Spin potential in and out of equilibrium

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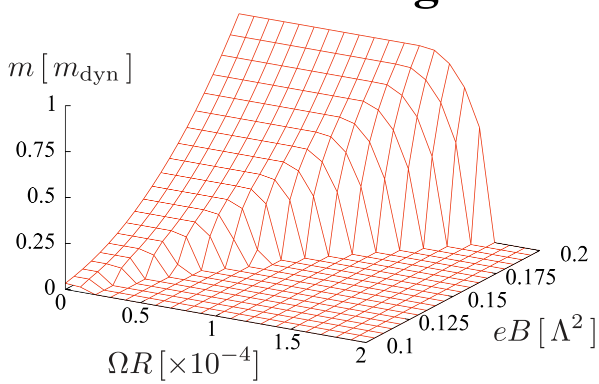
— 9th Int. Conf. on Chirality, Vorticity and Magnetic Fields in Quantum Matter —

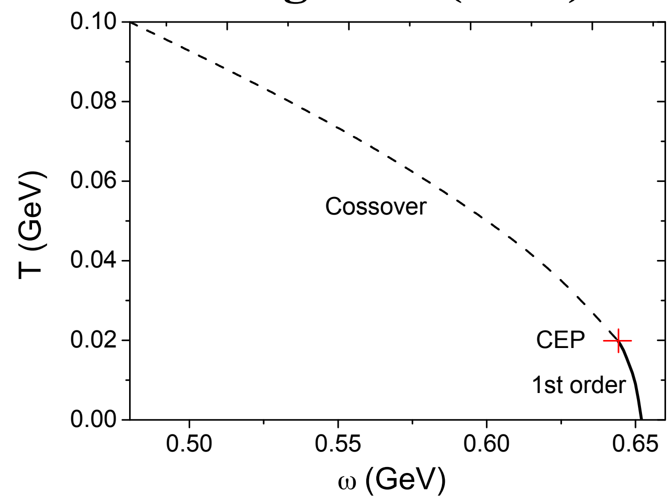
Quick Review: Equilibrium

At Rotation Center (or with *B*)

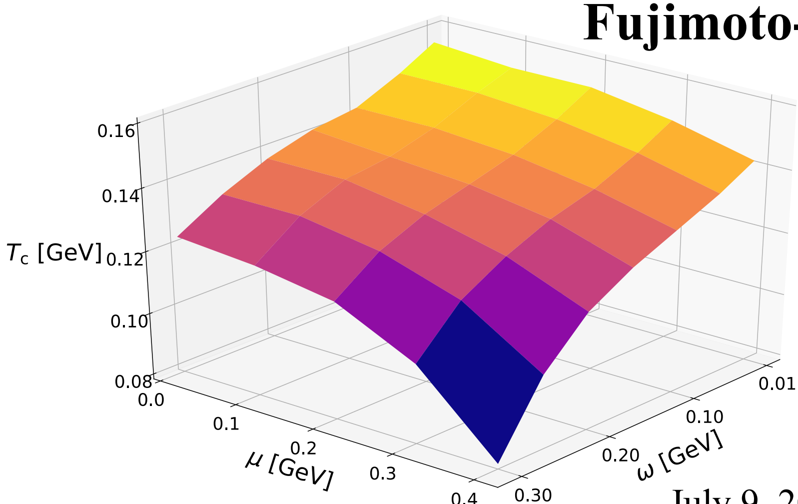
Chen-KF-Huang-Mameda (2015)



Jiang-Liao (2017)



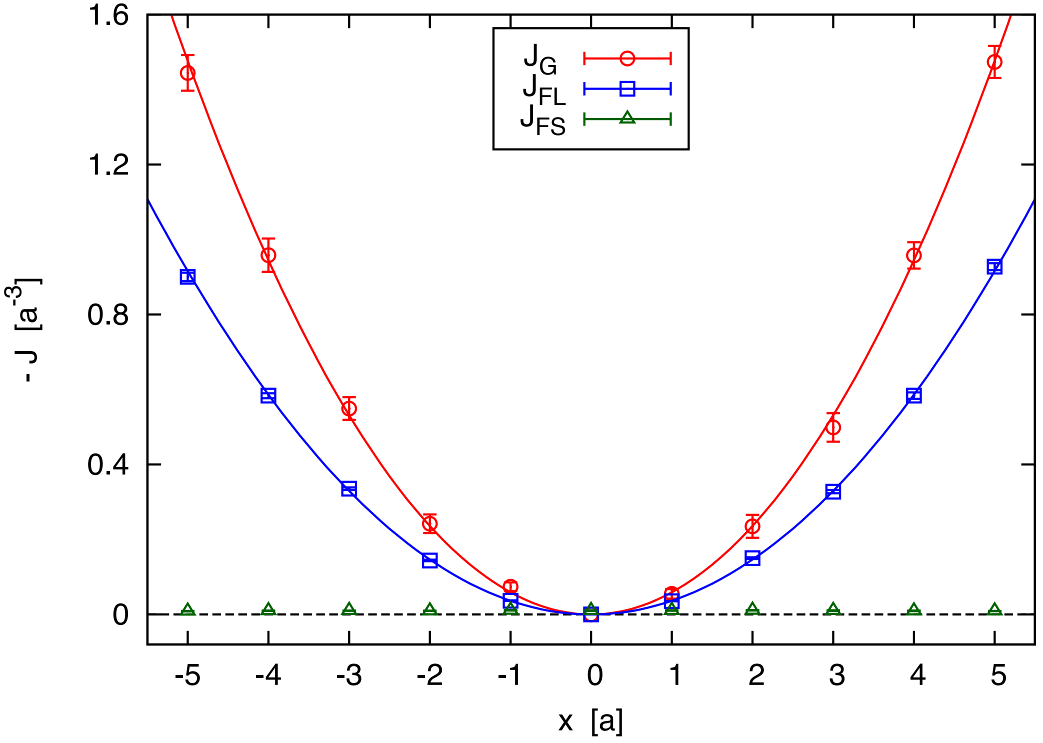
Fujimoto-KF-Hidaka (2021)

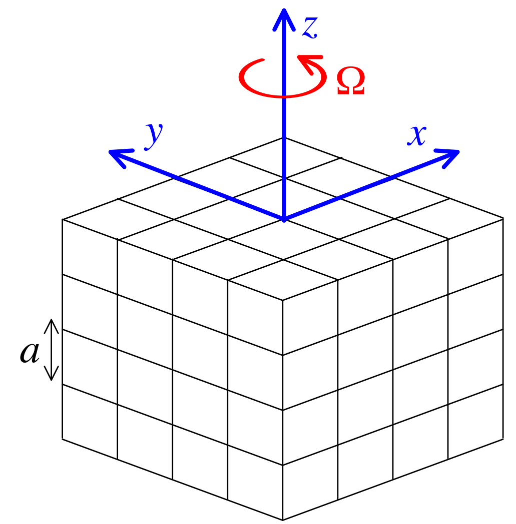


Rotation restores symmetry ($T_c \downarrow$).

Quick Review: Equilibrium

First Lattice: Yamamoto-Hirono (2013)

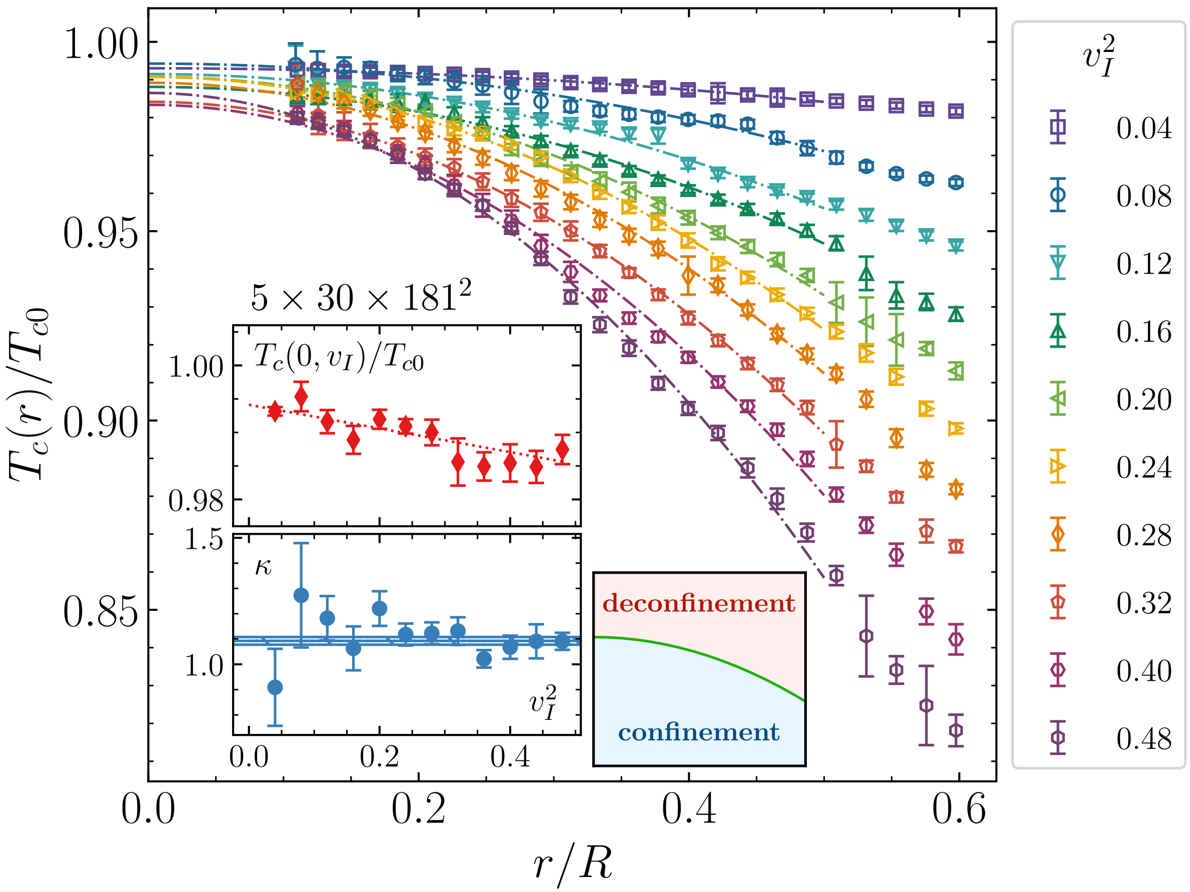




J_{FS} is small but nonzero… independent of the distance.

Quick Review: Equilibrium

Lattice: Braguta-Chernodub-Roenko (2023)



For imaginary Ω
$T_c \downarrow$

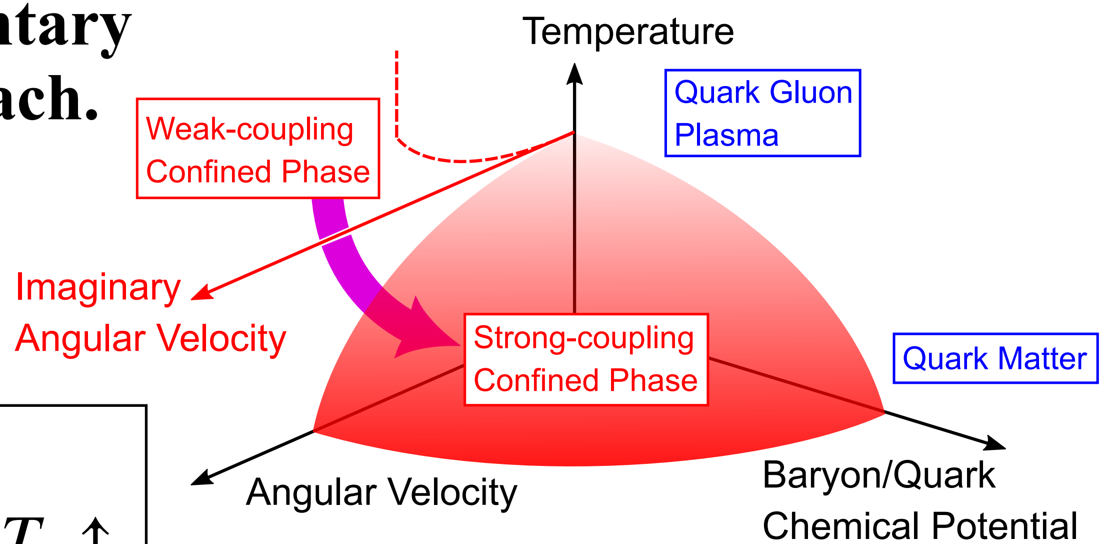
Real rotation breaks more symmetry?

For larger r, real v is larger, and $T_c \uparrow$

Quick Review: Equilibrium

pQCD: Chen-Fukushima-Shimada (2022,2024)

pQCD is a complementary first-principles approach.



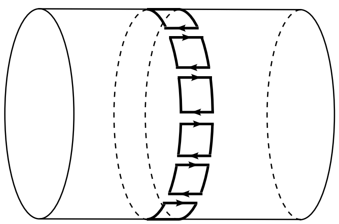
Real rotation $T_c \downarrow$
Imaginary rotation $T_c \uparrow$

Imaginary rotation at $r = 0$ makes the gluonic system center symmetric (confining) at arbitrary high T.

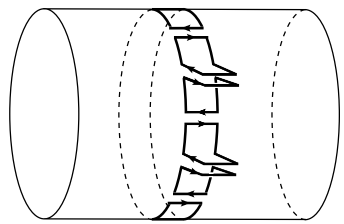
Quick Review: Equilibrium

Strong coupling expansion on the lattice:
Fukushima-Shimada (2025)

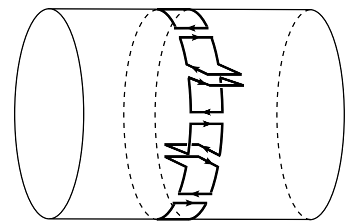
cf. Wang et al. (2025)



$O(\beta^{N_\tau}\Omega^0)$

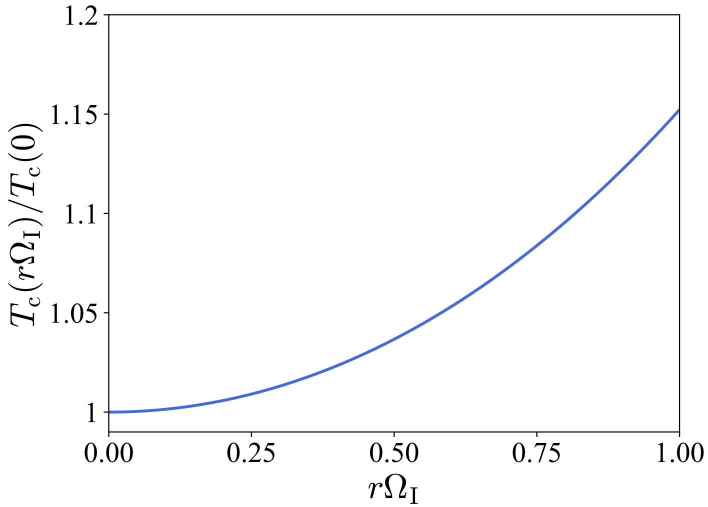


$O(\beta^{N_\tau}\Omega^4)$



$O(\beta^{N_\tau}\Omega^4)$

$$\beta = \frac{2N_c}{g^2} \ll 1$$



For imaginary Ω, $T_c \uparrow$

Consistent with pQCD/models

Gauge inv. problem of lattice formulation similar to finite μ

Hasenfratz-Karsch (1983)

Thermodynamics with Spin

We need for the spin hydro: not *J* but *S*

$$de = Tds + \mu dn + \omega_{\mu\nu} dJ^{\mu\nu}$$

$$dp = sdT + nd\mu + J_{\mu\nu} d\omega^{\mu\nu}$$

$$\langle n \rangle = \frac{\partial p}{\partial \mu}$$

$$\langle J^z \rangle = \frac{\partial p}{\partial \omega^z}$$

$$de = Tds + \mu dn + \omega_{\mu\nu} dS^{\mu\nu}$$

$$dp = sdT + nd\mu + S_{\mu\nu} d\omega^{\mu\nu}$$

How to define $p(\omega)$ for S ?

Thermodynamics with Spin

Fermionic sector as a warm-up:

$$J^{0\mu\nu} = \underbrace{i\psi^\dagger(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi}_{L^{\mu\nu}} + \underbrace{\psi^\dagger\Sigma^{\mu\nu}\psi}_{S^{\mu\nu}}$$

$$\psi^\dagger\mathscr{H}\psi \;\to\; \psi^\dagger(i\gamma^0\gamma^i D^i + \gamma^0 m - \frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\psi$$

$$\Rightarrow\; \mathscr{L} = \bar{\psi}(i\gamma^\mu D_\mu - m + \underbrace{\frac{1}{2}\omega_{\mu\nu}\gamma^0\Sigma^{\mu\nu}}_{-\frac{1}{2}\omega\gamma_5\gamma^3})\psi$$

Spin ~ Axial Vector Current

Thermodynamics with Spin

Fermionic sector as a warm-up:

$$\mathscr{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m \ - \ \frac{1}{2}\omega\gamma_5\gamma^3)\psi$$

$\omega \sim A_5^z$
$\sim \partial_z \theta$

Energy-dispersion can be explicitly obtained as

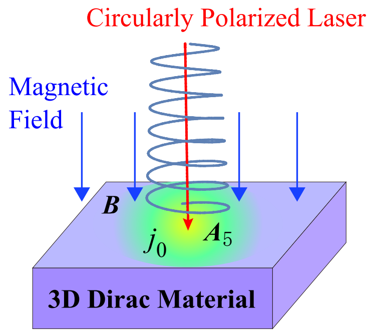
$$\varepsilon_\pm^2 = p_\perp^2 + \left(\sqrt{p_z^2 + m^2} \pm \frac{1}{2}\omega\right)^2$$

Talk by R. Farias

Thermodynamic properties get complicated because $\hat{S}$ does not commute with $\hat{H}$ unlike $\hat{J}$.

Thermodynamics with Spin

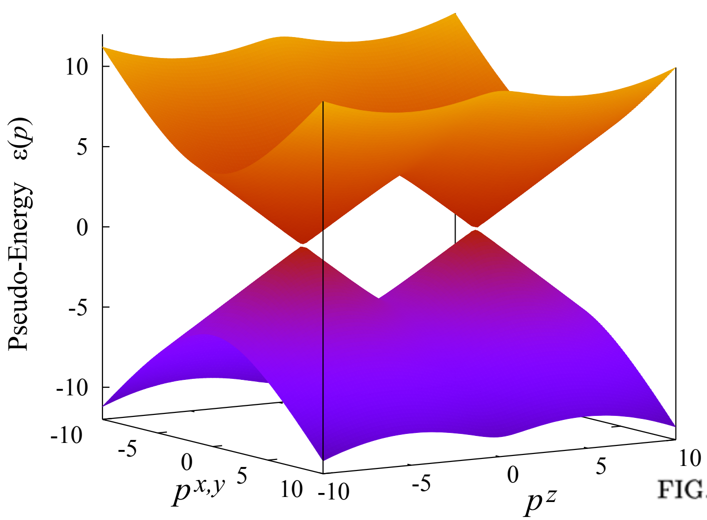
Matter with spin is topological if ω is large…



$$\varepsilon_\pm^2 = p_\perp^2 + \left(\sqrt{p_z^2+m^2} \pm \frac{1}{2}\omega\right)^2$$

C. Anomalies and magnetotransport in Weyl semi-metals

Ebihara-Fukushima-Oka (2015)



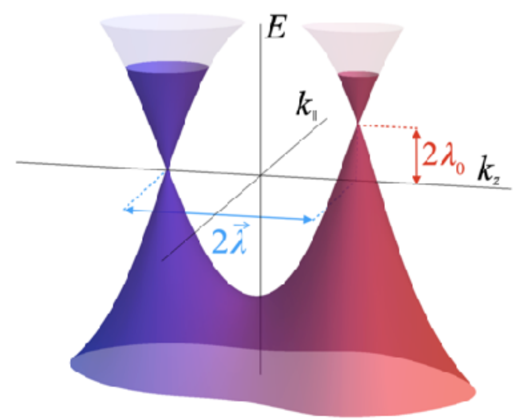


FIG. 4. A schematic view of a Weyl semimetal's band structure.

Chernodub-Ferreiros-
-Grushin-Landsteiner-
-Vozmediano
Phys.Rept. (2022)

Thermodynamics with Spin

More fun with gauge particles: **Chernodub-Fukushima (soon)**

Jaffe-Manohar decomposition:

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} + \underbrace{\boldsymbol{E}\times\boldsymbol{A}}_{\Delta G} - \underbrace{i\psi^\dagger(\boldsymbol{x}\times\nabla)\psi}_{L^q} + \underbrace{\boldsymbol{E}(\boldsymbol{x}\times\nabla)\boldsymbol{A}}_{L^g}$$

$$\boldsymbol{E}\cdot\boldsymbol{B} = \partial_t(\boldsymbol{A}\cdot\boldsymbol{B}) + \nabla\cdot(A_0\boldsymbol{B} + \boldsymbol{E}\times\boldsymbol{A}) \sim \partial_\mu K^\mu$$

We are proposing the gauge-spin coupling as

$$\boxed{\Delta\mathscr{L} = N\int d^4x\, \boldsymbol{\omega}\cdot\boldsymbol{K}}$$

Spatial counterpart of the coupling to μ_5

Thermodynamics with Spin

More fun with gauge particles: **Chernodub-Fukushima (soon)**

This is the spatial version of the chiral chemical pot.:

On chiral magnetic effect and holography

V.A. Rubakov

arXiv:1005.1888 [hep-ph]

Institute for Nuclear Research of the Russian Academy of Sciences,
60th October Anniversary Prospect, 7a, 117312 Moscow, Russia

$$\boxed{S_{eff} = 3\kappa\mu_A \int d^4x \, \epsilon^{ijk} A_i^V F_{jk}^V}$$

Thermodynamics with Spin

More fun with gauge particles: **Chernodub-Fukushima (soon)**

More generally:

ω_μ

$$\Delta\mathscr{L} = N\int d^4x\, \boxed{(\partial_\mu\theta)}\, K^\mu$$

$$= -N\int d^4x\, \theta\, \partial_\mu K^\mu$$

Zero-th component is μ_5 (relevant to CME)

Introducing the spin potential in the gauge sector is equivalent to introducing inhomogeneous θ angle!

See: Vazifeh-Franz (2013) for applications in cond-mat.

Thermodynamics with Spin

More fun with gauge particles: Chernodub-Fukushima (soon)

We already know the answer: $b = -\partial_z \theta$

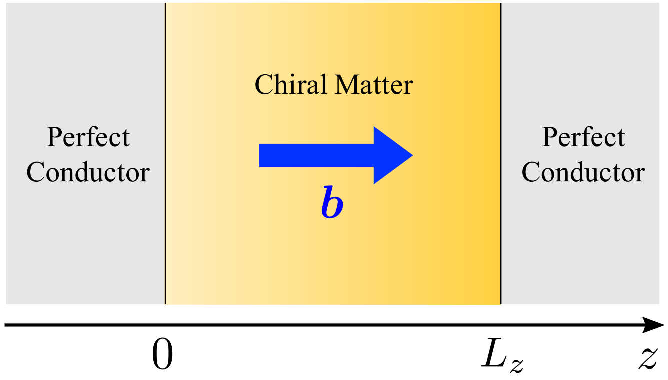
Qiu-Cao-Huang (2016)

$$\omega_{1,2}^2 = \boldsymbol{k}^2 \qquad \omega_\pm^2 = k_x^2 + k_y^2 + \left(\sqrt{k_z^2 + \frac{b^2}{4}} \pm \frac{b}{2}\right)^2$$

For the application to the Casimir force, see: Fukushima-Imaki-Qiu (2019)

cf. Jiang-Wilczek (2019)

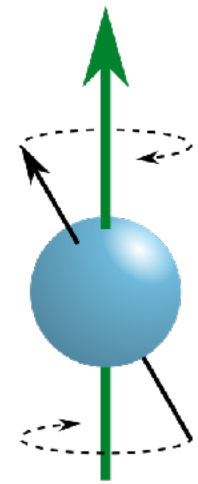
Not only the attractive but also the repulsive force!



Known about Spin Dynamics

Precession $$H = -\gamma \boldsymbol{B} \cdot \boldsymbol{S}$$

$$\frac{\partial \boldsymbol{S}}{\partial t} = -\gamma \boldsymbol{B} \times \boldsymbol{S}$$



If there is no relaxation, the spin is never aligned to the magnetic field, but it continues the precession forever…

Landau-Lifshitz

$$\frac{\partial \boldsymbol{S}}{\partial t} = -\gamma \boldsymbol{B} \times \boldsymbol{S} + \alpha \frac{\gamma}{S} \left[\boldsymbol{S} \times (\boldsymbol{S} \times \boldsymbol{B}) \right]$$

Known about Spin Dynamics

Precession

$$\frac{\partial \boldsymbol{S}}{\partial t} = -\gamma \boxed{\boldsymbol{B} \times \boldsymbol{S}} + \alpha \frac{\gamma}{S} \left[\boldsymbol{S} \times \underline{(\boldsymbol{S} \times \boldsymbol{B})} \right]$$

$$\simeq -\gamma \boldsymbol{B} \times \boldsymbol{S} + \frac{\alpha}{S} \left(\boldsymbol{S} \times \frac{\partial \boldsymbol{S}}{\partial t} \right)$$

Relaxation Term

Landau-Lifshitz-Gilbert (LLG) eq.

(1935, 1955)

Known about Spin Dynamics

Relativistic Generalization

Pauli-Lubański vector: $$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma$$

One particle states are classified by the representation theory (Casimir eigenvalues) of the Poincare group.

Spin four-vector: $$S^\mu = \frac{W^\mu}{(-W^2)^{1/2}}$$

(1926)

$$S\cdot P = 0 \quad \Rightarrow \quad \boxed{S\cdot u = 0}$$ **Frenkel-Mathisson-Pirani**

Under this condition, the spin has only 3 spatial d.o.f. instead of 6 rotation generators.

Known about Spin Dynamics

Relativistic Generalization

$$\frac{dS^\mu}{d\tau} = \gamma F^{\mu\nu} S_\nu + \kappa\, u^\mu \quad \textbf{contracted with} \quad u_\mu$$

$$\Rightarrow \quad u \cdot \frac{dS}{d\tau} = -S \cdot \frac{du}{d\tau} = \gamma\, u_\mu F^{\mu\nu} S_\nu + \kappa$$

Here, $u \cdot S = 0$ is used. Now, κ is solved.

$$\frac{dS^\mu}{d\tau} = -u^\mu S \cdot \frac{du}{d\tau} + \gamma \left(F^{\mu\nu} S_\nu + u^\mu S_\rho F^{\rho\sigma} u_\sigma \right) + \text{dissipative terms}$$

Relativistic Term **Precession Term**

Bargmann-Michel-Telegdi (BMT) equation

Conclusion First

From "our" spin hydro: **S.Fang-KF-S.Pu-D.L.Wang: 2506.20698**

$$\dot{s}^{\mu} = -\, u^{\mu} s^{\nu} \dot{u}_{\nu} + (\epsilon^{\mu\nu\rho\sigma} s_{\nu} u_{\rho} - 2\beta\gamma_{\phi} g^{\mu\sigma})(2\omega_{\sigma} - \mathfrak{w}_{\sigma})$$

Relativistic Term **Dissipation** **Precession**

$$-s_{\nu}\partial^{<\mu}u^{\nu>} - \left(\frac{1}{3} + 2v_n^2\right) s^{\mu}(\partial \cdot u)$$

New Hydro Terms

Surprisingly, these were missing in "old" spin hydro…

$$\mathfrak{w}^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} u_{\nu} \partial_{\rho} u_{\sigma} \xrightarrow[\text{Equilibrium}]{} 2\omega^{\mu}$$

Hydrodynamics with Spin

Some Common Notations

Noether current from rotational symmetry

$$J^{\lambda\mu\nu} = \underbrace{x^{\mu}\Theta^{\lambda\nu} - x^{\nu}\Theta^{\lambda\mu}}_{\text{Orbital}} + \underbrace{\Sigma^{\lambda\mu\nu}}_{\text{Spin}}$$

$$\partial_{\mu}\Theta^{\mu\nu} = \partial_{\lambda}J^{\lambda\mu\nu} = 0 \longrightarrow \boxed{\partial_{\lambda}\Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}}$$

Spin Equation of Motion

At global equilibrium **Spin potential ~ Thermal vorticity**

$$\beta\omega_{\mu\nu} = \frac{1}{2}\Omega^{\mu\nu} = -\frac{1}{2}\partial_{[\mu}\beta_{\nu]}$$

Hydrodynamics with Spin

Spin Hydrodynamics (Hattori-Hongo-Huang-Matsuo-Taya 2019)

Assuming:

$$\Sigma^{\lambda\mu\nu} = u^{\lambda} S^{\mu\nu} + \Sigma^{\lambda\mu\nu}_{(1)}$$

Entropy principle gives:

$$\Theta^{[\mu\nu]} = 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}$$

In analogy to

$$\Theta^{(\mu\nu)}_{(1)} = 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu}$$

In equilibrium

$$\beta\omega_{\mu\nu} = \frac{1}{2}\Omega^{\mu\nu} = -\frac{1}{2}\partial_{[\mu}\beta_{\nu]}$$

$$q^{\mu} = 2T\lambda_q(\Omega^{\mu\nu} - 2\beta\omega^{\mu\nu})u_{\nu}$$ **Boost Heat Conductivity**

$$\phi^{\mu\nu} = -\gamma_{\phi}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\Omega_{\alpha\beta} - 2\beta\omega_{\alpha\beta})$$ **Rotational Viscosity**

Antisymmetric Spin Tensor

Hongo-Huang-Kaminski-Stephanov-Yee (2022)

QFT + torsion background → Spin Hydrodynamics

we microscopically know the spin current for QFTs with Dirac fermions is totally anti-symmetric so that the spin density for such QFTs satisfies …

Cao-Hattori-Hongo-Huang-Taya (2022)

Spin density ~ $\mathcal{O}(1)$ → 3 bulk viscosities, 4 shear viscosities, 3 rotational viscosities, 4 cross viscosities, 3 conductivities,…

cf. Daher-Sheng-Wagner-Becattini (2025) found 23…

Antisymmetric Spin Tensor

Daher-Das-Ryblewski (2023)

Stability studies of first order spin-hydrodynamic frameworks

Frenkel condition to solve the problem of instability

$$\boxed{\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + u^\mu S^{\nu\lambda} + u^\nu S^{\lambda\mu} + \Sigma^{\lambda\mu\nu}_{(1)}}$$

Totally antisymmetric $\rightarrow u_\lambda u_\mu \Sigma^{\lambda\mu\nu} = 0$

$\underbrace{\textbf{3 rotations}}_{\textbf{Hermitian}} + \cancel{\underbrace{\textbf{3 boosts}}_{\textbf{non-Hermitian}}}$

Antisymmetric Spin Tensor

From "our" spin hydro: **S.Fang-KF-S.Pu-D.L.Wang: 2506.20698**

Differences:

q^μ **is already solved as** λ_q **does not appear!**

$$q^\mu = -\frac{1}{2} u_\nu \partial_\lambda \Sigma^{\lambda\mu\nu} = \frac{1}{2}(S^{\mu\nu}\dot{u}_\nu + \Delta^\mu_\nu \partial_\lambda S^{\nu\lambda})$$

Extra terms in the entropy production

$$-\partial_\mu (u^\mu S^{\rho\sigma})\beta\omega_{\rho\sigma} \neq 2\Theta^{[\rho\sigma]}\beta\omega_{\rho\sigma}$$

Antisymmetric Spin Tensor

From "our" spin hydro: **S.Fang-KF-S.Pu-D.L.Wang: 2506.20698**

Extra terms can be "renormalized" into

$$h^\mu \to h^\mu + h_s^\mu$$ **← Anomalous Hall effect**

$$\pi^{\mu\nu} \to \pi^{\mu\nu} + \pi_s^{\mu\nu}$$

$$\phi^{\mu\nu} \to \phi^{\mu\nu} + \phi_s^{\mu\nu}$$ **← Most interesting for the spin dynamics**

$$\phi_s^{\mu\nu} = -2S^{[\mu\lambda}\omega^{\nu]}_{\ \lambda} - v_n^2(\partial\cdot u)S^{\mu\nu}$$

Antisymmetric Spin Tensor

From "our" spin hydro: **S.Fang-KF-S.Pu-D.L.Wang: 2506.20698**

We can derive the spin equation of motion:

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -\,2(2q^{[\mu}u^{\nu]} + \phi^{\mu\nu})$$

$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + u^\mu S^{\nu\lambda} + u^\nu S^{\lambda\mu}$$

$$\Delta_{\mu\alpha}\Delta_{\nu\beta}\partial_\lambda \Sigma^{\lambda\alpha\beta}$$

$$= 2\gamma_\phi \Delta^{\mu\alpha}\Delta^{\nu\beta}(\Omega_{\alpha\beta} - 2\beta\omega_{\alpha\beta}) - 2\phi_s^{\mu\nu}$$

$$s^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu S_{\rho\sigma}$$ **Similar notations for ω, Ω**

After several minutes, you will find…

Conclusion Again

From "our" spin hydro: **S.Fang-KF-S.Pu-D.L.Wang: 2506.20698**

$$\dot{s}^\mu = -u^\mu s^\nu \dot{u}_\nu + (\epsilon^{\mu\nu\rho\sigma} s_\nu u_\rho - 2\beta\gamma_\phi g^{\mu\sigma})(2\omega_\sigma - \mathfrak{w}_\sigma)$$

Relativistic Term **Dissipation** **Precession**

$$-s_\nu \partial^{<\mu} u^{\nu>} - \left(\frac{1}{3} + 2v_n^2\right) s^\mu (\partial \cdot u)$$

New Hydro Terms

Extended BMT eq. with $\omega_{\mu\nu} \sim F_{\mu\nu}$ **(remember** $p \sim S \cdot \omega$**)**

$$\frac{dS^\mu}{d\tau} = -u^\mu S \cdot \frac{du}{d\tau} + \gamma \left(F^{\mu\nu} S_\nu + u^\mu S_\rho F^{\rho\sigma} u_\sigma\right) + \text{dissipative terms}$$

Summary

- **Spin hydrodynamics based on the entropy principle has natural terms to describe the dynamics of spin and thermal vorticity.**
  - □ Spin (finite dim.) has 6 or 3 charge observables?
  - □ Anti-symmetrized form should be physical.
- **Antisymmetric spin tensors lead to known terms in the BMT equation as well as new terms.**
  - □ Spin induced terms in the entropy production.
  - □ "Renormalized" in various currents.
- **Phenomenological implications?**
  - □ Extended BMT will be numerically solved (soon).