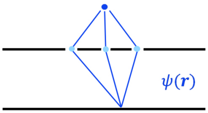


- two notions of classicality in many-body systems:

  $S/\hbar \longrightarrow \infty$ and $N = 1/\hbar_{\rm eff} \longrightarrow \infty$

Wave and quantum interference

single-particle wave interference:



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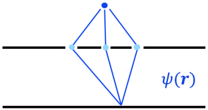
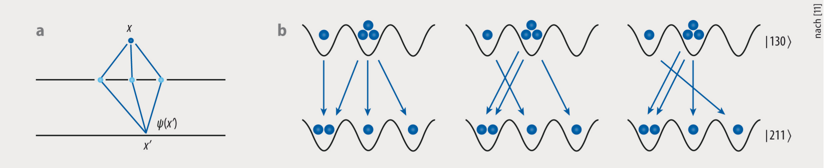


many-body quantum interference:

- Fock state: $|\mathbf{n}\rangle = |n_1, n_2, n_3, \ldots\rangle$, composed of single-particle states
- example: $|1,2,0\rangle \xrightarrow{K(t)} |1,1,1\rangle$

adapted from M Tichy et al., NJP 2012



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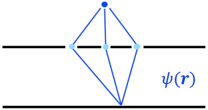

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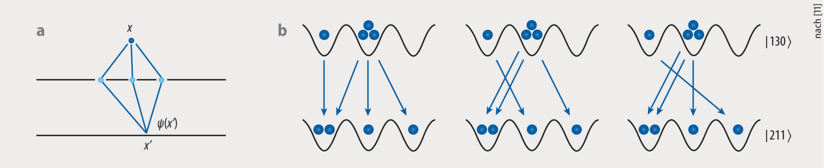
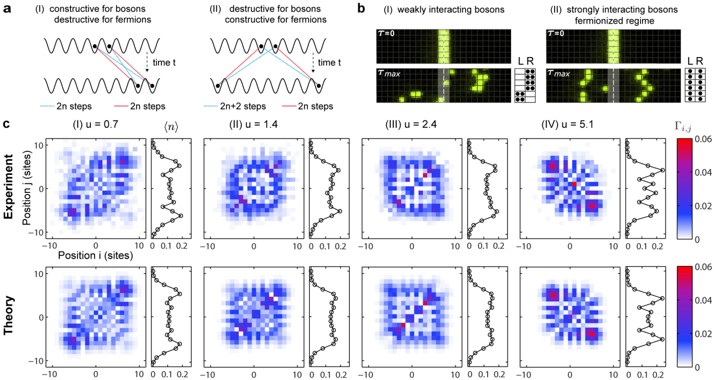


- N identical particles

  $\longrightarrow$ **interference and interactions** in (high-dim.) Fock space

Many-Body Interference of Bosons

Dynamics of 2 bosonic atoms in an optical lattice:



(average over 3200 realizations)

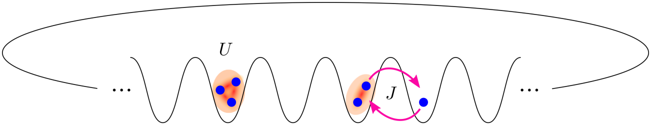
Ph. M. Preiss et al., Science (2015)

MB Gutzwiller-van Vleck propagators

Bosonic many-body dynamics

- **1D-Hubbard model** for N interacting bosons (e.g. cold atoms):

$$\hat{H} = \sum_{\ell=1}^{L} \left[-J \left(\hat{b}_\ell^\dagger \hat{b}_{\ell-1} + \hat{b}_{\ell-1}^\dagger \hat{b}_\ell \right) + \mathcal{E}_\ell \hat{n}_\ell \; + \; \frac{U}{2} \, \hat{n}_\ell (\hat{n}_\ell - 1) \right] \quad ; \quad \hat{n}_\ell = \hat{b}_\ell^\dagger \hat{b}_\ell$$



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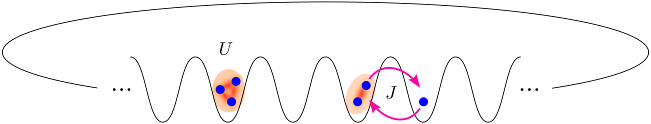

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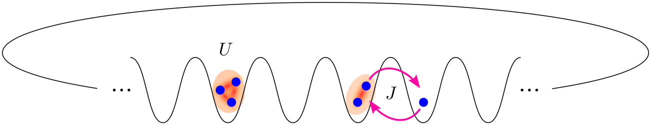

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- Fock state $|\mathbf{n}\rangle = |n_1, n_2, \ldots, n_l, \ldots\rangle$

Gutzwiller propagator for discrete quantum fields

- **semiclassical approach** to many-body propagator for $\hbar_{\mathrm{eff}} = 1/N \ll 1$:

$$K\left(\mathbf{n}^f, \mathbf{n}^i, t\right) = \langle \mathbf{n}^f | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} | \mathbf{n}^i \rangle$$

- **wave equation** for particles and $\hbar \to 0$: use **classical trajectories**
- **quantum dynamics of fields** and $N \to \infty$: use solutions of **classical field equations**
- start with a path integral and proceed in analogy to Gutzwiller's derivation

T Engl, PhD thesis;

T Engl, J Dujardin, A Arguelles, P Schlagheck, KR, J D Urbina, Phys. Rev. Lett. 2014

see also: L Simon, W T Strunz, Phys. Rev. A 2014

Gutzwiller propagator for discrete quantum fields

- introduce quadratures:

$$\hat{q}_i = \frac{\hat{b}_i^\dagger + \hat{b}_i}{\sqrt{2}} \ , \ \hat{p}_i = \frac{\hat{b}_i^\dagger - \hat{b}_i}{\sqrt{2}i}$$

- formaly analogous to position and momentum:

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \ , \ [\hat{q}_i, \hat{p}_j] = i\delta_{i,j} \quad ; \quad \langle \mathbf{q} | \mathbf{p} \rangle = \frac{\mathrm{e}^{i\mathbf{q}\cdot\mathbf{p}}}{(2\pi)^{d/2}}$$

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- proceed as in Gutzwiller's derivation; propagator:

$$K^{\mathrm{BH}}(\mathbf{q}^i, \mathbf{q}^f, t) = \int \mathcal{D}[\mathbf{q}(t)]\mathrm{e}^{\frac{i}{\hbar}R[\mathbf{q}(s),\mathbf{p}(s)]}$$

- the classical limit is given by the mean-field equations with action

$$R[\mathbf{q}(s), \mathbf{p}(s)] = \int_0^t [\dot{\mathbf{q}}(s)\cdot\mathbf{p}(s) - \mathcal{H}_{\mathrm{cl}}(\mathbf{q}(s), \mathbf{p}(s))]\, ds$$

Gutzwiller propagator for discrete quantum fields

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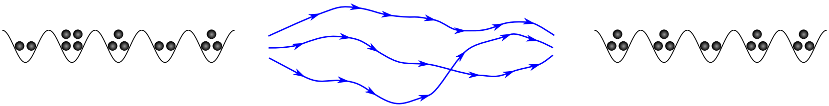
- for a **Fock space representation of the propagator** project onto $|n\rangle$ with $\langle q|n\rangle \sim H_{n\gg 1}(q) \to \cos\left(F(q,n)\right)$ with Hermite polynomial $H_n(q)$ and canonical trafo $F(q,n)$ between pairs (q,p) and (n,θ) .

Semiclassical (bosonic) many-body propagator

- stationary phase approximation
  $\rightarrow$ MB version of semiclassical Van Vleck-Gutzwiller propagator:

$$K^{sc}(\mathbf{n^f}, \mathbf{n^i}, T) \simeq \sum_\gamma A_\gamma \exp\left[\frac{i}{\hbar} R_\gamma(\mathbf{n}^f, \mathbf{n}^i, T)\right] \quad ; \quad |\mathbf{n}\rangle = |n_1, n_2, n_3, \ldots\rangle$$



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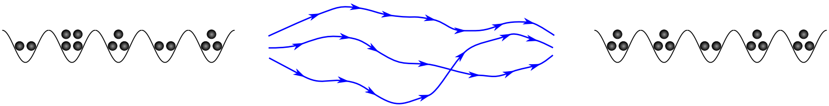

- ”classical” paths $\gamma : \mathbf{n}^i \longrightarrow \mathbf{n}^f$ in L -dimensional Fock space
  with $|\psi_l(0)|^2 = n_l^i + 1/2$; $|\psi_l(T)|^2 = n_l^f + 1/2$ and $\psi_l(t) = \sqrt{n_l(t)}\mathrm{e}^{i\theta_l(t)}$
- and action: $R_\gamma = \int_0^T \left[\hbar\boldsymbol{\Theta}_\gamma(t)\cdot\dot{\mathbf{n}}_\gamma(t) - \mathcal{H}_{cl}\right] dt$

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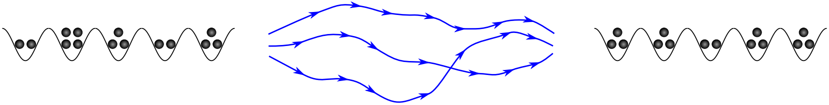

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- $\psi_1^{(\gamma)}(t), \ldots, \psi_L^{(\gamma)}(t)$: solutions of L discrete nonlinear Schrödinger Eqs.
  (discrete Gross-Pitaevskii equation; in the MF-limit $N/L \rightarrow \infty$):

$$i\hbar \frac{d}{dt}\psi_l(t) = \frac{\partial \mathcal{H}_{\mathrm{cl}}}{\partial \psi_l^*} = -J[\psi_{l+1}(t) + \psi_{l-1}(t)] + \mathcal{E}_l \psi_l(t) + U(|\psi_l(t)|^2 - 1)\psi_l(t)$$

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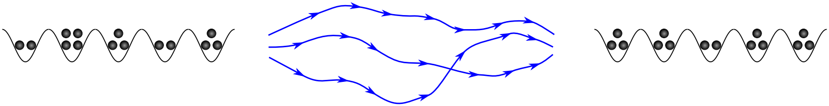

- **nonlinear** mean-field dynamics

  $\longrightarrow$ Lyapunov exponent λ for unstable mean-field modes γ

  $\longrightarrow$ **MB quantum chaos** and **scrambling**

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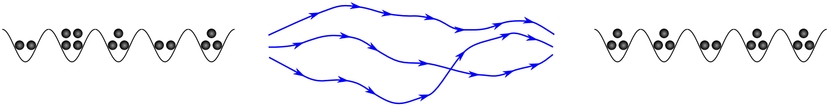

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- **deviations from classical mean field dynamics** at post-Ehrenfest time ($t_E = (1/\lambda) \ln N$) scales

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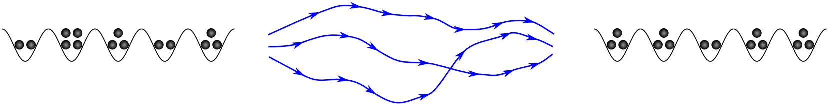

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- **MB interference** and **entanglement** between mean-field modes γ

Periodic Orbit Theory for Quantum Fields ?

Periodic Orbits and Classical Quantization Conditions

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IBM Watson Laboratory, Columbia University, New York, New York 10025

(Received 27 August 1970)

The relation between the solutions of the time-independent Schrödinger equation and the periodic orbits of the corresponding classical system is examined in the case where neither can be found by the separation of variables. If the quasiclassical approximation for the Green's function is integrated over the coordinates, a response function for the system is obtained which depends only on the energy and whose singularities give the approximate eigenvalues of the energy. This response function is written as a sum over all periodic orbits where each term has a phase factor containing the action integral and the number of conjugate points, as well as an amplitude factor containing the period and the stability exponent of the orbit. In terms of the approximate density of states per unit interval of energy, each stable periodic orbit is shown to yield a series of δ functions whose locations are given by a simple quantum condition: The action integral differs from an integer multiple of h by half the stability angle times $\hbar$. Unstable periodic orbits give a series of broadened peaks whose half-width equals the stability exponent times $\hbar$, whereas the location of the maxima is given again by a simple quantum condition. These results are applied to the anisotropic Kepler problem, i.e., an electron with an anisotropic mass tensor moving in a (spherically symmetric) Coulomb field. A class of simply closed, periodic orbits is found by a Fourier expansion method as in Hill's theory of the moon. They are shown to yield a well-defined set of levels, whose energy is compared with recent variational calculations of Faulkner on shallow bound states of donor impurities in semiconductors. The agreement is good for silicon, but only fair for the more anisotropic germanium.

M. Gutzwiller, Journ. Math. Phys. **12**, 343, (1971)

Bosonic Many-Body Trace Formula à la Gutzwiller

- take again limit $\hbar_{\text{eff}} = \frac{1}{N} \to 0$
- assume chaotic mean-field dynamics / nonlinear waves
- take trace of semiclassical MB propagator

Many-body density of states (MB DoS) (for bosons):

$$\rho_N(E) = \bar{\rho}_N(E) + \sum_{\substack{\text{periodic}\\ \text{orbits}}} \mathcal{D}_{\text{po}} \cos\left(\frac{1}{\hbar} S_{\text{po}}(E) + \sigma\frac{\pi}{2}\right) \; ; \; S_{\text{po}} = \hbar \int_0^{t_{\text{po}}} \mathbf{\Theta}(t)\dot{\mathbf{n}}(t)\mathrm{d}t$$

- interfering periodic mean-field solutions generate MB DoS !
- applicable to MB DoS close or at ground states
- $\Rightarrow$ universality in MB spectral statistics $\to$ Lecture III

T. Engl, J.D. Urbina, K. Richter, Phys. Rev. E (2015); R. Dubertrand, S. Müller, NJP (2016)

Applications: Many-Body Echoes

$$K^{sc}(\mathbf{n^f}, \mathbf{n^i}, T) \simeq \sum_\gamma A_\gamma \exp\left[\frac{i}{\hbar} R_\gamma(\mathbf{n^f}, \mathbf{n^i})\right] \quad ; \quad |\mathbf{n}\rangle = |n_1, n_2, n_3, \ldots\rangle$$

1. in Lecture III: **OTOCs:**

   ergodic averaging for **chaotic** MB systems $\rightarrow$ **universal results**

Applications: Many-Body Echoes

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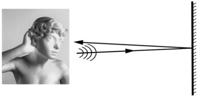
1. in Lecture III: **OTOCs:**

   ergodic averaging for **chaotic** MB systems $\rightarrow$ **universal results**

2. here: **Echoes:**
   - coherent backscattering in Fock space
     ergodic averaging for bosonic MB systems
   - autocorrelation functions:
     full evaluation for bosonic MB systems

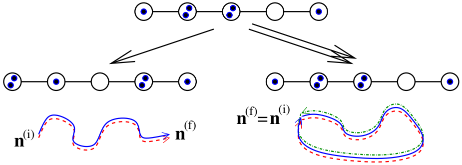


Many-body coherent backscattering

- quantum probability:

$$P\left(\mathbf{n}^{(f)},\mathbf{n}^{(i)},t\right)=\left|K\left(\mathbf{n}^{(f)},\mathbf{n}^{(i)},t\right)\right|^2\simeq\sum_{\gamma,\gamma':\mathbf{n}^{(i)}\rightarrow\mathbf{n}^{(f)}}\mathcal{A}_\gamma\mathcal{A}^*_{\gamma'}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(R_\gamma-R_{\gamma'}\right)}$$

- diagonal approximation $\gamma'=\gamma \;\longrightarrow P_{\mathrm{class}}\left(\mathbf{n}^{(f)},\mathbf{n}^{(i)},t\right)$
- $\mathbf{n}^{(f)}=\mathbf{n}^{(i)}$: additional term from time-reversed paths $\gamma'=\mathcal{T}\gamma$:



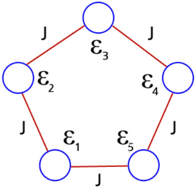
Semiclassical prediction for return probability in Fock space:

$$P_{\mathrm{quant}}(\mathbf{n}^f,\mathbf{n}^i;t)\simeq P_{\mathrm{class}}(\mathbf{n}^f,\mathbf{n}^i;t)(1+\delta_{\mathbf{n}^f,\mathbf{n}^i})$$

Bosonic many-body dynamics

Realization: Bose-Hubbard ring with 5 sites:

- ring topology allows for destroying TRS
- dim $H \sim 10^4$, classically: d = 5
- on-site interaction $U = 4J$
- average over on-site disorder: $\epsilon_\ell \in [0, 10J]$



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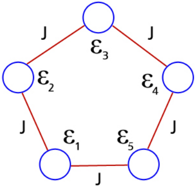
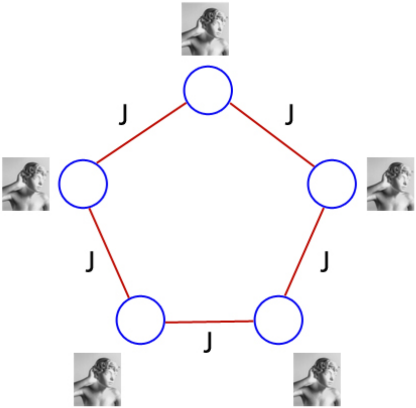


Numerical procedure:

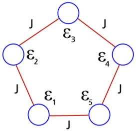
- Distribute particles to initial single-particle states $|\mathbf{n}^i\rangle = |n_1^i, n_2^i, n_3^i, \ldots\rangle$
- Evolve $|\mathbf{n}^i\rangle \xrightarrow{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}} |\mathbf{n}^f\rangle = \sum_{\mathbf{n}} c_{\mathbf{n}^\mathbf{f}} |\mathbf{n}\rangle$
- Probability to measure $|\mathbf{n}^f\rangle$: $P_{\text{quant}}(\mathbf{n}^f, \mathbf{n}^i; t) = |c_{\mathbf{n}^\mathbf{f}}|^2 = \left|K\left(\mathbf{n}^f, \mathbf{n}^i; t\right)\right|^2$

Quantum dynamical echo in Fock space

- let initial state $|\mathbf{n}^i\rangle$ propagate ...
- **listen in Fock space !** (without partial tracing)



Quantum dynamical echo in Fock space

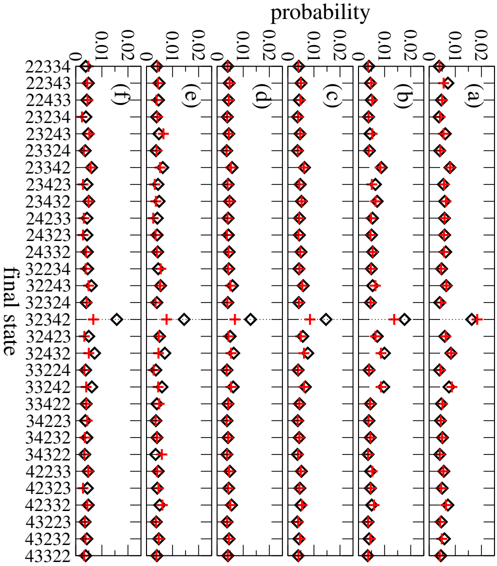


initial state:

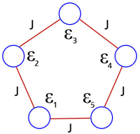
$|\mathbf{n}^i\rangle = |3,2,3,4,2\rangle \;\rightarrow$

Compute:

$$|\mathbf{n}^i\rangle \xrightarrow{e^{-\frac{i}{\hbar}\hat{H}t}} |\mathbf{n}^f\rangle$$



Quantum dynamical echo in Fock space

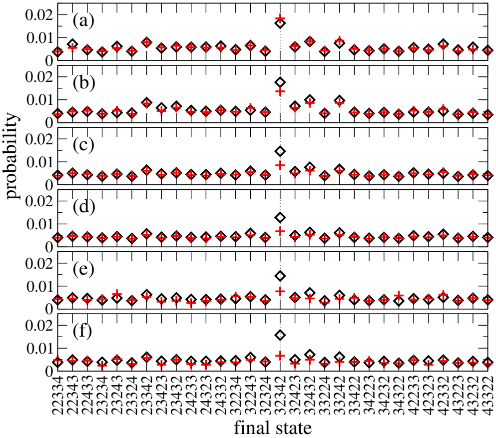


initial state:

$|\mathbf{n}^i\rangle = |3, 2, 3, 4, 2\rangle$

◊: average quantum

+: classical



$\frac{t}{\tau} = \frac{t}{\hbar/J}$

1.5 2.5

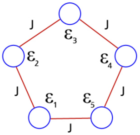
5

10

20

50

Quantum dynamical echo in Fock space



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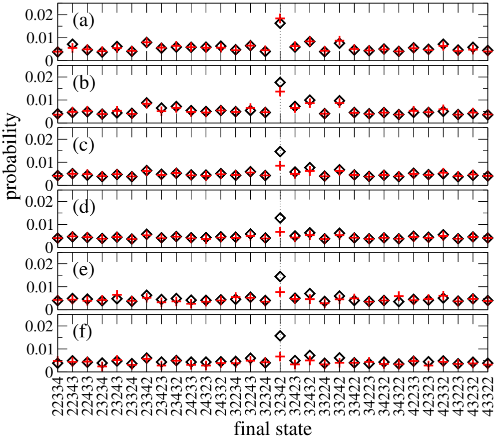


1.5 2.5

5

10

20

50

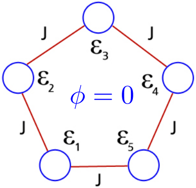
⟹ **Coherent backscattering in Fock space**

⟹ MB quantum interference inhibits quantum ergodicity

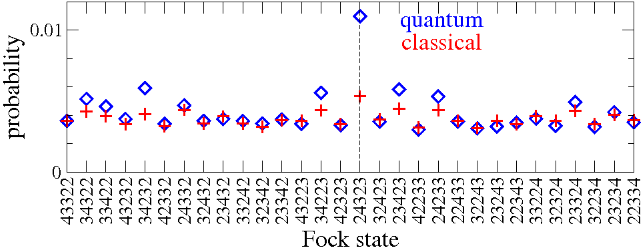
T Engl, J Dujardin, A Arguelles, P Schlagheck, K Richter, J D Urbina, PRL 2014

Coherent backscattering in Fock Space

check interference through dephasing:
breaking time-reversal symmetry (TRS)
through an artificial gauge field



transistion probability **without** flux:



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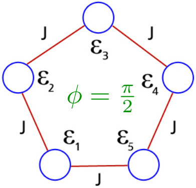

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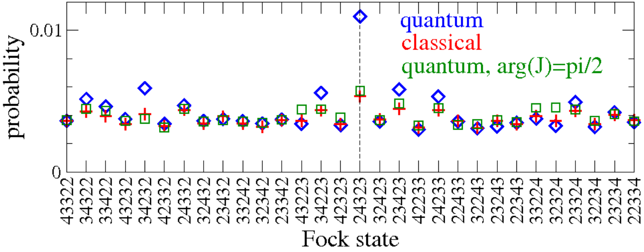
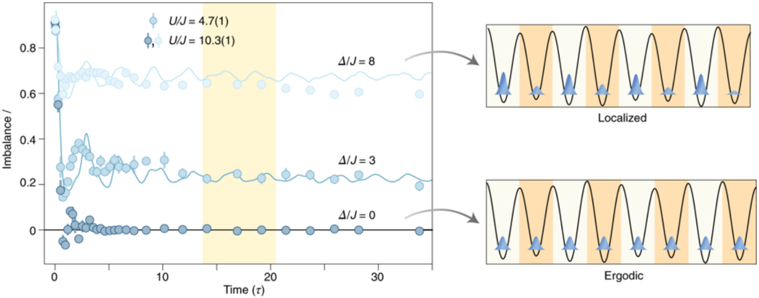


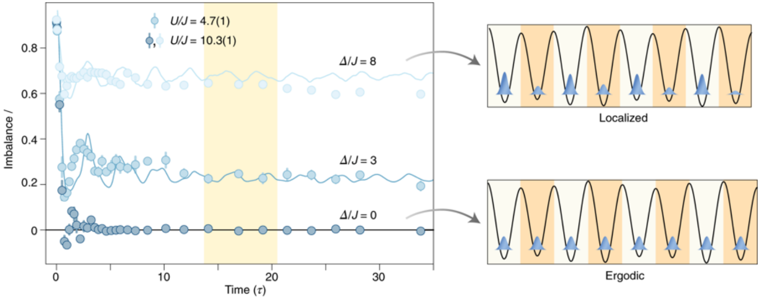
MB dynamics of ultracold atoms



M. Schreiber et al., Science (2015)

- far-out-of-equilibrium dynamics for 1D Bose Hubbard hamiltonians

MB dynamics of ultracold atoms



M. Schreiber et al., Science (2015)

- far-out-of-equilibrium dynamics for 1D Bose Hubbard hamiltonians
- consider **autocorrelator** $\mathcal{A}(t) = \langle \mathbf{n} | e^{-i\hat{H}t} | \mathbf{n} \rangle$
- full semiclassical treatment ($N \gg 1$) without ergodic average

Time evolution of a coherent state

- consider coherent state $|\Phi_{\mathbf{q}^i,\mathbf{p}^i}\rangle$ with

$$\Phi_{\mathbf{q}^i,\mathbf{p}^i}(\mathbf{q}) \equiv \langle \mathbf{q}|\Phi_{\mathbf{q}^i,\mathbf{p}^i}\rangle \sim e^{-\frac{1}{2}(\mathbf{q}-\mathbf{q}^i)^2+i\mathbf{p}^i\cdot(\mathbf{q}-\mathbf{q}^i)}$$

centred about the condensate wavefunction $(\psi_1^i,\ldots,\psi_L^i)$ with $\psi_l^i = (q_l^i + i p_l^i)/\sqrt{2}$

E. H. Lieb, R. Seiringer, and J. Yngvason, Rep. Math. Phys. (2007)

- semiclassical time evolution:

$$\langle \mathbf{q}|\hat{U}(t)|\Phi_{\mathbf{q}^i,\mathbf{p}^i}\rangle = \int d\mathbf{q}' \sum_\gamma A_\gamma(\mathbf{q},\mathbf{q}',t) e^{iR_\gamma(\mathbf{q},\mathbf{q}',t)/\hbar} \Phi_{\mathbf{q}^i,\mathbf{p}^i}(\mathbf{q}')$$

- Expectation values of operators: $\hat{f} \equiv f(\hat{\mathbf{q}},\hat{\mathbf{p}})$:

$$\langle \hat{f} \rangle = \int d\mathbf{q}\ \langle \mathbf{q}|\hat{U}(t)|\Phi_{\mathbf{q}^i,\mathbf{p}^i}\rangle^* \ f\left(\mathbf{q}, -i\frac{\partial}{\partial \mathbf{q}}\right)\ \langle \mathbf{q}|\hat{U}(t)|\Phi_{\mathbf{q}^i,\mathbf{p}^i}\rangle$$

- wildly oscillating in the semiclassical limit with phases

$$\exp[i[R_\gamma(\mathbf{q},\mathbf{q}',t) - R_{\gamma'}(\mathbf{q},\mathbf{q}',t)]/\hbar]$$

Time evolution of a coherent state

- perform diagonal approximation ($\gamma = \gamma'$):

$$\langle \hat{f} \rangle = \frac{1}{\pi^L} \int \int d\mathbf{q}' \, d\mathbf{p}' \; f(\mathbf{q}', \mathbf{p}'; t) \; e^{-(\mathbf{q}' - \mathbf{q}^i)^2 - (\mathbf{p}' - \mathbf{p}^i)^2}$$

$\rightarrow$ equivalent to **Truncated Wigner Approximation** (TWA)
M. J. Steel et al., Phys. Rev. A (1998), A. Sinatra, C. Lobo, Y. Castin, J. Phys. B (2002)

$\rightarrow$ valid for some effective average (e.g. for (one-body) observable or for through disorder)

$\rightarrow$ hard to account for **genuine quantum interference** beyond the diagonal approximation in this framework.

Semiclassical propagation ”à la Maslov”

- consider the intial wave function

$$\Psi_0(\mathbf{q}) = e^{iS_0(\mathbf{q})/\hbar}$$

- compute the semiclassical time evolution:

$$\Psi(\mathbf{q},t) = \sum_\gamma A_\gamma(\mathbf{q},t) e^{iS_\gamma(\mathbf{q},t)/\hbar}$$

  with
  - classical trajectory $\gamma : (\mathbf{q}_\gamma, \mathbf{p}_\gamma)(t)$
  - time evolved action $S_\gamma(\mathbf{q},t)$
  - momentum $\mathbf{p}_\gamma(t) = \frac{\partial S_\gamma}{\partial \mathbf{q}}(\mathbf{q}_\gamma(t),t)$

  V. P. Maslov and M. V. Fedoriuk, Semiclassical Approximations in Quantum Mechanics (1981)

- Here:

$$S_0(\mathbf{q})/\hbar = -i(\mathbf{q}-\mathbf{q}^i)^2/2 + \mathbf{p}^i \cdot (\mathbf{q}-\mathbf{q}^i)$$

$\Rightarrow$ complex trajectories

D. Huber, E. J. Heller, and R. G. Littlejohn, J. Chem. Phys. (1988);

H. Pal, M. Vyas, and S. Tomsovic, Phys. Rev. E (2016)

Echoes: far-out-of-equilibrium MB dynamics

- **case study**: 4-site Bose-Hubbard model; initial state: $|\mathbf{n}\rangle = |20, 0, 20, 0\rangle$
- consider **survival (return) probability** $|\mathcal{A}(t)|^2 = |\langle\mathbf{n}|e^{-i\hat{H}t}|\mathbf{n}\rangle|^2$

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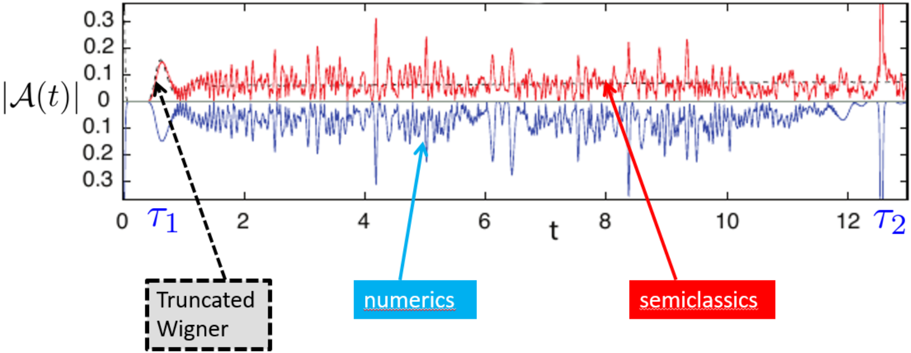

S Tomsovic, P Schlagheck, D Ullmo, J D Urbina, KR, Phys Rev A 2018, Phys Rev Lett 2019

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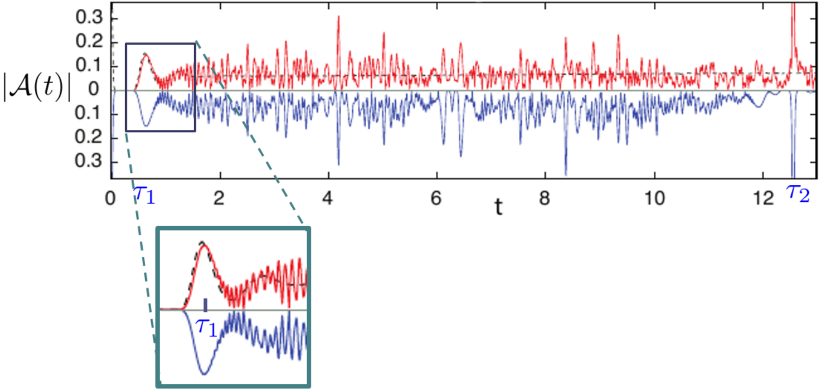

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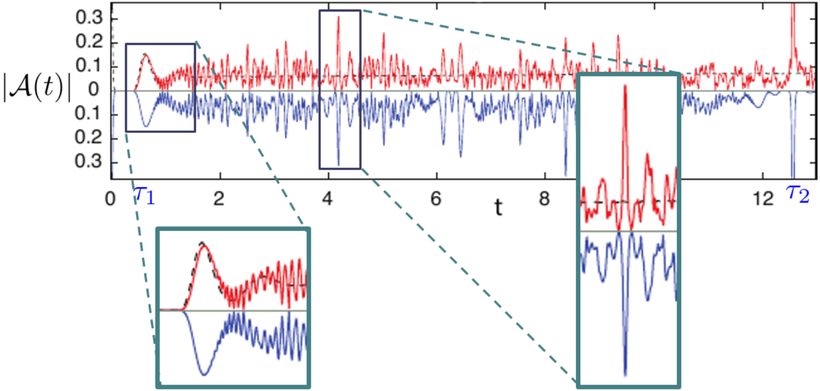
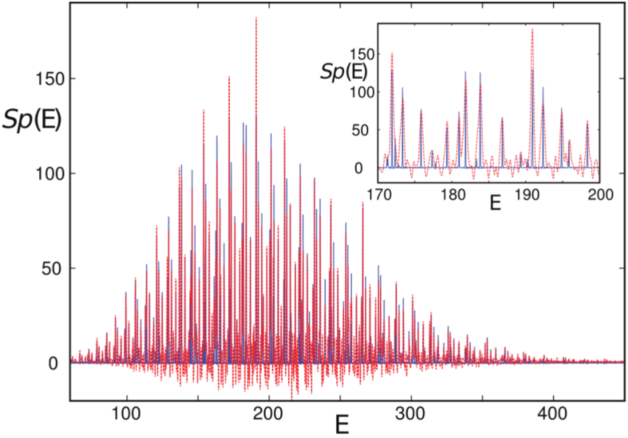


S Tomsovic, P Schlagheck, D Ullmo, J D Urbina, KR, Phys Rev A 2018, Phys Rev Lett 2019

Many-body spectroscopy

- spectrum from Fourier transform: $\delta\mathcal{P}(E) \sim \int \mathrm{d}\tau \exp\left(iE\tau\right)\mathcal{A}(t)$



Outlook:

Controlling MB quantum chaos

Brief outlook: Controlling MB quantum chaos

Facets of chaotic many-body quantum dynamics:

- **Scrambling / quantum butterfly effect / random interference**:
  ultimate form of thermalization: → Lecture III
- **MB quantum scars**:
  deviations from universality in regime that globally exhibits thermalization
- **Controlled coherent quantum targeting:**
  Hamiltonian chaos as a resource

Case study: quantum kicked rotor

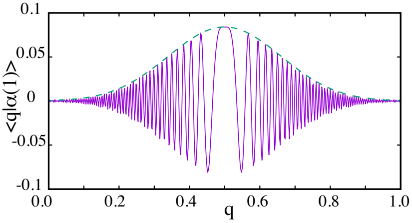
• classical hamiltonian (strong chaos):

$$H(q,p) = \frac{p^2}{2} - \frac{K}{4\pi^2}\cos(2\pi q)\sum_{n=-\infty}^{\infty}\delta(t-n)$$

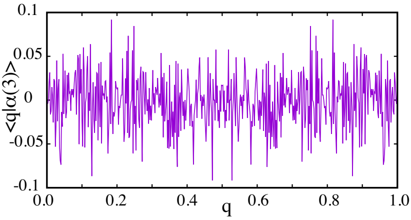
- **without control:**

  fast spreading of initial state towards ergodic distribution

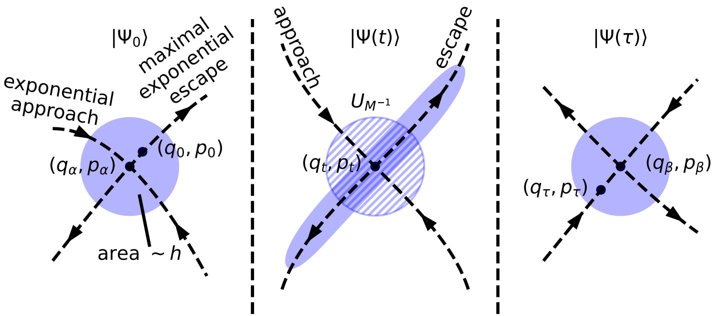
after one iteration:



after 3 iterations:



Scheme of optimal coherent quantum targeting



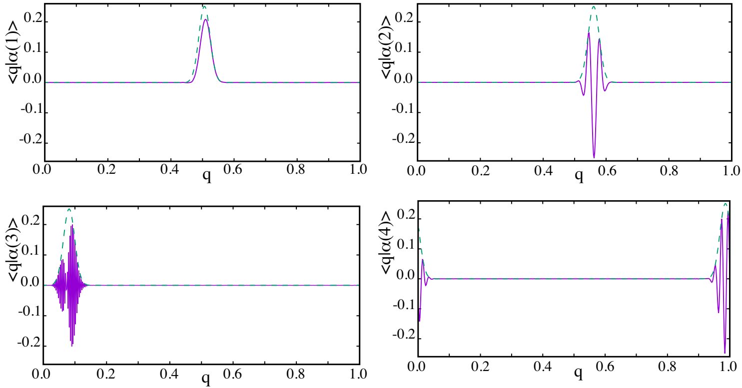
1. slightly shift minimal uncertainty state, via an operator U_s, to an optimal (heteroclinic) trajectory starting at (q_0, p_0).
2. density propagates (exponentially fast) towards predetermined target state
3. local spreading must be counteracted by contractions $U_{M^{-1}}$

Case study: quantum kicked rotor

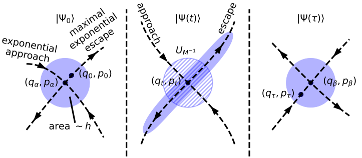
- **controlled quantum dynamics:**

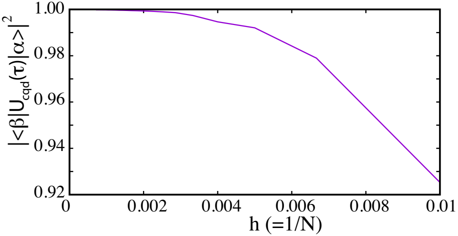
  quantum guiding along heteroclinic classical paths:

$$U_{\mathrm{cqd}}(\tau) = U_s \prod_{j=1}^{n} \{U_{M^{-1}} U(t)\}_j U_s \,,$$



Case study: quantum kicked rotor





controlled quantum dynamics:

- **quantum kicked rotor:**

  overlap: propagated with target state $\longrightarrow 100\%$ in the semiclassical limit!

  S Tomsovic, JD Urbina, & KR, Phys. Rev. Lett. (2023)

- **quantum coherent control and targeting for Bose-Hubbard systems?!** $\rightarrow$ poster by Lukas Beringer

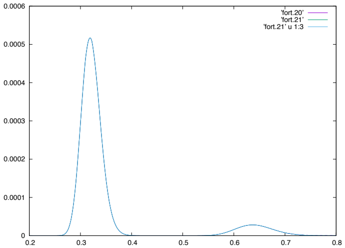
Semiclassical foundations of MB quantum chaos

- **Lecture I: The semiclassical limit $\hbar \to 0$**
  - Quantum chaos and semiclassical methods: A brief history
  - Semiclassical quantization of single-particle dynamics
  - The mean density of states: from few to many particles
  - Brief outlook: N-particle Scattering
- **Lecture II: The semiclassical limit $\hbar_{\text{eff}} = 1/N \to 0$**
  - Concepts of MB semiclassics
  - MB Gutzwiller-van Vleck propagator
  - Applications: Many-Body Echoes
  - Brief outlook: Controlling MB quantum chaos
- **Lecture III: Semiclassical roots of universality**
  - Spectral statistics and encounters
  - Rewinding time: OTOCs
  - Brief outlook: A semiclassical way towards (quantum) gravity ?!

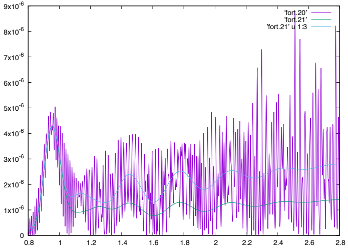
KR, J D Urbina, S Tomsovic, *Semiclassical roots of universality in MB quantum chaos*, J. Phys. A 2022

Semiclassical autocorrelation function for large particle numbers

- 160 particles, 8 sites, i.e. 16 phase space dimensions



short times



long times

- entering regimes beyond range of applicability of full quantum mechanical calculations

Many-body autocorrelation function

- far-out-of-equilibrium dynamics for 1D Bose Hubbard hamiltonians
- consider **autocorrelator** $\mathcal{C}(t) = |\mathcal{A}|(t)|^2 \quad ; \quad \mathcal{A}(t) = \langle \mathbf{n}|\hat{U}(t)|\mathbf{n}\rangle$

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- **challenge**: possibly huge phase space (dim $= 2\#$ of sites)

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- semiclassical treatment ($N \gg 1$) without ergodic average
- **challenge**: possibly huge phase space (dim $= 2\#$ of sites)
- **solution**: explore predominantly unstable directions !

$$\begin{pmatrix} \delta\mathbf{p}_t \\ \delta\mathbf{q}_t \end{pmatrix} = \begin{pmatrix} \mathbf{M}_t^{11} & \mathbf{M}_t^{12} \\ \mathbf{M}_t^{21} & \mathbf{M}_t^{22} \end{pmatrix} \begin{pmatrix} \delta\mathbf{p}_0 \\ \delta\mathbf{q}_0 \end{pmatrix}$$