ICTP Notes

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1 Homogeneous and Isotropic Space Time

The Einstein field equation contains nonlinear partial differential equation. We will solve the Field equations for the whole Universe which is homogeneous and isotropic. Homogeneous means that the Universe looks same at every point in Space. Isotropic means that the Universe looks very much the same whatever direction we look. The universe is also expanding which means that the distant galaxies were closer to us than they are today. We introduce a scale factor to connect the coordinate distance with the physical distance. More generally, Coordinate distance ⇒metric ⇒ physical distance.

The first question one can ask what is the effect of the isotropy and homogeneous on the metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{1}$$

The effect of the symmetry leads to the Robertson-Walker line element. We can break the above equation:

$$ds^2 = g_{00}dt^2 + 2g_{0i}dx^i dt + g_{ij}dx^i dx^j;\ x^i = (x,\ y\ z) \tag{2}$$

Now isotropy implies spherical symmetry, this means

$$g_{00} = g_{00}(r,t);\ \ r = \sqrt{x^2+y^2+z^2} \tag{3}$$

o, g_{0i} can not depend on any preferred vector a^i to carry the "i" index or it would not be isotropic. Thus it must have the form

$$g_{0i} = \frac{g_{0r}x^i}{r} \tag{4}$$

Since $x^i dx^i = rdr$. We can write

$$ds^2 = g_{00}dt^2 + 2g_{0r}drdt + g_{ij}dx^i dx^j \tag{5}$$

. We can now simplify by making coordinate transformation that eliminates the cross-term between dr and dt. Let

$$t = t' + \phi(r', t),\ r = r',\ \phi(0, t') = 0 \tag{6}$$

Now in the new frame consider $x^\alpha \equiv (x^0, r)$ 2 dim subspace.

$$g'_{0r} = \frac{\partial x^\alpha}{\partial x^{0'}}\frac{\partial x^\beta}{\partial r'} g_{\alpha\beta},\;\; g_{\alpha\beta} = \{g_{00}, g_{0r},\, g_{rr}\} \tag{7}$$

$$g'_{0r} = (1+\frac{\partial \phi}{\partial x^{0'}})\frac{\partial \phi}{\partial r'} g_{00} + (1+\frac{\partial \phi}{\partial x^{0'}}) g_{0r} \tag{8}$$

$$\frac{\partial \phi}{\partial r'} = -\frac{g_{0r}}{g_{00}} = \Phi(r,t) = \Phi(r', t' + \phi(r',t')) \tag{9}$$

Here we have chosen ϕ to make $g'_{0r} = 0$ This is a first order differential equation to determine ϕ, and in general will always have a solution.

The metric now reads

$$ds^2 = g_{00}(r,t)dt^2 - g_{ij}dx^i dx^j = g_{00}dt^2 - d\sigma^2 \tag{10}$$

We now need to impose isotropy the special components $d\sigma^2$. To see what this means, recall the ordinary flat space in sphericalcoordinates $x^3 = rcos\theta$, $x^2 = rsin\theta sin\phi$, $x^1 = rsin\theta cos\phi$. Then for flat space: $d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dr^2 + r^2 d\Omega^2$ where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$.

Since isotropy implies spherical symmetry, the general form for $d\sigma^2$ is

$$d\sigma^2 = F(r,t)dr^2 + G(r,t)d\Omega^2 \tag{11}$$

We now need to impose homogeneity, which is more complicated constraint. Consider a time interval dt at fixed r, θ, ϕ. Then

$$T(r,t) = (ds)_{dx^i=0} = \sqrt{g_{00}(r,t)}dt \tag{12}$$

where T is what clock is at rest w.r.t. frame will measure. Now homogenieity means that it should bbe possible to find a frame where clocks tick at the same rate at all in space. This means T should be at most a function of t independent of r. Hence $g_{00} = \phi(t)$. Then we can make a coordinate transformation in these coordinates. Our metric then simply reads

$$ds^2 = dt^2 - d\sigma^2 \tag{13}$$

Now our universe is expanding and so let us apply homogeneity to the expansion. Consider two infinitesimal close by points. The invariant distance is

$$l_r = F^{1/2}(r,t)dr = \text{invariant distance} \tag{14}$$

and the radial expansion rate is

$$\frac{\dot{l}_r}{l_r} = \frac{1}{2}\frac{\dot{F}(r,t)}{F(r,t)} \tag{15}$$

Homogeniety now implies the expansion must look the same at every point so

$$\frac{\dot{l}_r}{l_r} = \frac{1}{2}\frac{\dot{F}(t)}{F(t)} = \text{function of t only} \tag{16}$$

Similarly $l_\theta = G(\theta)^{1/2} d\theta$ and $l_\phi = sin\theta G^{1/2} d\phi$, one has

$$\frac{\dot{l}_\theta}{l_\theta} = \frac{1}{2}\frac{\dot{G}(t)}{G(t)} = \frac{\dot{l}_\phi}{l_\phi} = \text{function of t only} \tag{17}$$

Now by isotropy, the expansion in different directions must be equal. Hence

$$\frac{\dot{F}(t)}{F(t)} = \frac{\dot{G}(t)}{G(t)} = \Phi(t) \tag{18}$$

One can integrate to get

$$F(r,t) = R^2(t) f(r) \,\text{with}\, f(0) = 1 \tag{19}$$

$$G(r,t) = R^2(t) g(r), \;\; 2\frac{\dot{R}}{R} = \Phi(t) \tag{20}$$

and $f(r)$, $g(r)$ are integration constants. The choice $f(0) = 1$ fixes the scale of $R(t)$.

Returning now to Eqn.5.11, we see that G plays the role ofr^2 in flat space and so it is convenient to make a coordinate transformation

$$r'^2 = g(r) \tag{21}$$

which then reduces the metric to

$$ds^2 = (dx^0)^2 - R^2(t) d\sigma^2;\, d\sigma^2 = f(r) dr^2 + r^2 d\Omega^2 \tag{22}$$

We have not yet imposed the full content of homogeneity which means that the universe looks the same from any point. Thus if we make a translation of origin to a new origin things should look the same and this condition should restrict the form of $f(r)$. To see this consider a spatial transformation of coordinate

$$x^i = x^{i'} + \xi^i(x');\, \xi^i = infinitesimal \tag{23}$$

Now we know

$$g'_{ij}(x') = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} g_{kl}(x) \tag{24}$$

and expanding out to first order in ξ^i:

$$g'_{ij}(x') = g_{ij}(x) + g_{ik}\xi^k_{,j} + g_{jk}\xi^k_{,i} \tag{25}$$

We can also expand on LHS

$$g'_{ij}(x') = g'_{ij}(x^i - \xi^i(x')) \simeq g'_{ij}(x) - g'_{ij,k}(x)\xi^k(x) \tag{26}$$

$$g'_{ij}(x) = g_{ij}(x) + g_{(ik}\,\xi^k_{,j)} + g_{ij,k}\xi^k \tag{27}$$

where $A_{(ij)} = A_{ij} + A_{ji}$ Now the condition of homogeneity that we will take is following: If we translate to a new frame with new origin, the metric in new frame at any fixed numerical values of coordinate should look identical to metric old frame at same numerical values of coordinate, i.e.,

$$g'_{ij}(x) = g_{ij}(x) \tag{28}$$

This equation implies that you cannot tell in which frame you are in, i.e., everything looks the same. We can write

$$g_{(ik}\xi^k_{,j)} + g_{ij,k}\xi^k = 0 \tag{29}$$

This equation is called the Killing equation and ξ^k is the Killing vector.

In flat space one has that the coordinate transformation would be

$$x^i = x^{i'} + \epsilon^i;\ \epsilon^i = infinitesimal\, constant \tag{30}$$

However, Eqn.29 things are much more complicated in curved space.Eq.29 is a very powerful equation in that it not only determines the form of ξ^k but also restricts the form of the metric so that the there is invariance.

As a simple example of Eqn.29, consider a flat 3-space where

$$g_{ij} = \eta_{ij} = -\delta_{ij} \tag{31}$$

Then the Eqn.29 reads

$$\xi_{i,j} + \xi_{j,i} = 0,\ \xi = \eta_{ik}\xi^k \tag{32}$$

We can expand $\xi_i = \epsilon_i + \epsilon_{im}x^m + \frac{1}{2}\epsilon_{imn}x^m x^n + \cdots$. Then we get

$$\epsilon_i = arbitrary,\ \ \epsilon_{ij} = -\epsilon_{ji},\ \epsilon_{imn}\, etc = 0 \tag{33}$$

That is $\xi_i = \epsilon_i + \epsilon_{ij}x^j$ Which just are rigid translations and rotations. These are, of course, the basic symmetries of a Euclidean flat space.

Returning now to our metric, we have

$$g_{rr} = f(r),\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2 sin^2\theta \tag{34}$$

and we can write down what Eq.29 means for different components

$$i = j = r,\ \ 2f(r)\xi^r_{,r} + f_{,r}\xi^r = 0 \tag{35}$$

$$i = j = \theta,\ \ r\xi^\theta_{,\theta} + \xi^r = 0 \tag{36}$$

$$i = r, j = \theta,\ \ f(r)\xi^r_{,\theta} + r^2\xi^\theta_{,r} = 0 \tag{37}$$

We can integrate Eq.35 to give

$$(\xi^r f^{1/2}(r))_{,r} = 0 \tag{38}$$

or

$$\xi^r = \frac{C(\theta, \phi)}{f^{1/2}(r)} \tag{39}$$

If we take $r\frac{\partial}{\partial r}(36) - \frac{\partial}{\partial \theta}(37)$ then we can eliminate ξ^θ to get

$$r\xi^\theta_{,\theta} + r\xi^r_{,r} - (f\xi^r_{,\theta})_{,\theta} = 0 \tag{40}$$

and using Eqn.36 to eliminate ξ^θ_θ gives

$$-\xi^r + r\xi^r_{,r} - (f\xi^r_{,\theta})_{,\theta} = 0 \tag{41}$$

Inserting ξ^r from Eq39 gives

$$-\frac{1}{f} - \frac{1}{2}\frac{r}{f^2}f_{,r} = \frac{C_{,\theta\theta}}{C} \Rightarrow a \tag{42}$$

Here $-\frac{1}{f} - \frac{1}{2}\frac{r}{f^2}f_{,r}$ is a function of r only and $\frac{C_{,\theta\theta}}{C}$ is a function of θ only and a is a constant of integration. Which integrates to

$$f = \frac{1}{a}\frac{1}{1-kr^2},\ k = constant\,of\,integration \tag{43}$$

The condition $f(0) = 1$ implies $a = 1$ and from the right hand side of Eq.42 we get

$$C(\theta) = \epsilon cos\theta,\ \ \epsilon = infinitesimal\,amplitude \tag{44}$$

To summarize then our metric is

$$ds^2 = (dx^0)^2 - R^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2] \tag{45}$$

This is called Robertson-Walker metric with symmetry under

$$\xi^r(r,\theta) = \epsilon cos\theta(1-kr^2)^{1/2} \text{ and } \xi^\theta = \frac{\epsilon sin\theta}{r}(1-kr^2)^{1/2} \tag{46}$$

to represent homogeneity.

Note that in flat space a translation of the origin reads (for infinitesimal transformation)

$$\vec{r} = \vec{r'} + \vec{\epsilon},\ |\vec{r}| = |\vec{r'}| + \hat{r}.\vec{\epsilon}, \tag{47}$$

. For this case then

$$\xi^r = \hat{r}.\epsilon = \epsilon cos\theta \tag{48}$$

Comparing wit Eq.46 we see that for r small the two results agree. They differ only by $O(r^2)$ as expected by SPE. However for large r, the curvature of space effects what represents a translation of origin. In fact one can calculate the curvature scalar for the 3-space. One finds

$$^3R = \frac{k}{R^2(t)} \tag{49}$$

showing $k \neq 0$ implies that curvature is present.

1.1 Properties of Robertson-Walker Metric

The R-W metric depends on the function $R(t)$ and the parameter k. There are 3 classes of solutions depending on whether $k > 0$, $k = 0$, $k < 0$. One can rescale the radial coordinate

$$r = \lambda r' \tag{50}$$

So that

$$k' = \lambda^2 k,\ {R'}^2 = \lambda^2 R^2 \tag{51}$$

In this way one can reduce k' to

$$(i)k' = +1,\ (ii)k' = 0\ (iii)k' = -1 \tag{52}$$

Then R' carries the dimension of length. We can drop the "*prime*".

$$R(t) = \text{"cosmic scale factor"} \tag{53}$$

For cases $k = 0, -1$ we see the metric is regular for any r and so we can let the range of coordinate be

$$0 \le r < \infty,\ 0 \le \theta < \pi;\ 0 \le \phi < 2\pi;\ k = 0,\ -1 \tag{54}$$

But for $k = +1$, there is a singularity at $r = 1$, which we need to investigate. The case $k = 0$ is a "flat universe" and $k = -1$ and "open universe".

To see some of the geometry, we calculate the circumference for a circle of coordinate radius r at $\theta = \pi/2$:

$$\bar{C}(r) = \int d\sigma|_{\theta=\pi/2,\, dr=0=d\theta} = R(t)\int [d\theta^2 + sin^2\theta d\phi^2]^{1/2} r = R(t) r \int_0^{2\pi} d\phi \tag{55}$$

or

$$\bar{C}(r) = 2\pi R(t) r \tag{56}$$

On the other hand the proper radius of the circle from the origin is

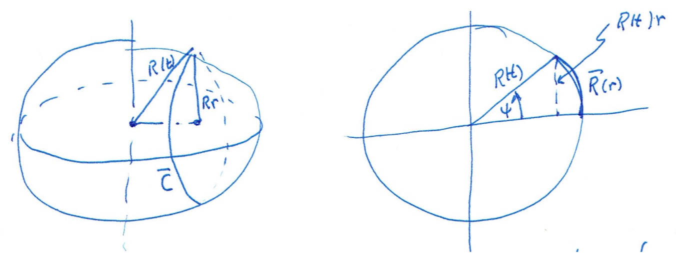
$$\bar{R}(r) = \int d\sigma|_{d\phi=0=d\theta} = R(t)\int_0^r \frac{dr'}{\sqrt{1 - k{r'}^2}} \tag{57}$$

and integrating

$$\bar{R}(r) = R(t)\begin{cases} sin^{-1} r; & k = 1 (r \le 1) \\ r & k = 0 \\ sinh^{-1} r; & k = -1 \end{cases} \tag{58}$$

We see the non-Euclidean nature of the space when $k \ne 0$ i.e.,

$$\frac{\bar{C}(r)}{\bar{R}(t)} = 2\pi \begin{cases} \frac{r}{sin^{-1} r}; & k = 1 (r \le 1) \\ 1 & k = 0 \\ \frac{r}{sinh^{-1} r}; & k = -1 \end{cases} \tag{59}$$



We see that for coordinate $r \ll 1$), all 3 cases give the Euclidean result $\bar{C}/R = 2\pi$. But large r there are major deviations for $k \neq 0$, e.g.,

$$k = 1: \ \frac{\bar{C}}{\bar{R}} = 2\pi \frac{1}{\pi/2} = 4 \ at \, r = 1 \tag{60}$$

$$k = -1: \ \frac{C}{\bar{R}} \simeq 2\pi \frac{r}{Lnr} \ as \, r \to \infty \tag{61}$$

Let us now look at the significance of the singularity at $r = 1$ for the $k = 1$ case. $\bar{C}(r)$ and $\bar{R}(r)$ are the circumference and radius of a circle of coordinate radius r. To see the meaning of the result consider a circle drawn on sphere of radius $R(t)$. Now the circumference in this construction is $\bar{C}(r) = 2\pi Rr$ and ψ is the angle

$$sin\psi = \frac{R(t)r}{R(t)} = r \tag{62}$$

On the other hand, the radius measured on the sphere is $\bar{R}(r)$ given by

$$\psi = \frac{\bar{R}}{R(t)} \tag{63}$$

$$\bar{R}(r) = R(t)\psi = R(t)sin^{-1}r \tag{64}$$

which is precisely what we got from our metric. Thus the physical space corresponds to the surface of the sphere. The coordinate radius measures the distance from axis up to sphere and the radius of sphere $R(t)$ is that distance when r is its maximum, i.e., $r = 1$ at North Pole.

Now as ψ increases $\bar{R}(r)$ increases until $\psi = \pi/2$ and $r = 1$; when $\bar{R} = \frac{\pi}{2}R$. As ψ continues to increase r decreases until $\psi = \pi$ and $r = 0$ with $\bar{R} = \bar{R}_{max} = \pi R(t)$. Thus r is a singular coordinate in that it is doubled values as one covers the full surface of the sphere. We can eliminate this singularity by introducing ψ to replace the r coordinate

$$\psi(r) = sin^{-1}r \tag{65}$$

Then our metric becomes

$$ds^2 = (dx^0)^2 - R^2(t)[d\psi^2 + sin^2\psi(d\theta^2 + sin^2\theta d\phi^2)] \tag{66}$$

and now σ^2 is precisely the line element for a 3-sphere embedded in a fictitious Euclidean 4 space i.e., let

$$x^1,\, x^2,\, x^3,\, x^4 = Euclidean\, coordinate \tag{67}$$

Then spherical coordinates are

$$x^4 = \rho cos\psi,\, x^3 = \rho sin\psi cos\theta,\, x^2 = \rho sin\psi sin\theta cos\phi,\, x^1 = \rho sin\psi sin\theta sin\phi \tag{68}$$

with $0 \leq \psi, \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \rho < \infty$ ($\rho^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$).

The Euclidean distance in this 4-space is

$$d\sigma_4^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 = d\rho^2 + \rho^2[d\psi^2 + sin^2\psi(d\theta^2 + sin^2\theta d\phi^2)] \tag{69}$$

The 3-space of radius R is given by fixing ρ

$$\rho = R(t),\;\; d\rho = 0 \tag{70}$$

Which reduces down to

$$d\sigma_3^2 = R^2[d\psi^2 + sin^2\psi d\Omega^2] \tag{71}$$

Which is precisely the $d\sigma^2$ for R-W with $k = +1$. Thus the R-W metric is precisely the metric for a 3-sphere of radius $R(t)$ embedded in a fictitious 4-dimensional Euclidean space.

Since we now have a non-singular coordinate system, we can use it to calculate the 3-volume of the sphere. Our metric is

$$-g_{\psi\psi} = R^2,\, -g_{\theta\theta} = R^2 \sin^2\psi,\, -g_{\phi\phi} = R^2 sin^2\psi sin^2\theta \tag{72}$$

and the proper (invariant) volume is

$$V_3 = \int \sqrt{-g} d\psi d\theta d\phi = R^3 \int_0^{\pi} d\psi sin^2\psi \int_0^{\pi} d\theta sin\theta \int_0^{2\pi} d\phi \tag{73}$$

or

$$V_3 = 2\pi^2 R^3 \tag{74}$$

The volume is finite and scaled by $R(t)$

One can do a similar analysis for the case $k = -1$. Here the space is characterized by a hyperboloid embedded in a fictitious 4-dim space with Lorentzian metric. Thus define

$$d\sigma_4^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2 \tag{75}$$

and parametrize the space with

$$x^4 = \rho cosh\chi,\ x^3 = \rho sinh\chi cos\theta,\ x^2 = \rho sinh\chi sin\theta cos\phi,\ x^1 = \rho sinh\chi sin\theta sin\phi \tag{76}$$

Then one finds

$$d\sigma_4^2 = d\rho^2 + \rho^2[(d\chi)^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \tag{77}$$

and the R-W metric occurs when we set

$$\rho = R(t),\ d\rho = 0 \tag{78}$$

reducing to

$$d\sigma_4^2 = R^2(t)[(d\chi)^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)] \tag{79}$$

This is just the R-W metric for the $k = -1$ case with a change of variables

$$\psi = sinh^{-1}r \tag{80}$$

In general we ave

$$ds^2 = (dx_0)^2 - R^2(t)[(d\psi)^2 + r^2(\psi)(d\theta^2 + sin^2\theta d\phi^2)] \tag{81}$$

where

$$r(\psi) = \begin{cases} sin\psi; & k = 1\ closed \\ \psi; & k = 0\ flat \\ sinh\psi; & k = -1\ open \end{cases} \tag{82}$$

1.2 Motion in a Robertson-Walker Metric

To get some insight as to the meaning of the R-W metric, let us consider the motion of a test particle (e.g., a galaxy) subject to the gravitational field produced by the R-W metric. Recall that a particle equation of motion is

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0;\ u^\alpha = \frac{dx^\alpha}{ds} \tag{83}$$

Suppose we place a particle initially at rest w.r.t to the R-W frame and ask what is its further motion. We have

$$u^i(0) = 0,\ u^0 = \frac{dx^0}{ds} = \frac{1}{\sqrt{g_{00}}} = 1 \tag{84}$$

Our equation reduces initially to

$$\left.\frac{d^2x^\mu}{ds^2}\right)_{s=0} + \Gamma^\mu_{00} = 0 \tag{85}$$

But

$$\Gamma^\mu_{00} = \frac{g^{\mu\alpha}}{2}[g_{\alpha 0,0} + g_{0\alpha,0} - g_{00,\alpha}] \tag{86}$$

and since for R-W

$$g_{0\alpha} = \eta_{0\alpha} \tag{87}$$

we have

$$\Gamma^\mu_{00} = 0 \tag{88}$$

Thus

$$\frac{du^\mu}{ds}|_{s=0} = \frac{d^2x^\mu}{ds^2}|_{s=0} = 0 \tag{89}$$

Hence since $u^\mu(s)$ a first order equation of motion, it implies a particle initially at rest w.r.t. the (RW) reference frame will stay at rest. One can in fact go further. If we assume a particle has a small velocity with respect RW frame, one finds it rapidly approaches rest for an expanding universe.

Experimentally, one finds that galaxies are moving slowly w.r.t. the cosmic frame. Thus, the motion of solar system relative to CMB is

$$v_{\odot} = (370 \pm 10) km/s \tag{90}$$

and other galaxies have smaller velocities, i.e., with $v/c \ll 1$. Thus galaxies do appear to have been made of material originally at rest w.r.t. cosmic frame and the small velocities seen are due to local gravitational forces. These small velocities are referred to as "peculiar velocities".

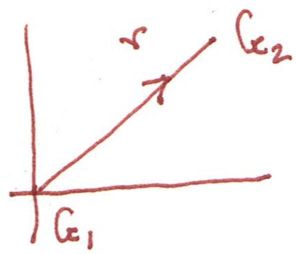
However the expansion of the universe does indeed mean that galaxies are moving apart. Thus the proper distance between the galaxies

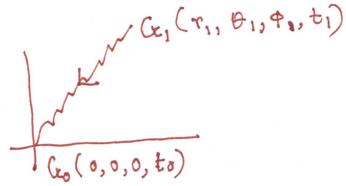
$$l = \int d\sigma]_{\theta,\,\phi=const} = \int_0^r dr\sqrt{g_{rr}} \tag{91}$$

Thus

$$l(r,t) = R(t)\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = R(t)\psi(r)\begin{cases} sin^{-1}r; & k=1 (r \le 1) \\ r & k=0 \\ sinh^{-1}r; & k=-1 \end{cases} \tag{92}$$

We saw that if initially, G_1 and G_2 are at rest w.r.t. cosmic frame, they will not move and will stay at $r = const$ at all time. Thus the t-dependence of the separation totally from $R(t)$ and galaxies move apart in an expanding universe or together in a contracting universe. The "fabric of space" appears to expand pulling galaxies apart. Thus the situation is similar to galaxies on the surface of a balloon that is blowing up.





1.3 Cosmological Red shift

The fundamental cosmological law was the discovery by Hubble of the redshift-distance of Cosmology. We will see that the this a direct consequence of the R-W metric and does not even use the Einstein's equation. To consider the redshift, let us assume we have a galaxy G_1 at pt $\vec{r_1}$ which emits an e.m. wave at time t_1, which arrives at our galaxy G_0 at a later time t_0 The e.m. wave travels with velocity c and hence moves along a null geodesic

$$ds^2 = c^2 dt^2 - \frac{R^2 dr^2}{1 - kr^2} = 0 \tag{93}$$

hence

$$dt = -\frac{1}{c} \frac{R(t)dr}{\sqrt{1 - kr^2}} \tag{94}$$

(where the minus sign occurs because t is increasing, $dt > 0$, but r is decreasing, $dr < 0$). The front of the wave that arrives at G_0 at time t_0 where

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = -\frac{1}{c} \int_{r_1}^{0} \frac{dr'}{\sqrt{1 - kr'^2}} = \frac{1}{c} \int_{0}^{r_1} \frac{dr'}{\sqrt{1 - kr'^2}} \tag{95}$$

Now let T_1= Period of wave emitted by G_1. The end of the wave is emitted by G_1 at time $t = t_1 + T_1$. It will arrive at some later time $t = t_0 + T_0$ and since T_0 is the time interval that one wavelength is seen one has T_0 = period of observed wave at G_0.

Now the end of the wave also travels on a null geodesic and so

$$\int_{t_1+T_1}^{t_0+T_0} \frac{dt}{R(t)} = \frac{1}{c} \int_{0}^{r_1} \frac{dr'}{\sqrt{1 - kr'^2}} \tag{96}$$

We will assume here that both G_0 and G_1 are stationary w.r.t the R-W coordinate frame, i.e., we will neglect the peculiar velocities. We find that characteristically these were of order 100km/s relative to the R-W frame. and since the Hubble constant is

$$H \simeq 100 \frac{km}{sec} \frac{1}{Mpc} \tag{97}$$

objects 100Mpc away will have a Hubble expansion velocity of

$$100 \times 100 \frac{km}{sec} = 10^4 \frac{km}{s} \tag{98}$$

and hence the peculiar velocities will be negligible correction in comparison to the expansion velocity. (Galactic clusters are characteristically $\simeq 10 - 20$ Mpc away from each other.

In this approximation, G_1 is at a fixed value of r and r is not a function of time. Hence subtracting Eq. 95 and Eq.96 gives

$$\int_{t_1+T_1}^{t_0+T_0} \frac{dt}{R(t)} - \int_{t_1}^{t_0} \frac{dt}{R(t)} = 0 \tag{99}$$

or

$$\int_{t_1+T_1}^{t_1} \frac{dt}{R(t)} + \int_{t_1}^{t_0+T_0} \frac{dt}{R(t)} = 0 \tag{100}$$

Now T is a very smal number, i.e.,

$$T = 2\pi \frac{\lambda}{c} \simeq 10^{-14}\, sec,\ \lambda = 5000 A^0 \tag{101}$$

and since $R(t)$ is a slowly varying function, we approximate Eq.100 by

$$\frac{T_0}{R(t_0)} - \frac{T_1}{R(t_1)} = 0 \tag{102}$$

and using $\nu = \frac{2\pi}{T}$ =frequency we have

$$\frac{\nu_0}{\nu_1} - \frac{R(t_1)}{R(t_0)} \tag{103}$$

Now ν_1 is emitted frequency at rest w.r.t to G_1 and since this frame is instantaneously inertial, it is the same frequency an atom at rest w.r.t. to inertial frame on Earth would be. Thus ν_1 is the standard spectral frequency seen in laboratories on Earth and ν_0 is what we observe this frequency to be at G_0, and is the red shifted frequency due to the expansion during the time of travel, i.e., $R(t_0) > R(T_1)$, for an expanding universe.

We introduce the parameter z:

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\frac{c}{\nu_0} - \frac{c}{\nu_1}}{\frac{c}{\nu_1}} = \frac{\nu_1}{\nu_0} - 1 \tag{104}$$

. Hence from Eq.103

$$z = \frac{R(t_0)}{R(t_1)} - 1 \tag{105}$$

This will be a "red-shift" in wavelength if $R(t_0) > R(t_1)$ (i.e., $\lambda_0 > \lambda_1$) i.e., if the universe is expanding, or a "blue-shift" if $R(t_0) < R(t_1)$ if the universe is contracting. Experimentally, all measurements of galaxies sufficiently distant that peculiar velocities can be neglected show a red-shift, so that the universe is expanding.

1.4 Definition of Measures

The phenomenological Hubble law was a relation between z and distance. We need therefore a definition of distance. In special relativity that is not a problem since

$$ds^2 = (dx^0)^2 - (dr^2 + r^2 d\Omega^2) = (dx^0)^2 - (d\sigma)^2 \tag{106}$$

and so the distance is just the invariant length $\int d\sigma$

$$d = \int_0^{r_1} d\sigma]_{d\theta=0=d\phi} = r_1 \tag{107}$$

which is just the coordinate distance. In general relativity, things are more complicated even if space is flat. Here the R-W ds^2 is

$$ds^2 = (dx^0)^2 - R^2(t)(dr^2 + r^2 d\Omega^2) \tag{108}$$

Now to get the distance we need to go into a local inertial frame which we can do at any fixed time. For example at time of emission one can transform to the inertial frame

$$r' = R(t_1)r,\ ds^2 = (dx^0)^2 - ({dr'}^2 + {r'}^2 d\Omega^2) \tag{109}$$

and so measurement would give for distance

$$d(t_1) = R(t_1)r_1 \tag{110}$$

Similarly one might ask for distance at time the light arrives at G_0. Then

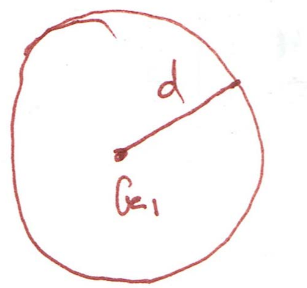
$$d(t_0) = R(t_0)r_1 \tag{111}$$

One could even consider a more complicated distance measure such as

$$d = \frac{R^2(t_0)}{R(t_1)} r_1 \tag{112}$$

How does one know operationally which distance one is talking about?

As one example, we consider the "luminosity distance" d_L. In non-relativistic physics one defines the absolute luminosity as L=absolute luminosity of a source=energy



emitted/sec. This energy spreads out over a sphere of radius d at time $t = d/c$, so the flux of energy observed at distance d is

$$l = \frac{L}{4\pi d^2} = \text{energy/time} \times \text{area observed over a distance d} \qquad (113)$$

In general relativity, the situation is more complicated as space is expanding. However, one may still define the luminosity distance by

$$d_L^2 \equiv \frac{L}{4\pi l} \qquad (114)$$

Both L and l are physical quantities, and so this is a well defined measure of distance. Let us calculate what the measure is for the R-W metric.

$$d\sigma^2 = R^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2] \qquad (115)$$

Now suppose our receiver in G_0 is a telescope of radius b. At time t_0 the energy is received. In inertial coordinates at t_0 one has

$$r_1' = R(t_0) r_1 \qquad (116)$$

and hence the solid angle subtended by the telescope is

$$\Omega = \frac{\pi b^2}{r_1'^2} = \frac{\pi b^2}{R^2(t_0) r_1^2} = \frac{A}{R^2(t_0) r_1^2} \qquad (117)$$

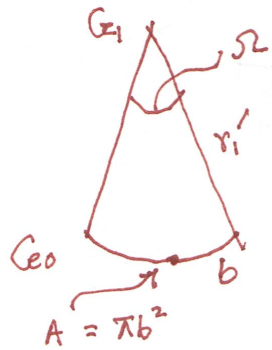
Now the power received is

$$P = \frac{energy}{time} \qquad (118)$$

For every photon emitted with frequency ν_1 $h\nu_1$, is red shifted to energy $h\nu_0$ where we had

$$h\nu_0 = h\nu_1 \frac{R(t_1)}{R(t_0)} \qquad (119)$$

Further let δt the time interval for emission of photon and δt_0 the time interval during which it arrives. At the beginning of emission t_1, the wave



arrives at t_0 which can be written as Eqn95

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \frac{1}{c}\int_0^{r_1} \frac{dr'}{\sqrt{1-kr'^2}} \tag{120}$$

and at the end of interval

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{R(t)} = \frac{1}{c}\int_0^{r_1} \frac{dr'}{\sqrt{1-kr'^2}} \tag{121}$$

and subtracting gives

$$\delta t_0 = \delta t_1 \frac{R(t_0)}{R(t_1)} \tag{122}$$

Thus the power of energy received is for N photons is

$$P = \frac{Nh\nu_0 A}{\delta t_0 (R_0^2 r_1^2 4\pi)} = \frac{Nh\nu_1}{\delta t_1}\frac{R^2(t_1)}{R^2(t_0)}\frac{A}{4\pi R^2(t_0) r_1^2} \tag{123}$$

The flux of energy received is

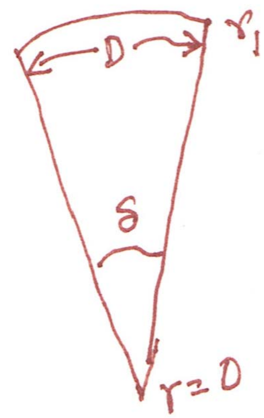
$$l = \frac{P}{A} = \frac{L}{4\pi R_0^2 r_1^2}\frac{1}{(1+z)^2} \tag{124}$$

we have then

$$d_L = (R_0 r_1)(1+z) = \frac{R^2(t_0)}{R(t_1)} r_1 \tag{125}$$

Note that this formula holds even for $k \neq 0$. There are other measures of distance one use:
(i) Angular size of source: One defines

$$d_A \equiv \frac{D}{\delta} \tag{126}$$



which be the distance non-relativistically one finds for R-W

$$d_A = R(t_1)r_1 = \frac{R_0 r_1}{1+z} \tag{127}$$

$$\frac{d_L}{d_A} = (1+z)^2 \tag{128}$$

and if z is large, these two distance measures can differ considerably.

(ii) Proper motion of a source: If a source is moving with transverse velocity v_T to an observer non-rel., the line of sight angle will change by an amount in time δt

$$\delta = \frac{v_T \delta t}{d} \tag{129}$$

One defines the proper motion distance to be

$$d_M = \frac{v_T \delta t_0}{\delta} \tag{130}$$

where δt_0 is the time interval measured by the observer and δ is the angle measured by the observer. For R-W metric, one finds

$$d_M = R(t_0)r_1 \tag{131}$$

and hence

$$\frac{d_L}{d_M} = 1+z \tag{132}$$

It is conventional to think of the red-shift as Doppler shift due to the receding motion of the distant galaxy G_1 and in part this is true. However, the above shows that general relativity contributes to the effect in a unique way. For non-relativistic motion the Doppler motion is

$$z = \frac{\delta\nu}{\nu} = \frac{v}{c} < 1 \tag{133}$$

However one can use $z > 1$ and in fact galaxies with $z \sim 10$ have been observed. Thus gravitational effects play am important part in the red-shift.

1.5 Hubble Law

Having now understood how to define distance in R-W metric, we are in a position to deduce the Hubble law which relates red-shift to distance.

We have

$$z = \frac{R(t_0)}{R(t_1)} - 1 \tag{134}$$

For not too distant galaxies we can expand the denominator and the present time t_0 of our galaxy G_0

$$R(t_1) = R(t_0 - \Delta t) = R(t_0) - \dot{R}(t_0)\Delta t + \frac{1}{2}\ddot{R}(t_0)(\Delta t)^2 + \cdots \tag{135}$$

where $\Delta t = t_0 - t_1$. We define

$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \text{Hubble constant at time } t_0 = \text{rate of expansion} \tag{136}$$

$$q_0 = -\frac{\ddot{R}_0}{R_0 H_0^2} = -\frac{\ddot{R}_0 R_0}{\dot{R}_0^2} = \text{deceleration parameter} \tag{137}$$

Then

$$\frac{R(t_1)}{R(t_0)} = 1 - H_0\Delta t - \frac{1}{2}H_0^2 q_0 (\Delta t)^2 + \cdots \tag{138}$$

and inverting gives

$$\frac{R(t_0)}{R(t_1)} = 1 + H_0\Delta t + (1 + \frac{1}{2}q_0)H_0^2(\Delta t)^2 + \cdots \tag{139}$$

and hence

$$z = H_0(t_0 - t_1) + (1 + \frac{1}{2}q_0)H_0^2(t_0 - t_1)^2 + \cdots \tag{140}$$

And we can invent this to get the time interval in terms of z:

$$t_0 - t_1 = H_0^{-1}z - (1 + \frac{1}{2}q_0)H_0(t_0 - t_1)^2 + \cdots \tag{141}$$

and iterating gives

$$t_0 - t_1 = H_0^{-1}[z - (1 + \frac{1}{2}q_0)z^2 + \cdots] \tag{142}$$

We really want however the distance as a function of z, we can relate time to distance

$$\int_{t_1}^{t_0} dt \frac{R(t_0)}{R(t)} = \frac{R(t_0)}{c}\int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} = \frac{R_0}{c}\chi(r_1) = \frac{R_0}{c}\begin{cases} sin^{-1}r_1; & k = 1 \\ r_1; & k = 0 \\ sinh^{-1}r_1; & k = -1 \end{cases} \tag{143}$$

For the L.H.S, we insert in the expansion

$$L.H.S = \int_{t_1}^{t_0} dt[1+H_0(t_0-t)+(1+\frac{1}{2}q_0)H_0^2(t_0-t)^2+\cdots] = (t_0-t_1)+\frac{1}{2}H_0^2(t_0-t_1)^2+\cdots \tag{144}$$

And for the R.H.S., we have

$$\text{R.H.S.} = \frac{R_0}{c}\begin{cases} r_1 + \frac{r_1^3}{6} + \cdots \\ r_1 \\ r_1 - \frac{r_1^3}{6} + \cdots \end{cases} = \frac{R_0}{c}(r_1 + \frac{k}{6}r_1^3 + \cdots \tag{145}$$

We can now solve for r_1 in terms of t_0 again by iterating. The r_1^3 terms give contributions of $O(\Delta t^3)$ so that we get

$$r_1 = \frac{c}{R_0}[(t_0 - t_1) + \frac{1}{2}H_0^2(t_0 - t_1)^2 + \cdots] \tag{146}$$

We then get

$$r_1 = \frac{c}{R_0}H_0^{-1}[z - \frac{1}{2}(1 + q_0)H_0^2 z^2 + \cdots] \tag{147}$$

Note that the curvature term involving k does not enter until $O(z^3)$. We can write

$$R(t_0)r_1 = \frac{d_L}{1+z} \tag{148}$$

which allows us to write

$$\frac{d_L}{1+z}\frac{H_0}{c} = z - \frac{1}{2}(1 + q_0)H_0^2 z^2 + \cdots \tag{149}$$

or

$$d_L\frac{H_0}{c} = z + \frac{1}{2}(1 - q_0)H_0^2 z^2 + \cdots \tag{150}$$

For small z, we have precisely Hubble's law that the redshift is linear in the distance. For larger z, however we expect deviation from the linear law unless $q_0 = 1$. Note however, that it is d_L that enters into the previous equation. If for example, we use d_A as our measure of distance, then we get

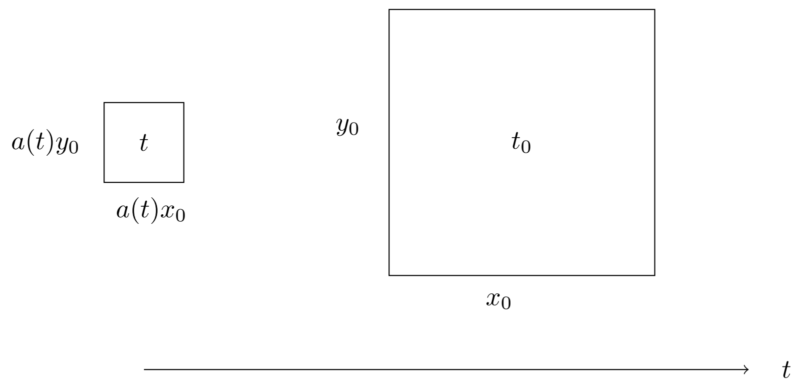
$$d_A(1 + z)^2\frac{H_0}{c} = z + \frac{1}{2}(1 - q_0)z^2 \tag{151}$$

or

$$d_A\frac{H_0}{c} = z - \frac{1}{2}(3 + q_0)z^2 \tag{152}$$

and the quadratic(and higher terms) get modified.

2 Flat Universe



The metric for a flat universe

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2] \tag{153}$$

$a(t)$is the scale factor whic is defined as $a(t) \equiv \frac{R(t)}{R(t_0)}$ with $a(t_0) = 1$ and t is the physical time.

The energy momentum tensor also satisfies homogeneous and isotropic condition

$$T^{\alpha\beta} = (\rho + P)U^\alpha U^\beta - Pg^{\alpha\beta} \tag{154}$$

$U^\alpha = (1, 0, 0, 0)$, ρ (density) and P (pressure) are function of time

$$g_{00} = 1,\ g_{ij} = -a^2(t)\delta_{ij},\ T_{00} = \rho,\ T_{ij} = a^2(t)P\delta_{ij} \tag{155}$$

Now we solve Einstein equation

$$G^{\mu\nu} = 8\pi GT^{\mu\nu} \tag{156}$$

Using the metric for the flat Universe and $T^{\alpha\beta}$ defined above.

$$\Gamma^i_{jk} = 0,\ \Gamma^i_{00} = 0,\ \Gamma^0_{ij} = a\dot{a}\delta_{ij},\ \Gamma^0_{00} = 0,\ \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j,\ \Gamma^0_{0i} = \Gamma^0_{i0} = 0 \tag{157}$$

$$R_{00} = -3\frac{\ddot{a}}{a},\ R_{0i} = 0;\ R = R_{00} - \frac{1}{a^2}R_{ii} = -[6\frac{\ddot{a}}{a} + 6(\frac{\dot{a}}{a})^2],\ R_{ij} = (a\ddot{a} + 2\dot{a}^2)\delta_{ij} \tag{158}$$

$$G_{00} = R_{00} - \frac{1}{2}Rg_{00} = 8\pi GT_{00} \tag{159}$$

$$\Rightarrow 3(\frac{\dot{a}}{a})^2 = 8\pi G\rho \tag{160}$$

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij} \Rightarrow -2\ddot{a}a - a^2 = 8\pi Ga^2P \tag{161}$$

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3P) \tag{162}$$

Eqn160 is Friedmann-Robertson-Walker equation. Eqn162 is Raychaudhuri equation. We can solve these two equations for the evolution for $a(t)$.

Both of these equations are sourced by ρ and P. These equations satisfy the conservation equation.

$$T^{\alpha\beta}_{;\beta} = 0 \tag{163}$$

Using homogeneous and isotropic case

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{164}$$

Now let us use the equation of state $P = \omega\rho$ where ω is a constant.

2.1 Non relativistic Matter

The energy in a volume V is given by $E = M$, $\rho = \frac{E}{V}$ where ρ is the mass density. In the evolving Universe $V \propto a^3$ and $\rho \propto \frac{1}{a^3}$ and

$$P \simeq nk_BT \ll nMc^2 \simeq \rho c^2 \tag{165}$$

so $P \simeq 0$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0 \tag{166}$$

Solving this equation $\rho \propto \frac{1}{a^3}$. We can solve the equation

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^3},\ a^{1/2}\dot{a} = (\frac{8\pi G\rho_0}{3})^{1/3} \tag{167}$$

to find $a \propto t^{2/3}$. If $a(t_0) = 1$ where t_0: today, we have $a = (\frac{t}{t_0})^{2/3}$. At $t = 0$, we have $a = 0$, i.e., there is an initial singularity: Big Bang. Finally we find that $\ddot{a} < 0$, i.e., the Universe is decelerating.

2.2 Relativistic Matter

These are massless photons and neutrinos. recall that their energy is given by $E = h\nu = h2\pi/\lambda$ where ν is the frequency and λ is the wavelength. Since each length is stretched by the scale factor a the λa, the energy is shifted by $E \propto \frac{1}{a}$. The mass density$\rho = \frac{E}{V} \propto \frac{1}{V\lambda} \propto \frac{1}{a^3a} = \frac{1}{a^4}$. The energy of radiation decreases far more quickly than that of non-relativistic matter. Also we can use the equation of state for radiation $P = \frac{\rho}{3}$

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = \frac{1}{a^4}\frac{d}{dt}(\rho a^4) = 0 \Rightarrow \rho \propto \frac{1}{a^4} \tag{168}$$

We get

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^4},\ a\dot{a} = (\frac{8\pi G\rho_0}{3})^{1/2} \tag{169}$$

which gives us

$$a \propto t^{1/2}, \text{ or } a = (\frac{t}{t_0})^{1/2} \tag{170}$$

Once again the universe is decelerating and $H = \frac{\dot{a}}{a} = \frac{1}{2t}$. For general equation of state $P = \omega\rho$

$$\dot{\rho} + 3(1+\omega)\frac{\dot{a}}{a}\rho = \frac{1}{a^{3(1+\omega)}}\frac{d}{dt}(\rho a^{3(1+\omega)}) = 0 \tag{171}$$

As ω gets smaller and more negative ρ decreases more slowly, we can solve the previous equations

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^{3(1+\omega)}},\ a^{\frac{1+3\omega}{2}}\dot{a} = (\frac{8\pi G\rho_0}{3})^{1/2} \tag{172}$$

which gives us

$$a = (\frac{t}{t_0})^{\frac{2}{3(1+\omega)}} \tag{173}$$

which is valid for $\omega > -1$. For $\omega < \frac{-1}{3}$, the expansion rate is accelerating. For the special case $\omega = -\frac{1}{3}$, $a \propto t$. For cosmological constant $P = -\rho$. In such a scenario ae^{Ht}, ρ is constant, $\frac{\dot{a}}{a}$ is constant.

So far we have considered only one type of matter but in general there is a mix, e.g.,

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}(\frac{\rho_{M_0}}{a^3} + \frac{\rho_{R_0}}{a^4}) \tag{174}$$

In fact the true picture should involve all 3 types, i.e., relativistic, non-relativistic and cosmological constant or vacuum energy (Λ era). Before we get into more, we first consider scenarios which are not flat.

Let us now consider a 3D surface that is positively curved. It is the surface of a 3D hyper-surface in a fictitious space with 4D. We have already seen tha the equation for the surface of a sphere in this 4D space with coordinates (x, y, z, w) is $x^2+y^2+z^2+w^2 = R^2$. We can similarly define a surface with negative curvature

$$x^2 + y^2 + z^2 - w^2 = -R^2 \tag{175}$$

We have seen before for such surfaces

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)] \tag{176}$$

where k is positive, zero or negative for spherical, flat or hyperbolic geometries $|k| = \frac{1}{R^2}$

We can repeat our calculations we did for flat geometry in these new cases. The metric

$$g_{\alpha\beta} = diag(+1, -\frac{-a^2}{1-kr^2}, -a^2r^2, -a^2r^2sin^2\theta) \tag{177}$$

i, j runs over r, $theta$, ϕ

$$\Gamma^0_{ij} = -a\dot{a}\tilde{g}_{ij},\ \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j,\ \Gamma^i_{jk} = \tilde{\Gamma^i}_{jk} \tag{178}$$

Here $\tilde{g}_{ij}$ and $\tilde{\Gamma^i}_{jk}$ are the metric and connection coefficients of the conformal 3-space (that is of the 3 space with the conformal factor a is divided out)

$$\tilde{\Gamma}^r_{rr} = \frac{kr}{1-kr^2},\ \tilde{\Gamma}^r_{\theta\theta} = -r(1-kr^2),\ \tilde{\Gamma}^r_{\phi\phi} = -(1-kr^2)rsin^2\theta,\ \tilde{\Gamma}^\theta_{\phi\phi} = -\frac{sin2\theta}{2},\ \tilde{\Gamma}^\theta_{\theta r} = \frac{1}{r},\ \tilde{\Gamma}^\phi_{\theta\phi} = \frac{1}{tan\theta} \tag{179}$$

The Ricci tensor and scalar can be combined to form the Einstein tensor

$$G_{00} = \frac{3\dot{a}^2 + k}{a^2},\ G_{ij} = (2a\ddot{a} + \dot{a}^2 + k)\tilde{g}_{ij} \tag{180}$$

While the energy momentum tensor is

$$T_{00} = \rho,\ T_{ij} = -a^2 P\tilde{g}_{ij} \tag{181}$$

Combining we get

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \text{ and } 3\frac{\ddot{a}}{a} = \frac{4\pi G}{3}(\rho + 3P) \tag{182}$$

Let us explore the components of the overall geometry of the Universe, i.e., the term proportional to k in the F.R.W equations.

For example consider a non-relativistic matter filled equation We can see that the term proportional to K will only be important at late times when it dominates over energy density of non-relativistic matter. In other words, we can say that curvature dominates over the non-relativistic matter.

This means that the curvature dominates at later times. Let us consider now two possibilities: $k < 0$ and $k > 0$

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} + \frac{|k|}{a^2} \tag{183}$$

when curvature dominates

$$(\frac{\dot{a}}{a})^2 = \frac{|k|}{a^2} \tag{184}$$

So $a \propto t$ In this case the scale factor grows at the speed of light.

$K > 0$: From the F.R.W. equations we see that this is a point when

$$\frac{8\pi G\rho}{3} = \frac{k}{a^2} \tag{185}$$

and therefore $\dot{a} = 0$, when the Universe stops expanding. At this point the Universe starts contracting and evolves to a Big crunch. If $k = 0$, there is a strict relationship between $H = \frac{\dot{a}}{a}$ and ρ

$$H^2 = (\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} \Rightarrow \rho = \rho_c = \frac{3H^2}{8\pi G} \tag{186}$$

$\rho_c = 1.9 \times 10^{-26} h^2 kgm^{-3}$ = critical density Using $H_0 = 100hkms^{-1}Mpc^{-1}$ It is convenient to define a a move compact notation. We define the fractional energy density $\Omega = \frac{\rho}{\rho_c}$. Ω is a function of a and we express its value today as Ω_0. There are various contributions to the energy density

$$\Omega_R = \frac{\rho_R}{\rho_c} : \text{ Radiation [Relativistic]} \tag{187}$$

$$\Omega_M = \frac{\rho_M}{\rho_c} : \text{ Matter [Non − Relativistic]} \tag{188}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} : \ \Lambda \text{ [Cosmological Constant]} \tag{189}$$

$$\Omega_k = -\frac{k}{a^2H^2} : \text{ Curvature} \tag{190}$$

$$\Omega = \Omega_R + \Omega_M + \Omega_\Lambda \Rightarrow H^2(1-\Omega) = -\frac{k}{a^2} \tag{191}$$

$$\Omega < 1 : \rho < \rho_c, \, k < 0 : \text{ Universe is open} \tag{192}$$

$$\Omega = 1 : \rho = \rho_c, \, k = 0 : \text{ Universe is flat} \tag{193}$$

$$\Omega > 1 : \rho > \rho_c, \, k > 0 : \text{ Universe is closed} \tag{194}$$

We can write

$$H^2(a) = H_0^2[\frac{\Omega_{M_0}}{a^3} + \frac{\Omega_{R_0}}{a^4} + \frac{\Omega_{k_0}}{a^2} + \Omega_\Lambda] \tag{195}$$

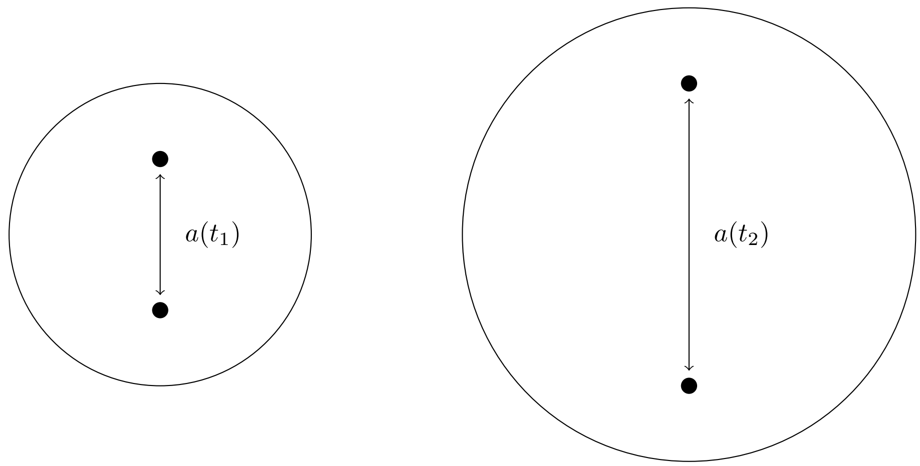
"0" indicates the quantities evaluated at t_0.

How does Ω evolve? If the Universe is dominated by matter then

$$\Omega - 1 = \frac{k}{a^2H^2} \propto kt^{2/3} \tag{196}$$

i.e., if $\Omega \neq 1$, it is unstable and driven away from 1. The same is true for a radiation dominated Universe and for any decelerating Universe $\Omega = 1$ as an unstable fixed point and we saw that curvature dominate at late time.

Let us examine $H \equiv \frac{\dot{a(t)}}{a(t)}$. Suppose we place a galaxy ate r_1, θ_1, ϕ_1. As the Universe expands, the galaxy stays at the same location, all the cosmological distances get stretched by an amount $a(t)$. For example, we can use the surface of a balloon to describe this phenomenon. We see that the two galaxies are located at the same location of the r, θ, ϕ coordinate system at t_1 and t_2. However, the distances between them are stretched by the ratio of the scale factors $a(t_2)$ and $a(t_1)$.



Suppose that the 2 galaxies are separated by a distance $d_1 = a(t_1)s$ where s is the distance between these galaxies are normalized (co-moving) coordinates. At the time t_2, the distance is $d_2 = a(t_2)s$. So the necessary velocity

$$v = \frac{d_2 - d_1}{t_2 - t_1} = \frac{a(t_2) - a(t_1)}{t_2 - t_1} s \tag{197}$$

using $\Delta t = t_2 - t_1 \to 0$

$$v = \frac{\dot{a}}{a} as = Hd \tag{198}$$

H_0 is the Hubble's constant and we can define

$$t_H = \frac{1}{H_0} = 9.78 \times 10^9 h^- 1yr \tag{199}$$

The Hubble distance

$$D_H = \frac{1}{H_0} = 300h^{-1}Mpc \tag{200}$$

Let us choose our local coordinates such that we are at $T = 0$. Consider a light ray that moves radially towards us, that is θ, ϕ=constants. If this light ray was emitted from $r = r_E$ and $t = t_E$ it will reach us at a time t_0 given by

$$c \int_{t_E}^{t_0} \frac{dt}{a(t)} = c \int_0^{r_E} \frac{dr}{\sqrt{1 - kr^2}} \tag{201}$$

Using $-k = \frac{\Omega_{k_0}}{D_H^2}$

$$\int_0^{r_E} \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} \frac{D_H}{\sqrt{\Omega_{k_0}}} sinh^{-1}[\sqrt{\Omega_{k_0}} \frac{r_E}{D_H}] \ for \ \Omega_{k_0} > 0 \\ r_E \ for \ \Omega_{k_0} = 0 \\ \frac{D_H}{\sqrt{|\Omega_{k_0}|}} Sin^{-1}[\sqrt{|\Omega_{k_0}|} \frac{r_E}{D_H}] \ for \ \Omega_{k_0} < 0 \end{cases} \tag{202}$$

The furthest physical distance d_h, we can observe today is given by $\int_0^{r_E} \frac{dr}{\sqrt{1-kr^2}}$ scaled by the physical scale factor $a(t_0)$

$$d_h(t_0) = a(t_0) \int_0^{r_E} \frac{dr}{\sqrt{1-kr^2}} = a(t_0) \int_0^{t_0} \frac{dt}{a(t)} \tag{203}$$

We can calculate $\int \frac{dt}{a(t)}$ for different era

$$d = \int_t^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a^2(t')H(a')} \tag{204}$$

Using $\frac{da}{dt} = aH$

For matter domination $H \propto a^{-3/2}$

$$d(a) = \frac{2}{H_0}[1 - \sqrt{a}] \tag{205}$$

$$d(z) = \frac{2}{H_0}[1 - \frac{1}{\sqrt{1+z}}] \tag{206}$$

For small z, $d \to \frac{z}{H_0}$ and for large z, $d \to \frac{2}{H_0}$

We also can define lookback time t_L

$$t_L(a) = \int_{t(a)}^{t} dt' = \int_a^1 \frac{da'}{a(t')H(a')} \tag{207}$$

For flat matter domination

$$t_L(a) = \frac{2}{3H_0}[1 - (1+z)^{-3/2}] \tag{208}$$

For very large $z \to \infty$

$$t_L = \frac{2}{3H_0} \tag{209}$$

Now using

$$H^2 = (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}(\rho_m + \rho_{vac}) - \frac{k}{a^2} \tag{210}$$

$$H^2 = H_0^2[\Omega_{M_0}(1+z)^3 + \Omega_{k_0}(1+z)^2 + \Omega_\Lambda] \tag{211}$$

[We assume that the radiation is neglected now]

$$H = \frac{d}{dt} Log(\frac{a(t)}{a_0}) = \frac{d}{dt} Log(\frac{1}{1+z}) = -\frac{1}{1+z}\frac{dz}{dt} \tag{212}$$

$$\frac{dt}{dz} = -\frac{(1-z)^{-1}}{H_0[\Omega_{M_0}(1+z)^3 + \Omega_{k_0}(1+z)^2 + \Omega_\Lambda]^{1/2}} \tag{213}$$

The look back time from the present

$$t_0 - t_1 = H_0^{-1} \int_0^z \frac{dz}{(1+z)[\Omega_{M_0}(1+z)^3 + \Omega_{k_0}(1+z)^2 + \Omega_\Lambda]^{1/2}} \tag{214}$$

Choose $t_1 = 0$ ($z = \infty$) we obtain the present age of the Universe If $\Omega_{M_0} = 1$, $\Omega_{k_0} = 0$ $\Omega_\Lambda = 0$[Today]

$$H_0 t_0 = \int_0^\infty \frac{1}{(1+z)^{5/2}} dz = \frac{2}{3} \Rightarrow t_0 = \frac{2}{3H_0} \tag{215}$$

Using $H_0 = 70 kms^{-1} Mpc^{-1}$ $t_0 = 9.3$ billion years

3 Thermal History of the Universe

How are the contents of the universe affected by the expansion? The universe expands and its contents cool down. Let us focus on the radiation now. The energy density of the radiation $\rho \propto \frac{1}{a^4}$. Radiation is in thermal equilibrium and acts like a black body. The occupation number/mode

$$F(\nu) = \frac{2}{e^{h\nu/k_B T} - 1} \tag{216}$$

ν is the frequency. The corresponding energy density/mode

$$\epsilon(\nu) d\nu = \frac{8\pi\nu^3 d\nu}{c^3} \frac{h}{e^{h\nu/k_B T} - 1} \tag{217}$$

We use the natural unit, i.e., $k_B = 1$, $c = 1, h = 1$

Integrating over all frequencies

$$\rho_r = \frac{\pi^2}{15}(k_B T)(\frac{k_B T}{hc})^3 \Rightarrow \rho_r \propto T^4 \Rightarrow T \propto \frac{1}{a} \tag{218}$$

Is T the temperature of the Universe? Everything else has to feel the temperature. This means they have to interact (even if only indirectly) with photons, e.g., the scattering of photons by electrons and positrons through the emission and absorption of photons.

We also need the radiation to dominate in the early time. We know $\rho_{\mathrm{non-relativistic}} \propto a^{-3}$ while $\rho_r \propto a^{-4}$. So even if ρ_r dominates in the early universe, it may be negligible today.

However the number density of photons $n_\gamma \propto a^{-3}$.Experimentally, we found the number density n_B is very small (n_B is the number density of Baryons). Compared to the number density of photons

$$\eta_B = \frac{n_B}{n_\gamma} \simeq 10^{-10} \tag{219}$$

There are more photons than protons, neutrons. Temperature of the photon sets the temp. of the universe. The temp. decreases as the inverse of the scale factor.

For ideal gas of Bosons or Fermions the occupation/mode

$$F(\vec{p}) = \frac{g}{exp(\frac{E-\mu}{T}) \pm 1} \tag{220}$$

μ is the chemical potential which leads to chemical equilibrium in an interaction for

$$i + j = k + l \Rightarrow \mu_i + \mu_j = \mu_k + \mu_l \tag{221}$$

Chemical potentials are described in terms of some conserved quantities, μ_B, etc. If $\mu = 0$, then we have equal numbers of particles and anti-particles

Numbers of chemical potentials compared to the numbers of conserved particle numbers. $E = \sqrt{p^2 + m^2}$, g is the degeneracy factor. $+1(-1)$ corresponds to Fermi-Dirac (Bose-Einstein) distribution. We can use this distribution to calculate some macroscopic quantities. The number density

$$n = \frac{g}{(2\pi)^3} \int \frac{d^3p}{exp(\frac{E-\mu}{T}) \pm 1} \tag{222}$$

The energy distribution

$$\rho = \frac{g}{(2\pi)^3} \int \frac{E(\vec{p})d^3p}{exp(\frac{E-\mu}{T}) \pm 1} \tag{223}$$

The pressure

$$P = \frac{g}{(2\pi)^3} \int \frac{p^2}{3E} \frac{d^3p}{exp(\frac{E-\mu}{T}) \pm 1} \tag{224}$$

Let us consider two limits $T \gg M$ and $T \ll M$ with $\mu = 0$

For $T \gg M$,

$$n = \frac{\zeta(3)}{\pi^2} g T^3, \text{ B.E.} \tag{225}$$

$$n = \frac{3\zeta(3)}{4\pi^2} g T^3, \text{ F.D.} \tag{226}$$

With $\zeta(3) = 1.2$

$$\rho = \frac{g\pi^2}{30} T^4, \text{ B.E.} \tag{227}$$

$$\rho = \frac{7}{8} g \frac{\pi^2}{30} T^4, \text{ F.D.} \tag{228}$$

The pressure satisfies $P = \frac{\rho}{3}$.

For $T \ll M$

$$n = g(2\pi)^{3/2}(MT)^{3/2}e^{-\frac{M}{T}}, \ \rho = Mn \tag{229}$$

$$P = nT \ll nM \Rightarrow P \ll \rho \tag{230}$$

The pressure is negligible for nonrelativistic case.

For the average particle energy in the relativistic case

$$\langle E \rangle = \frac{\rho}{n} = \begin{array}{ll} \frac{7\pi^4}{180\xi(3)} T \simeq 3.15T & \text{F.D.} \\ \frac{\pi^4}{30\xi(3)} T \simeq 2.701T & \text{B.E.} \end{array} \tag{231}$$

If the chemical potential $\mu = 0$ then there are equal numbers of particles and anti-particles. If $\mu \neq 0$, we find for fermions in the ultrarelativistic limit T

$$n - \bar{n} = \frac{g}{(2\pi)^3} \int dp 4\pi p^2 (\frac{1}{exp(\frac{p-\mu}{T}) \pm 1} - \frac{1}{exp(\frac{p+\mu}{T}) \pm 1}) = \frac{gT^3}{6\pi^2} (\pi^2 (\frac{\mu}{T}) + (\frac{\mu}{T})^3) \quad (232)$$

The total energy density

$$\rho + \bar{\rho} = \frac{g}{(2\pi)^3} \int_0^\infty dp 4\pi p^2 (\frac{1}{exp(\frac{p-\mu}{T}) \pm 1} + \frac{1}{exp(\frac{p+\mu}{T}) \pm 1}) = \frac{7}{8} g \frac{\pi^2}{15} T^4 (1 + \frac{30}{7\pi^2} (\frac{\mu}{T})^2 + \frac{15}{7\pi^4} (\frac{\mu}{T})^4) \quad (233)$$

For the non-relativistic case

$$e^{(E-\mu)/T} \pm 1 \simeq e^{(E-\mu)/T} \quad (234)$$

$$n = g(\frac{mT}{2\pi})^{3/2} e^{-\frac{m-\mu}{T}} \quad (235)$$

$$\rho = n(m + \frac{3T}{2}), \text{Since E = m} + \frac{\text{p}^2}{2\text{m}}, \; P = nT \ll \rho, \; \langle E \rangle = m + \frac{3T}{2} \quad (236)$$

$$n - \bar{n} = 2g(\frac{mT}{2\pi})^{3/2} e^{\frac{-m}{T}} sinh \frac{\mu}{T} \quad (237)$$

We now need to understand the problem of calculating the total contribution to the energy and number density of all kinds of particles in the early universe

Let us now consider entropy:

$$dS(V,T) = \frac{1}{T}[d(\rho V) + P(T)dV] \Rightarrow dS = \frac{\partial S}{\partial V}(V,T)dV + \frac{\partial S}{\partial T}(V,T)dT \quad (238)$$

$$\frac{\partial S}{\partial V} = \frac{1}{T}(\rho(T) + P(T)) \quad (239)$$

$$\frac{\partial S}{\partial T} = \frac{V}{T}\frac{(d\rho(T)}{dT} \quad (240)$$

Equality of the mixed derivation

$$\frac{\partial^2 S}{\partial V \partial T}(V,T) = \frac{\partial^2 S}{\partial T \partial V}(V,T) \quad (241)$$

$$\frac{\partial}{\partial T}(\frac{1}{T}(\rho(T) + P(T)) = \frac{\partial}{\partial V}(\frac{V}{T}\frac{d\rho}{dT}(T)) \quad (242)$$

$$\Rightarrow \frac{dP}{dT} = \frac{1}{T}(\rho + P) \quad (243)$$

Use this to write $TdS = d(\rho V) + d(PV) - VdP$

$$dS = \frac{1}{T} d[(\rho + P)V] - \frac{V}{T^2}(\rho + P)dT \Rightarrow S = \frac{V}{T}(\rho + P) \quad (244)$$

Use

$$T^{\mu\nu}_{;\nu} = 0 \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \Rightarrow \frac{d}{dt}(\rho a^3) = -P\frac{da^3}{dt} \quad (245)$$

We can write $a^3\frac{dP(T)}{dt} = \frac{d}{dt}(a^3(\rho + P))$ Now use $\frac{dP}{dT} = \frac{1}{T}(\rho + P)$ to write

$$\frac{d}{dt}(\frac{a^3}{T}(\rho + P)) = 0 \quad (246)$$

One defines

$$s = \frac{S}{V} = (\frac{\rho + P}{T}) \quad (247)$$

where $V = a^3$. In the early universe both the energy density ρ and pressure P were dominated by the relativistic particles with equation of state $P = \rho/3$ and $s = \frac{2\pi^2}{45}g^s_{eff}T^3$ where g_{eff} is the effective number of degrees of freedom.

For the relativistic particles

$$\rho_{Re} = \frac{\pi^2}{30}g_{eff}T^4,\ P_{Re}(T) = \frac{1}{3}\rho_{Re}(T) = \frac{\pi^2}{90}g_{eff}(T)T^4 \quad (248)$$

$g_{eff}(T)$ is the total numbers of internal degrees of freedom (e.g., spin, color etc) of the particles that are relativistic and in thermal equilibrium at temp T. For example, in the Standard Model of particle physics we have, γ, g, $W^{\pm}$, Z, H, u, d, c, s, t, b, e, μ, τ, ν_e, ν_μ, ν_τ

$$g_{eff}(TeV) = 28 + \frac{7}{8}90 = 106.75 \quad (249)$$

Here, γ(photon): spin 1, $W^{\pm}$, Z: massive gauge boson: spin 1, quarks (u, d, c, s, t, b): colored and spin 1/2, leptons (e, μ, τ, ν_e, ν_μ, ν_τ) colorless and spin 1/2, H (Higgs boson): spin 0.

If the interaction rate becomes smaller than the expansion rate, then the particles will have lower temperature than the photons, but still can be relativistic (e.g., neutrinos) and this temperature will be unaffected by the heating takes for photons after the particles are decoupled.

This situation is handled by introducing a specific temperature T for each kind of relativistic particle which can be included in the effective g_i

$$g_{eff} = i = boson\Sigma g_i(\frac{T_i}{T})^4 + \frac{7}{8}i = fermions\Sigma g_i(\frac{T_i}{T})^4 \quad (250)$$

Inserting this in the FRW equation

$$H^2 = \frac{8\pi G}{3}\rho_{Re} = \frac{8\pi G}{3}\frac{\pi^2}{30}g_{eff}T^4 = 2.76\frac{g_{eff}}{M^2_{Planck}}T^4 \Rightarrow H = 1.66\frac{\sqrt{g_{eff}}}{M_{Planck}}T^2 \quad (251)$$

. You have noticed that we used g^s_{eff} for entropy density expression while g_{eff} for energy density. They are different (we will discuss more later) since

$$g^s_{eff} = i = boson\Sigma g_i(\frac{T_i}{T})^3 + \frac{7}{8}i = fermions\Sigma g_i(\frac{T_i}{T})^3 \quad (252)$$

.

Let us first realize that the relativistic particles contribute to the entropy density. We can write the entropy density in terms of energy density.

$$TdS = dU + PdV,\ U : \text{internal Energy} \tag{253}$$

$$Vd\rho = \frac{H}{T}dT = (U + PV)\frac{dT}{T} \tag{254}$$

$H = U + PV$: Enthalpy

$$dS = d\frac{(U + PV)}{T} \Rightarrow S = \frac{U + PV}{T} + constant \tag{255}$$

We can choose the integration constant such that $S = 0$ for the absolute 0 temp U, P constants all the particles of the universe

$$U = U_{rel} + U_{non-rel},\ \rho = \rho_{rel} + \rho_{non-rel} \tag{256}$$

For relativistic particle

$$U_R = \rho_R V,\ P_R = \frac{\rho_R}{3} \tag{257}$$

$$S_R = \frac{4\rho_R V}{3T},\ \text{using}\ \langle E_R \rangle = \frac{\rho_R}{n_R} \tag{258}$$

$$S_R = n_R V 4\frac{E_R}{3T} \simeq 4n_R V \Rightarrow \frac{\pi^2}{30} g T^3 V \tag{259}$$

The effective number of relativistic degrees of freedom g can change with time. The entropy conservation S_R=constant $V \propto a^3(t)$ gives $T \propto \frac{1}{g^{1/3}a(t)}$ where g is constant.

For non-relativistic

$$U_M = \frac{3}{2}n_M VT,\ P_M = n_M T,\ S_M = \frac{5}{2}n_M V \tag{260}$$

n_M is exponentially suppressed. It does not contribute to the effective g calculation

3.1 Electron-positron annihilation into photons

A good example of temperature change due to the change in g is the e^+e^- annihilation

$$e^+ + e^- \rightarrow \gamma + \gamma \tag{261}$$

When the temp. was greater than the rest mass of an electron

$$\gamma + \gamma \rightarrow e^+ + e^- \tag{262}$$

i.e., the pair creation occurs.

Also the particle behaves relativistically when the temp. is greater than $\simeq m/3$. The entropy conservation

$$T_2 = T_1(\frac{g_1}{g_2})^{1/3} \tag{263}$$

T_1 and T_2 are photon temp. before and after annihilation

$$g_1 = 2 + \frac{7}{8} \times 4 = \frac{11}{2},\ g_2 = 2 \tag{264}$$

Therefore we conclude that the annihilation increases the photon temp. by $(\frac{11}{4})^{1/3}$. After this the photon temp. decreases $T \propto \frac{1}{a(t)}$.

While there are about equal number of electrons, positrons and photons before the annihilation epoch, the number electrons after the annihilation is about 2 billion times smaller than photons as most of the electrons annihilate with positrons (the tiny excess is a mystery!). However the tiny excess is enough to keep the universe opaque. In order to make the scattering efficient the scattering rate needs to be larger than the expansion rate, i.e., $\sigma_T n_e > H$ where σ_T is Thompson scattering cross-section, n_e: number density of free electrons, H: Hubble expansion rate, σ_T=6.65×10^{-25} cm^2.Since the scattering is efficient and the universe remains opaque in the matter dominated regions with $H = H_0\sqrt{\Omega_M(1+z)^3}$

$$\frac{H}{\sigma_T n_e} = \frac{H_0\sqrt{\Omega_M(1+z)^3}}{\sigma_T n_{CMB}}\frac{n_{CMB}}{n_e} \tag{265}$$

Here $n_{CMB} = 410(1+z)^3\text{cm}^{-3}$ are the numbers of cosmic microwave background photons

$$\frac{n_{CMB}}{n_e} \simeq 2 \times 10^9 \tag{266}$$

$$\frac{c}{H_0} = 2.998h^{-1}Mpc = 9.25h^{-1} \times 10^{27}cm \tag{267}$$

$$\frac{H_T}{\sigma_T n_e} \simeq 0.9 \times 10^{-2}(\frac{1000}{1+z})^{3/2}(\frac{\frac{n_{CMB}}{n_e}}{2 \times 10^9}) \tag{268}$$

at $z \simeq 10^3$, the mean free time of photon was still only 1% of the Hubble time and universe was still opaque with $H < \sigma_T n_e$ We can also write $\frac{d_{\sigma_T n_e}}{d_H} \sim 10^{-2}$ with $d_H \sim \frac{1}{H}$ and $d_{\sigma_T n_e} \sim \frac{1}{\sigma_T n_e}$.

3.2 Recombination and Decoupling

At around $z \simeq 10^3$ or $T_{CMB} \simeq 3000K$, the electron number density rapidly fall relative to n_{CMB} resulting in the decoupling of photons from the electron scattering. At this temperature, the Universe is cool enough for electrons to be coupled by protons forming neutral Hydrogen atoms.

$$p + e^- \rightarrow H + \gamma \tag{269}$$

Once started, this process removes electrons rapidly, reducing their number density and thus allowing for photons to propagate freely.

As the ionization energy of the hydrogen atom is 13.6 eV, one might think that the neutral hydrogen begins to form when the temperature falls below 13.6eV$\simeq 1.6 \times 10^5$K. However in reality, the formation of Hydrogen atoms is delayed until $T \sim 3700$K. When the temperature is $T = 1.6 \times 10^5$K, only 15% of photons energies lower than 13.6 eV. When the temperature drops to $T =$ 70000K, about half of photons have energies lower than 13.6eV. Still there are so many photons per hydrogen atom to begin with and thus roughly speaking, the ratios of the number pf photons to the number of photons to the number of electrons give a logarithmic correction to the temperature of the hydrogen formation epoch as $T \simeq \frac{70,000}{Log 10^9} \simeq 3400$.

Finally when a significant amount of hydrogen atoms are formed at the temperature, photons do not decouple from the plasma until the universe cools down to $T \simeq 3000$K

The first approximation will be to assume that protons, electrons, hydrogen atoms are in thermal equilibrium. At this temperature all these species are non-relativistic and their equilibrium densities are given as

$$n_p = 2\int \frac{d^3p}{(2\pi)^3} exp[\frac{-m_p + \frac{p^2}{2m_p} + \mu}{T}] = 2e^{\frac{\mu_p - m_p}{T}}(\frac{m_p T}{2\pi}))^{3/2} \tag{270}$$

$$n_e = 2e^{\frac{\mu_e - m_e}{T}}(\frac{m_e T}{2\pi})^{3/2} \tag{271}$$

$$n_H = 2e^{\frac{\mu_H - m_p}{T}}(\frac{m_H T}{2\pi})^{3/2} \tag{272}$$

Now we assume that the protons, electrons and hydrogen atoms are in ionization equilibrium, which means that for

$$p + e^- \to H + \gamma,\ \mu_p + \mu_e = \mu_H \tag{273}$$

We write the Saha equation

$$\frac{n_p n_e}{n_H} = 2e^{\frac{-(m_p + m_e - m_H)}{T}}(\frac{m_p}{m_H}\frac{m_e T}{2\pi}))^{3/2} \tag{274}$$

Define binding energy

$$B_H \equiv (m_p + m_e - m_H) = 13.6eV \tag{275}$$

$m_e = M_p/2000$, $m_p = 1$ GeV, $m_p = m_H$.

For charge neutrality, $n_e = n_p$ we get

$$\frac{n_p^2}{n_H} = e^{\frac{-B_H}{T}}(\frac{m_e T}{2\pi}))^{3/2} \tag{276}$$

We define the ionization fraction

$$X \equiv \frac{n_p}{n_p + n_H}, \quad \begin{array}{ll} X = 1 & \text{fully ionized hydrogen} \\ X = 0 & \text{fully neutral hydrogen} \end{array} \tag{277}$$

The Saha equation becomes

$$\frac{X^2}{1-X} = \frac{1}{n_p + n_H}(\frac{m_e T}{2\pi})^{3/2} e^{-\frac{B_H}{T}} \tag{278}$$

We need to solve for X as a function of T. For convenience, let us define $n_p + n_H$ to the baryon mass density of the universe. We use this result from the Big Bang Nucleosynthesis. 76% of the baryonic mass in the universe after the BBN is contained in the protons (and the rest in the Helium nuclei), i.e., $m_p(n_p + n_H) = 0.76\rho_b$. The time independent baryon to photons ratio

$$\eta \equiv \frac{\rho_b}{m_p n_{CMB}} = 273.9(\Omega_b h^2) \times 10^{-10} \Rightarrow \eta = 6.3 \times 10^{-10},\ \Omega_\mathrm{b}\mathrm{h}^2 = 0.023 \tag{279}$$

We get

$$n_{CMB} = 410cm^{-3}(\frac{T}{T_0})^3\ \mathrm{with T_0 = 2.725K} \tag{280}$$

Grouping all these numbers

$$\frac{X^2}{1-X} = \frac{2.5 \times 10^{-6}}{\eta}(\tilde{T})^{3/2} e^{-\frac{1}{\tilde{T}}},\ \tilde{T} = \frac{T}{\beta_H} \tag{281}$$

We get

$$X(T) = \frac{2}{1 + \sqrt{1 + (1.6 \times 10^{-6} \tilde{T}^{3/2} e^{-\frac{1}{\tilde{T}}}}} \tag{282}$$

We can find an approximate temperature at which the universe is half neutral

$$X \equiv \frac{1}{2},\ \mathrm{then}\ \tilde{T}^{3/2} e^{-\frac{1}{\tilde{T}}} = 5 \times 10^6/\eta \Rightarrow \tilde{T} = 0.0237\ \mathrm{or}\ T = 3740 \tag{283}$$

For $\eta = 1$(i.e., equal numbers of photons and baryons), T can be found T=7900K.

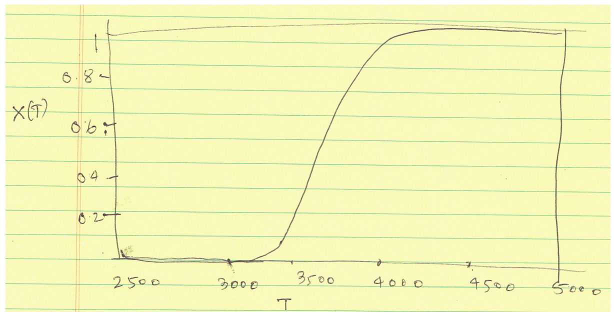
Now we can go back to the ionization history and recalculate $H/(\sigma_T n_e)$

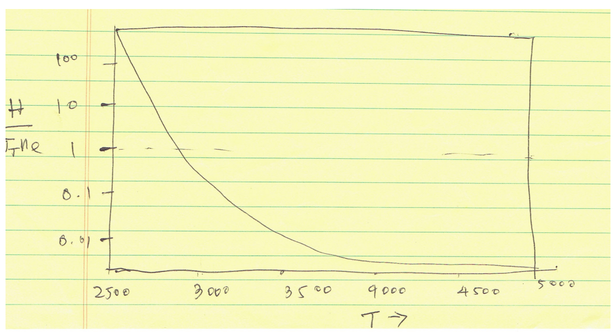
$$\begin{aligned} \frac{H}{\sigma_T n_e} &= \frac{H_0\sqrt{\Omega_m(1+z)^3}}{\sigma_T n_{CMB}} \frac{1}{0.76\eta X(Z)} \\ &= \frac{0.94 \times 10^{-2}}{X(z)}(\frac{1000}{1+z})^{3/2}(\frac{6.3 \times 10^{-10}}{\eta}) = \frac{0.94 \times 10^{-2}}{X(z)}(\frac{2725K}{T})^{3/2}(\frac{6.3 \times 10^{-10}}{\eta}) \end{aligned} \tag{284}$$

$$\frac{H}{\sigma_T n_e} = 1,\ T = 3000K,\ z = 1100 \tag{285}$$

Here we define the decoupling temperature $T_{dec} = 3000K$, i.e., when $H = \sigma_T n_e$.

For lower temperature $H > \sigma_T n_e$, Expansion rate is larger than the photons scattering off electron and photons are set free.





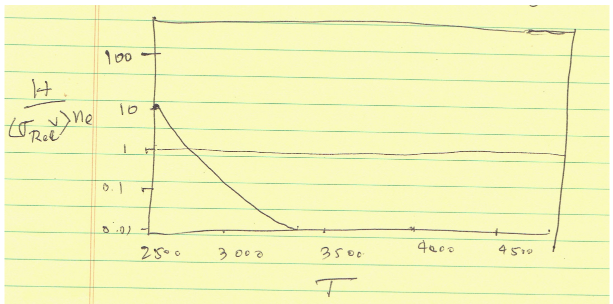
3.3 Freeze-out of recombination

The above calculation shows that all of the electrons will eventually be captured by protons leaving no free electrons at low temperature. However as the recombination rate is proportional to $n_e n_p$. The rate quickly falls quickly as the number densities go down within the expansion of the universe. Eventually the recombination stops. This is the epoch of recombination freeze-out.

The recombination rate is $\langle \sigma_{rec} v \rangle$ is

$$\langle \sigma_{rec} v \rangle = 2.33 \times 10^{-14} \frac{ln(1/\tilde{T})}{\tilde{T}^{1/2}} cm^3 s^{-1} = 7.77 \times 10^{-25} \frac{ln(1/\tilde{T})}{\tilde{T}^{1/2}} cm^2 \quad (286)$$

[In natural unit].



$\langle \sigma_{rec} v \rangle$ is of the same order as σ_T: Thompson scattering

$$\frac{H}{\langle \sigma_{rec} v \rangle n_e} = \frac{H_0\sqrt{\Omega_m(1+z)^3}}{\langle \sigma_{rec} v \rangle n_{CMB}} \frac{1}{0.76\eta X(z)} \tag{287}$$
$$= \frac{1.06 \times 10^{-3}}{X(T) ln(157894/T)} (\frac{2725K}{T})(\frac{6.3 \times 10^{-10}}{\eta})$$

The above rate crosses unity as $T_{freeze-out} = 2700K$ which is lower than the decoupling temperature. The residual ionization fraction of the recombination, i.e., the ionization fraction left after the recombination freeze-out by evaluating $X(T)$ at $T = 2700K$. The small amount of X means small amount of residual electrons which is needed for forming hydrogen molecules via

$$H + e^- \to H^- + \gamma,\ H^- + H \to H_2 + e^- \tag{288}$$

4 Dark Matter

We want to calculate the current density of dark matter particles. Suppose X is a neutral DM particles. In that early universe t was large

$$X^0 + X^0 \leftrightarrow f + \bar{f} \tag{289}$$

Suppose X^0 is also fermion.

$$f + \bar{f} \leftrightarrow \gamma + \gamma\ etc. \tag{290}$$

Similarly, X^0 cannot decay [it is stable]. However two of them collide with each other and annihilate. X^0 is in thermal equilibrium with other matter and hence with photons. Thermal equilibrium is maintained if the reaction rate is faster than the Hubble expansion rate

$$\Gamma_{X^0 X^0 \to f\bar{f}} > H \tag{291}$$

However the universe catches up

$$\Gamma_{X^0 X^0 \to f \bar{f}} \simeq H \tag{292}$$

t_D is the temperature. X^0 decouples from the plasma. For

$$H > \Gamma_{X^0 X^0 \to f \bar{f}} \tag{293}$$

, the X^0's cease to annihilate. Thus the number of X^0 at that time remains unchanged and form "relic density". This is $\Omega_{X^0} h^2$. We now need to provide a quantitative picture.

Boltzman equation describes the time evolution of the distribution function is phase space. For non-relativistic system this is given by the function $f(\vec{r}, \vec{p}, t)$. The change in the function in course of its time motion is

$$\frac{Df}{dt} = \frac{\partial f}{\partial t} + \frac{d\vec{r}}{dt}.\vec{\nabla}_r f + \frac{d\vec{p}}{dt}.\vec{\nabla} f \tag{294}$$

where $\frac{d\vec{p}}{dt} = \vec{F}$ we get

$$\frac{Df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{m}\vec{p}.\vec{\nabla}_r f + \vec{F}.\vec{\nabla} f \tag{295}$$

For the relativistic case, we generalize this

$$f = f(x^\alpha, p^\alpha) \tag{296}$$

The motion in phase space is defined via proper time τ with $d\tau = \frac{1}{c} ds$ =proper time.

The change of f is now

$$\frac{Df}{d\tau} = v^\alpha \frac{\partial f}{\partial x^\alpha} + \frac{dp^\alpha}{d\tau}\frac{\partial f}{\partial p^\alpha} \tag{297}$$

Writing $p^\alpha = m v^\alpha$, the geodesic equation reads

$$\frac{dv^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu} v^\mu v^\nu \tag{298}$$

$$\frac{Df}{d\tau} = v^\alpha \frac{\partial f}{\partial x^\alpha} - m\Gamma^\alpha_{\mu\nu} v^\mu v^\nu \frac{\partial f}{\partial p^\alpha} \tag{299}$$

$$\frac{Df}{d\tau} = \frac{dt}{d\tau}\frac{Df}{dt} = \frac{dx^0}{d\tau}\frac{Df}{dt} = v^0 \frac{Df}{dt} = \frac{p^0}{m}\frac{Df}{dt} \tag{300}$$

Writing $p^0 = E$

$$\frac{Df}{dt} = \frac{m}{p^0}(v^\alpha \frac{\partial f}{\partial x^\alpha} - m\Gamma^\alpha_{\mu\nu} v^\mu v^\nu \frac{\partial f}{\partial p^\alpha}) \tag{301}$$

We use Robertson-Walker metric, the space is homogeneous and isotropic. We use $f = (E, t)$ where $|\vec{p}| = \sqrt{E^2 - m^2}$

$$\frac{Df}{dt} = \frac{m}{E}(v^0 \frac{\partial f}{\partial x^0} - m\Gamma^0_{\mu\nu} v^\mu v^\nu \frac{\partial f}{\partial E}) \tag{302}$$

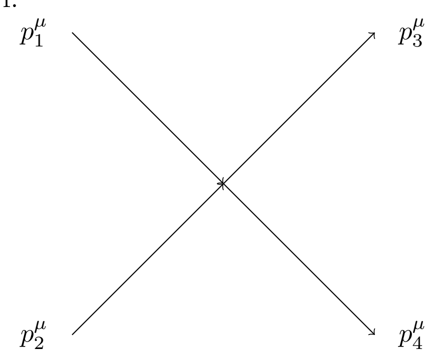
where $v^0 = \frac{p^0}{m}$. For R-W metric

$$\Gamma^0_{ij} = -\frac{\dot{a}}{a} g_{ij},\, \Gamma^0_{00} = 0 = \Gamma^0_{0i},\, g_{ij} = -\delta_{ij} \tag{303}$$

$$\frac{Df}{dt} = \frac{\partial f}{\partial t} - \frac{\dot{a}}{a} \frac{\vec{p}.\vec{p}}{E} \frac{\partial f}{\partial E} \tag{304}$$

What cause the distribution to change in time. If there is no collision, the f is constant

Scattering of particles from one momentum state to another leads to change in f:



$$a_1 + a_2 \leftrightarrow a_3 + a_4 \tag{305}$$

Let us look at the distribution function particles

$$f = f(\vec{p}, t) = f(E, t) = f_1 \tag{306}$$

Similarly we have functions for a_2, a_3, a_4

$$d\Gamma(i \to f) = \frac{(2\pi)^4}{V} |M_{fi}|^2 \delta^4(p_3 + p_4 - p_1 - p_2) \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3} = \text{transition probability/time for i} \to \text{f} \tag{307}$$

with a_3, a_4 with final particles cell d^3p_3, d^3p_4 at momenta $\vec{p}_3$, $\vec{p}_4$ and spin $s_f = s_3 s_4$. V is the box normalization and M_{fi} is the matrix element of the transition from $i \to f$. If the Hamiltonian is

$$H = H_0 + H_1 \tag{308}$$

We can generate

$$\langle p_3,\, p_4,\, out|H|p_1\, p_2,\, in > \tag{309}$$

Let us go to the Lab frame

$$\vec{v}_2 = 0,\, \vec{v}_1 \neq 0 \tag{310}$$

Incident flux $\rho_1 = \frac{1}{V} v_1$ =Incident flux of a_1

$$d\sigma(i \to f) = \frac{d\Gamma(i \to f)}{\rho_1} = \frac{(2\pi)^4}{v_1} |M_{fi}|^2 \delta^4(p_3 + p_4 - p_1 - p_2) \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3} = \text{transition probability/time for i} \to \text{f} \tag{311}$$

If we wish to go to any other frame, we do this by requiring $d\sigma$ = Lorentz scalar. If this turns out that this is achieved by

$$v = \frac{\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2}}{E_1 E_2} \tag{312}$$

and one has

$$v = v_1,\, p_2^\mu = (0,\, m_2) : \text{Laboratoryframe} \tag{313}$$

We can write

$$v = |\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}| = |\vec{v}_1 - \vec{v}_2| \tag{314}$$

$\vec{p}_1 + \vec{p}_2 = 0$: Center of mass frame.

However we can go to any other frame. Usually the CM frame is the easiest one to use.

Now returning to get the number of particles scattered out of initial state, we multiply the probability $d\Gamma$ by the number in initial state X_2 and sum over initial X_2 and sum over final X_3, X_4 and multiply no. of X_1

$$N(i \to f) = s_3,\, s_4,\, s_2 \sum \frac{1}{V} p_2 \sum \int \frac{d^3p_3 d^3p_4}{(2\pi)^6} (2\pi)^4 M_{fi}|^2 \delta^4(p_3 + p_4 - p_1 - p_2) f_1 f_2 (1 \pm f_3)(1 \pm f_4) \tag{315}$$

where $\frac{1}{V} p_2 \sum = \int \frac{d^3p_2}{(2\pi)^3}$ with +: Bose-Einstein and −: Fermi-Dirac. $(1 \pm f_{3,4})$ account for Pauli suppression if a Fermi state is already filled or a BE enhancement.

Similarly, we have an enhancement of the initial state since thermal equilibrium allows the inverse process $f \to i$

$$N(f \to i) = s_1,\, s_2 \sum \frac{1}{V} p_2 \sum \int \frac{d^3p_3 d^3p_4}{(2\pi)^6} (2\pi)^4 |M_{fi}|^2 \delta^4(p_3 + p_4 - p_1 - p_2) f_3 f_4 (1 \pm f_1)(1 \pm f_2) \tag{316}$$

Our balance is

$$\frac{Df}{dt} = N(i \to f) - N(f \to i) \tag{317}$$

We now make some reasonable assumptions:

- T invariance (or PC invariance) implies

$$|M_{fi}|^2 = |M_{is_i \to fs_f}|^2 = |M_{is_i \leftarrow fs_f}|^2 = |M_{s_i s_f}|^2 \tag{318}$$

  This is true except for certain weak interactions

- We will assume that in the vicinity of freezeut, the particles are non-relativistic. Then

$$f_i \simeq e^{-E/T} \ll 1;\, E/T \ll 1 \tag{319}$$

and we can neglect the BE and FD enhancement and suppression, i.e

$$1 \pm f_i \sim 1 \tag{320}$$

In general our particles have spin and different spin states. We will assume the distribution functions don't depend on the sin quantum number. Then for

$$n(t) = s \sum \int \frac{d^3p}{(2\pi)^3} f(p,t) = numbers/volume \tag{321}$$

We have

$$n(t) = g \int \frac{d^3p}{(2\pi)^3} f(p,t) \tag{322}$$

This is true for systems with isotropy. Our basic equation then simplifies

$$\frac{\partial f_1}{\partial t} - \frac{\dot{a}}{a}\frac{\vec{p}_1^2}{E_1}\frac{\partial f_1}{\partial E_1} = -s_2, s_3, s_4 \sum \int \frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3}\frac{d^3p_4}{(2\pi)^3}(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)|M_{s_i s_f}|^2(f_1f_2-f_3f_4) \tag{323}$$

We can integrate over p_1 to get an equation for the number density n_1:

$$\frac{dn_1(t)}{dt} - \frac{\dot{a}}{a}g_1 \int \frac{d^3p_1}{(2\pi)^3}\frac{\vec{p}_1^2}{E_1}\frac{\partial f_1}{\partial E_1} = -s_1, s_2, s_3, s_4 \sum \int i = 14\Pi \frac{d^3p_i}{(2\pi)^3}|M_{s_3 s_4 s_1 s_2}|^2(f_1f_2-f_3f_4) \tag{324}$$

The second term in left hand side reduces to

$$\begin{aligned} g_1 \int \frac{d^3p_1}{(2\pi)^3}\frac{\vec{p}_1^2}{E_1}\frac{\partial f_1}{\partial E_1} &= \frac{g_1}{(2\pi)^3}\int d\Omega \int_{m_1}^{\infty} \frac{\vec{p}_1^4}{E_1}\frac{dp_1}{dE_1}dE_1\frac{\partial f_1}{\partial E_1} \\ &= \frac{g_1}{(2\pi)^3}\int d\Omega \int_{m_1}^{\infty} \vec{p}_1^3 dE_1 \frac{\partial f_1}{\partial E_1} \\ &= -\frac{3g_1}{(2\pi)^3}\int d\Omega \int_{m_1}^{\infty} dE_1 \vec{p}_1^2 \frac{dp_1}{dE_1} f_1 = -3g_1 \int \frac{d^3p_1}{(2\pi)^3} f_1 = -3n_1(t) \end{aligned} \tag{325}$$

In general f_i are spin independent, the matrix element $|M|^2$ will in general have spin independent. However we can define the spin averaged matrix element $|M|^2$ will in general have spin independent. However we can define the spin averaged matrix element

$$\frac{1}{g_1g_2g_3g_4} s_1, s_2, s_3, s_4 |M_{s_3 s_4 s_2 s_1}|^2 = |M|^2 \tag{326}$$

$|M|^2$ is the spin averaged matrix element and depend only on the momenta of the process

$$\frac{dn_1(t)}{dt} + 3\frac{\dot{a}}{a}n_1 = -\int i = 14\Pi g_i \frac{d^3p_i}{(2\pi)^3}|M|^2(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)(f_1f_2-f_3f_4) \tag{327}$$

We can write down similar equations for f_2, f_3, f_4 and we would have a complete set to try to solve we will make some physically valid approx. to simplify the analysis.

When the X_3, X_4 are created by X_1, X_2. We can assume that they interact rapidly with the plasma and quickly thermalize.

Thus f_3, f_4 can have their equilibrium values, i.e., the Boltzmann distribution

$$f_3 = f_3^{eq} = e^{-E_3/T},\ f_4 = f_4^{eq} = e^{-E_4/t} \tag{328}$$

where $E_3\sqrt{p_3^2 + m_3^2}$. Hence we can write

$$\delta^4 f_3 f_4 \simeq \delta^4 f_3^{eq} f_4^{eq} = \delta^4 e^{-(E_3+E_4)/T} = \delta^4 e^{-(E_1+E_2)/T} = \delta^4 f_1^{eq} f_2^{eq} \tag{329}$$

$$\int i = 1, 2\Pi g_i \frac{d^3 p_i}{(2\pi)^3} \left[\int i = 3, 4\Pi g_i \frac{d^3 p_i}{(2\pi)^4} |M|^2 (2\pi)^4 \delta^4 (p_3 + p_4 - p_1 - p_2) \right] (f_1 f_2 - f_3^{eq} f_4^{eq}) \tag{330}$$

$[\cdots]$ is related to cross-section, i.e.,

$$\int i = 3, 4\Pi g_i \frac{d^3 p_i}{(2\pi)^3} \sigma_{i \to f} v (f_1 f_2 - f_3^{eq} f_4^{eq}) \tag{331}$$

We can define the thermal average of the σv as

$$\langle \sigma_{i \to f} v \rangle = \frac{\int i = 1, 2\Pi \frac{g_i dp_i}{(2\pi)^3} \sigma_{i \to f} v f_1 f_2}{n_1 n_2} \tag{332}$$

For the equilibrium case this means

$$\langle \sigma_{i \to fv} \rangle^{eq} = \frac{\int i = 1, 2\Pi \frac{g_i d^3 p_i}{(2\pi)^3} \sigma_{i \to f} v e^{-(E_1+E_2)/T}}{\int \frac{g_1 d^3 p_1}{(2\pi)^3} e^{-E_1/T} \int \frac{g_2 d^3 p_2}{(2\pi)^3} e^{-E_2/T}} \tag{333}$$

We are averaging over the Boltzmann distribution. The particles are close to equilibrium, but the expansion of the universe changes $n_{1,2}$, i.e., allows a chemical potential to grow while the scattering can change the momentum distribution. We can write the ansatz for $f_{1,2}$ as

$$f_{1,2} = e^{\frac{E_{1,2}}{T} - \frac{\mu_{1,2}(t)}{T(t)}} \tag{334}$$

$$\langle \sigma_{i \to f} v \rangle = \frac{e^{\frac{\mu_1+\mu_2}{T}} i = 1, 2 \int \Pi \frac{g_i d^3 p_i}{(2\pi)^3} \sigma_{i \to f} v e^{-(E_1+E_2)/T}}{e^{\frac{\mu_1+\mu_2}{T}} \int \frac{g_1 d^3 p_1}{(2\pi)^3} e^{-E_1/T} \int \frac{g_2 d^3 p_2}{(2\pi)^3} e^{-E_2/T}} \tag{335}$$

or

$$\langle \sigma_i v \rangle = \langle \sigma_{i \to f} v \rangle^{eq} \tag{336}$$

Our Boltzmann equation now reads

$$\frac{dn_1}{dt} + 3\frac{\dot{a}}{a} n_1 = -\langle \sigma_{i \to f} v \rangle^{eq} [n_1 n_2 - n_1^{eq} n_2^{eq}] \tag{337}$$

This is called Lee-Weinberg equation.

In our analysis we have considered only a single annihilation process

$$X_1 + X_2 \to X_3 + X_4 \tag{338}$$

There could be many final states each contributing to the annihilation, e.g.,

$$\begin{aligned} X_1 + X_2 &\to f + \bar{f} \\ &\to W^+ + W^- \\ &\to Z^0 + Z^0 \end{aligned} \tag{339}$$

Writing $n_1(t) = n_2(t) = n_X$

$$\frac{dn_X}{dt} + 3H(t)n_X = -\langle\sigma_{i\to f}v\rangle^{eq}[n_X^2 - n_X^{eq2}] \tag{340}$$

σ_A is the annihilation cross-section

$$\sigma_A = \sum \sigma_{X_1+X_2\to f_i} \ f_i = \text{final state} \tag{341}$$

In this approximation, we can write

$$\frac{dn_X}{dt} + 3H(t)n_X = \frac{1}{a^3}\frac{d}{dt}(a^3(t)n_X) \tag{342}$$

If there is no scattering, then $n_X = \frac{const}{a^3(t)}$ and the number/volume decreases as volume increases. The first term of the R.H.S of the Eqn340 is the depletion of n_X due to

$$X_1 + X_1 \to f_i \tag{343}$$

The second term is the increase of n_X due to the creation of X by the inverse reaction $f_i \to X_1X_1$

4.1 relic abundance calculation

The basic equation looks complicated since it is a non-linear equation. If we know the particle physics interactions giving rise to $\langle\sigma_A v\rangle$, then we can calculate $\langle\sigma_A v\rangle^{eq}$ and express it as a function of $T(t)$.

We can solve the equation numerically however some simple analytic approximations can be made. In the early universe when one has radiation domination

$$T(t) = (\frac{45}{2\pi^2 k g_*})^{1/4}\frac{1}{t^{1/2}} = a^{1/4}\frac{1}{t^{1/2}} \tag{344}$$

g_*=multiplicative factor associated with the number of relativistic particles. The Hubble constant is $H = \frac{1}{2t}$

$$H(T) = (\frac{\pi k g_*}{2(45)})^{1/2}T^2 = \frac{1}{2}a^{-1/2}T^2 \tag{345}$$

$k^{-1/2} = (\frac{hc}{8\pi G_N})^{1/2} = 2.44 \times 10^8 GeV$ Hence $H(T) = 0.33 g_*^{1/2} \frac{T^2}{k^{-1/2}}$, $M_{Pl} = k^{-1/2} = 2.44 \times 10^{18} GeV$. Also

$$t = (\frac{45k^{-1}}{2\pi^2 g_*})^{1/2} \frac{1}{T^2} = \frac{1.51}{g_*^{1/2}} \frac{k^{-1/2}}{T^2} = \frac{1.51}{g_*^{1/2}} \frac{M_{Pl}}{T^2} \tag{346}$$

We can use temperature to represent time. We also have that at early times when our particles are relativistic $T > m_X$, $n_X = \frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3 \sim T^3$. The Hubble term

$$3Hn_X \sim T^5 \tag{347}$$

The collision term $n_X \sim T^6$. Thus the collision is dominant in the early universe

$$\frac{dn_X}{dt} = \langle \sigma v \rangle^{eq} [n_X^2 - {n_X}^{eq2}] \tag{348}$$

$$\frac{dn_X}{dt} = \frac{dT}{dt}\frac{dn_X}{dT} \sim T^3 T^2 \sim \frac{1}{t^{3//2}} T^2 \tag{349}$$

$$\frac{dn_X}{dt} = \langle \sigma v \rangle [n_X^2 - {n_X}^{eq2}] \tag{350}$$

Where $\frac{dn_X}{dt} \propto T^5$ while $n_X^2 \sim T^6$ This means n_X must be very close to n_X^{eq} to cancel the extra factor of T.

$$n_X^2 = {n_X}^{eq2} + O(T^5) \tag{351}$$

$$n_X = {n_X}^{eq}(1 + O(\frac{1}{T})) \tag{352}$$

Thus the number density is mostly given by the equilibrium distribution. More clearly: Let us define $Y = \frac{n_X}{s}$, $s = entropy/volume$. $Y=$ numbers of particle in $V = R^3(t)$. Here $s = \frac{2\pi^2}{45} g_* T^3 = bT^3$, $b = \frac{2\pi^2}{45} g_*$. Change $n_X \rightarrow Y(X)$, $t \rightarrow X = \frac{m_X}{T}$

$$\frac{dn_X}{dt} = \frac{dX}{dt}\frac{d}{dX}(Ys) = \frac{dX}{dt} s \frac{dY}{dX} + \frac{dX}{dt}\frac{ds}{dX} Y \tag{353}$$

We get

$$\frac{dX}{dt} = -\frac{m_X}{T^2}\frac{dT}{dt} = -\frac{m_X}{T^2}(-\frac{1}{2}\frac{a^{1/4}}{t^{1/2}}) = \frac{m_X}{2a^{1/2}} T = \frac{m_X^2}{2a^{1/2} X} \tag{354}$$

$$\frac{ds}{dX} = \frac{dT}{dX}\frac{ds}{dT} = -3\frac{m_X}{X^2} bT^2 = -3\frac{m_X^3 b}{X^4} \tag{355}$$

We get

$$\frac{dX}{dt}\frac{ds}{dX} Y = -3\frac{m_X^5 bY}{2a^{1/2} X^5} \tag{356}$$

Also we can write

$$3Hn_X = \frac{3}{2} a^{-1/2} T^2 bT^3 Y = \frac{3}{2} a^{-1/2} b \frac{m_X^5}{X^5} Y \tag{357}$$

So $3Hn_X$ cancels $Y\frac{dX}{dt}\frac{ds}{dX}$. We can write

$$\frac{dY}{dX} = -2a^{1/}Xs\frac{\langle\sigma_A v\rangle^{eq}}{m_X^2}[Y_X^2 - {Y_X}^{eq2}] \tag{358}$$

Define $H(m_X) = \frac{1}{2}a^{-1/2}m_X^2$; $H(X) = H(m_X)X^{-2}$. $H(m_X)$ is Hubble constant at temperature $=m_X$

$$\frac{dY}{dX} = -Xs\frac{\langle\sigma_A v\rangle^{eq}}{H(m_X)}[Y_X^2 - {Y_X}^{eq2}] \tag{359}$$

Using $y^{eq} = n^{eq}/s$, we can write

$$\frac{X}{Y^{eq}}\frac{dY}{dX} = -n^{eq}\frac{\langle\sigma_A v\rangle^{eq}}{H(X)}[\frac{Y_X^2}{{Y_X}^{eq2}} - 1] \tag{360}$$

We can write $\Gamma_A = n^{eq}\langle\sigma_A v\rangle^{eq}$=numbers of annihilation/time (n=number/volume and $\langle\sigma_A v\rangle^{eq}$ annihilation volume/time)

If $\Gamma^A/H \gg 1$ then $Y(x) \to Y^{Eq}$. If $\Gamma^A/H \ll 1$ the R.H.S. becomes negligible and $Y(x) =$ constant. The number of particles in $V = R^3(t)$ becomes constant and we have "freeze-out". The particles remained are the relics of Big Bang.

We can solve the Boltzmann equation. Let us examine $\langle\sigma_A v\rangle$ first

$$\langle\sigma_{i\to f}v\rangle = \frac{\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\sigma_{i\to f}ve^{-(E_1+E_2)/T}}{\int\frac{d^3p_1d^3p_2}{(2\pi)^6}e^{-(E_1+E_2)/T}} \tag{361}$$

Since we are in the non-relativistic regime

$$E_{1,2} = m + \frac{p_{1,2}^2}{2m};\ m_1 = m_2 \tag{362}$$

The $e^{-m/T}$ cancels out between numerator and denominator

$$\langle\sigma_{i\to f}v\rangle = \frac{\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\sigma_{i\to f}ve^{-(p_1^2+p_2^2)/(2mT)}}{\int\frac{d^3p_1d^3p_2}{(2\pi)^6}e^{-(p_1^2+p_2^2)/(2mT)}} \tag{363}$$

Going to the center of mass frame

$$P = \frac{1}{2}(p_1+p_2),\ p = p_1 - p_2,\ d^3p_1d^3p_2 = d^3Pd^3p,\ p_1^2+p_2^2 = 2P^2 + \frac{1}{2}p^2 \tag{364}$$

$$\langle\sigma_{i\to f}v\rangle = \frac{\int\frac{d^3Pd^3p}{(2\pi)^6}\sigma_{i\to f}ve^{-P^2/(mT)}e^{-p^2/(4mT)}}{\int\frac{d^3Pd^3p}{(2\pi)^6}e^{-P^2/(mT)}e^{-p^2/(4mT)}} \tag{365}$$

It is convenient to do the cross-section calculations in C.M. frame. Then σ_A depends only on p (One can go to $\vec{P} = 0$), $\sigma_A = \sigma_A(v)$; $p = mv$

$$\langle\sigma_{i\to f}v\rangle = \frac{\int_0^\infty v^2dv\sigma_{i\to f}ve^{-mv^2/(4T)}}{\int_0^\infty v^2dve^{-mv^2/(4T)}} \tag{366}$$

where $d^3p = p^2 dp d\omega = m^3 v^2 dv d\Omega$

$$\sigma_A v \sim \frac{|M|^2 v}{J_{inc}} = \frac{|M|^2 v}{\rho v} \tag{367}$$

In general, $\sigma_A v$ is regular at $v = 0$

$$\sigma_A v = a + \frac{1}{6} b v^2 + \cdots, a,\ b = constant,\ \langle \sigma_A v \rangle = a + \frac{1}{6} b \langle v^2 \rangle \tag{368}$$

The thermal average is being taken W.R.T Boltzmann distribution.

From kinetic theory we know that for Boltzmann distribution of mass μ

$$\langle \frac{1}{2} \mu v^2 \rangle = \frac{3}{2} T,\ \mu = \frac{m}{2} \tag{369}$$

$$\langle v^2 \rangle = \frac{3}{\mu T} = \frac{6T}{m} \tag{370}$$

$$\langle \sigma_A v \rangle = a + \frac{bT}{m} \tag{371}$$

This simple approximation does not work always.

4.2 Approximate solution for relic $\mathbf{X}^0$

We saw that $Y = \frac{n_X}{s} \sim \frac{n_X}{T^3 g_{*s}}$ eliminate the Hubble expansion from the Boltzmann equation. Define

$$\phi(t) = \frac{n_X(t)}{T^3(t) g_{*s}},\ \ Z(t) = \frac{T(t)}{m},\ \ m = m_X \tag{372}$$

$$T(t) = a^{1/4} \frac{1}{t^{1/2}},\ a = \frac{45}{2\pi^2 k g_{*s}} \tag{373}$$

The entropy $S = sR^3$ =constant where $s \sim T^3 g_{*s}$ with $T^3 g_{as} R^3 = c$

$$\frac{d}{dt}(T^3 g_{as}) = -\frac{3cR}{R^4} = -3c\frac{H}{R^3} = -3T^3 g_{*s} H \tag{374}$$

$$\frac{d\phi}{dt} = \frac{1}{T^3 g_{as}} \frac{dn}{dt} + 3H \frac{n}{T^3 g_{*s}} \tag{375}$$

Using the Boltzmann equation

$$\frac{d\phi}{dt} = \frac{1}{T^3 g_{*s}} \langle \sigma v \rangle [n^2 - n_{eq}^2] = -T^3 g_{*s} \langle \sigma v \rangle [\phi^2 - \phi_{eq}^2] \tag{376}$$

where $\phi_{eq} = \frac{n^{eq}}{T^3 g_{*s}}$

Using $\frac{d\phi}{dt} = \frac{dT}{dt} \frac{d\phi}{dT}$

$$\frac{dT}{dt} = -\frac{1}{2} \frac{a^{1/4}}{t^{3/2}} + \frac{1}{4} \frac{a^{-3/4}}{t^{1/2}} (-a) \frac{1}{g_*} \frac{dg_*}{dT} \frac{dT}{dt} = -\frac{1}{2} \frac{T^3}{a^{1/2}} - \frac{1}{4} \frac{T}{g_*} \frac{dg_*}{dt} \frac{dT}{dt} \tag{377}$$

$$\frac{dT}{dt} = \frac{1}{1+\frac{1}{4}\frac{dlng_*}{dlnT}}(-\frac{1}{2}\frac{T^3}{a^{1/2}}) \tag{378}$$

In general g_* is constant except when one particle drops out from being relativistic. We will neglect $\frac{dg_*}{dT}$ in the denomination

$$\frac{d\phi}{dt} \simeq (-\frac{1}{2}\frac{T^3}{a^{1/2}})\frac{1}{m}\frac{d\phi}{dz} \tag{379}$$

$$\frac{d\phi}{dz} = 2ma^{1/2}g_{*s}\langle\sigma v\rangle[\phi^2 - \phi_{eq}^2] \tag{380}$$

$$2a^{1/2} = 2(\frac{45}{2\pi^2 kg_*})^{1/2};\ \frac{1}{k} = \frac{1}{8\pi G_N} = M_{pl}^2 \tag{381}$$

$$\frac{d\phi}{dz} = m(\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}\langle\sigma v\rangle[\phi^2 - \phi_{eq}^2] \tag{382}$$

For $\Gamma_A \ll H$, the Boltzmann equation pushes ϕ to equilibrium

$$\phi(z) \simeq \phi^{eq}(z),\ \Gamma_A \gg H \tag{383}$$

At freeze-out, the species decouples from the plasma so that the back scattering-ϕ_{eq}^2 R.H.S. is no longer important and so in the other region

$$\frac{d\phi}{dz} = m(\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}\langle\sigma v\rangle\phi^2 \tag{384}$$

for $\Gamma_A \gg H$ with $z_f = T_f/m$. For $\Gamma_A \simeq H$, one still has $\phi \simeq \phi^{eq}$ $(z = z_f)$

$$(\frac{d\phi}{dz})_{z_f} \simeq m(\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}\langle\sigma v\rangle\phi_{eq}^2 \tag{385}$$

Use

$$\phi_{eq} = \frac{g_X}{T^3}\frac{1}{g_{*s}}(\frac{mT}{2\pi})^{3/2}e^{\frac{-m}{T}} \tag{386}$$

Inserting

$$\phi_{eq} = \frac{2}{g_{*s}}(\frac{1}{2\pi z})^{3/2}e^{\frac{-1}{z}},\ g_X = 2 \tag{387}$$

$$\frac{2}{g_{*s}}(\frac{1}{2\pi z})^{3/2}e^{\frac{-1}{z}}[\frac{1}{z^2} - \frac{3}{2}\frac{1}{z}] = m(\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}\langle\sigma v\rangle\frac{4e^{2/z}}{g_{*s}^2(2\pi z)^3} \tag{388}$$

Solving the above equation at $z = z_f$

$$e^{\frac{1}{z_f}} = (2\pi)^3(\frac{2}{45}G_N g_*)^{1/2}\frac{1}{m\langle\sigma v\rangle z_f^{1/2}}(1 - \frac{3}{2}z_f) \tag{389}$$

$$z_f^{-1} = Ln[z_f^{1/2}m\langle\sigma v\rangle\frac{1}{2\pi^3}(\frac{45}{4\pi^2 kg_*})^{1/2}\frac{1}{1-\frac{3}{2}z_f}] \tag{390}$$

we will see now that $z_f = \frac{1}{20}$(i.e., our particles are freezing out nonrelativistically) and so we can approximate

$$\frac{1}{1-\frac{3}{2}z_f} \simeq 1 \tag{391}$$

$$z_f^{-1} = Ln[z_f^{1/2} m\langle\sigma v\rangle(\frac{1}{G_N g_*})^{1/2} 0.0765] \tag{392}$$

A more rigorous numerical solution in the vicinity of freeze-out gives

$$z_f^{-1} = Ln[z_f^{1/2} \cdots 0.0765] + Ln[c(c+2)] \tag{393}$$

where $c \simeq 0.5$. This produces a negligible correction to $z_f^{-1/2}$. Thus

$$z_f^{-1} = Ln[z_f^{1/2} \cdots 0.0765] + c(c+2) \tag{394}$$

and the correction of size $\frac{Lnc(c+2)}{z_f^{1/2}} = Ln\frac{0.525}{20} = 0.011 \sim 1\%$ correction

$$z_f^{-1} = Ln[m\langle\sigma v\rangle(\frac{1}{G_N g_*})^{1/2} 0.0765] + Lnz_f^{1/2} \tag{395}$$

To the zeroth approximation we can neglect $Lnz_f^{1/2}$]

We need an annihilation cross-section. We use an example without getting into any detailed calculation. Assume Dark matter couples to the SM particles with weak coupling:

$$\sigma v \sim \frac{\alpha_2^2 m_X^2}{4\pi m_N^4} \tag{396}$$

Here m_N is a new particle which shows up in the dark matter annihilation diagram and $\alpha_2 \sim 0.03$. We can write

$$Ln[z_f^{1/2} m\langle\sigma v\rangle(\frac{1}{G_N g_*})^{1/2} 0.0765] \simeq Ln[\frac{\alpha_2^2 m_X^3 M_{pl}}{g_*^{1/2} m_N^4} 0.0765] \tag{397}$$

Assume $m_X = m_N = 100$ GeV, $M_{pl} = 2.44 \times 10^{18}$ GeV, $g_* = 100$, $z_f^{-1/2} = 23$, $LnZ_f^{1/2} = Ln\sqrt{23} = 1.56$. We get $\frac{Lnz_f^{1/2}}{z_f^{-1}} \simeq 0.068 = 7\%$.

To do the calculation correctly one needs to accurately calculate σv and take the thermal average $\langle\sigma v\rangle$ and put the correct g_*. The above estimate shows that

$$z_f = \frac{T_f}{m_X} \simeq \frac{1}{20} \tag{398}$$

Freeze-out occurs nonrelativistically, for $z < z_f$

$$\frac{d}{dz}(-\frac{1}{\phi(z)}) = m(\frac{45}{4\pi^2 k g_*})^{1/2} g_{*s}\langle\sigma v\rangle \tag{399}$$

$$-\frac{1}{\phi(z_0)}+\frac{1}{\phi(z_f)}=m\int_{z_f}^{z_0}(\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}\langle\sigma v\rangle dz \tag{400}$$

with $z_0=\frac{T_0}{m}$, $T_0=2.73^{\rm deg}$K. Let us set $z_0=0$, we can write

$$\phi(z_0)=\frac{\phi(z_f)}{1+m\phi(z_f)\int_{z_f}^{z_0}(\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}\langle\sigma v\rangle dz} \tag{401}$$

In order to estimate the size of the denominator $\phi(z_f)=\phi^{eq}(z_f)$

$$\phi(z_f)\int_{z_f}^{z_0}(\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}\langle\sigma v\rangle dz=\frac{2}{g_{*s}}(\frac{1}{2\pi z_f})^{3/2}e^{\frac{1}{z_f}}m\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}z_f\langle\sigma v\rangle \tag{402}$$

$$=z_f^{-1/2}e^{\frac{1}{z_f}}(0.0765)\frac{m_x M_{pl}}{\sqrt{g_*}}\frac{\alpha_2^2}{4\pi}\frac{m_X^2}{m_N^4}=60\gg 1 \tag{403}$$

So we neglect "1" in the denominator and we write

$$\phi(z_0)=\frac{\phi(z_f)}{m\phi(z_f)\int_{z_f}^{z_0}(\frac{45}{4\pi^2 kg_*})^{1/2}g_{*s}\langle\sigma v\rangle dz} \tag{404}$$

The number density of X^0 today

$$n_X\simeq(\frac{4\pi^2 kg_*}{45})^{1/2}\frac{t_0^3 g_{*s}(0)}{mg_{*s}(z_f)}\frac{1}{\int_0^{z_f}\langle\sigma v\rangle dz} \tag{405}$$

The relic density is now

$$\rho_{X^0}=m_X n_X\simeq(\frac{4\pi^2 G_N}{45})^{1/2}\frac{(g_{*s}(0)/g_{*s}(z_f))}{J(z_f)}T_0^3 g_*^{1/2} \tag{406}$$

Where $J(z_f)=\int_0^{z_f}\langle\sigma v\rangle dz$, $\Omega_{X^0}=\frac{\rho_{X^0}}{\rho_c}$, $\rho_c=\frac{3H_0^2}{8\pi G_N}=1.878\times 10^{-29}$ gm cm^{-3}h^2. We therefore can write

$$\Omega_{X^0}h^2=(\frac{4\pi^2 G_N}{45})^{1/2}\frac{g_{*s}(0)}{g_{*s}(z_f)}\frac{T_0^3}{J(z_f)}g_*(z_f)^{1/2}\frac{1}{1.878\times 10^{-29}gmcm^{-3}} \tag{407}$$

$N_f=g_*(z_f)=$ number of degrees of freedom at freeze-out.

$$(\frac{T_{X^0}}{T_0})^3=\frac{g_{*s}(0)}{g_{*s}(z_f)}=\text{“reheating factor”} \tag{408}$$

Since when particle becomes non-relativistic and drops out of g_* and its entropy is not lost and it reheats the photon temperature (associated with γ. g, e, μ, τ, ν_e, ν_μ, ν_τ, u, d, c, s, t, b

$$N_f=g_*(z_f)=2+8\times 2+\frac{7}{8}\times[3\times 4+3\times 2+4\times 4\times 4\times 3]=\frac{303}{4} \tag{409}$$

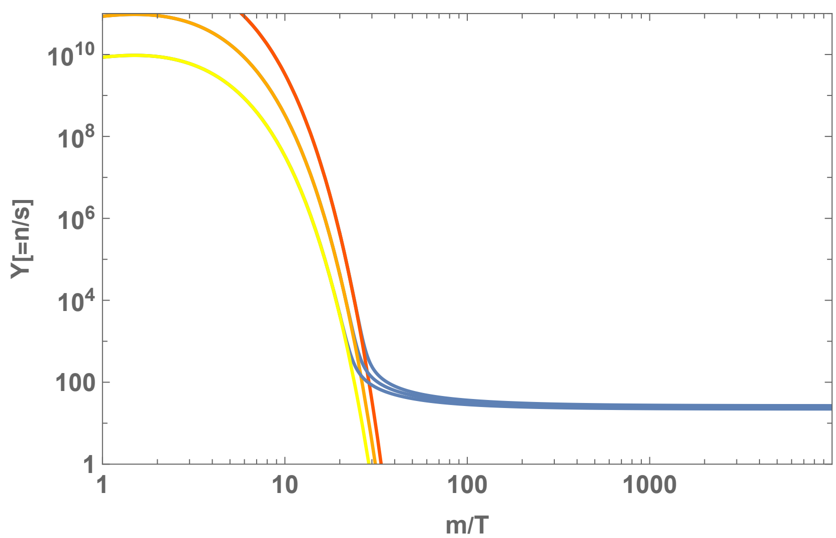


Figure 1: Numerical solution of Boltzmann equation ($Y \equiv \phi$ vs m/T) assuming 3 different cross-section values, $3 * 10^{-(23+n)}$ cm^3/sec for n=1, 2 and 3

We use

$$\frac{g_{*s}(0)}{g_{*s}(z_f)} = (19.4)^{-1},\ (\frac{T_{X^0}}{T^0})N_f^{1/2} = 0.449 \tag{410}$$

Use $\sigma v = \frac{\alpha_2^2}{4\pi}\frac{m_{X^0}^2}{m_N^4}v$, $\frac{v}{c} = \sqrt{z_f}$, $J(z_f) = \int_0^{z_f}\langle\sigma v\rangle dz = \langle\sigma v\rangle z_f = \frac{\alpha_2^2}{4\pi}\frac{m_{X^0}^2}{m_N^4}(z_f)^{3/2}$. Use $m_N = 100$ GeV, $m_{X^0} = 60$GeV, $z_f = \frac{1}{20}$, $\alpha_2 = 0.03$, $J(z_f) = 3.49 \times 10^{-11}$ GeV^{-2} to calculate $\Omega_{X^0}h^2$.

4.3 Relativistic dark matter

$$Y_X(\infty) = Y_X^{eq}(X_f) = \frac{45\zeta(3)}{2\pi^4}\frac{g_{DM}}{g_{*s}(z_f)} \tag{411}$$

We get

$$n_{X^0} = s_0 Y_\infty = 6.3 \times 10^{-39}\frac{g_{DM}}{g_{*s}(z_f)}GeV^3 \tag{412}$$

Use $s_0 = \frac{2\pi^2}{45}g_{*s_0}T_0^3$, $g_{*s_0} = 3.91$

Precise value of X_f is unimportant since Y^{eq} is constant. The species which are relativistic at freeze-out are called hot relics.

$$\Omega = \frac{\rho_{X^0}}{\rho_c},\ \rho_{X^0} = m_{X^0}n_{X^0} \tag{413}$$

If we use neutrinos as hot relics then $g_{DM} = 2 \times \frac{3}{4} = 1.5$ for one neutrino type and we can find $\Omega_\nu h^2$ in terms of m_ν

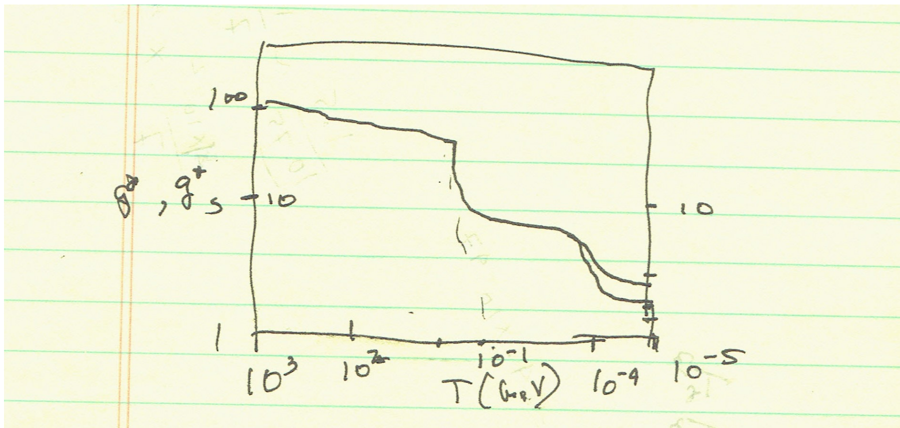


Figure 2: g_*, g_{*s} vs T

4.4 calculation of relativistic degrees of freedom for g_* and g_{*s}

$$g_* = i = boson \sum g_i (\frac{T_i}{T})^4 + \frac{7}{8} i = fermion \sum g_i (\frac{T_i}{T})^4 \tag{414}$$

$$g_{*s} = i = boson \sum g_i (\frac{T_i}{T})^3 + \frac{7}{8} i = fermion \sum g_i (\frac{T_i}{T})^3 \tag{415}$$

Using $T_\nu = (4/11)^{1/3} T_\gamma$ we determine for $T \ll MeV$

$$g_* = 2 + \frac{7}{8} \times 6 \times (\frac{4}{11})^{4/3} = 3.36 \tag{416}$$

Similarly,

$$g_{*s} = 2 + \frac{7}{8} \times 6 \times (\frac{4}{11})^{3/3} = 3.91 \tag{417}$$

Since $t_\nu \neq T_\gamma$ $g_* \neq g_{*s}$.

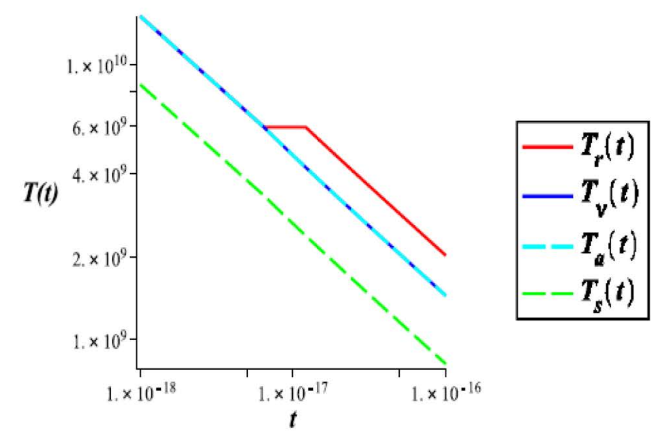


Figure 3: T vs t(in H_0^{-1} unit) for Majorana, active(Dirac) and sterile(Dirac) neutrinos