

I.9 Solving the Boltzmann Equation

Solve numerically the Boltzmann equation for the dark matter (DM) number density n

$$\frac{dn}{dt} + 3Hn = -\langle\sigma v\rangle (n^2 - n_{\text{eq}}^2), \quad (\text{I.15})$$

corresponding to the case of 2-to-2 DM annihilations into standard model (SM) states. Assume *i*) a Universe dominated by SM radiation, *ii*) conservation of the SM entropy, and *iii*) an *s*-wave (i.e. constant) thermally-averaged annihilation cross section $\langle\sigma v\rangle$.

Hints:

1. *i*) implies that the Hubble expansion rate as a function of the SM temperature T is

$$H(T) = \sqrt{\frac{\rho_R(T)}{3M_P^2}}, \quad (\text{I.16})$$

with the SM energy density is given by

$$\rho_R(T) = \frac{\pi^2}{30} g_\star(T) T^4, \quad (\text{I.17})$$

$M_P \simeq 2.4 \times 10^{18}$ GeV is the reduced Planck mass and $g_\star(T)$ is the number of relativistic degrees of freedom that contributes to the energy density of the SM. As a first approximation, take, e.g. $g_\star = 106.75$. It is a good number for temperatures higher than the top mass.

2. For the DM number density in equilibrium (without chemical potential), use the Maxwell-Boltzmann approximation

$$n_{\text{eq}}(T) = \frac{g}{2\pi^2} m^2 T K_2\left(\frac{m}{T}\right), \quad (\text{I.18})$$

where m is the mass of the DM, g is the number of internal degrees of freedom of the DM particle, and K_i the modified Bessel function of i^{th} order.

3. Given *i*) and *ii*), Eq. (I.15) can be conveniently rewritten as

$$\frac{dY}{dx} = -\frac{\langle\sigma v\rangle s}{xH} [Y^2 - (Y^{\text{eq}})^2], \quad (\text{I.19})$$

where $x \equiv m/T$, $Y(T) \equiv n(T)/s(T)$, and $Y_{\text{eq}}(T) \equiv n_{\text{eq}}(T)/s(T)$, with

$$s(T) = \frac{2\pi^2}{45} g_{\star s}(T) T^3 \quad (\text{I.20})$$

is the SM entropy density and $g_{\star s}(T)$ the number of relativistic degrees of freedom contributing to the DM entropy. Take $g_{\star s}(T) = g_\star(T)$, which is a good approximation for temperatures higher than a few MeVs.

4. For wisely chosen values of m and $\langle\sigma v\rangle$,¹⁶ solve Eq. (I.19). Recall that observations of the DM relic density require that at low temperatures (i.e. $x \gg 100$)

$$Y \times m \simeq 4.3 \times 10^{-10} \text{ GeV}. \quad (\text{I.21})$$

For the initial conditions (yes, a first-order differential equation requires an initial condition!), take $x_{\text{ini}} = 15$ and $Y(x_{\text{ini}}) = Y_{\text{eq}}(x_{\text{ini}})$. For a given DM mass, what is the required value of $\langle\sigma v\rangle$? Calculate $\langle\sigma v\rangle$ in GeV^{-2} , picobarns, and cm^3/s .

5. How does the relic abundance (i.e. Y) evolves with an increase / decrease of $\langle\sigma v\rangle$?
 6. Play wildly with the initial conditions x_{ini} and $Y(x_{\text{ini}})$.
 7. Repeat point 4 for other values of m , and plot $\langle\sigma v\rangle$ versus m .
 8. Considering the simple parametrization

$$\langle\sigma v\rangle = \frac{\alpha^2}{m^2}, \quad (\text{I.22})$$

with α a dimensionless coupling, plot α versus m . What is the maximal WIMP DM mass compatible with perturbativity?

9. One can go one step forward and take into account the temperature dependence of g_\star and $g_{\star s}$. Use the tabulated values of g_\star and $g_{\star s}$ in the file `Data/hgEff/DHS.thg` of [MicrOMEGAs](#). In their notation $\text{heff} = g_{\star s}$ while $\text{geff} = g_\star$. Plot again $\langle\sigma v\rangle$ versus m taking now into account the variation of g_\star and $g_{\star s}$.
 10. Decrease the (constant) value of $\langle\sigma v\rangle$ without fear. Use the initial conditions $x_{\text{ini}} = 10^{-1}$ and $Y(x_{\text{ini}}) = 0$. Discover a second solution satisfying Eq. (I.21) and corresponding to ultraviolet freeze-in (UV FIMP) paradigm. Why is this freeze-in mechanism called *ultraviolet*?
 11. How does the relic abundance (i.e. Y) evolves with an increase / decrease of $\langle\sigma v\rangle$?
 12. Play wildly with the initial conditions x_{ini} and $Y(x_{\text{ini}})$.
 13. Plot $\langle\sigma v\rangle$ versus m for UV FIMPs.
 14. Repeat the WIMP analysis, now for DM semi-annihilation (i.e. reactions $\text{DM} + \text{DM} \rightarrow \text{DM} + \text{SM}$)

$$\frac{dY}{dx} = -\frac{\langle\sigma v\rangle s}{x H} [Y^2 - Y Y^{\text{eq}}]. \quad (\text{I.23})$$

15. Repeat the WIMP analysis, now for 3-to-2 SIMP DM self-annihilation (i.e. reactions $\text{DM} + \text{DM} + \text{DM} \rightarrow \text{DM} + \text{DM}$)

$$\frac{dY}{dx} = -\frac{\langle\sigma v^2\rangle_{3\text{-to-2}} s^2}{x H} [Y^3 - Y^2 Y^{\text{eq}}]. \quad (\text{I.24})$$

¹⁶Wisely means masses in the GeV to TeV ballpark and cross sections at the picobarn scale!

16. Considering the simple parametrization

$$\langle \sigma v^2 \rangle_{3\text{-to-}2} = \frac{\alpha^3}{m^5}, \quad (\text{I.25})$$

with α a dimensionless coupling, plot α versus m . What is the maximal SIMP DM mass compatible with perturbativity?

17. Forget assumptions *i*) and *ii*). Assume instead the existence of a long-lived non-relativistic particle ϕ of mass m_ϕ , that only decays into SM particles with a total decay width Γ_ϕ corresponding to a lifetime $\tau_\phi = 1/\Gamma_\phi \sim 1$ s. The evolution of the number density n_ϕ of ϕ and the SM radiation energy density ρ_R can be tracked by the system of coupled Boltzmann equations

$$\frac{dn_\phi}{dt} + 3Hn = -\Gamma_\phi n_\phi, \quad (\text{I.26})$$

$$\frac{d\rho_R}{dt} + 4H\rho_R = +\Gamma_\phi n_\phi m_\phi, \quad (\text{I.27})$$

and the Friedmann equation

$$H = \sqrt{\frac{\rho_R + m_\phi n_\phi}{3M_P^2}}, \quad (\text{I.28})$$

where $\rho_\phi = m_\phi n_\phi$ is the energy density of ϕ . Rewrite Eqs. (I.26), (I.27) and (I.28) in terms of the comoving quantities $\Phi \equiv n_\phi \times a^3$, $R \equiv \rho_R \times a^4$, and the cosmic scale factor a . Solve numerically the new equations, looking for periods where ϕ dominates the total energy density of the Universe. Extract the evolution of the SM temperature using Eq. (I.17). Use the initial condition $a_{\text{ini}} = 1$ with $\rho_R(a_{\text{ini}}) = 0$ and $\rho_\phi(a_{\text{ini}}) \neq 0$ for a reheating-like scenario or $\rho_R(a_{\text{ini}}) \gg \rho_\phi(a_{\text{ini}}) > 0$ and for an early matter domination era. Recall that $H(T) < 2 \times 10^{-5} M_P$.

18. Rewrite Eq. (I.15) using $N \equiv n \times a^3$ and a . Solve it in the previously discussed background. Understand WIMPs during low-temperature reheating, and WIMPs during an early matter dominated era.