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Pseudo-gauge invariant local thermodynamic equilibrium density operator

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OUTLINE

- Introduction
- Quantum states in relativistic heavy ion collisions
- A pseudo-gauge invariant local thermodynamic equilibrium state
- Conclusions

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Introduction

In QFT in flat space-time, from translational and Lorentz invariance one obtains two conserved Noether currents:

$$\widehat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^a)} \partial^\nu \Psi^a - g^{\mu\nu} \mathcal{L}$$

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Psi^a)} D^A (J^{\mu\nu})^a_b \Psi^b$$

$$\partial_\mu \widehat{T}^{\mu\nu} = 0$$

$$\partial_\lambda \widehat{\mathcal{J}}^{\lambda,\mu\nu} = \partial_\lambda \left(\widehat{\mathcal{S}}^{\lambda,\mu\nu} + x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right) = \partial_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu} + \widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu} = 0$$

However, the Lagrangian density can be changed and so, are those tensors objectively defined? (well known problem already for the EM stress-energy tensor)

Spin and pseudo-gauge invariance

F. Halbwachs, *Theorie relativiste des fluids a spin*, Gauthier-Villars (1960)

F. W. Hehl, Rept. Math. Phys. 9 (1976) 55 (see also F. B., L. Tinti Phys. Rev. D 84 (2011) 025013)

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha\left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu}\right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$

Φ = superpotential

$$\partial_\mu \widehat{T}'^{\mu\nu} = 0 \qquad \int_\Sigma \mathrm{d}\Sigma_\mu\, \widehat{T}^{\mu\nu} = \int_\Sigma \mathrm{d}\Sigma_\mu\, \widehat{T}'^{\mu\nu}$$

$$\partial_\lambda \widehat{\mathcal{J}}'^{\lambda,\mu\nu} = 0 \qquad \int_\Sigma \mathrm{d}\Sigma_\mu\, \widehat{\mathcal{J}}^{\mu\lambda\nu} = \int_\Sigma \mathrm{d}\Sigma_\mu\, \widehat{\mathcal{J}}'^{\mu\lambda\nu}$$

EXAMPLE: Belinfante symmetrization

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha\left(\widehat{\mathcal{S}}^{\alpha,\mu\nu} - \widehat{\mathcal{S}}^{\mu,\alpha\nu} - \widehat{\mathcal{S}}^{\nu,\alpha\mu}\right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

Free Dirac field:

$$\widehat{T}^{\mu\nu} = \frac{1}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi$$

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \frac{1}{2}\overline{\Psi}\{\gamma^\lambda, \Sigma^{\mu\nu}\}\Psi = \frac{i}{8}\overline{\Psi}\{\gamma^\lambda[\gamma^\mu, \gamma^\nu]\}\Psi$$

Canonical pseudo-gauge

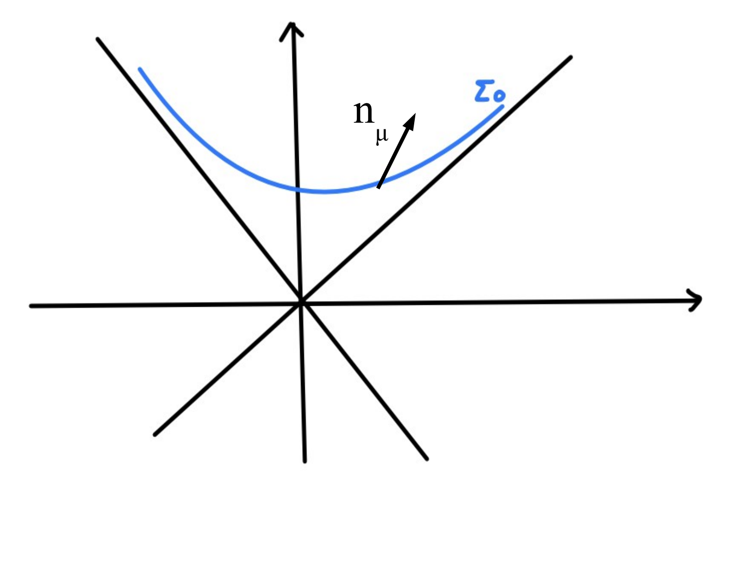
$$\widehat{T}'^{\mu\nu} = \frac{i}{4}\left[\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi + \overline{\Psi}\gamma^\nu \overleftrightarrow{\partial}^\mu \Psi\right]$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

Belinfante pseudo-gauge

Morale: cannot uniquely separate orbital from spin angular momentum

What has this to do with heavy ion collisions?



In relativistic heavy ion collisions, we usually assume that the sytems achieves Local Thermodynamic Equilibrium (LTE) over some hypersurface Σ_0

LTE: maximize entropy $S = -\mathrm{Tr}(\widehat{\rho}\log\widehat{\rho})$

with constrained *densities of conserved currents*

$$n_\mu \mathrm{Tr}(\widehat{\rho}\widehat{T}^{\mu\nu}) = n_\mu T^{\mu\nu} \qquad n_\mu \mathrm{Tr}(\widehat{\rho}\widehat{j}^{\mu}) = n_\mu j^{\mu} \qquad n_\mu \mathrm{Tr}(\widehat{\rho}\widehat{\mathcal{J}}^{\mu\lambda\nu}) = n_\mu \mathcal{J}^{\mu\lambda\nu}$$

The solution is:

$$\widehat{\rho}_{\mathrm{LE}} = \frac{1}{Z}\exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{\mathcal{S}}^{\mu,\lambda\nu} - \zeta\widehat{j}^{\mu}\right)\right].$$

$\Omega_{\lambda\nu} \equiv$ reduced spin potential

Pseudo-gauge dependent LTE

The density operator which represents the initial state of the plasma is NON INVARIANT under a pseudo-gauge transformation
(F.B., W. Florkowski, E. Speranza, Phys.Lett. B 789 (2019) 419)

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{\mathcal{S}}^{\mu,\lambda\nu} - \zeta \widehat{j}^\mu \right)\right].$$

For instance, for a canonical to Belinfante transformation

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}_B\beta_\nu - \frac{1}{2}(\Omega_{\lambda\nu} - \varpi_{\lambda\nu})\widehat{\mathcal{S}}^{\mu,\lambda\nu} + \frac{1}{2}\xi_{\lambda\nu}\left(\widehat{\mathcal{S}}^{\lambda,\mu\nu} + \widehat{\mathcal{S}}^{\nu,\mu\lambda}\right) - \zeta \widehat{j}^\mu \right)\right],$$

It is invariant only if $\varpi = \Omega$ and $\xi_{\mu\nu} = \frac{1}{2}(\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu) = 0$

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}_B\beta_\nu - \zeta \widehat{j}^\mu \right)\right],$$

GLOBAL THERMODYNAMIC EQUILIBRIUM

Consequence

Every measured quantity (mean, fluctuations) in quantum mechanics can be written:

$$\mathrm{Tr}(\widehat{\rho}\widehat{X}) = \langle \widehat{X} \rangle$$

Operators ^X corresponding to measurable observables in relatistic heavy ion collisions (momentum spectra, moments ,spin polarization etc.) are PSEUDO-GAUGE INDEPENDENT.

However, the initial local equilibrium state is PSEUDO-GAUGE DEPENDENT and, as a result,

$$\mathrm{Tr}(\widehat{\rho}\widehat{X}) = \langle \widehat{X} \rangle$$

PG dependent

PG dependent

PG independent

Spin polarization vector of fermions: leading order expressions at LTE at freeze-out

M. Buzzegoli, Phys. Rev. C 105 (2022) 4, 044907

1) Belinfante PG

$$S^\mu_{\rm B}(k) \simeq S^\mu_\varpi(k) + S^\mu_\xi(k);$$

$$S^\mu_\varpi(k) = -\frac{1}{8m}\epsilon^{\mu\rho\sigma\tau}k_\tau \frac{\int_\Sigma {\rm d}\Sigma\cdot k\, n_{\rm F}\,(1-n_{\rm F})\,\varpi_{\rho\sigma}}{\int_\Sigma {\rm d}\Sigma\cdot k\, n_{\rm F}},$$

$$S^\mu_\xi(k) = -\frac{1}{4m}\epsilon^{\mu\lambda\sigma\tau}\frac{k_\tau k^\rho}{\varepsilon_k}\frac{\int_\Sigma {\rm d}\Sigma\cdot k\, n_{\rm F}\,(1-n_{\rm F})\,\hat{t}_\lambda\xi_{\rho\sigma}}{\int_\Sigma {\rm d}\Sigma\cdot k\, n_{\rm F}},$$

2) Canonical PG

$$S^\mu_{\rm C}(k) \simeq S^\mu_\varpi(k) + S^\mu_\xi(k) + \Delta^{\rm C}_\Theta S^\mu(k).$$

$$\Delta^{\rm C}_\Theta S^\mu(k) = \frac{\epsilon^{\lambda\rho\sigma\tau}\hat{t}_\lambda(k^\mu k_\tau - \eta^\mu_\tau m^2)}{8m\varepsilon_k} \times \frac{\int_\Sigma {\rm d}\Sigma(x)\cdot k\, n_{\rm F}\,(1-n_{\rm F})\,(\varpi_{\rho\sigma}-\Omega_{\rho\sigma})}{\int_\Sigma {\rm d}\Sigma\cdot k\, n_{\rm F}}.$$

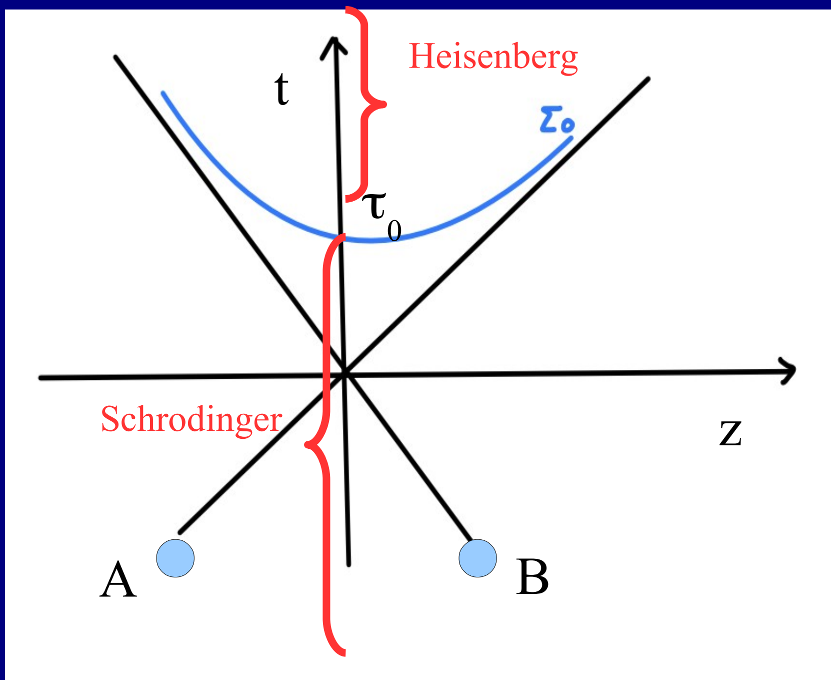
3) GLW-HW PG (spin tensor conserved)

$$S^\mu_{\rm GLW,HW}(k) = -\frac{1}{8m}\epsilon^{\mu\rho\sigma\tau}k_\tau \frac{\int_\Sigma {\rm d}\Sigma\cdot k\, n_{\rm F}\,(1-n_{\rm F})\,\Omega_{\rho\sigma}}{\int_\Sigma {\rm d}\Sigma\cdot k\, n_{\rm F}}.$$

The problem

LTE density operator is our assumption of the initial QCD plasma state. Strictly speaking, the actual quantum state (in the Heisenberg representation) are the two colliding nuclei, which is a pseudo-gauge invariant quantum state

$$|P_A\rangle|P_B\rangle$$



Evolving the initial actual quantum state in the Schroedinger representation

$$\lim_{t\to-\infty} \exp[-i\widehat{H}(\tau_0 - t)]|P_A\rangle|P_B\rangle$$

still yields a pseudo-gauge invariant state!

Basic (tacit) assumption in heavy ion collisions:

$$\lim_{t\to-\infty} \exp[-i\widehat{H}(\tau_0 - t)]|P_A\rangle|P_B\rangle\langle P_B|\langle P_A|\exp[i\widehat{H}(\tau_0 - t)] \Rightarrow \widehat{\rho}_{\rm LE} = \frac{1}{Z}\exp\left[-\int_{\Sigma_0} {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{\mathcal{S}}^{\mu,\lambda\nu} - \zeta\widehat{j}^\mu\right)\right].$$

Pseudo-gauge invariant state | Non pseudo-gauge invariant state

The proposed solution

We should replace the non PG-invariant LTE with a PG-invariant LTE to make our approximation fulfill the requirement that the actual state is PG-invariant!

Construct an LTE-like density operator requiring

1) Minimal deviation from the known form: a linear combination of stress-energy and spin tensor
2) PG invariance
3) Reproducing the correct global equilibrium form

1) $\Rightarrow$ $\widehat{\rho}_{\rm LE} = \frac{1}{Z}\exp[-\widehat{\Upsilon}]$

$$\widehat{\Upsilon} = \int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} X_\nu + Y_{\lambda\nu}\widehat{\mathcal{S}}^{\mu,\lambda\nu} + Z_{\lambda\nu}\widehat{\mathcal{S}}^{\lambda,\mu\nu}\right)$$

2) $\Rightarrow$ under a PG

$$\delta\widehat{\Upsilon} = -\int_\Sigma {\rm d}\Sigma_\mu \left[\widehat{\Phi}^{\mu,\lambda\nu}\left(Y_{\lambda\nu} - \frac{1}{4}\partial_\lambda X_\nu + \frac{1}{4}\partial_\nu X_\lambda\right) + \widehat{\Phi}^{\lambda,\mu\nu}\left(Z_{\lambda\nu} + \frac{1}{2}\partial_\lambda X_\nu + \frac{1}{2}\partial_\nu X_\lambda\right)\right] . = 0$$

3)

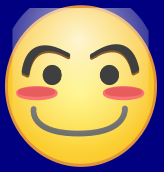
$$\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \frac{1}{2}\varpi_{\lambda\nu}\widehat{\mathcal{S}}^{\mu,\lambda\nu} - \xi_{\lambda\nu}\widehat{\mathcal{S}}^{\lambda,\mu\nu} \right)\right]$$

$$\varpi_{\mu\nu} = \frac{1}{2}(\partial_\nu\beta_\mu + \partial_\mu\beta_\nu) \qquad \xi_{\mu\nu} = \frac{1}{2}(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu)$$

This is invariant under PG transformations!

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha\left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu} \right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$



No matter the stress-energy tensor/spin tensor chosen, the state is the same

Consequences

- With this operator, not only at global but also at local thermodynamic equilibrium must the reduced spin potential be equal to thermal vorticity

$$\Omega = \varpi$$

- One can choose the Belinfante PG and reduce the LTE to the simplest form:

$$\boxed{\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \widehat{T}^{\mu\nu}_B \beta_\nu \right]}$$

$\Rightarrow$ all “correct” spin polarization expressions are those calculated thus far in the Belinfante PG

What about spin hydrodynamics?

It looks like spin hydro is immaterial because the identity $\Omega = \varpi$ deprives the equation

$$\partial_\mu \mathcal{S}^{\mu\lambda\nu}[\beta,\zeta] = T^{\nu\lambda}[\beta,\zeta] - T^{\lambda\nu}[\beta,\zeta]$$

of any dynamical content, i.e. it must be an identity. The equations

$$\partial_\mu T^{\mu\nu}[\beta,\zeta] = 0 \qquad \partial_\mu j^\mu[\beta,\zeta] = 0$$

suffice to solve the dynamical problem.

Yet, are the fields β and ζ pseudo-gauge invariant if the stress-energy tensor is changed in relativistic hydrodynamics?

Summary

- We have derived a pseudo-gauge invariant local thermodynamic equilibrium density operator

- It entails that at local equilibrium: reduced spin potential=thermal vorticity

- Invariance of the expressions of mean values under a pseudo-gauge transformation: for the spin polarization, the expressions obtained in the Belinfante PG are the correct ones

- Would spin hydrodynamics still be relevant?

Pseudo-gauge transformations of currents

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{\mathcal{S}}^{\mu,\lambda\nu} - \zeta \widehat{j}^\mu \right) \right].$$

$$\widehat{j}'^\mu = \widehat{j}^\mu + \partial_\lambda \widehat{M}^{\lambda\mu}$$

The LE quantum state depends on the PG unless ζ =const

Example for the Dirac field

$$\bar{\Psi}\gamma^\mu\Psi \to \bar{\Psi}\gamma^\mu\Psi + C\partial_\lambda \left(\bar{\Psi}[\gamma^\lambda, \gamma^\mu]\Psi \right)$$

QED is a gauge theory which is not invariant under a PG transformation of the currents. For instance, the Hamiltonian, integrating the gauge-invariant stress-energy tensor, is not PG invariant

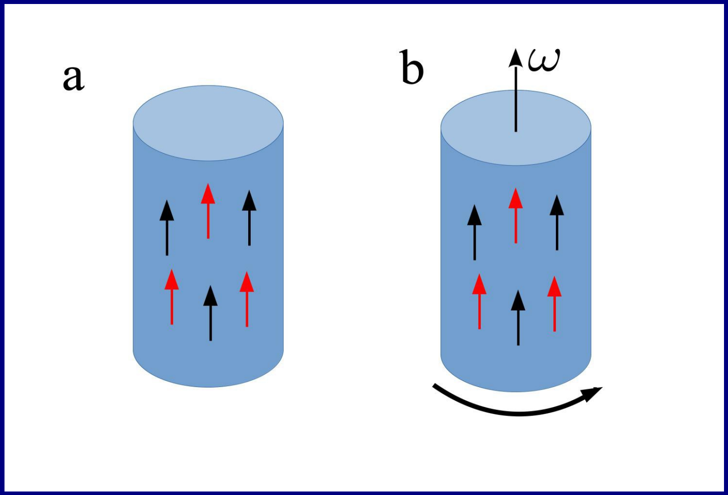
$$\widehat{T}^{\mu\nu} = \frac{i}{4}\bar{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi - \frac{q}{2}\bar{\Psi}\gamma^\mu\Psi\widehat{A}^\nu + (\mu \leftrightarrow \nu) \equiv \widehat{T}_F^{\mu\nu} - \frac{1}{2}\left(\widehat{j}^\mu \widehat{A}^\nu + \widehat{j}^\nu \widehat{A}^\mu \right)$$

In fact, there is a measurable field which is not invariant under a PG transformation of the current:

$$\partial_\lambda \widehat{F}^{\lambda\nu} = \widehat{j}^\nu$$

and singles out a UNIQUE current operator: $\bar{\Psi}\gamma^\mu\Psi$

What does the previous LTE physically represent?



a) Non-rotating globally neutral meta-stable state with both particles and anti-particles polarized and zero velocity

b) Global equilibrium state

F. B., W. Florkowski and E. Speranza, Phys. Lett. B 789 (2019), 419-425

M. Hongo, X. G. Huang, M. Kaminski, M. Stephanov and H. U. Yee, JHEP 11 (2021), 150

For a quantum state to represent a), a spin tensor is needed.

Question: can we prepare a state like a *???*

Spin polarization and spin hydrodynamics

There is no direct relation between spin tensor and spin of the particles.
Spin polarization can be non-vanishing even if there is no spin tensor contributing to the angular momentum current (that is, Belinfante PG).

The spin polarization vector *operator* does NOT depend on the pseudo-gauge

$$S^\mu(p) = \sum_{i=1}^{3} [p]_i^\mu \operatorname{tr}(D^{1/2}(J_i)\Theta(p)) \qquad \Theta(p)_{sr} = \frac{\operatorname{Tr}(\widehat{\rho}\widehat{a}^\dagger(p)_r \widehat{a}(p)_s)}{\sum_t \operatorname{Tr}(\widehat{\rho}\widehat{a}^\dagger(p)_t \widehat{a}(p)_t)}$$

Spin polarization acquires a dependence on the pseudo-gauge choice because the *quantum state* depends on the pseudo-gauge: this is the problem

Local thermodynamic equilibrium density operator:

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z}\exp\left[-\int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{\mathcal{S}}^{\mu,\lambda\nu} - \zeta\widehat{j}^\mu\right)\right].$$

$$\Omega_{\lambda\nu} \equiv \text{reduced spin potential}$$