Critical dynamics of natural swarms

Tomás S. Grigera

Instituto de Física de Líquidos y Sistemas Biológicos — CONICET and Universidad Nacional de La Plata, Argentina

São Paulo, October 4, 2023

Natural swarms: experiments

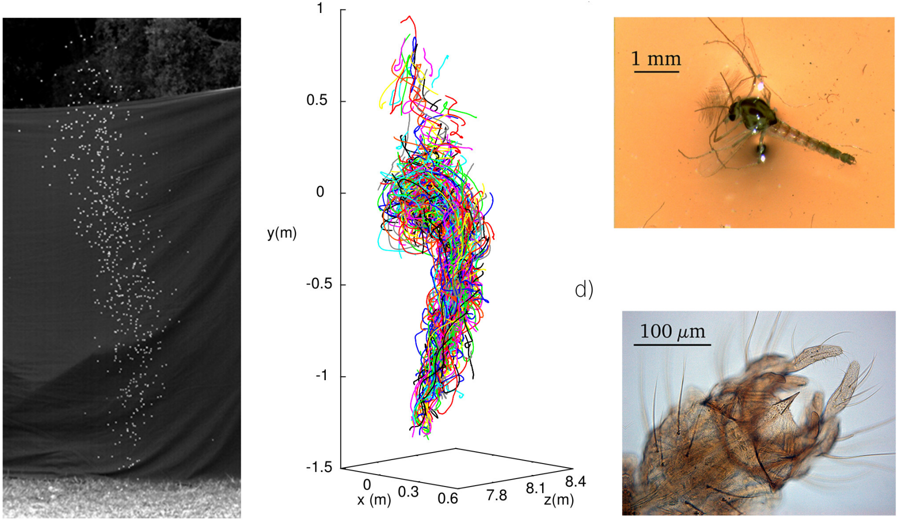
- Statics
- Dynamics

Theory

- Field theories with alignment
- Field theory with alignment and activity
- The missing ingredient: inertia

Swarms of midges

Diptera: Chironomidae and *Diptera: Ceratopogonidae*

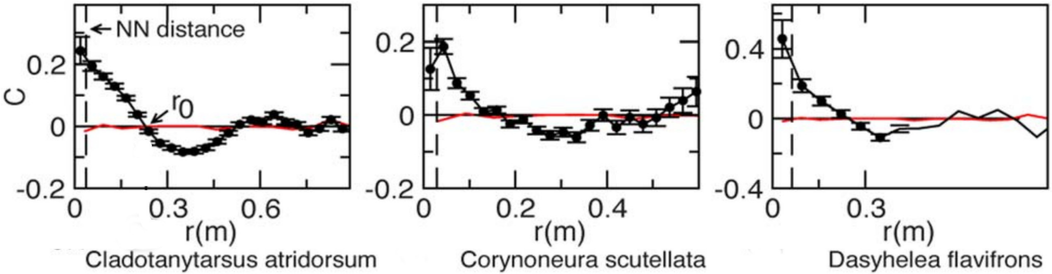


[Attanasi *et al.* PLOS Comp. Biology 2014]

Space correlations

Swarms are not a gas of insects

$$C(r) = \frac{\sum_{i,j} \delta \boldsymbol{v}_i \cdot \delta \boldsymbol{v}_j \delta(r - r_{ij})}{\sum_{i,j} \delta(r - r_{ij})}, \qquad \delta \boldsymbol{v} = \boldsymbol{v}_i - \frac{1}{N} \sum \boldsymbol{v}_i$$



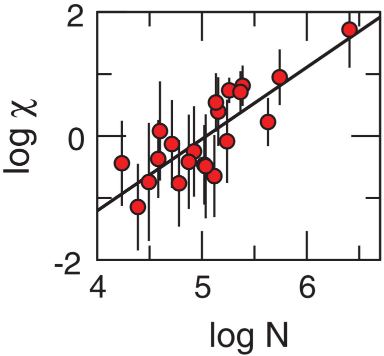
[Attanasi *et al.* PLOS Comp. Biology 2014]

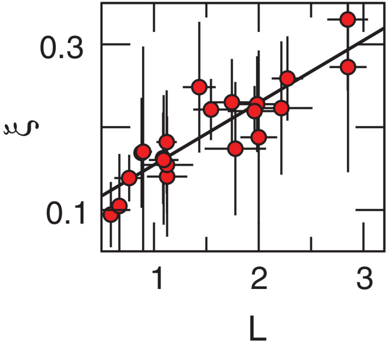
- $r_0 \sim 20$–$30\,$cm, $r_1 \sim 5\,$cm, $r_c \sim 2$–$5\,$cm, swarm size $\sim 1\,$m
- r_0 is not quite ξ
- Is r_0 large?

Space correlations are scale-free

When looking at swarms of different size, r_0 grows as L

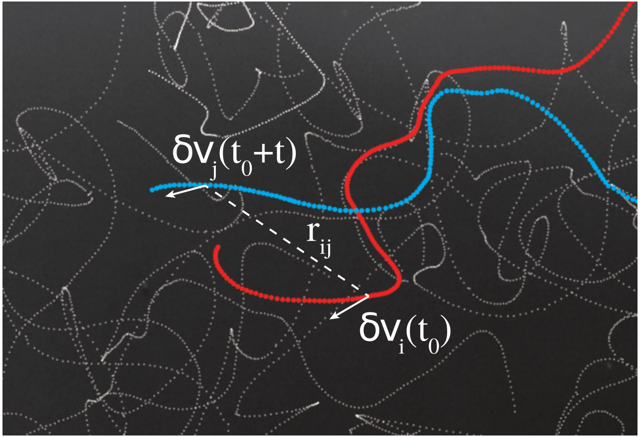
Midges





[Attanasi *et al.* Phys. Rev. Lett. 2014]

Space-time correlations



$$C(r,t) = \langle \delta \boldsymbol{v}(r_0, t_0) \cdot \delta \boldsymbol{v}(r_0 + r, t_0 + t) \rangle$$

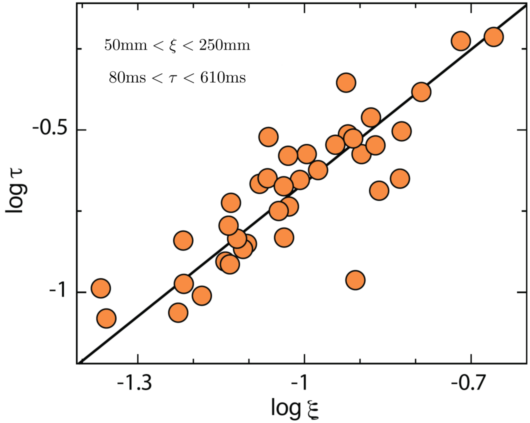
$$= \left\langle \sum_{i,j}^{N} \frac{\delta \boldsymbol{v}_i(t_0) \cdot \delta \boldsymbol{v}_j(t_0 + t) \delta \left[r - r_{ij}(t_0, t) \right]}{\sum_{k,l} \delta \left[r - r_{kl}(t_0, t) \right]} \right\rangle_{t_0},$$

$$r_{ij}(t_0, t) = |\boldsymbol{x}_i(t_0) - \boldsymbol{x}_j(t_0 + t)|$$

Dynamic scaling

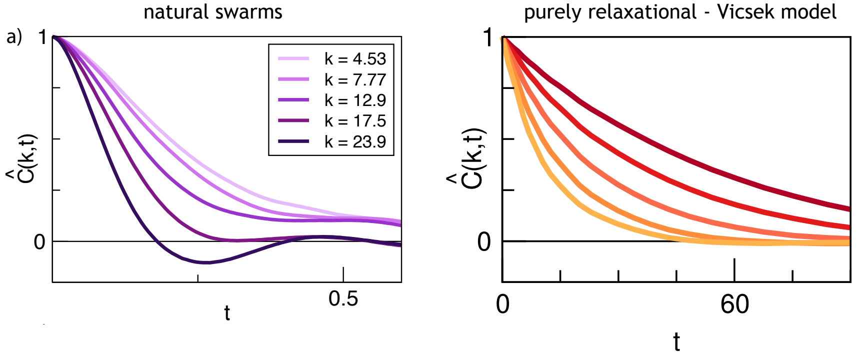
Critical slowing down

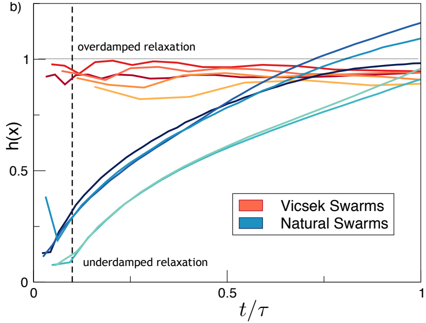
$$\tau \sim k^{-z}, \qquad \text{or} \qquad \tau \sim \xi^{z}$$



- Swarms obey scaling hypothesis $\tau_k = k^{-z} g(k\xi) = \xi^z (\xi k)^{-z} g(k\xi)$
- $z \approx 1.37 \pm 0.11 \Rightarrow$ unknown universality class

Underdamped relaxation





$$h(x) = -\frac{1}{x} \log C(x) \Big|_{x=t/\tau}$$

Natural swarms: experiments

- Statics
- Dynamics

Theory

- Field theories with alignment
- Field theory with alignment and activity
- The missing ingredient: inertia

A theory of natural swarms

Must account for:

- Scale-free correlations (static scaling)
- Critical slowing down (dynamic scaling) with $z \approx 1.37 \pm 0.11$
- Underdamped dynamics (damped oscillations)

Required ingredients:

1. Ferromagnetism (alignment)
2. Activity
3. Inertia

Ferromagnetism

classical ferromagnet

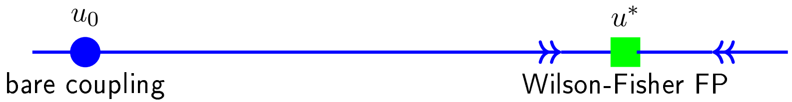
$$\frac{d\boldsymbol{\sigma}_i}{dt} = J \sum_{ij} n_{ij} \boldsymbol{\sigma}_j + \xi_i, \qquad |\boldsymbol{\sigma}| = 1, \quad \xi : \text{ Gaussian noise}$$

$$\Big\downarrow \text{ coarse-graining}$$

model A:

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = -\Gamma \frac{\delta \mathcal{H}}{\delta \boldsymbol{\psi}(x,t)} + \xi(x,t),$$

$$\mathcal{H} = \int d^d x \left\{ (\nabla \boldsymbol{\psi})^2 + r\psi^2 + u\psi^4 \right\}, \qquad \text{(Landau-Guinzburg)}$$



$z \approx 2$ vs $z_{\text{exp}} = 1.37 \pm 0.11$

[Hohenberg, Halperin, Ma 1972]

Activity

Vicsek: active ferromagnet

$$\frac{d\boldsymbol{\sigma}_i}{dt} = J \sum_{ij} n_{ij}(\{\boldsymbol{x}\})\boldsymbol{\sigma}_j + \xi_i,$$

$$\frac{d\boldsymbol{x}_i}{dt} = v_0 \boldsymbol{\sigma}_i.$$

$\downarrow$ coarse-graining

Toner-Tu (1998) field theory (active model A)

$$D_t \boldsymbol{v} = -\Gamma \frac{\delta \mathcal{H}}{\delta \psi(x,t)} - \nabla P + \xi(x,t),$$

$$\partial_t \rho + \nabla(\rho \boldsymbol{v}) = 0$$

Material derivative $D_t = \partial_t + \lambda(\boldsymbol{v} \cdot \nabla)$.

Incompressible Toner-Tu

continuous transition

The field-theory version of the (incompressible) Vicsek model

$$\frac{\partial \boldsymbol{v}}{\partial t} + \lambda(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \Gamma \nabla^2 \boldsymbol{v} + \Gamma r \boldsymbol{v} + \Gamma u v^2 \boldsymbol{v} + \boldsymbol{\nabla} P + \boldsymbol{\xi},$$

$$\text{noise: } \langle \xi_\alpha(\boldsymbol{x}, t) \xi_\beta(\boldsymbol{x}', t') \rangle = 2\tilde{\Gamma} \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t') \delta_{\alpha\beta}$$

Incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ fixes pressure P.

[Chen, Toner, Lee, New J Phys 2015]

Incompressible Toner-Tu

continuous transition

The field-theory version of the (incompressible) Vicsek model

$$\frac{\partial \boldsymbol{v}}{\partial t} + \lambda(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \Gamma \nabla^2 \boldsymbol{v} + \Gamma r \boldsymbol{v} + \Gamma u v^2 \boldsymbol{v} + \boldsymbol{\nabla} P + \boldsymbol{\xi},$$

noise: $\langle \xi_\alpha(\boldsymbol{x}, t) \xi_\beta(\boldsymbol{x}', t') \rangle = 2\tilde{\Gamma} \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t') \delta_{\alpha\beta}$

Incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ fixes pressure P.

Hohenberg-Halperin model A plus activity

Incompressible Toner-Tu

continuous transition

The field-theory version of the (incompressible) Vicsek model

$$\frac{\partial \boldsymbol{v}}{\partial t} + \lambda(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \Gamma \nabla^2 \boldsymbol{v} + \Gamma r \boldsymbol{v} + \Gamma u v^2 \boldsymbol{v} + \boldsymbol{\nabla} P + \boldsymbol{\xi},$$

$$\text{noise: } \langle \xi_\alpha(\boldsymbol{x}, t) \xi_\beta(\boldsymbol{x}', t') \rangle = 2\tilde{\Gamma}\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')\delta_{\alpha\beta}$$

Incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ fixes pressure P.

stirred fluid [Forster, Nelson and Stephen PRA 1977] plus alignment

Incompressible Toner-Tu

continuous transition

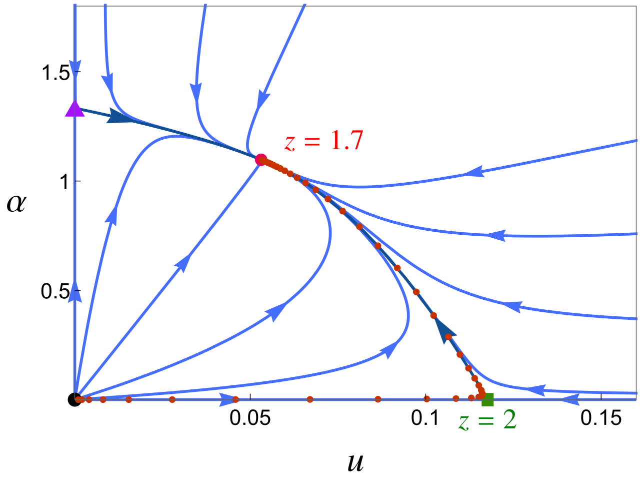
The field-theory version of the (incompressible) Vicsek model

$$\frac{\partial \boldsymbol{v}}{\partial t} + \lambda(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \Gamma \nabla^2 \boldsymbol{v} + \Gamma r \boldsymbol{v} + \Gamma u v^2 \boldsymbol{v} + \boldsymbol{\nabla} P + \boldsymbol{\xi},$$

noise: $\langle \xi_\alpha(\boldsymbol{x}, t) \xi_\beta(\boldsymbol{x}', t') \rangle = 2\tilde{\Gamma} \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t') \delta_{\alpha\beta}$

Incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ fixes pressure P.

Note $\lambda \neq 1$, $\tilde{\Gamma} \neq \Gamma$



[Chen, Toner, Lee, New J Phys 2015]

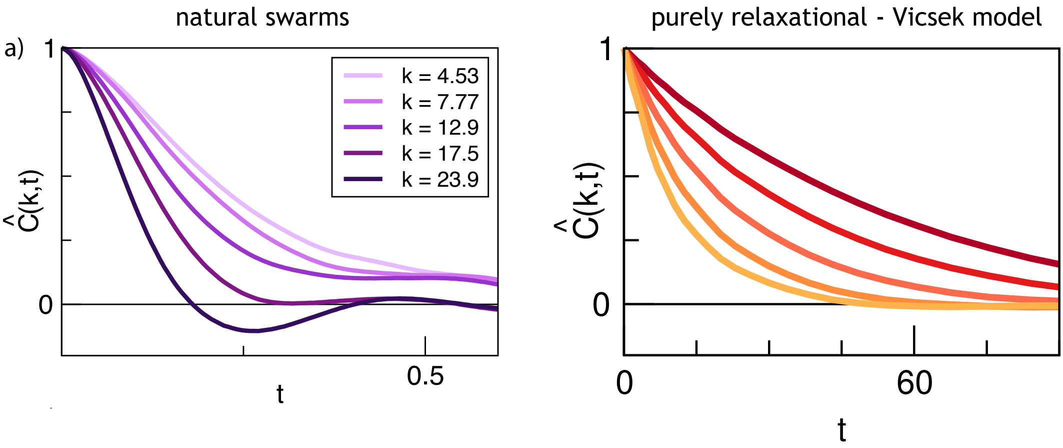
$z \approx 2 \xrightarrow{\text{activity}} z \approx 1.73$

BUT $z_{\text{exp}} = 1.37 \pm 0.11$

activity: $\alpha = \lambda^2 \left(\frac{\tilde{\Gamma}}{\Gamma^3} \right) \Lambda^{-\epsilon}$, alignment: $\hat{u} = u \left(\frac{\tilde{\Gamma}}{\Gamma} \right) \Lambda^{-\epsilon}$

Dispersion relations

Clue to missing inertia



$$\omega \sim ck + iDk^2, \qquad \text{vs} \qquad \omega \sim iDk^2$$

Theory of dynamic critical phenomena

P. C. Hohenberg

Bell Laboratories, Murray Hill, New Jersey 07974
and Physik Department, Technische Universität München, 8046, Garching, W. Germany

B. I. Halperin*

Department of Physics, Harvard University, Cambridge, Mass. 02138

Reviews of Modern Physics, July 1977

An introductory review of the central ideas in the modern theory of dynamic critical phenomena is followed by a more detailed account of recent developments in the field. The concepts of the conventional theory, mode-coupling, scaling, universality, and the renormalization group are introduced and are illustrated in the context of a simple example—the phase separation of a symmetric binary fluid. The renormalization group is then developed in some detail, and applied to a variety of systems. The main dynamic universality classes are identified and characterized. It is found that the mode-coupling and renormalization group theories successfully explain available experimental data at the critical point of pure fluids, and binary mixtures, and at many magnetic phase transitions, but that a number of discrepancies exist with data at the superfluid transition of ^{4}He.

CONTENTS

theory with purely *irreversible* terms
- $z \approx 2$
- $\omega = iDk^2$

theories with *reversible* mode-coupling terms
- $z = 1.5$
- $\omega = iDk^2 \pm ck$

Inertia

Model A vs model G

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\Gamma \frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} + \boldsymbol{\xi}(\boldsymbol{x}, t),$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = +g\boldsymbol{v} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{s}} - \Gamma \frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} + \boldsymbol{\xi}_v(\boldsymbol{x}, t),$$
$$\frac{\partial \boldsymbol{s}}{\partial t} = -g\boldsymbol{v} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} - \Lambda \frac{\delta \mathcal{H}}{\delta \boldsymbol{s}} + \boldsymbol{\xi}_s(\boldsymbol{x}, t),$$

Reversible dynamics generated by Poisson brackets

$$\{s_\mu, v_\nu\} = g\epsilon_{\mu\nu\gamma} v_\gamma, \qquad \{s_\mu, s_\nu\} = g\epsilon_{\mu\nu\gamma} w_\gamma,$$

The spin $\boldsymbol{s}(\boldsymbol{x}, t)$ is the generator of the rotations of the primary field $\boldsymbol{v}(\boldsymbol{x}, t)$.
Spin is conserved $\{\boldsymbol{s}, \mathcal{H}\} = 0$, $\Lambda = \lambda \nabla^2$.

Model G has $z = 3/2$ and $\omega = iDk^2 \pm ck$.

Active model G

or coarse-grained inertial spin model (ISM)

Add activity to model G:

$$\frac{\partial \boldsymbol{v}}{\partial t} = +g\boldsymbol{v} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{s}} - \Gamma \frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} + \boldsymbol{\xi}_v(\boldsymbol{x}, t),$$
$$\frac{\partial \boldsymbol{s}}{\partial t} = -g\boldsymbol{v} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} - \Lambda \frac{\delta \mathcal{H}}{\delta \boldsymbol{s}} + \boldsymbol{\xi}_s(\boldsymbol{x}, t),$$

$$\partial_t \boldsymbol{v} \to D_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \gamma_v (\boldsymbol{v} \cdot \nabla)\boldsymbol{v},$$
$$\partial_t \boldsymbol{s} \to D_t \boldsymbol{s} = \partial_t \boldsymbol{s} + \gamma_v (\boldsymbol{v} \cdot \nabla)\boldsymbol{s},$$

Our theory:

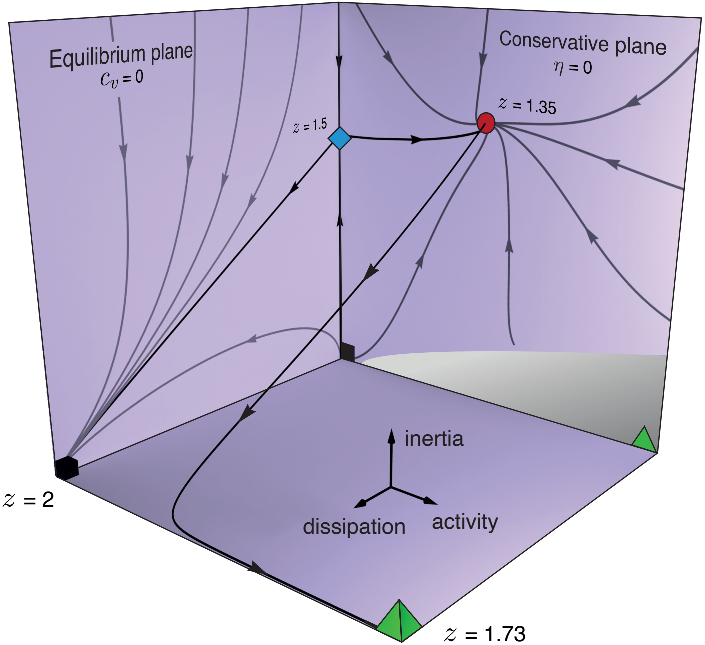
$$D_t \boldsymbol{v}(\boldsymbol{x}, t) = +g\boldsymbol{v} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{s}} - \Gamma \frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} - \nabla P + \boldsymbol{\xi}_v(\boldsymbol{x}, t),$$
$$D_t \boldsymbol{s}(\boldsymbol{s}, t) = -g\boldsymbol{v} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} - \Lambda \frac{\delta \mathcal{H}}{\delta \boldsymbol{s}} + \boldsymbol{\xi}_s(\boldsymbol{x}, t),$$
$$\nabla \cdot \boldsymbol{v} = 0$$

(Small) dissipation $\Lambda = \eta + \lambda \nabla^2$

$$\mathcal{H} = \int d^d x \left[\frac{1}{2} (\nabla \boldsymbol{v})^2 + \frac{1}{2} r \boldsymbol{v}^2 + \frac{u}{4!} \boldsymbol{v}^4 + \frac{1}{2} s^2 \right]$$

RG flow

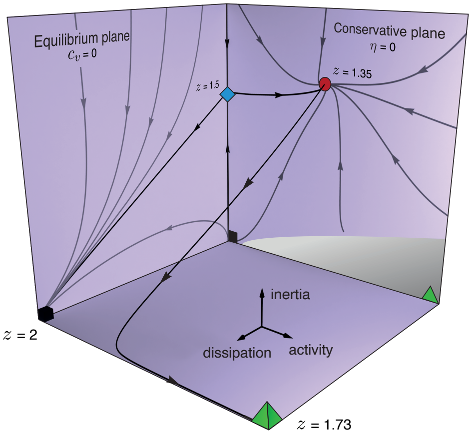
At 1 loop



New dynamic critical exponent

Alignment + activity + inertia

$$z \approx 2 \xrightarrow{\text{activity}} z \approx 1.73 \xrightarrow{\text{inertia}} z \approx 1.35$$



$$z_{\text{exp}} = 1.37 \pm 0.11$$

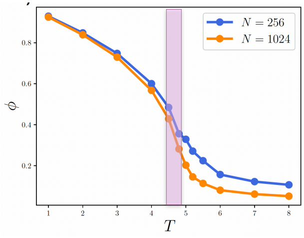
Numerical simulations

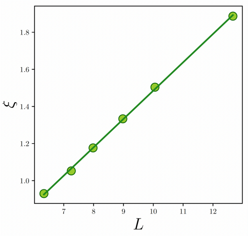
Inertial spin model

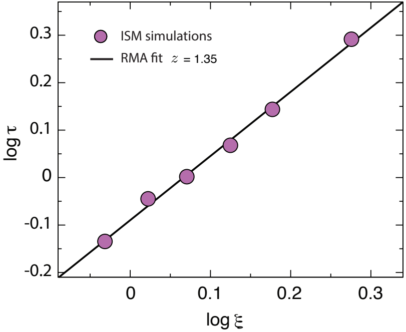
$$\frac{d\boldsymbol{v}_i}{dt} = \frac{1}{\chi}\boldsymbol{s}_i \times \boldsymbol{v}_i,$$

$$\frac{d\boldsymbol{s}_i}{dt} = \boldsymbol{v}_i \times \sum_j n_{ij}(t)\boldsymbol{v}_j - \frac{\eta}{\chi}\boldsymbol{s}_i + \boldsymbol{v}_i \times \boldsymbol{\xi}_i,$$

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i.$$







Conclusions

- Midge swarms live near a critical point:
  - velocity correlation length scales with size,
  - velocities are disordered.
- Swarms obey dynamic scaling: the correlation length rules dynamic behavior.
- The dynamic critical exponent z is different from known dynamic models.
- Values of z:
  - $z_{\text{exp}} = 1.37 \pm 0.11$
  - $z_{\text{RG}} = 1.35$
  - $z_{\text{sim}} = 1.35 \pm 0.04$

Collaborators

Istituto Sistemi Complessi and/or *Sapienza* University (Roma):

- Andrea Cavagna (ISC Roma)
- Irene Giardina (Sapienza Roma)
- Stefania Melillo (ISC Roma)
- Leonardo Parisi (ISC Roma)
- Luca Di Carlo (Sapienza Roma $\longrightarrow$ Princeton)
- Giulia Pisegna (Sapienza Roma $\longrightarrow$ Göttingen)
- Mattia Scandolo (Sapienza Roma)

Natural swarms in 3.99 dimensions, *Nature Physics* 2023