# The coevolution of opinions and networks

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Workshop on Dynamical Processes on Complex Networks, ICTP-SAIFR, IFT-UNESP, 2024

# Problems in Opinion Dynamics

Typical problems in Opinion Dynamics:

- Consensus: Will it emerge?
- Time to consensus?
- Will a society remain polarized?
- Extremism: what creates it?
- How to avoid bad consequences from extreme opinions?

Problems with extreme opinions

- Violent behavior
- Lack of trust and break of democratic debate
- Non acceptance of strong scientific consensus, such as vaccinations or climate change

# Causes for polarized debates

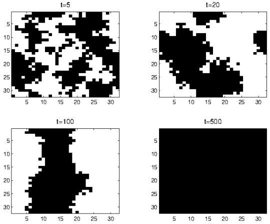
It is crucial to understand what causes polarization

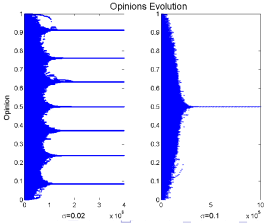
- Interactions with similar individuals (static networks)
- Confirmation bias (even in full graphs)
- Opinion-based Trust
- Search for similar minds (dynamics networks)

# Structure of the Presentation:

# Opinion Dynamics Models

- Different models are used for different problems.
  - Discrete Opinions: few options, can represent decisions well.
  - Continuous Opinions: more useful to represent choices of numerical values. Defining strength of opinion is easier.
- What about decisions when opinion strength matters?

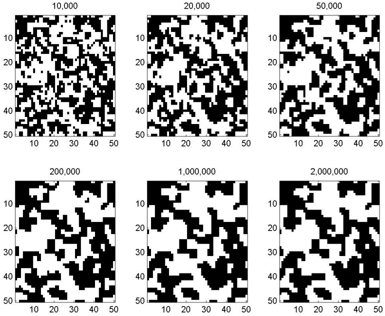


# Continuous Opinions and Discrete Actions (CODA)

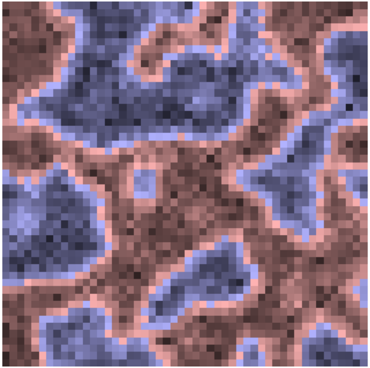
doi.org/10.1142/S0129183108012339

- Looking only at the dynamics, it is an additive model
- Agent $i$ have a choice $\sigma_i$, obtained from their internal opinion $\nu_i$ by $\sigma_i = sign(\nu_i)$
- $\nu_i$ is updated by $\nu_i(t+1) = \nu_i(t) \pm 1$, the sign depends on observing the neighbors.



# Demonstrating CODA from metal models

- CODA was first obtained from probabilistic calculations.
- There are two possible choices, $A$ and $B$.
- Agent $i$ assigns a probability $p_i(A)$ that option $A$ is better.
- The observable actions are a function of the internal opinion, that is $\sigma_i = +1$ if $p_i > 0.5$ and $\sigma_i = -1$ otherwise.



# How do my neighbors make their choices

- We need an update rule, how $p_i(t+1)$ is obtained from $p_i(t)$ when agent $i$ is influenced by its neighbor $j$ (or a collection of neighbors).
- To get CODA, I assumed the simplest possibility. Assuming $A$ is better, there is a fixed probability $\alpha = P(\sigma_j = +1|A) > 0.5$ that the agent $j$ will support for $A$.
- This might be the simplest possible mental model, too simple it does not need to be called a mental model.
- Even here, there might be assymetrical choices, that is $\beta = P(\sigma_j = -1|B)$ does not need to be equal to $\alpha$.
- As long as $\alpha \neq \beta$, we can have, instead of $\alpha > 0.5$, the more general rule $\alpha > 1 - \beta$.

# Bayes theorem and updating

- We have $p_i(A)$ and $\alpha = P(\sigma_j = +1|A)$ (and similar versions for $B$).
- The update rule becomes a simple case of using Bayes theorem. In this case, if agent $i$ observes choice $A$, we have

$$p_i(t+1|\sigma_j = +1) = \frac{p_i(A)P(\sigma_j = +1|A)}{N} = \frac{p_i(t)\alpha}{N}, \qquad (1)$$

  where $N$ is a normalizing constant given by $N = p_i(t)\alpha + (1 - p_i(t))(1 - \beta)$.
- Similarly, for $q_i(B) = 1 - p_i(B)$, we can write

$$q_i(t+1|\sigma_j = +1) = \frac{q_i(A)P(\sigma_j = +1|B)}{N} = \frac{q_i(t)(1-\beta)}{N} \quad (2)$$

## Simplifying

- Remebering that we don't need $q$, we can get rid of the normalization constant if we introduce the odds ratio, $o_i(t)$ as $o_i(t) = \frac{p_i(t)}{1-p_i(t)}$, so that dividing Equations 1 and 2, we get

$$o_i(t+1) = \frac{p_i(t)\alpha}{(1-p_i(t))(1-\beta)} = o_i(t)\frac{\alpha}{1-\beta}, \qquad (3)$$

- We can calculate the log-odds $\nu_i = \ln(o_i)$ and applying the logarithm to both sides of Equation 3, we have an additive model

$$\nu_i(t+1) = \nu_i(t) + \ln(\frac{\alpha}{1-\beta}), \qquad (4)$$

where $C = \ln(\frac{\alpha}{1-\beta})$ is a fixed term that does not change during computations.

# Simplifying part II

- When $p_i = 0.5$, we have $\nu_i = 0$ so the rule for observation is simply $\sigma_i = sign(\nu_i)$.
- Let us also assume that $\alpha = \beta$.
- Notice that the dynamics of choices depend only on the sign and not the value of $\nu_i$. If we care only about that dynamics, we can further simplify Equation 4 by deviding it by $C$ and using a normalized $\nu_i^* = \nu_i / C$, so that, for the general case
$$\nu_i^*(t+1) = \nu_i^*(t) \pm 1, \tag{5}$$
where the sign in the addition depends on whether $A$ or $B$ is supported by the neighbors.

# General CODA

- The value of $\alpha$ and, as a consequence, $C$ are irrelevant to the dynamics of $\sigma_i$ and are only needed if we want to translate $\nu_i^*$ back to a probability value.
- If $\alpha \neq \beta$, there is no natural renormalization, as the size steps will be different depending on whether the neighbor chooses $A$ or $B$.

# General CODA II

- For example, let us assume that $\alpha = 0.8$ and $\beta = 0.4$.
- That means agents expect their neighbors to choose $A$ more often than $B$ even when $B$ is the best choice ($\beta = 0.4$).
- In this case we will have $\nu_i(t+1) \approx \nu_i(t) + 0.288$ when the neighbor chooses $A$ and $\nu_i(t+1) \approx \nu_i(t) - 0.693$ when the neighbor chooses $B$.
- The ratio between size steps is about 2.4. Suprise matters!

# Structure of the Presentation:

1. Opinion Dynamics and the CODA model

2. Theoretical Framework

3. Opinions and networks

# A Theoretical Framework for Update Rules

doi.org/10.1063/1.4759605

- Define an opinion as the subjective probability on the debated issue.
- Bayes Theorem can provide rules for changing the opinion.

# Details
## What is the issue?

- Assign a variable $x$ to the issue (continuous or discrete? what range? one issue or a cultural problem, with several dimensions?).
- Each agent makes inferences about $x$.
- Each agent $i$ needs to have a subjective opinion about $x$, represented by a probability distribution $f_i(x)$ .
- The function indicates agent $i$ belief on how likely each possible value of $x$ is.

# Details
## Communicating

- Communication depends on the agent opinion $f_i(x)$: a functional $A[f_i]$.
- Communication does not need to be intentional, it can be an observed behavior of $j$, that I will refer to as $A_j$, for simplification.

# Details
## Mental Models: arXiv:2106.00199

- The agents must have a model about how likely other agents will pick each possible observable value. That is, they need a relationship between the each possible true value of $x$, $x^*$, and each possible observation $A_j$, given by a likelihood distribution $p(A_j|x^*)$.
- That likelihood is a probability distribution stating, assuming $x^*$ were the correct value, how likely it would be that the neighbor $j$ would comunicate $A_j$.

# Details
## Updating

- The probability distribution $p(A_j|x)$ plays the role of a likelihood of the observation $A_j$ and thus defines a Bayesian update rule.
- Agent $i$ already had a prior opinion $f_i(x)$, obtaining its posterior opinion $f_i(x|A_j)$ is a simple task of applying Bayes Theorem.
- That is, if agent $i$ observes choices $A_j$, we will have

$$f_i(x, t+1) \propto f_i(x, t)p(A_j|x) \tag{6}$$

- Renormalize or transform to easier variables as needed or possible.

# Discrete models as limit cases

doi.org/10.1016/j.physa.2013.10.009

- Agent *i* can include in its mental model for CODA the fact it influences its neighbors.
- In this case, we would have $\alpha = P(\sigma_j = +1|A)$ replaced by

$$a = P(\sigma_j = +1|A, \sigma_i = +1)$$

$$\neq P(\sigma_j = +1|A, \sigma_i = -1) = c$$

  and a similar pair instead of $\beta$.
- This leads to assymetrical steps. In the limit where its own influence approaches certainty, we recover spin-like models update rules.

# Other applications, for heterogeneous agents

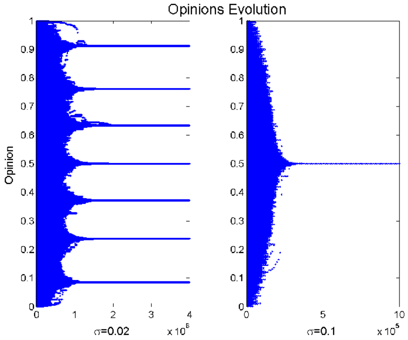
- Contrarians: Agents expect their neighbors to be wrong more often than right. doi.org/10.1142/S0219525910002773
- Inflexibles: Following Galam unifying frame, agents are influenced by a random group and inflexibility emerges as consequence. doi.org/10.1103/PhysRevE.87.042807

# Bounded Confidence

doi.org/10.1088/1742-5468/2009/02/P02017

- BC-like model can be obtained for a continuous variable in $[0, 1]$
- Mental model: a Normal distribution around the right value plus a uniform term corresponding to no information.

$$f(x_j|\theta) = pN(\theta, \sigma_j^2) + (1-p)U(0,1)$$



# From CODA to Bounded Confidence

- Bounded Confidence: continuous opinion over range 0 to 1. Tendency to moderate opinions.
- CODA: Internal probability (0 to 1), observed choice (A or B). Tendency to extremism.

# What drives extremism?

doi.org/10.3389/fphy.2016.00007

- One obvious difference: communication - discrete versus continuous
- One subtle difference: mental model - choosing sides versus mixing choices. In CODA, agents look for the ONE best alternative (wishers). They could also look for the right proportion of $A$ and $B$, instead (mixers).

# What drives extremism?

| | **Certainty wishers** | **Mixers** |
|---|---|---|
| **Discrete observation** | CODA | New model 2 |
| **Continuous observation** | New model 1 | Bounded Confidence (BC) |

# Wishers with Continuous observations (New model 1)

- Estimate $p$ - probability that $A$ is best choice.
- Agent observes opinion $p_j$ of agent $j$.
- Likelihood: $Be(p_j|\alpha,\beta,A)=\frac{1}{B(\alpha,\beta)}p_j^{\alpha-1}(1-p_j)^{\beta-1}$.

# Mixers with Discrete Observations (New Model 2)

- Estimate *f* - optimal proportion for *A*.
- Requires a continuous opinion probability distribution over $0 \leq f \leq 1$.
- Agent observes only if *j* thinks there should be more *A* or more *B*.
- Likelihood: Binomial distribution.

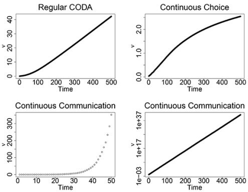



Figure: Strength of the opinion as a function of time. Upper left: wishers with discrete communication, that is, the regular CODA model; Upper right: mixers with with discrete communication (new model 2); Lower line: wishers with continuous communication (new model 1); at left, just its evolution to 1/10 of the time the previous cases evolved; At lower right, the evolution of opinions is shown for the whole range with a logarithmic scale of the opinions.

# CODA with $M$ choices

- We need a likelihood matrix $\mathbb{L}_{mn} = p(A_j = m | x^* = n)$ connecting each possibility to the chance neighbor $j$ will pick it.
- It is possible to obtain an additive model if we choose pairwise logodds as variables

$$\nu_{q(q+1)} = \ln \frac{f(q)}{f(q+1)}, \tag{7}$$

  where $q$ assumes values in the range $1, \cdots, M-1$ for $M$ possible choices.
- However, while $\nu_{q(q+1)}$ are convenient for efficient simulation, for interpreting the data it is more convenient to look at $\nu_{q_o} = \ln(\frac{f(o)}{1-f(o)})$.

# Symmetrical choices versus choices over a one-dimensional axis

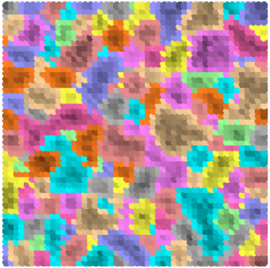


Figure: Symmetrical case, $M = 10$

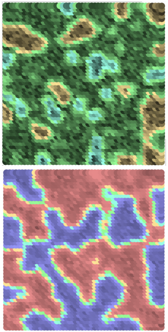


Figure: Choices over an one-dimensinal axis, $M = 15$

# Structure of the Presentation:

# CODA and Clustering

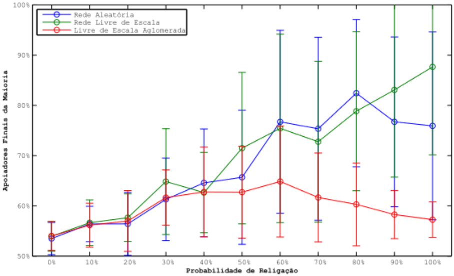



**Figura 21:** *Percentual de apoiadores finais da opinião majoritária em redes interpoladas no Modelo CODA, média de 20 realizações.*

Figure: Average proportiong of the population following the majority as a function of the rewiring probability.

# Trust

doi.org/10.1016/j.physleta.2013.07.007

- Agents can consider how much they trust each neigbor.
- Assume agents think there are trustworthy (T) and untrustworthy (U) agents.
- In this case, we can have (here, I assume the symmetries from $\alpha = \beta$) that

$$\alpha = P(\sigma_j = +1 | A, T) > 0.5$$

$$\mu = P(\sigma_j = +1 | A, U) < 0.5$$

- Agents can update their opinions $p_i$ together with trust matrix $\tau_{ij}$ that corresponds to $i$ estimate of the chance $j$ is of the $T$ type.

# Trust II

- Updating $p$ and $\tau$ means one can no longer obtain a simplified model for log-odds. Instead, we must write

$$p_i(t+1) = \frac{p_i\left[\tau_{ij}\alpha + (1-\tau_{ij})\mu\right]}{p_i\left[\tau_{ij}\alpha + (1-\tau_{ij})\mu\right] + (1-p_i)\left[\tau_{ij}(1-\alpha) + (1-\tau_{ij})(1-\mu)\right]},$$

and

$$\tau_{ij}(t+1) = \frac{\tau_{ij}\left[p_i\alpha + (1-p_i)(1-\alpha)\right]}{\tau_{ij}\left[p_i\alpha + (1-p_i)(1-\alpha)\right] + (1-\tau_{ij})\left[p_i\mu + (1-p_i)(1-\mu)\right]}.$$

# Trust and consensus

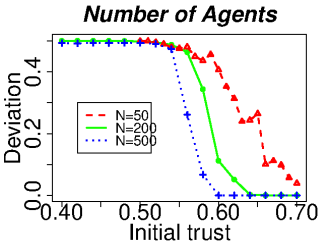



Figure: Standard deviation of final opinions for several population sizes

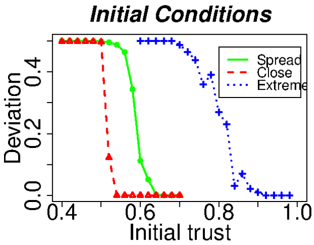



Figure: Standard deviation of final opinions for distinct initial conditions

# Network of Trust

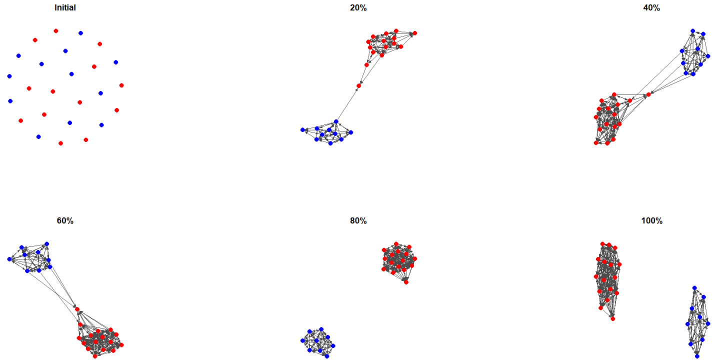



Figure: Evolution of the network of trust, trust larger than 0.65

# Opinions and Networks

doi.org/10.1142/S0129183119500773

- Opinions evolve by CODA algorithm
- Network changes by using an energy function:
  - Only spatial components: $H = \beta \sum_E d_{ij}$
  - Spatial and opinions: $H = \beta \sum_E (d_{ij} - J\sigma_i\sigma_j)$
- Implemented using Metropolis: Randomly choose an edge to be eliminated and a new one to be created. Accept change with probability $P = \exp -\beta[d_{34} - d_{12} - J\Delta(\sigma_i\sigma_j)]$

# Agents located over a lattice

- Depending on $\beta$, ordered or disordered states.
- Order can come mostly from position or opinion, depending on $J$:

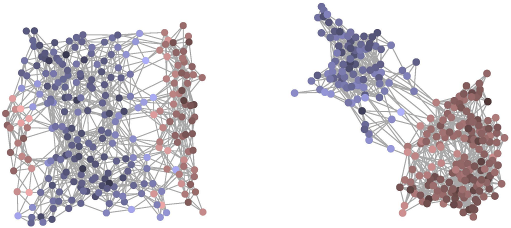
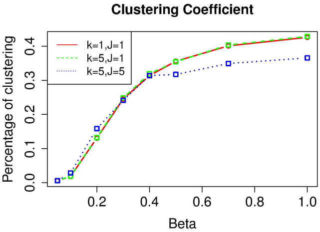
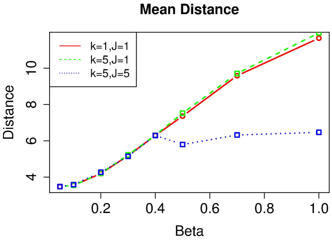
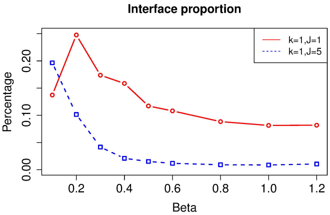


Figure: $\beta = 1.00$. Left panel: $J = 1$. Right panel: $J = 5$.

# Network characteristics



# Thank you!