Anomalous transport phenomena on the lattice

Gergely Endrődi

ELTE Eötvös University Budapest

Chirality, vorticity and magnetic fields in quantum matter

ICTP SAIFR São Paulo, July 8, 2025

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Gergely Endrődi

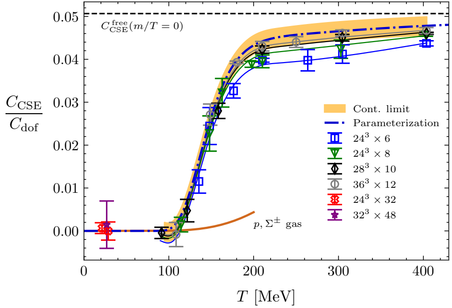
in collaboration with:

Bastian Brandt, Eduardo Garnacho, Gergely Markó, Dean Valois

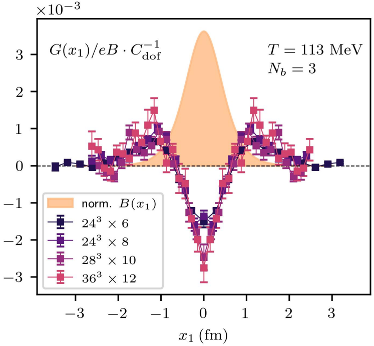
Appetizer

first fully non-perturbative determination
of in-equilibrium anomalous transport coefficients

chiral separation effect



(local) chiral magnetic effect



Brandt, Endrődi, Garnacho, Markó, JHEP 02 142

Brandt, Endrődi, Garnacho, Markó, Valois, 2409.17616

Brandt, Endrődi, Garnacho, Markó, JHEP 09 092

Outline

- introduction, anomalous transport phenomena
- in-equilibrium chiral magnetic effect
- in-equilibrium chiral separation effect
- local in-equilibrium chiral magnetic effect
- summary

Introduction, anomalous transport

Magnetic fields in lattice QCD

- many groups Bali et al. D'Elia, Bonati et al. Braguta, Chernodub et al. Cea et al. Alexandru et al. Ding et al. Buividovich et al. Yamamoto Detmold et al. and more

- phase diagram

  Endrődi, JHEP 07 (2015)

- equation of state

  Bali et al. JHEP 07 (2020)

- fluctuations

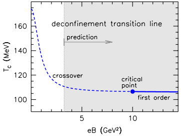
  Ding et al. PRL 132 (2024)

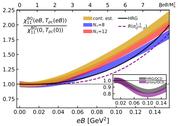
- transport phenomena

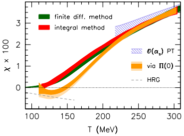
  Astrakhantsev et al. PRD 102 (2020)

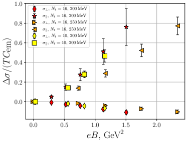
- anomalous transport phenomena

recent review Endrődi, PPNP 141 (2025)









Anomalous transport

- usual transport:
  vector current due to electric field

$$\langle \boldsymbol{J} \rangle = \sigma \cdot \boldsymbol{E}$$

- chiral magnetic effect (CME)
  Kharzeev, McLerran, Warringa, NPA 803 (2008)   Fukushima, Kharzeev, Warringa, PRD 78 (2008)
  vector current due to chirality and magnetic field

$$\langle \boldsymbol{J} \rangle = \sigma_{\mathrm{CME}} \cdot \boldsymbol{B}$$

- chiral separation effect (CSE)
  Son, Zhitnitsky, PRD 70 (2004)   Metlitski, Zhitnitsky, PRD 72 (2005)
  axial current due to baryon number and magnetic field

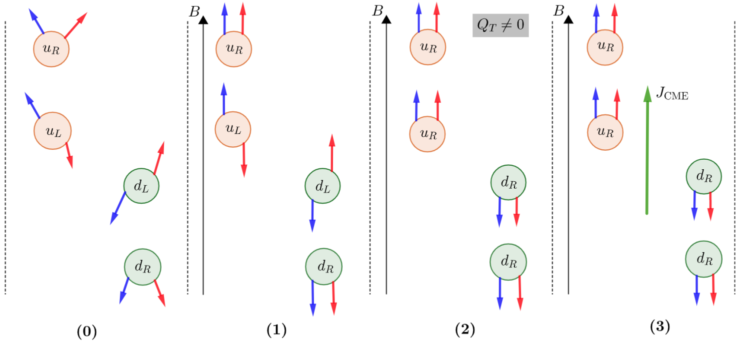
$$\langle \boldsymbol{J}_5 \rangle = \sigma_{\mathrm{CSE}} \cdot \boldsymbol{B}$$

Phenomenological and theoretical relevance

- experimental observation of CME in condensed matter systems
  Li, Kharzeev, Zhan et al., Nature Phys. 12 (2016)
- experimental searches for CME and related observables in heavy-ion collisions
  STAR collaboration, PRC 105 (2022)
- serves as indirect way to probe topological fluctuations and CP-odd domains in heavy-ion collisions
- recent reviews: Kharzeev, Liao, Voloshin, Wang, PPNP 88 (2016)
  Kharzeev, Liao, Tribedy, IJMPE 33 (2024) Feng, Voloshin, Wang, 2502.09742

General (handwaving) argument

- spin, momentum   chiral magnetic effect



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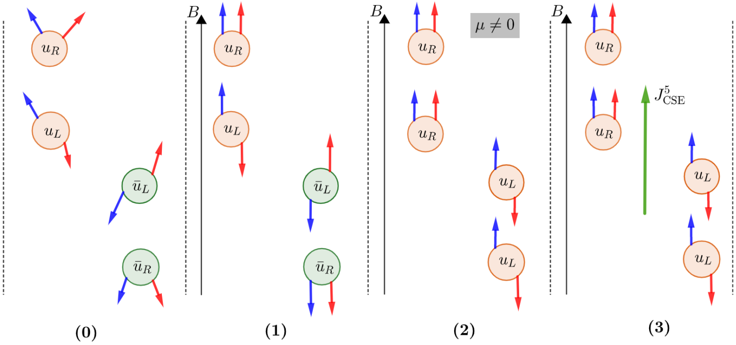

General (handwaving) argument – issues

- quantum theory requires ultraviolet regularization
- massless vs. massive fermions
- strong interactions between fermions
- in-equilibrium vs. out-of-equilibrium nature

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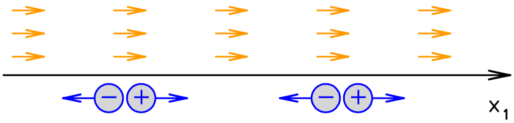
In-equilibrium vs. out-of-equilibrium

- example: charge transport due to electric field $\boldsymbol{E} \parallel \boldsymbol{e}_1$



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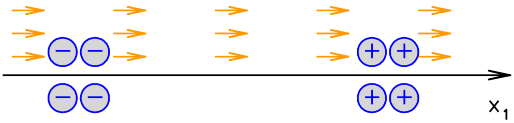

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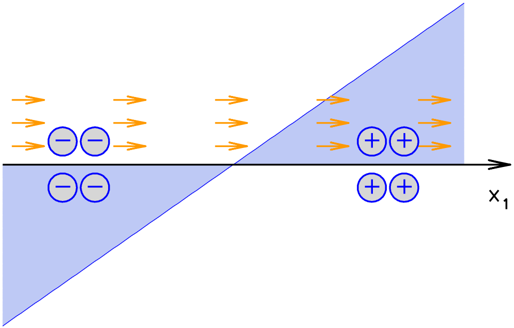

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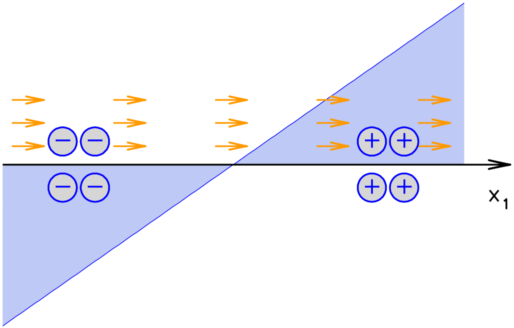

- out-of equilibrium linear response:
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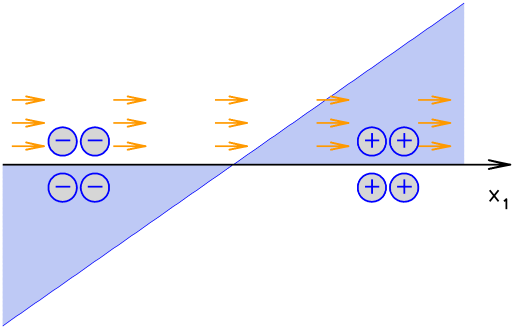

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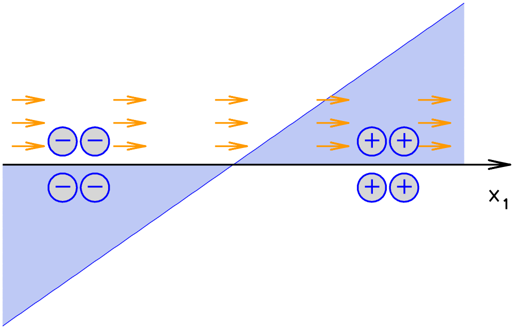

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- leading to an equilibrium distribution (electric polarization/susceptibility)
- same story can be told for CME

No global currents in equilibrium

- **Bloch's theorem**: Bohm Phys. Rev. 75 (1949) N. Yamamoto, PRD 92 (2015)
  persistent electric currents do not exist in ground state of quantum systems

- applies to conserved currents
- applies to global (spatially averaged) currents
- applies in the thermodynamic limit ($V \to \infty$)

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- in-equilibrium CSE is possible
- in-equilibrium local CME currents are possible

Chiral magnetic effect in equilibrium

CME and inconsistencies

- parameterize chiral imbalance $J_{05} = \int \bar{\psi}\gamma_0\gamma_5\psi$ by a chiral chemical potential μ_5
  Fukushima, Kharzeev, Warringa, PRD 78 (2008)
- CME for weak chiral imbalance ($\boldsymbol{B} = B\boldsymbol{e}_3$)

$$\langle J_3 \rangle = \sigma_{\mathrm{CME}}\, B = C_{\mathrm{CME}}\, \mu_5 B + \mathcal{O}(\mu_5^3)$$

- from Bloch's theorem it follows that in equilibrium

$$C_{\mathrm{CME}} = 0 \checkmark$$

- several results in the literature give incorrectly

$$C_{\mathrm{CME}} = \frac{1}{2\pi^2} \lightning$$

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careful regularization is required

One typical issue with regularization

- vacuum thermodynamic potential from exact energies at $\mu_5 > 0$ and $B > 0$

unregularized $$\Omega(B, \mu_5) = \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_3}{2\pi} E_{p_3} + \text{ higher Landau levels}$$

with energy $E_{p_3} = \sqrt{(p_3 + \mu_5)^2 + m^2}$

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unregularized $$\langle J_3 \rangle = \left. \frac{\partial \Omega}{\partial A_3} \right|_{A_3=0} = \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_3}{2\pi} \frac{\mathrm{d}}{\mathrm{d}p_3} E_{p_3}$$
$$= \frac{B}{4\pi^2} \left[\lim_{p_3 \to \infty} E_{p_3} - \lim_{p_3 \to -\infty} E_{p_3} \right]$$

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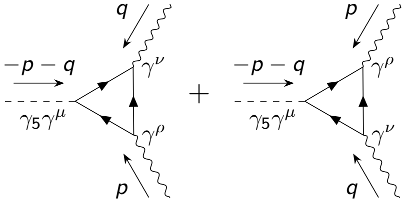
- this needs regularization and with cutoff

cutoff regularization $$\langle J_3\rangle = \frac{B}{4\pi^2}\left[(\Lambda+\mu_5)-(\Lambda-\mu_5)\right] = \underbrace{\frac{1}{2\pi^2}}_{C_{\mathrm{CME}}}\,B\mu_5 \;\lightning$$

Perturbation theory

- triangle diagram

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =$$



- gives in-equilibrium CME coefficient

$$C_{\text{CME}} = \lim_{p,q\to 0} \frac{1}{q_1} \Gamma^{023}_{AVV}(p+q,p,q)$$

- also gives the axial anomaly Peskin-Schroeder 19.2

$$\langle \partial_\mu J_5^\mu \rangle \;\sim\; (p+q)_\mu \Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) A_\nu A_\rho$$

Regularization sensitivity – anomaly

- naive regularization

$$(p+q)_\mu \Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q)A_\nu A_\rho = mP_5(p,q) \lightning$$

- Pauli-Villars regularization
  (regulator particles $s = 1, 2, 3$ with $c_s = \pm 1$ and $m_s \to \infty$)

$$(p+q)_\mu \Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q)A_\nu A_\rho = mP_5(p,q) + \sum_{s=1}^{3} c_s m_s P^{\nu\rho}_{5,s}(p,q)A_\nu A_\rho$$

$$\xrightarrow{m_s\to\infty} mP_5(p,q) + \frac{\epsilon^{\alpha\beta\nu\rho}F_{\alpha\nu}F_{\beta\rho}}{16\pi^2} \checkmark$$

Regulator sensitivity – CME

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$$C_{\text{CME}} = \lim_{p,q,p+q\to 0} \frac{1}{q_1} \Gamma^{023}_{AVV}(p+q,p,q) = \frac{1}{2\pi^2} \ \lightning$$

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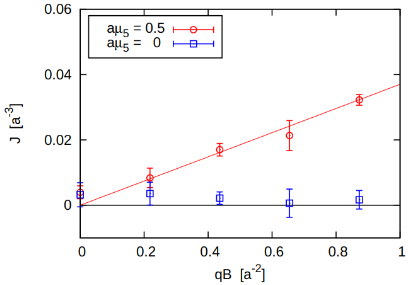
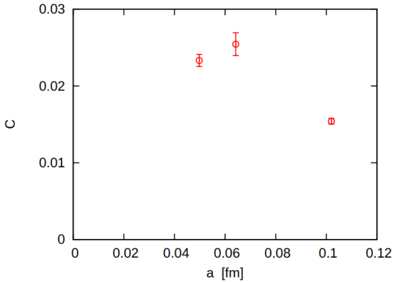
- in equilibrium, C_{CME} vanishes due to anomalous contribution

CME in equilibrium – lattice simulations

Regularization sensitivity on the lattice: quenched Wilson

- seminal lattice determination of $\langle J_3 \rangle$ at $B \neq 0$, $\mu_5 \neq 0$ A. Yamamoto, PRL 107 (2011)





- coefficient $C_{\mathrm{CME}} \approx 0.025 \approx 1/(4\pi^2)$ ⚡

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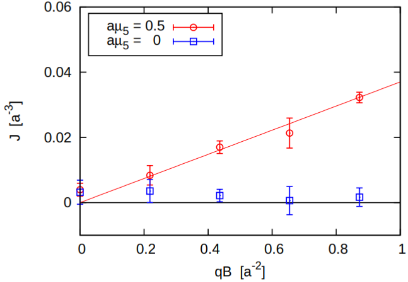

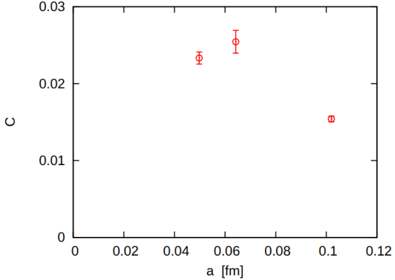

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$$J_\nu^{\mathrm{non-cons}} = \bar{\psi}(n)\gamma_\nu\psi(n)$$

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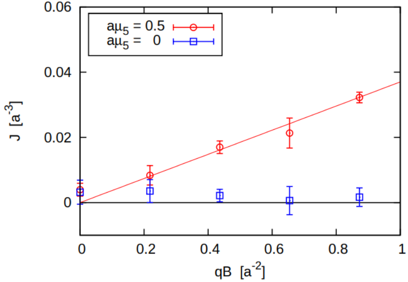

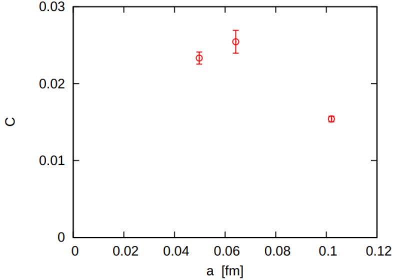

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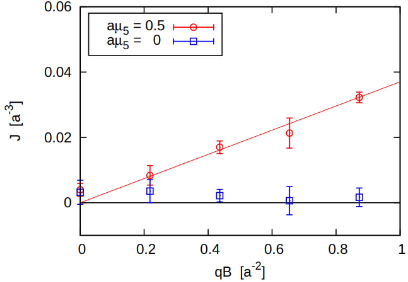

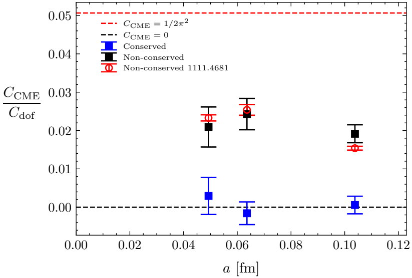

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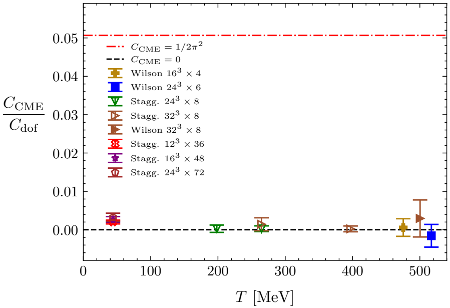
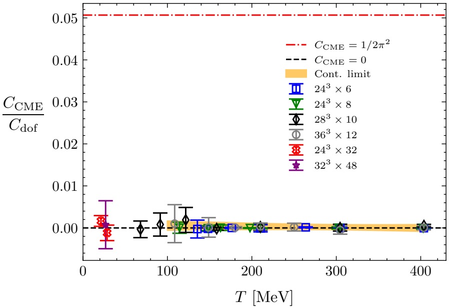
- conserved current: $C_{\rm CME} = 0$ ✓ Brandt, Endrődi, Garnacho, Markó, JHEP 09 (2024)

CME in equilibrium – final result

- quenched, heavier-than-physical Wilson quarks
- full QCD simulations with dynamical staggered quarks employing the conserved electric current
- global CME current vanishes in equilibrium

  🔗 Brandt, Endrődi, Garnacho, Markó, JHEP 09 (2024)





Chiral separation effect in equilibrium

Chiral separation effect

- axial current due to magnetic field and baryon density
  🔗 Son, Zhitnitsky, PRD 70 (2004) 🔗 Metlitski, Zhitnitsky, PRD 72 (2005)
- parameterize baryon density $J_0 = \int \bar{\psi}\gamma_0\psi$ by chemical potential μ
- CSE for small density ($\boldsymbol{B} = B\boldsymbol{e}_3$)

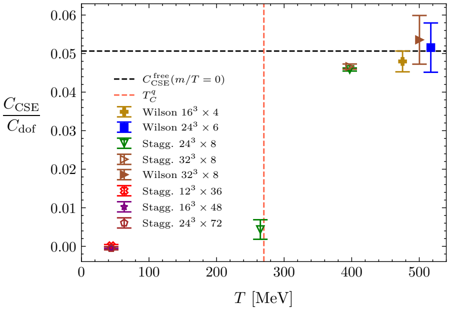
$$\langle J_{35} \rangle = \sigma_{\mathrm{CSE}}\, B = C_{\mathrm{CSE}}\, \mu B + \mathcal{O}(\mu^3)$$

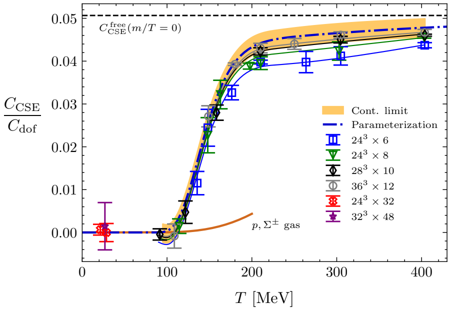
- Bloch's theorem allows in-equilibrium CSE $(\partial_\nu J_{\nu 5} \neq 0)$
- regularization less intricate, but conserved vector current on lattice is important
- previous lattice efforts 🔗 Puhr, Buividovich, PRL 118 (2017)
  🔗 Buividovich, Smith, von Smekal, PRD 104 (2021)

CSE in equilibrium – final result

- quenched, heavier-than-physical Wilson quarks
- full QCD simulations with staggered quarks
  at physical quark masses, extrapolated to the continuum limit
  employing the conserved electric current

  Brandt, Endrődi, Garnacho, Markó, JHEP 02 (2024)



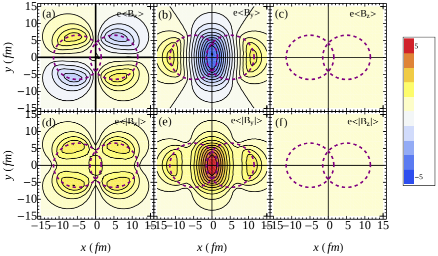


- comparison to baryon gas model at low T

Local chiral magnetic effect

Inhomogeneous magnetic fields

- up to now: homogeneous magnetic background
- off-central heavy-ion collisions: inhomogeneous fields Deng et al., PRC 85 (2012)



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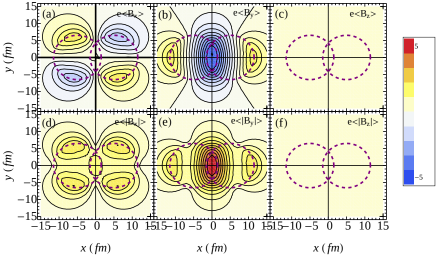

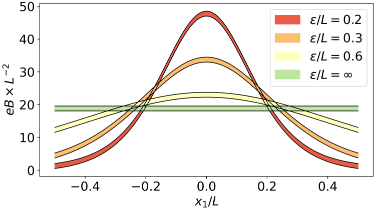

- consider profile $B(x) = B\cosh^{-2}(x/\epsilon)$ Dunne, hep-th/0406216
  with $\epsilon \sim 0.6$ fm

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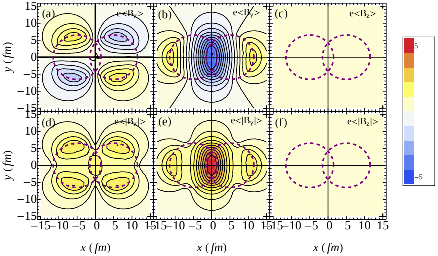

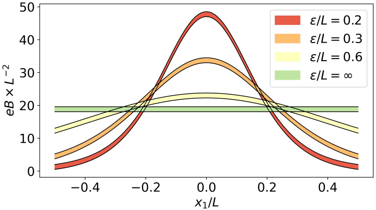

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  with $\epsilon \sim 0.6$ fm
- impact on thermodynamic observables in QCD and phase diagram
  Brandt, Endrődi, Markó, Sandbote, Valois, JHEP 11 (2023)
  Brandt, Endrődi, Markó, Valois, JHEP 07 (2024)

Local currents

- response for weak μ_5 for homogeneous B

$$\langle J_3 \rangle = C_{\mathrm{CME}}\, \mu_5 B$$

Local currents

- response for weak μ_5 for homogeneous and inhomogeneous B

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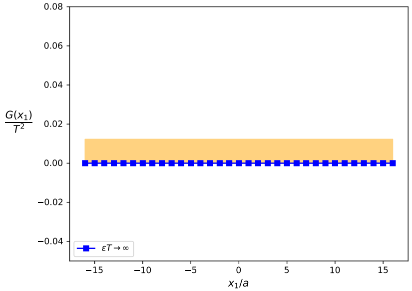

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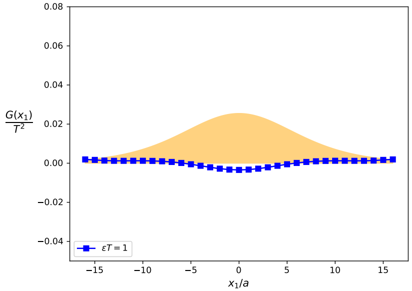

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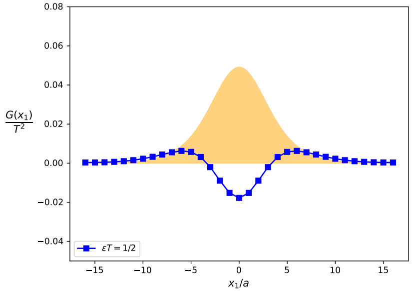

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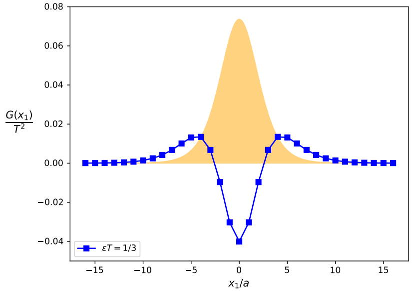

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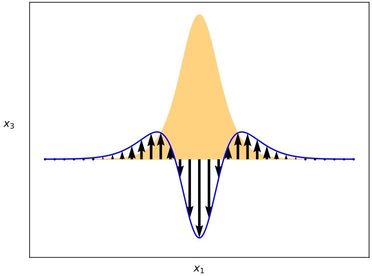

Local currents in QCD

- full QCD simulations with staggered quarks
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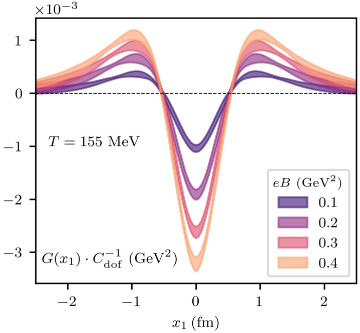
- non-trivial localized CME signal 🔗 Brandt, Endrődi, Garnacho, Markó, Valois, 2409.17616



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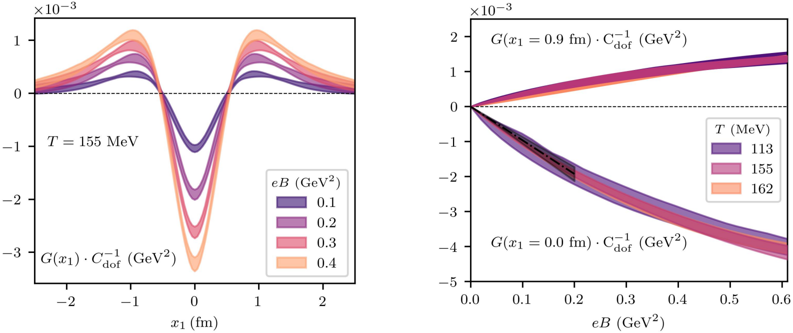

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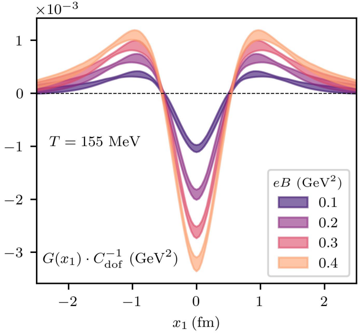
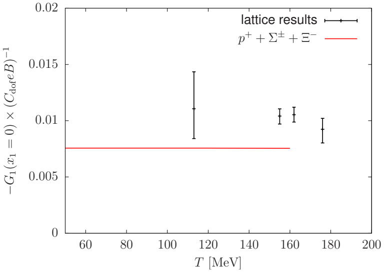
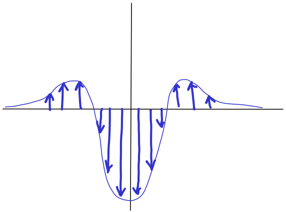




Impact for experiments

- local currents may serve as initial conditions in hydrodynamic expansion in transverse plane:



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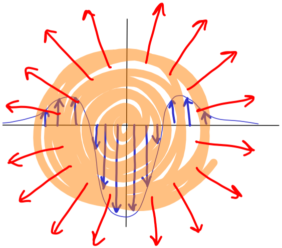

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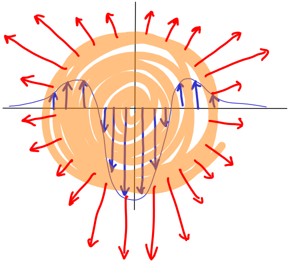

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Outlook

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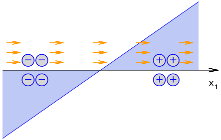
- all of the above was in equilibrium
  what about the out-of-equilibrium response?
  - E. Garnacho Tue 11:00
- all of the above CME studies involved μ_5
  what about $\boldsymbol{E} \cdot \boldsymbol{B}$ as CP-odd background?
  - J. Hernández Tue 11:30

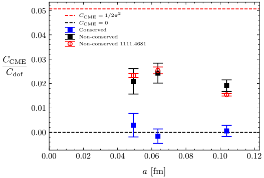
Summary

Summary

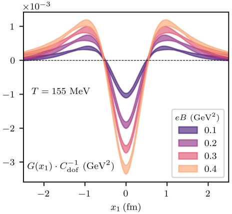
- CME subtleties:
  in- / out-of-equilibrium



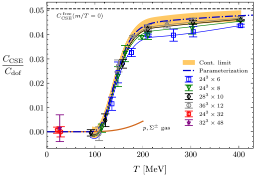
- careful regularization crucial
  in-equilibrium global CME vanishes



- in-equilibrium local CME in full QCD



- in-equilibrium CSE in full QCD



Backup

More on lattice currents

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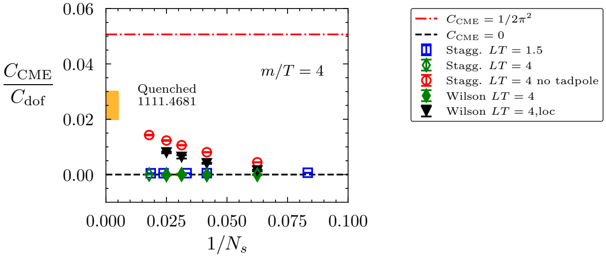
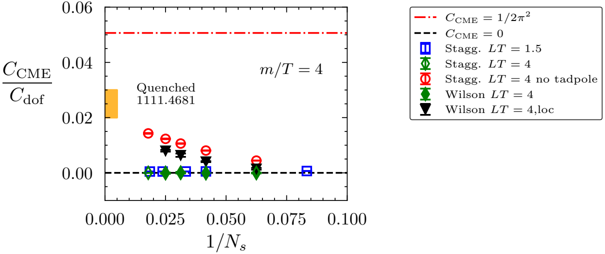
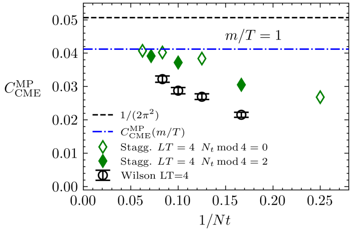


Benchmarks in the free case

- equilibrium CME



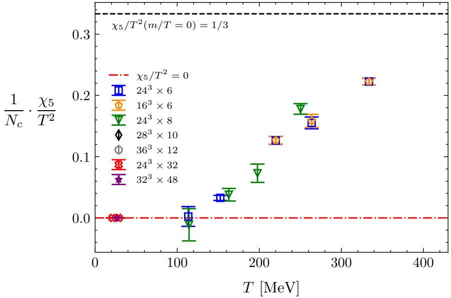
- CME midpoint estimate



Chiral density

- chiral density J_{05} is parameterized by chiral chemical potential μ_5

$$J_{05}(\mu_5) = \chi_5\, \mu_5 + \mathcal{O}(\mu_5^3), \qquad \chi_5 = \frac{T}{V}\, \frac{\partial^2 \log \mathcal{Z}}{\partial \mu_5^2}\bigg|_{\mu_5=0}$$



Inhomogeneous chiral imbalance

- inhomogeneous $B(x_1)$ *and* inhomogeneous $\mu_5(x_1)$

$$\langle J_3(x_1)\rangle = \int \mathrm{d}x_1' \underbrace{\mathrm{d}x_1'' \, \chi_{\rm CME}(x_1 - x_1', x_1 - x_1'') \, B(x_1'')}_{H(x_1, x_1')} \, \mu_5(x_1')$$