The coevolution of opinions and networks

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Problems in Opinion Dynamics

Typical problems in Opinion Dynamics:

- Consensus: Will it emerge?
- Time to consensus?
- Will a society remain polarized?
- Extremism: what creates it?
- How to avoid bad consequences from extreme opinions?

Problems with extreme opinions

- Violent behavior
- Lack of trust and break of democratic debate
- Non acceptance of strong scientific consensus, such as vaccinations or climate change
Causes for polarized debates

It is crucial to understand what causes polarization

- Interactions with similar individuals (static networks)
- Confirmation bias (even in full graphs)
- Opinion-based Trust
- Search for similar minds (dynamics networks)
Structure of the Presentation:

1. Opinion Dynamics and the CODA model
2. Theoretical Framework
3. Opinions and networks
Opinion Dynamics Models

- Different models are used for different problems.
  - Discrete Opinions: few options, can represent decisions well.
  - Continuous Opinions: more useful to represent choices of numerical values. Defining strength of opinion is easier.

- What about decisions when opinion strength matters?
Continuous Opinions and Discrete Actions (CODA)
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- Looking only at the dynamics, it is an additive model
- Agent $i$ have a choice $\sigma_i$, obtained from their internal opinion $\nu_i$ by $\sigma_i = \text{sign}(\nu_i)$
- $\nu_i$ is updated by $\nu_i(t + 1) = \nu_i(t) \pm 1$, the sign depends on observing the neighbors.
CODA was first obtained from probabilistic calculations.

There are two possible choices, $A$ and $B$.

Agent $i$ assigns a probability $p_i(A)$ that option $A$ is better.

The observable actions are a function of the internal opinion, that is $\sigma_i = +1$ if $p_i > 0.5$ and $\sigma_i = -1$ otherwise.
How do my neighbors make their choices

- We need an update rule, how $p_i(t+1)$ is obtained from $p_i(t)$ when agent $i$ is influenced by its neighbor $j$ (or a collection of neighbors).

- To get CODA, I assumed the simplest possibility. Assuming $A$ is better, there is a fixed probability $\alpha = P(\sigma_j = +1|A) > 0.5$ that the agent $j$ will support for $A$.

- This might be the simplest possible mental model, too simple it does not need to be called a mental model.

- Even here, there might be assymetrical choices, that is $\beta = P(\sigma_j = -1|B)$ does not need to be equal to $\alpha$.

- As long as $\alpha \neq \beta$, we can have, instead of $\alpha > 0.5$, the more general rule $\alpha > 1 - \beta$. 
Bayes theorem and updating

- We have \( p_i(A) \) and \( \alpha = P(\sigma_j = +1|A) \) (and similar versions for \( B \)).
- The update rule becomes a simple case of using Bayes theorem. In this case, if agent \( i \) observes choice \( A \), we have

\[
p_i(t + 1|\sigma_j = +1) = \frac{p_i(A)P(\sigma_j = +1|A)}{N} = \frac{p_i(t)\alpha}{N},
\]

where \( N \) is a normalizing constant given by

\[
N = p_i(t)\alpha + (1 - p_i(t))(1 - \beta).
\]
- Similarly, for \( q_i(B) = 1 - p_i(B) \), we can write

\[
q_i(t + 1|\sigma_j = +1) = \frac{q_i(A)P(\sigma_j = +1|B)}{N} = \frac{q_i(t)(1 - \beta)}{N}
\]
Simplifying

- Remembering that we don’t need $q$, we can get rid of the normalization constant if we introduce the odds ratio, $o_i(t)$ as $o_i(t) = \frac{p_i(t)}{1-p_i(t)}$, so that dividing Equations 1 and 2, we get

$$o_i(t + 1) = \frac{p_i(t)\alpha}{(1 - p_i(t))(1 - \beta)} = o_i(t)\frac{\alpha}{1 - \beta},$$

(3)

- We can calculate the log-odds $\nu_i = \ln(o_i)$ and applying the logarithm to both sides of Equation 3, we have an additive model

$$\nu_i(t + 1) = \nu_i(t) + \ln\left(\frac{\alpha}{1 - \beta}\right),$$

(4)

where $C = \ln\left(\frac{\alpha}{1 - \beta}\right)$ is a fixed term that does not change during computations.
Simplifying part II

- When $p_i = 0.5$, we have $\nu_i = 0$ so the rule for observation is simply $\sigma_i = \text{sign}(\nu_i)$.
- Let us also assume that $\alpha = \beta$.
- Notice that the dynamics of choices depend only on the sign and not the value of $\nu_i$. If we care only about that dynamics, we can further simplify Equation 4 by dividing it by $C$ and using a normalized $\nu_i^* = \nu_i / C$, so that, for the general case

$$\nu_i^*(t + 1) = \nu_i^*(t) \pm 1,$$

where the sign in the addition depends on whether $A$ or $B$ is supported by the neighbors.
Theoretical Framework

General CODA

- The value of $\alpha$ and, as a consequence, $C$ are irrelevant to the dynamics of $\sigma_i$ and are only needed if we want to translate $\nu_i^*$ back to a probability value.
- If $\alpha \neq \beta$, there is no natural renormalization, as the size steps will be different depending on whether the neighbor chooses $A$ or $B$. 
For example, let us assume that $\alpha = 0.8$ and $\beta = 0.4$.

That means agents expect their neighbors to choose $A$ more often than $B$ even when $B$ is the best choice ($\beta = 0.4$).

In this case we will have $\nu_i(t + 1) \approx \nu_i(t) + 0.288$ when the neighbor chooses $A$ and $\nu_i(t + 1) \approx \nu_i(t) - 0.693$ when the neighbor chooses $B$.

The ratio between size steps is about 2.4. Surprise matters!
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A Theoretical Framework for Update Rules

doi.org/10.1063/1.4759605

- Define an opinion as the subjective probability on the debated issue.
- Bayes Theorem can provide rules for changing the opinion.
Details
What is the issue?

- Assign a variable \( x \) to the issue (continuous or discrete? what range? one issue or a cultural problem, with several dimensions?).
- Each agent makes inferences about \( x \).
- Each agent \( i \) needs to have a subjective opinion about \( x \), represented by a probability distribution \( f_i(x) \).
- The function indicates agent \( i \) belief on how likely each possible value of \( x \) is.
Communication depends on the agent opinion $f_i(x)$: a functional $A[f_i]$. 

Communication does not need to be intentional, it can be an observed behavior of $j$, that I will refer to as $A_j$, for simplification.
The agents must have a model about how likely other agents will pick each possible observable value. That is, they need a relationship between the each possible true value of $x$, $x^*$, and each possible observation $A_j$, given by a likelihood distribution $p(A_j|x^*)$.

That likelihood is a probability distribution stating, assuming $x^*$ were the correct value, how likely it would be that the neighbor $j$ would communicate $A_j$. 

Mental Models: arXiv:2106.00199
Theoretical Framework

Details
Updating

- The probability distribution $p(A_j|x)$ plays the role of a likelihood of the observation $A_j$ and thus defines a Bayesian update rule.
- Agent $i$ already had a prior opinion $f_i(x)$, obtaining its posterior opinion $f_i(x|A_j)$ is a simple task of applying Bayes Theorem.
- That is, if agent $i$ observes choices $A_j$, we will have

$$f_i(x, t + 1) \propto f_i(x, t)p(A_j|x)$$

(6)

- Renormalize or transform to easier variables as needed or possible.
Agent $i$ can include in its mental model for CODA the fact it influences its neighbors.

In this case, we would have $\alpha = P(\sigma_j = +1 | A)$ replaced by

$$a = P(\sigma_j = +1 | A, \sigma_i = +1)$$

$$\neq P(\sigma_j = +1 | A, \sigma_i = -1) = c$$

and a similar pair instead of $\beta$.

This leads to asymmetrical steps. In the limit where its own influence approaches certainty, we recover spin-like models update rules.
Other applications, for heterogeneous agents

- Contrarians: Agents expect their neighbors to be wrong more often than right. doi.org/10.1142/S0219525910002773

- Inflexibles: Following Galam unifying frame, agents are influenced by a random group and inflexibility emerges as consequence. doi.org/10.1103/PhysRevE.87.042807
Bounded Confidence

doi.org/10.1088/1742-5468/2009/02/P02017

- BC-like model can be obtained for a continuous variable in [0, 1]
- Mental model: a Normal distribution around the right value plus a uniform term corresponding to no information.

\[ f(x_j|\theta) = pN(\theta, \sigma_j^2) + (1-p)U(0, 1) \]
From CODA to Bounded Confidence

- Bounded Confidence: continuous opinion over range 0 to 1. Tendency to moderate opinions.
- CODA: Internal probability (0 to 1), observed choice (A or B). Tendency to extremism.
What drives extremism?
doi.org/10.3389/fphy.2016.00007

- One obvious difference: communication - discrete versus continuous
- One subtle difference: mental model - choosing sides versus mixing choices. In CODA, agents look for the ONE best alternative (wishers). They could also look for the right proportion of A and B, instead (mixers).
What drives extremism?

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<th>Certainty wishers</th>
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Wishers with Continuous observations (New model 1)

- Estimate $p$ - probability that $A$ is best choice.
- Agent observes opinion $p_j$ of agent $j$.
- Likelihood: $Be(p_j|\alpha, \beta, A) = \frac{1}{B(\alpha, \beta)} p_j^{\alpha-1} (1 - p_j)^{\beta-1}$. 
Mixers with Discrete Observations (New Model 2)

- Estimate $f$ - optimal proportion for $A$.
- Requires a continuous opinion probability distribution over $0 \leq f \leq 1$.
- Agent observes only if $j$ thinks there should be more $A$ or more $B$.
- Likelihood: Binomial distribution.

**Figure**: Strength of the opinion as a function of time. Upper left: wishers with discrete communication, that is, the regular CODA model; Upper right: mixers with discrete communication (new model 2); Lower line: wishers with continuous communication (new model 1); at left, just its evolution to 1/10 of the time the previous cases evolved; At lower right, the evolution of opinions is shown for the whole range with a logarithmic scale of the opinions.
CODA with $M$ choices

- We need a likelihood matrix $L_{mn} = p(A_j = m | x^* = n)$ connecting each possibility to the chance neighbor $j$ will pick it.

- It is possible to obtain an additive model if we choose pairwise logodds as variables:

$$\nu_{q(q+1)} = \ln \frac{f(q)}{f(q + 1)},$$  

(7)

where $q$ assumes values in the range $1, \ldots, M - 1$ for $M$ possible choices.

- However, while $\nu_{q(q+1)}$ are convenient for efficient simulation, for interpreting the data it is more convenient to look at $\nu_{q_o} = \ln \left( \frac{f(o)}{1 - f(o)} \right)$. 
Symmetrical choices versus choices over a one-dimensional axis

**Figure**: Symmetrical case, $M = 10$

**Figure**: Choices over an one-dimensional axis, $M = 15$
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CODA and Clustering

Figure: Average proportion of the population following the majority as a function of the rewiring probability.

Figure 21: Percentual de apoiadores finais da opinião majoritária em redes interpoladas no Modelo CODA, média de 20 realizações.
Trust

doi.org/10.1016/j.physleta.2013.07.007

- Agents can consider how much they trust each neighbor.
- Assume agents think there are trustworthy (T) and untrustworthy (U) agents.
- In this case, we can have (here, I assume the symmetries from $\alpha = \beta$) that

\[
\alpha = P(\sigma_j = +1|A, T) > 0.5
\]

\[
\mu = P(\sigma_j = +1|A, U) < 0.5
\]

- Agents can update their opinions $p_i$ together with trust matrix $\tau_{ij}$ that corresponds to $i$ estimate of the chance $j$ is of the $T$ type.
Updating $p$ and $\tau$ means one can no longer obtain a simplified model for log-odds. Instead, we must write

$$p_i(t + 1) = \frac{p_i [\tau_{ij} \alpha + (1 - \tau_{ij}) \mu]}{p_i [\tau_{ij} \alpha + (1 - \tau_{ij}) \mu] + (1 - p_i) [\tau_{ij} (1 - \alpha) + (1 - \tau_{ij})(1 - \mu)]},$$

and

$$\tau_{ij}(t + 1) = \frac{\tau_{ij} [p_i \alpha + (1 - p_i)(1 - \alpha)]}{\tau_{ij} [p_i \alpha + (1 - p_i)(1 - \alpha)] + (1 - \tau_{ij}) [p_i \mu + (1 - p_i)(1 - \mu)]}.$$
Trust and consensus

**Figure:** Standard deviation of final opinions for several population sizes

**Figure:** Standard deviation of final opinions for distinct initial conditions
Network of Trust

Figure: Evolution of the network of trust, trust larger than 0.65

Coevolution of opinions and networks
Opinions and Networks

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- Opinions evolve by CODA algorithm
- Network changes by using an energy function:
  - Only spatial components: \( H = \beta \sum_E d_{ij} \)
  - Spatial and opinions: \( H = \beta \sum_E (d_{ij} - J\sigma_i\sigma_j) \)
- Implemented using Metropolis: Randomly choose an edge to be eliminated and a new one to be created. Accept change with probability \( P = \exp(-\beta[d_{34} - d_{12} - J\Delta(\sigma_i\sigma_j)]) \)
Agents located over a lattice

- Depending on $\beta$, ordered or disordered states.
- Order can come mostly from position or opinion, depending on $J$:

Figure: $\beta = 1.00$. Left panel: $J = 1$. Right panel: $J = 5$. 
Network characteristics

**Clustering Coefficient**
- Beta vs. Percentage of clustering for different values of k and J:
  - k=1, J=1
  - k=5, J=1
  - k=5, J=5

**Mean Distance**
- Beta vs. Distance for different values of k and J:
  - k=1, J=1
  - k=5, J=1
  - k=5, J=5

**Interface Proportion**
- Beta vs. Percentage for different values of k and J:
  - k=1, J=1
  - k=1, J=5
Thank you!