Effective action for relativistic hydrodynamics from Crooks fluctuation theorem

PRL **134**, no.23 (2025)

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In collaboration with Nicki Mullins and Jorge Noronha

9th Conference on Chirality, Vorticity and Magnetic Fields in Quantum Matter

CBPF
Centro Brasileiro de Pesquisas Físicas

Outline

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Motivation

Fluctuating relativistic hydrodynamics

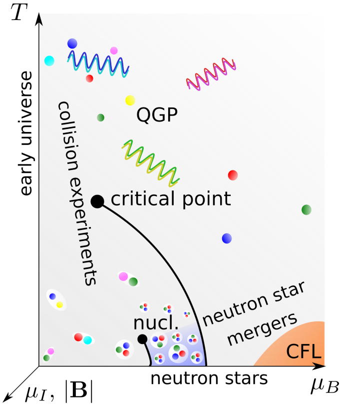
Relevant for the quark-gluon plasma

- Near the QCD critical point.
- In small collision systems.

Problems

- Relativistic causality.
- Stability in any reference frame.

L. Gavassino, PRX **12** (2022)
N. Mullins, MH and J. Noronha, PRD **108** (2023)
N. Mullins, MH, L. Gavassino, J. Noronha, PRD **108** (2023)

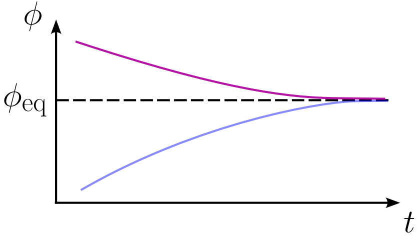


Goal: General formalism, causal at nonlinear level, in the presence of **external fields**.

Approach to equilibrium

- Relaxation equation, $\langle\phi\rangle \rightarrow \phi_{\rm eq}$

$$\frac{d\phi}{dt} + \lambda\left(\phi - \phi_{\rm eq}\right) = 0$$



"Statistical Physics: Volume 5,"
Landau, L.D. and Lifshitz, E.M.

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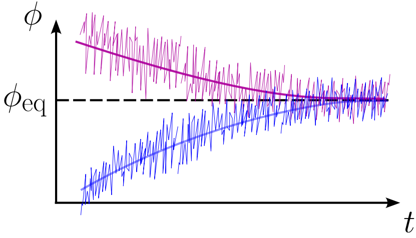
$$\frac{d\phi}{dt} + \lambda\,(\phi - \phi_{\rm eq}) = \xi$$

- Fluctuations from noise term,

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \Gamma\,\delta(t - t')$$



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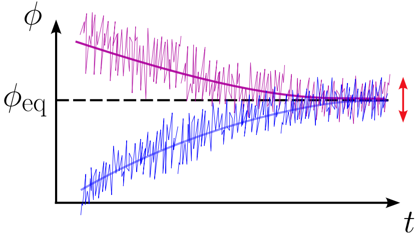
$$\langle\xi(t)\rangle = 0, \quad \langle\xi(t)\xi(t')\rangle = \Gamma\,\delta(t-t')$$

- Fluctuation-dissipation relation:

$$\Gamma = 2\lambda\,\langle\delta\phi^2\rangle_{\text{eq}}$$

- Theorem for linear response:

$$G_S(\omega) = -\frac{2T}{\omega}\,\mathrm{Im}\,G_R(\omega)$$



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Example: Non-relativistic diffusion

- Diffusion: Fick's law + stochastic noise with $\langle \boldsymbol{\xi} \rangle = \mathbf{0}$

$$\partial_t \delta n + \nabla \cdot \boldsymbol{j} = 0, \qquad \boldsymbol{j} = -D\, \nabla \delta n + \boldsymbol{\xi}$$

- Entropy density $s = \bar{s} - \bar{\mu}\, \delta n + \dfrac{\delta n^2}{2\chi} \Rightarrow$ **equal-time correlator**:

$$\mathcal{P} \propto \exp\left(- \int \frac{\delta n^2}{2\chi\, T}\, d^3x \right) \quad \Rightarrow \quad \langle \delta n(t, \boldsymbol{x}) \delta n(t, \boldsymbol{x}') \rangle = \Gamma\, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')$$

- **Fluctuation-dissipation relation**: $\Gamma = 2\chi\, T\, D$

- Relativistic flow $\Rightarrow$ relativistic causality.
  - × Relativistic Navier-Stokes acausal.

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  - $\rightarrow$ Nonlinear fluctuation-dissipation consistency.

    E. Wang and U. W. Heinz, PRD **66** (2002)
  - $\rightarrow$ Nonlinear causality conditions.

    F. S. Bemfica, M. M. Disconzi and J. Noronha, PRD **98** (2018)
    F. S. Bemfica, M. M. Disconzi and J. Noronha, PRD **100** (2019)

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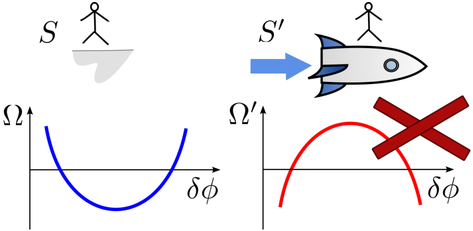
  - × Navier-Stokes stable only in local rest frame.



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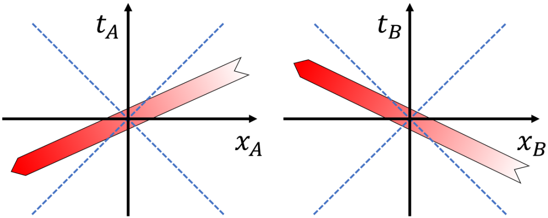
L. Gavassino, PRX **12** (2022)



Outline

Stochastic action

- Path integral for transition probability,

$$P[A \to B] = \int_A^B \mathcal{D}\phi\, \mathcal{D}\overline{\phi}\, e^{i\, S[\phi,\overline{\phi}]}$$

- Doubling of degrees of freedom
$\phi \to (\phi, \overline{\phi})$

P. C. Martin, E. D. Siggia and H. A. Rose, PRA **8** (1973)
C. R. Galley, PRL **110** (2013)

- Calculation of Green's functions by inserting sources.

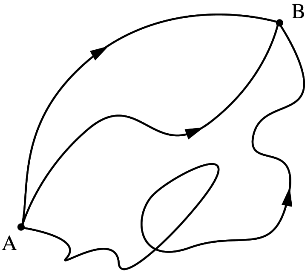


Fig.: Matt McIrvin, `https://commons.wikimedia.org/wiki/File:Three_paths_from_A_to_B.png`

Ideal hydrodynamics

- Conservation laws at fluctuation level with $\delta\left[\nabla_\mu J^\mu\right]$, $\delta\left[\nabla_\mu T^{\mu\nu}\right]$

$$\mathcal{L} = -\bar{\alpha}\,\nabla_\mu J^\mu - \bar{\beta}_\nu\,\nabla_\mu T^{\mu\nu}$$

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- Divergence-type theory with generating current $X^\mu(\phi)$, $\phi = (\alpha, \beta^\mu)$

$$J^\mu \equiv \frac{\partial X^\mu}{\partial \alpha}, \qquad T^{\mu\nu} \equiv \frac{\partial X^\mu}{\partial \beta_\nu} \qquad \Rightarrow \qquad \boxed{\mathcal{L} = -\overline{\phi}^a\,\nabla_\mu \frac{\partial X^\mu}{\partial \phi^a}}$$

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Thermodynamics

- With most general $X^\mu = \beta^\mu P(\alpha, \beta^2)$,

$$J^\mu = n\,u^\mu, \qquad T^{\mu\nu} = \varepsilon\,u^\mu u^\nu + P\,g^{\mu\nu}$$

- Thermodynamic relations with temperature β^{-1}, chemical potential α/β, velocity β^μ/β, pressure P, density n and energy density ε.

Dissipative divergence-type hydrodynamics

- Promote dissipative currents φ to dynamical degrees of freedom.

I. Muller, Z. Phys. **198** (1967), W. Israel, Annals Phys. **100** (1976)
W. Israel and J. M. Stewart, Annals Phys. **118** (1979)

- Lagrangian with $X^\mu(\Phi)$, with $\Phi \equiv (\phi, \varphi)$:

$$\mathcal{L} = -\overline{\Phi}^a \nabla_\mu \frac{\partial X^\mu}{\partial \Phi^a} + i\,\Xi(\phi^a, \varphi^a, \overline{\varphi}^a)$$

- Equations of motion from Euler-Lagrange:

$$\nabla_\mu \frac{\partial X^\mu}{\partial \phi^a} = 0, \qquad - \nabla_\mu \frac{\partial X^\mu}{\partial \phi^a} + i\,\frac{\partial \Xi}{\partial \overline{\varphi}^a} = 0$$

I-Shih Liu, I. Müller, T. Ruggeri, Annals Phys. **169** (1986)
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Dissipative potential $\Xi(\phi^a, \varphi^a, \overline{\varphi}^a)$

- Linear terms in $\overline{\varphi} \Rightarrow$ dissipation.
- Quadratic terms in $\overline{\varphi} \Rightarrow$ Gaussian noise conjugate to $\overline{\varphi}$.

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Outline

Equilibrium

- Second law of thermodynamics:

$$\nabla_\mu s^\mu = -i\,\varphi^a \frac{\partial \Xi}{\partial \overline{\varphi}^a} \geq 0 \quad \text{on-shell,} \quad \text{with} \quad s^\mu = X^\mu - \Phi^a \frac{\partial X^\mu}{\partial \Phi^a},$$

- Equilibrium = minimum free energy (- entropy + constraints)

$$\Omega[\Sigma] = \int_\Sigma d\Sigma_\mu\, \Omega^\mu = \int_\Sigma d\Sigma_\mu\, \left(-s^\mu - \alpha_*\, J^\mu - \beta_{*\nu}\, T^{\mu\nu}\right)$$

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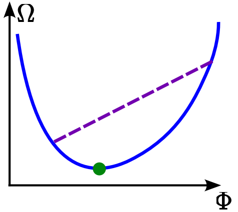
Causality and covariant stability

- Symmetric hyperbolic, **causal** equations if $n_\mu \Omega^\mu$ is convex $\forall$ timelike past-directed n^μ.
- Stability for any observer at nonlinear level.



Equilibrium distribution in covariant form

Fluctuations in equilibrium are given by

$$P_{\rm eq}^{(\Sigma)}(\Phi^a) \propto e^{-\Omega[\Sigma]} = e^{-\int_\Sigma d\Sigma_\mu \, \Omega^\mu}$$

for an arbitrary spacelike hypersurface Σ.

N. Mullins, MH and J. Noronha, PRD **108** (2023)
L. Gavassino, M. Antonelli and B. Haskell, PRL **128** (2022)

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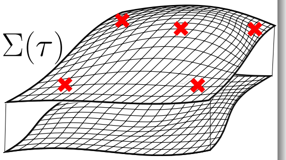
Relativity of simultaneity

- Arbitrary spacelike foliation $\Sigma(\tau)$.
- Equal-time → mutually spacelike-separated.
- Correlations from equilibrium physics.



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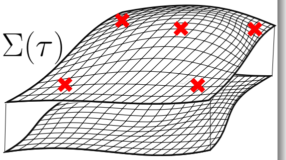
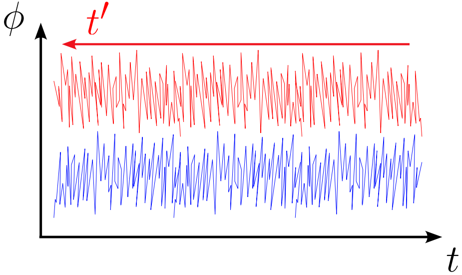


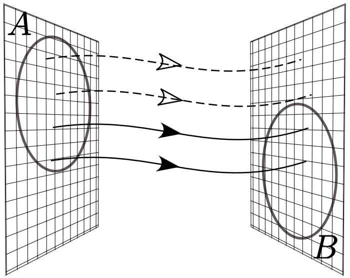
Detailed balance

Equilibrium distribution at long times ensured by detailed balance.
Same rates backward and forward in equilibrium ⇒ fixed point.



Detailed balance and microscopic reversibility

Remnant of microscopic time-reversal symmetry $\mathbf{\Theta}$.



Transition rate:

$$W(A \to B) = W(A)\, P(A \to B)$$

C. Stahl, M. Qi, P. Glorioso, A. Lucas and R. Nandkishore, PRB **108** (2023) X. Huang, J. H. Farrell, A. J. Friedman, I. Zane, P. Glorioso and A. Lucas, arXiv:2310.12233

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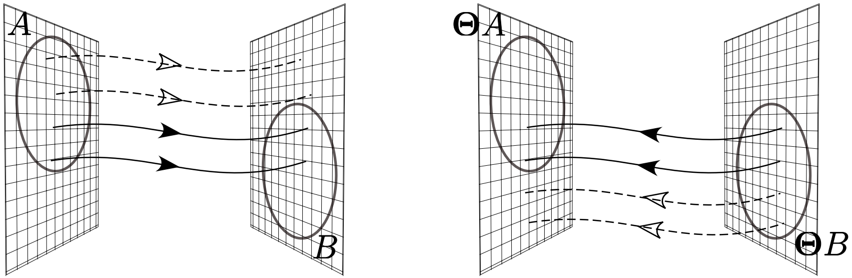

Transition rate:

$$W(\mathbf{\Theta}B \to \mathbf{\Theta}A) = W(B)\, P(\mathbf{\Theta}B \to \mathbf{\Theta}A)$$

C. Stahl, M. Qi, P. Glorioso, A. Lucas and R. Nandkishore, PRB **108** (2023) X. Huang, J. H. Farrell, A. J. Friedman, I. Zane, P. Glorioso and A. Lucas, arXiv:2310.12233

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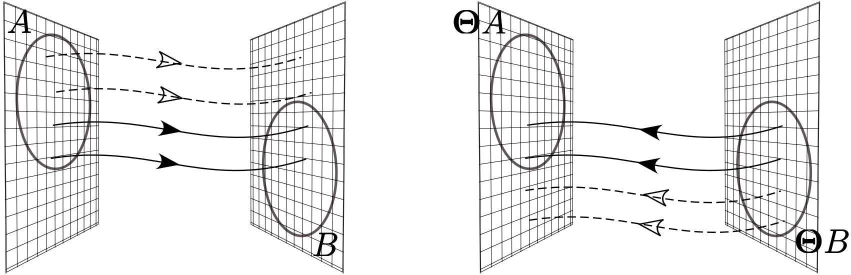


Transition rate:

$$W(\mathbf{\Theta}B \to \mathbf{\Theta}A) = W(B)\, P(\mathbf{\Theta}B \to \mathbf{\Theta}A)$$

Same number of trajectories connecting both states, $S = \log W$:

$$P(\mathbf{\Theta}B \to \mathbf{\Theta}A) = P(A \to B)\, e^{S_A - S_B} = P(A \to B)\, e^{\Omega_B - \Omega_A}$$

C. Stahl, M. Qi, P. Glorioso, A. Lucas and R. Nandkishore, PRB **108** (2023) X. Huang, J. H. Farrell, A. J. Friedman, I. Zane, P. Glorioso and A. Lucas, arXiv:2310.12233

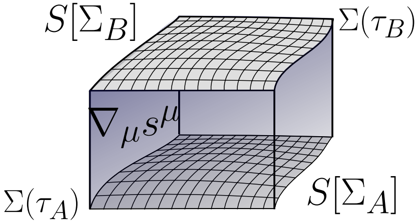
Detailed balance condition

Under time reversal + exchange initial and final states, $A \leftrightarrow B$,

$$\mathcal{L} \to \mathcal{L} - i\,\nabla_\mu s^\mu \qquad \Rightarrow \qquad iS \to iS + \Omega_A - \Omega_B.$$

Detailed balance for states in arbitrary spacelike hypersurfaces $\Sigma_{A,B}$.
Action invariant up to boundary term.

G. Torrieri, JHEP **02** (2021)
N. Mullins, MH, L. Gavassino and J. Noronha, PRD **108** (2023)



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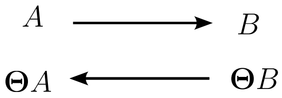
G. Torrieri, JHEP **02** (2021)
N. Mullins, MH, L. Gavassino and J. Noronha, PRD **108** (2023)

Discrete $\mathbb{Z}_2$ symmetry

- Boundary terms from ideal action if

$$\Phi \to \mathbf{\Theta}\Phi, \quad \overline{\Phi} \to \mathbf{\Theta}\left(\overline{\Phi} - i\,\Phi\right), \quad \Xi \to \Xi$$



- Dissipative potential Ξ must be invariant.

Entropy production from external work

- With spacetime dependent couplings or fields $\lambda^h(x)$,

$$iS \to iS + \int d^4x \sqrt{-g}\, \sigma$$

- Now $\sigma = \nabla_\mu s^\mu + \sigma_{\text{ext}}$ is the total entropy production.
- External work $\Rightarrow \sigma_{\text{ext}} = (\partial X^\mu / \partial \lambda^h) \nabla_\mu \lambda^h$.

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Crooks fluctuation theorem

Relationship between process and its inverse:

$$\frac{P(\Phi|\lambda)}{P(\boldsymbol{\Theta}\Phi|\boldsymbol{\Theta}\lambda)} = e^{+\int d^4x \sqrt{-g}\,\sigma} = e^{\beta W - (\Omega_B - \Omega_A)}$$

- Reversible/adiabatic when work equals free-energy difference.

G. E. Crooks, PRE **60** (1999)

Sources and external fields

- For a scalar field,

$$P \to P + \lambda\,\phi, \quad \mathcal{L} \to \mathcal{L} - \overline{\phi}\,\nabla_\mu(\beta^\mu \lambda) + \overline{\lambda}\,\phi, \quad \sigma_{\text{ext}} = \phi\,\beta^\mu \partial_\mu \lambda$$

- Effective action by integrating from $\tau \to -\infty$ to $\tau \to \infty$,

$$e^{i\,W[\lambda,\overline{\lambda}]} = \int \mathcal{D}\phi\,\mathcal{D}\overline{\phi}\,e^{i\,S[\phi,\overline{\phi};\lambda,\overline{\lambda}]}$$

- Green's functions → variation w.r.t. external field λ and source $\overline{\lambda}$.

$$\langle \phi(x) \rangle = \frac{\delta W}{\delta \overline{\lambda}(x)}, \quad G_R(x, x') = \frac{\delta}{\delta \lambda(x')} \langle \phi(x) \rangle = \frac{\delta^2 W}{\delta \lambda(x')\,\delta \overline{\lambda}(x)}, \quad \ldots$$

Fluctuation-dissipation theorem

KMS Symmetry

Variation of the action becomes boundary term again if

$$\Phi \to \boldsymbol{\Theta}\Phi, \quad \overline{\Phi} \to \boldsymbol{\Theta}\left(\overline{\Phi} - i\,\Phi\right), \quad \lambda \to \boldsymbol{\Theta}\lambda, \quad \overline{\lambda} \to \boldsymbol{\Theta}(\overline{\lambda} + i\,\pounds_\beta \lambda)$$

Symmetry equivalent to the Kubo–Martin–Schwinger condition for a Schwinger-Keldysh effective action.

M. Crossley, P. Glorioso and H. Liu, JHEP **09** (2017)
Sieberer, Chiocchetta, Gambassi, Täuber and S. Diehl, PRB **92** (2015)

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KMS symmetry guarantees Green's functions of arbitrary order satisfy generalized fluctuation-dissipation theorems.

E. Wang and U. W. Heinz, PRD **66** (2002)
M. Crossley, P. Glorioso and H. Liu, JHEP **09** (2017)
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Outline

Relativistic diffusion

Example of how to construct an effective action:

1. Choose degrees of freedom: $\alpha = \mu/T$, diffusive current j^μ, $\beta_\mu j^\mu = 0$.
2. Write generating current, $X^\mu = P(\alpha, j^2)\beta^\mu + \alpha j^\mu$,

N. Mullins, MH and J. Noronha, PRL **134**, no.23 (2025)

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3. Couple external sources according to KMS symmetry.

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2. Write generating current, $X^\mu = P(\alpha, j^2)\beta^\mu + \alpha j^\mu + \beta^{[\mu} J^{\nu]} A_\nu$,

3. Couple external sources according to KMS symmetry.

4. Write general Ξ with $\mathbb{Z}_2$ symmetry. Truncate in $\mathcal{O}(j^2, \bar{j}^2)$,

$$\Xi = \frac{1}{\kappa}\Delta^{\mu\nu}\bar{j}_\mu \left(\bar{j}_\nu + i\, j_\nu\right)$$

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Theory of relativistic diffusion

$$\mathcal{L} = -\bar{\alpha}\nabla_\mu J^\mu - \bar{j}_\nu \nabla_\mu \left(\frac{\tau}{\kappa} u^\mu j^\nu + \alpha \Delta^{\mu\nu}\right) + \frac{i}{\kappa}\bar{j}_\mu \left(\bar{j}^\nu + i\, j^\nu\right)$$
$$- \bar{j}_\nu \nabla_\mu \left(\beta^{[\mu}\Delta^{\lambda]\nu} A_\lambda\right) + \bar{A}_\mu J^\mu$$

leading to on-shell equations of motion:

$$u^\mu \partial_t n + \nabla_\mu j^\mu = 0, \qquad \tau\, u^\mu \nabla_\mu j^\nu + j^\nu + \kappa\, \nabla_\perp^\nu \alpha = \kappa\, \beta_\mu F^{\mu\nu}$$

Conditions

1. Causal if K^μ is timelike future-directed for all Φ, Φ', with

$$K^\mu(\Phi, \Phi') = \Omega^\mu(\Phi) - \Omega^\mu(\Phi') + (\Phi'^a - \Phi^a) \left.\frac{\partial \Omega^\mu}{\partial \Phi^a}\right|_\Phi$$

See supplemental material in N. Mullins, MH and J. Noronha, PRL **134**, no.23 (2025)

2. Total on-shell entropy production $\sigma = j^2/\kappa \geq 0 \Rightarrow \kappa \geq 0$.

Outline

Conclusions

- General formalism for **causal and stable** fluctuating relativistic hydrodynamics.
- Nonlinear fluctuation-dissipation relations at arbitrary order for any observer.
- Allows for **external fields** and evolving backgrounds.

Nicki Mullins' PhD work.

Outlook

- Chiral anomaly with chiral chemical potential μ_5 and current j_5^μ.
- Renormalization of transport coefficients.

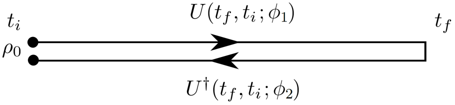
P. Kovtun, G. D. Moore and P. Romatschke, PRD **84** (2011)
Y. Akamatsu, A. Mazeliauskas and D. Teaney, PRC **95** (2017)

Supported by UERJ PAPD program.

N. Mullins, MH and J. Noronha, PRL **134**, no.23 (2025)

Backup slides...

Schwinger-Keldysh effective field theory

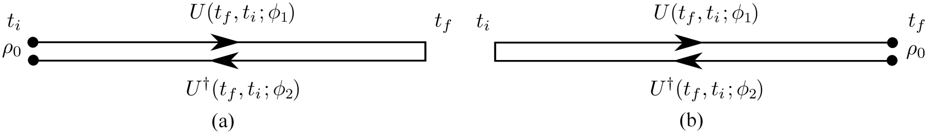


(a)

- Effective field theory on the closed time path:

$$\begin{aligned} e^{W[\boldsymbol{h}_1,\boldsymbol{h}_2]} &= \mathrm{tr}\left\{U_{(T,0)}[\boldsymbol{h}_1]\,\hat{\rho}_0\,U_{(0,T)}[\boldsymbol{h}_2]\right\} \\ &= \int \mathcal{D}\boldsymbol{\phi}_r^{\text{slow}}\,\mathcal{D}\boldsymbol{\phi}_a^{\text{slow}}\left(\int \mathcal{D}\boldsymbol{\phi}_r^{\text{fast}}\,\mathcal{D}\boldsymbol{\phi}_a^{\text{fast}}\,e^{i\,S_{\text{CTP}}[\boldsymbol{\phi}_{r,a}^{\text{fast,slow}};\,\boldsymbol{h}_{r,a}]}\right) \\ &= \int \mathcal{D}\boldsymbol{\phi}_r^{\text{slow}}\,\mathcal{D}\boldsymbol{\phi}_a^{\text{slow}}\,e^{i\,S_{\text{eff}}[\boldsymbol{\phi}_{r,a}^{\text{slow}};\,\boldsymbol{h}_{r,a}]} \end{aligned}$$

Schwinger-Keldysh effective field theory



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- Dynamical $\mathbb{Z}_2$ Kubo-Martin-Schwinger symmetry, equilibrium density operator ρ_0 = imaginary translation by $i\beta^\mu$:

$$\phi_r(x^\alpha) \to \Theta\, \phi_r(x^\alpha), \quad \phi_a(x^\alpha) \to \Theta\, (\phi_a(x^\alpha) + i\beta^\alpha \partial_\alpha \phi_r(x^\alpha)),$$

[H. Liu and P. Glorioso, PoS **TASI2017** (2018)]
[Sieberer, Chiocchetta, Gambassi, Täuber and S. Diehl, PRB **92** (2015)]

Conditions

1. $S^*_{\mathrm{EFT}}[-B_a, B_r] = -S_{\mathrm{EFT}}[B_a, B_r]$ ✓
2. Kubo-Martin-Schwinger (KMS) symmetry ∼ ✓
3. $S^*_{\mathrm{EFT}}[B_a = 0, B_r] = 0$ ✓

KMS symmetry

- Similar to detailed balance, but initial and final state always ρ_0 and traced over.

[H. Liu and P. Glorioso, PoS **TASI2017** (2018)]

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[H. Liu and P. Glorioso, PoS **TASI2017** (2018)]

Bemfica-Noronha-Disconzi-Kovtun (BDNK) hydrodynamics

- Causal first-order hydrodynamics.
- All possible first-order corrections to conserved currents.
- General *hydrodynamic frame*: u^μ does not track n or ε

BDNK diffusion

- EFT-like construction via derivative expansion:

$$\partial_\mu J^\mu = 0, \ ^{\text{(classical)}} \qquad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu,$$

$$\mathcal{N} = n + \lambda\, T\, u^\alpha \partial_\alpha \left(\mu/T\right), \qquad \mathcal{J}^\alpha = -T\kappa\, \Delta^{\alpha\nu} \partial_\nu \left(\mu/T\right).$$

Schwinger-Keldysh action for BDNK hydrodynamics

Current defined via variation wrt sources $J^\mu = \delta S_{\rm EFT}/\delta A_\mu$.

$$\mathcal{L}_{\rm EFT} = \chi\, u^\mu B_{a\mu}\, u^\nu B_{r\nu} + iT\,(\kappa\Delta^{\rho\nu} - \lambda u^\rho u^\nu)\, B_{a\rho}\,[B_{a\nu} + i\beta^\alpha \partial_\alpha B_{r\nu}]$$

where $B_{s\mu} \equiv A_{s\mu} + \partial_\mu \varphi_s$, with $s = r, a$.

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[H. Liu and P. Glorioso, PoS **TASI2017** (2018)]
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[H. Liu and P. Glorioso, PoS **TASI2017** (2018)]
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Conditions

1. $S^*_{\rm EFT}[-B_a, B_r] = -S_{\rm EFT}[B_a, B_r]$ ✓
2. Kubo-Martin-Schwinger (KMS) symmetry ✓
3. $S^*_{\rm EFT}[B_a = 0, B_r] = 0$ ✓
4. Im $S_{\rm EFT}[B_a, B_r] \geq 0 \Rightarrow \lambda \leq 0$ (acausal and unstable!)

[H. Liu and P. Glorioso, PoS **TASI2017** (2018)]
[Sieberer, Chiocchetta, Gambassi, Täuber and S. Diehl, PRB **92** (2015)]

Instability and regime of validity

Consider the magnitude of the SK path integral for $A_s^\mu = 0$:

$$|e^{iS_{\rm EFT}}| = \exp\left[-TV \int d^4x\, (\kappa\Delta^{\nu\rho} - \lambda u^\nu u^\rho)\partial_\nu\varphi^a\partial_\rho\varphi^a \right]$$
$$= \exp\left[-TV \int \frac{d^4k}{(2\pi)^4}\, (\kappa\Delta^{\nu\rho} - \lambda\, u^\nu u^\rho)k_\nu k_\rho\, |\varphi^a(k^\mu)|^2 \right].$$

The path integral can only be stable if

$$\boxed{(\kappa\Delta^{\nu\rho} - \lambda\, u^\nu u^\rho)\, k_\nu k_\rho \geq 0}$$

Causal choice $\lambda \geq \kappa$ only possible if one limits the path integral to $|\omega| \leq \sqrt{\kappa/\lambda}\, |\boldsymbol{k}|$!

Pushing the theory past this regime will lead to nonsense such as $\langle|\delta n(k^\mu)|^2\rangle < 0$!