Hot and dense perturbative QCD in a very strong magnetic background

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Based on work with L.F. Palhares & T. Restrepo:

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Strong motivation: in-medium strong interactions under extreme magnetic fields are:

- of experimental relevance
  - HICs, early universe, magnetars
- rich in new phenomenology
  - Chiral magnetic effect, new QCD phase diagram, vacuum SUC
- amenable to lattice simulations: new open channel for comparison!
  - model constraining, tests for pQCD and nonpert. methods, ...

However…

Strong motivation: in-medium strong interactions under extreme magnetic fields are:

- of experimental relevance
  - HICs, early universe, magnetars

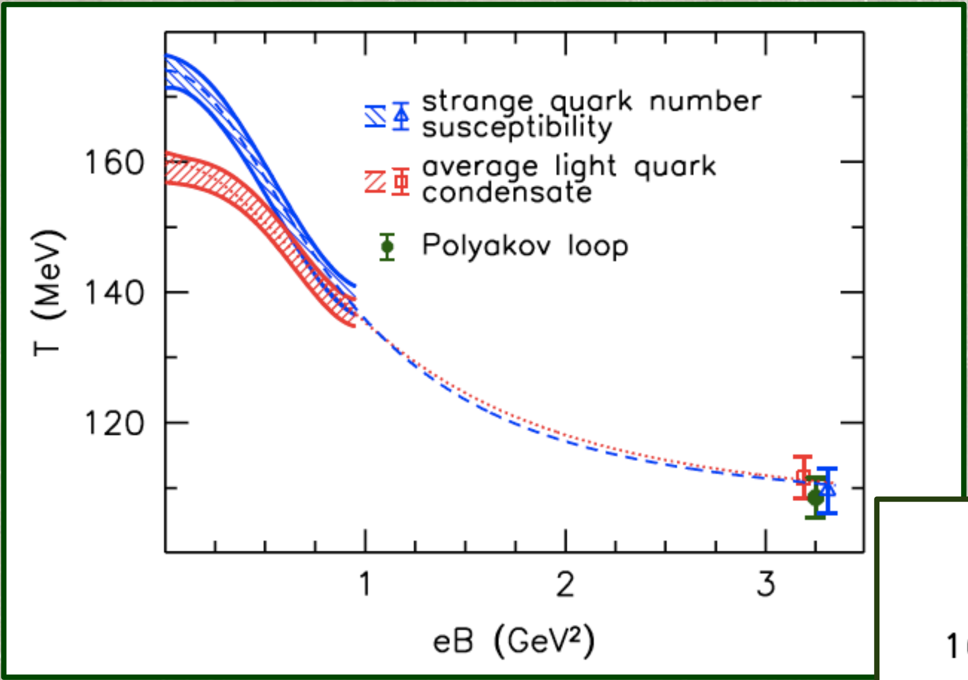
Real deal, BUT very hard for theory: difficult to compute with reasonable control over approximations & hypotheses!

- rich in new phenomenology
  - Chiral magnetic effect, new QCD phase diagram, vacuum SUC

- amenable to lattice simulations: new open channel for comparison!
  - model constraining, tests for pQCD and nonpert. methods, …

Here we can play!

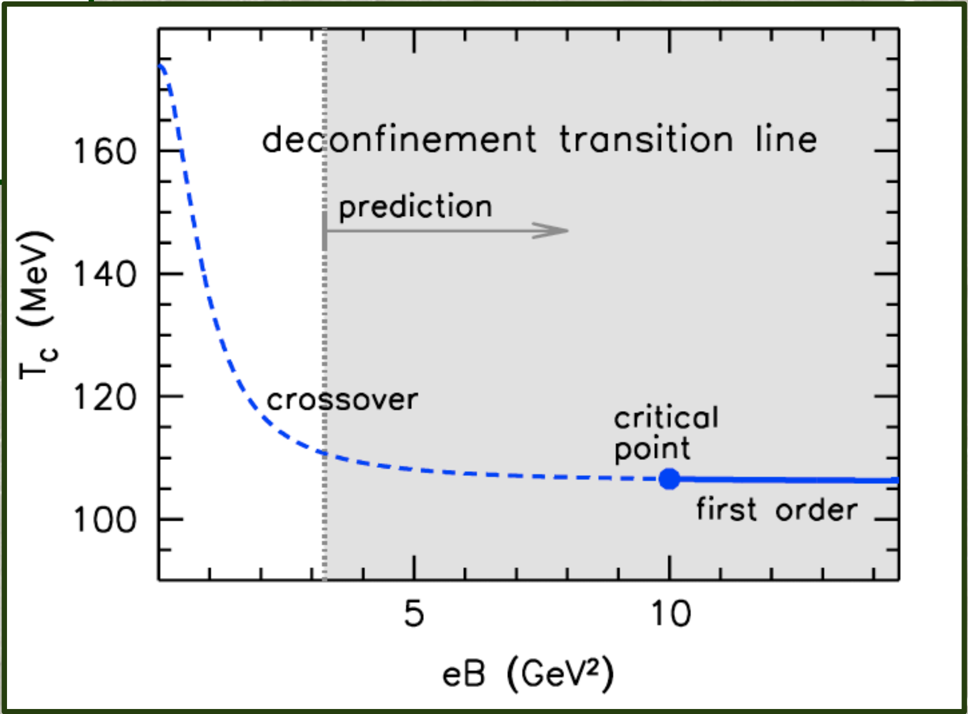
At extremely large values of B, interesting things might happen…



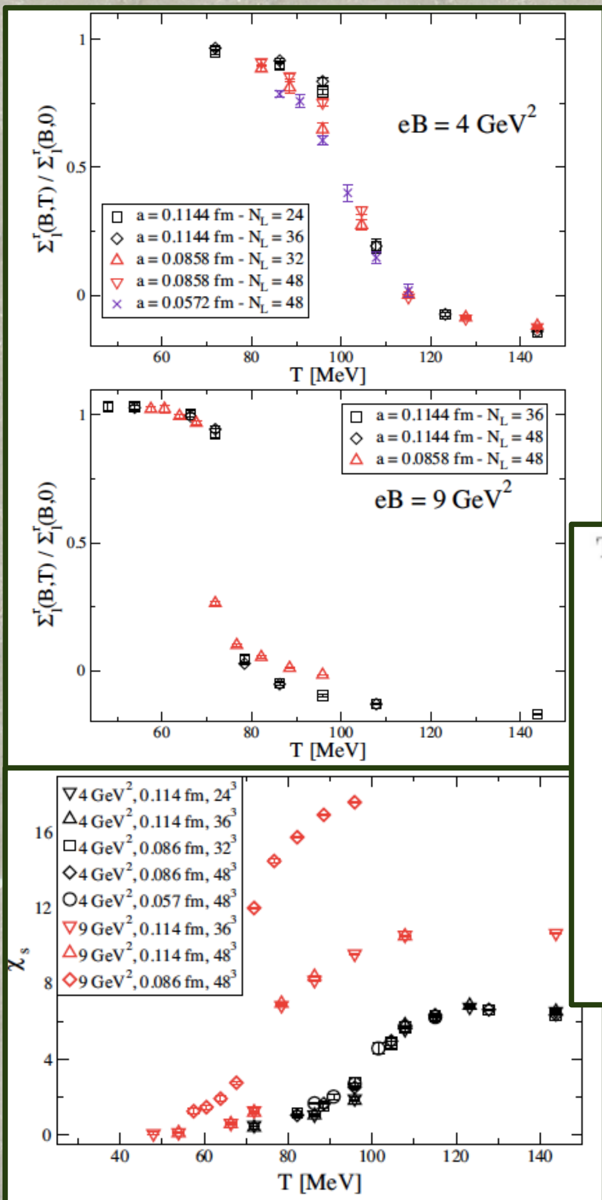
Very exciting lattice results at larger B and predictions for extremely large B !

[Endrodi (2015)]

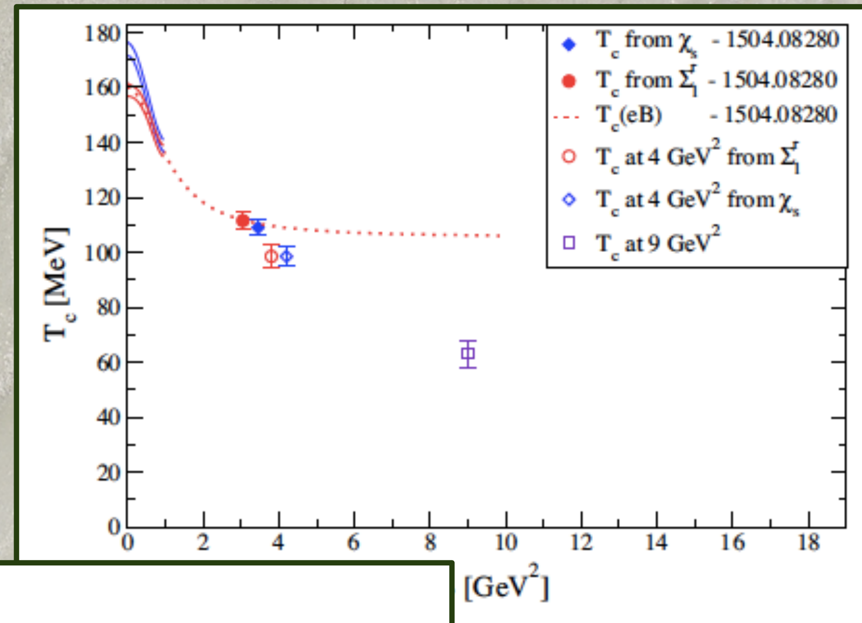
… as was discussed ~10 years ago in Endrodi's pioneering investigation.

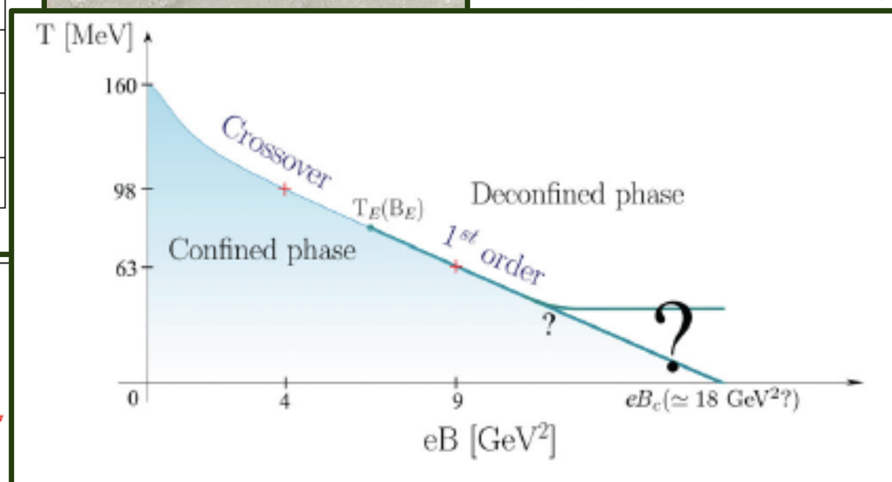


Almost ten years later, D'Elia et al revisited this problem… for higher fields!



[D'Elia et al (2022)]





$$m_s \ll T \ll \sqrt{eB}$$

Not quite yet, but we can play!

Besides Lattice QCD, which other tool can provide predictions from the actual fundamental gauge theory?

Perturbation theory!

pQCD to $O(g^2)$ in a nonperturbative magnetic background

[Blaizot, ESF & Palhares (2013)]

Framework: perturbative QCD in a nonperturbative magnetic background.

Quark-gluon interaction

up to $O(g^2)$

Exact quark propagator in a constant and uniform magnetic field:

$$\boxed{S_0 = \left[i\partial\!\!\!/ - q_f A\!\!\!/_{\rm cl}(x) - m_f \right]^{-1}}$$

$$A_{\rm cl}(x) = (0, \vec{A}(x)) \;\; | \;\; \nabla \times \vec{A} = B\hat{z}$$

Basic ingredients

- Exact fermion propagator [Schwinger (1951); Chodos et al (1990)]

$$S_0(x,y) \;=\; \Phi(x,y)\,\overline{S}_0(x-y)$$

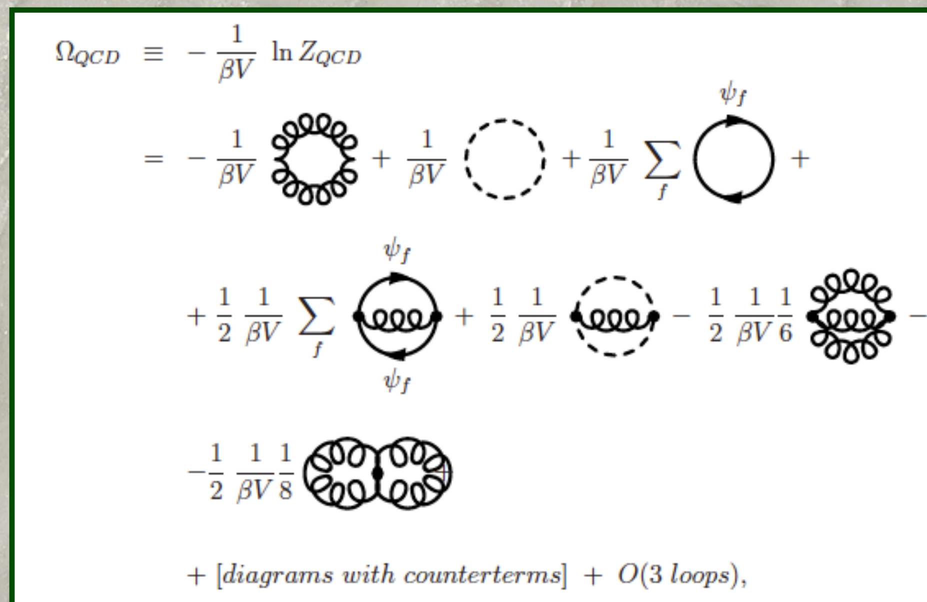
$$\Phi(x,y) \;\equiv\; \exp\left[iq\int_x^y dx'_\mu A^\mu_{\rm cl}(x')\right]$$

$$\overline{S}_0(P) \;=\; i\exp\left[-\frac{\mathbf{p}_T^2}{qB}\right]\sum_{n=0}^{\infty}(-1)^n\frac{D_n(qB,P)}{\mathbf{p}_L^2-m_f^2-2nqB}$$

=

- Thermodynamic potential:

(gluonic part from usual hot pQCD + magnetically-dressed quarks)



Exchange diagram in a magnetic background (in the LLL approx.):

[Palhares (2012); Blaizot, ESF & Palhares (2013)]

 $^{\mathrm{LLL}} = \left(\frac{q_f B}{2\pi}\right) \int \frac{dk_1 dk_2}{(2\pi)^2}\, \mathrm{e}^{-\frac{k_1^2+k_2^2}{2q_f B}}$  $^{\bar{d}=2;\; m_k^2=k_1^2+k_2^2}$.

Average over gluon "transverse mass"

exchange contribution in
dim. 2 hot QCD with a "massive gluon"

Results:

- Clear dimensional reduction in the quark dynamics.
- There are no UV divergences.
- In D=1+1, the Dirac trace is proportional to the quark mass: trivial chiral limit!

$$
\begin{aligned}
\frac{P_{\mathrm{exch}}}{N_c} = & -\frac{1}{2}\, g^2 \left(\frac{N_c^2-1}{2}\right) m_f^2 \times \\
& \left(\frac{q_f B}{2\pi}\right) \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2}\, \mathrm{e}^{-\frac{\mathbf{k}_\perp^2}{2q_f B}} \int \frac{dp_3 dq_3 dk_3}{(2\pi)^3}\, (2\pi)\delta(k_3-p_3+q_3) \times \\
& \frac{1}{\omega E_{\mathbf{p}} E_{\mathbf{q}}} \left\{ \frac{\omega\, \Sigma_+}{E_-^2-\omega^2} + \frac{\omega\, \Sigma_-}{E_+^2-\omega^2} + 2\left[\frac{E_+}{E_+^2-\omega^2} - \frac{E_-}{E_-^2-\omega^2}\right] n_B(\omega)\, N_F(1) \right. \\
& \left. -\left[\frac{2(E_{\mathbf{q}}+\omega)}{(E_- -\omega)(E_+ +\omega)}\right] N_F(1) - 2\,\frac{E_+}{E_+^2-\omega^2}\, n_B(\omega) - \frac{1}{E_+ +\omega} \right\},
\end{aligned}
$$

IR-divergent sum-integrals to deal with, BUT:

- NLO always 1-2 orders of magnitude below the leading contribution.
  ➡ Improved convergence of the perturbative series at high T and extremely large B?

- No IR divergence -> trivial chiral limit! [strong suppression for small masses]

At the time (2013):

[Blaizot, ESF & Palhares (2013)]

- No running coupling, no running mass, no bands.
- No other observables.
- No test for ultra-high B fields.

~10 years later, we revisited magnetic pQCD. Now with:

[ESF, Palhares & Restrepo (2023)]

- Running coupling, running mass & bands.
- Dependence on the choice of the renormalization scale.
- Pressure, quark condensate, strange quark number susceptibility.
- Tests for ultra-high B fields ("reached" on the lattice!).

$$\boxed{m_s \ll T \ll \sqrt{eB}}$$

Differently from what was done in 2013, it is more convenient to perform momentum integrals first, and leave one integral over the transverse mass and the Matsubara sums to the end (valid for $\mu = 0$). In the LLL limit:

[ESF, Palhares & Restrepo (2023)]

NLO (exchange) contribution to the pressure:

$$\frac{P_{\rm exch}^{\rm LLL}}{N_c} = \frac{1}{2} g^2 \left(\frac{N_c^2-1}{2N_c}\right) T^2 \sum_f m_f^2 \left(\frac{q_f B}{2\pi}\right) \sum_{\ell, n_2} \int \frac{dm_k}{2\pi} m_k e^{-\frac{m_k^2}{2q_f B}} \frac{\mathcal{E}_\ell - \mathcal{E}_{n_2}}{\mathcal{E}_\ell \mathcal{E}_{n_1} \mathcal{E}_{n_2} |\mathcal{E}_\ell - \mathcal{E}_{n_2}| (|\mathcal{E}_\ell - \mathcal{E}_{n_2}| + \mathcal{E}_{n_1})}$$

$$\mathcal{E}_\ell = \sqrt{\omega_\ell^2 + m_k^2},\ \mathcal{E}_{n_1} = \sqrt{(\omega_{n_2} + \omega_\ell)^2 + m_f^2},\ \text{and}\ \mathcal{E}_{n_2} = \sqrt{\omega_{n_2}^2 + m_f^2}$$

$$\omega_\ell = 2\pi \ell T \ \text{and}\ \omega_{n_2} = (2n_2+1)\pi T$$

We also need (LO, quarks):

$$\frac{P_{\rm free}^{\rm LLL}}{N_c} = -\sum_f \frac{(q_f B)^2}{2\pi^2} [x_f \ln \sqrt{x_f}] + T \sum_f \frac{q_f B}{2\pi} \int \frac{dp_z}{2\pi} \left\{ \ln\left(1 + e^{-\beta[E(0,p_z)-\mu_f]}\right) + \ln\left(1 + e^{-\beta[E(0,p_z)+\mu_f]}\right) \right\}.$$

$$P_{\rm free}^G = 2(N_c^2-1)\frac{\pi^2 T^4}{90}$$

(gluons)

$$P_2^G = -N_c(N_c^2-1)\frac{g^2 T^4}{144}$$

Chiral condensate

$$\langle\bar{\psi}_f\psi_f\rangle = -\frac{\partial P_f}{\partial m_f} = -\frac{\partial P^{\rm LLL}_{\rm free}}{\partial m_f} - \frac{\partial P^{\rm LLL}_{\rm exch}}{\partial m_f}$$

- Massless quarks: true order parameter for the chiral transition.
- Light quark masses: pseudo order parameter.
- Perturbative analysis reliable only for very large T & even larger B - cannot bring information near the phase transition or crossover.
- Nevertheless: there are lattice results for high T & B —> comparison of two first-principle calculations in this region is certainly relevant!

From magnetic pQCD

$$\frac{\partial P^{\rm LLL}_{\rm free}}{\partial m_f} = -N_c m_f \frac{q_f B}{(2\pi)^2}\left[1 + \ln x_f + \int dp_z \frac{2n_F(E_p)}{E_p}\right]$$

$$\frac{\partial P^{\rm LLL}_{\rm exch}}{\partial m_f} = -\frac{1}{2}g^2\left(\frac{N_c^2-1}{2}\right)T^2\left(\frac{q_f B}{2\pi}\right)\sum_{l,n_2}\int\frac{dm_k}{2\pi}m_k e^{-\frac{m_k^2}{2q_f B}}$$
$$\times\left\{\frac{m_f^3[\mathcal{E}_{n_2}|\mathcal{E}_l-\mathcal{E}_{n_2}| - \mathcal{E}_{n_1}(\mathcal{E}_l-\mathcal{E}_{n2})]}{\mathcal{E}_l\mathcal{E}_{n_1}^2\mathcal{E}_{n_2}^2(\mathcal{E}_l-\mathcal{E}_{n_2})(|\mathcal{E}_l-\mathcal{E}_{n_2}|+\mathcal{E}_{n_1})^2} - \frac{m_f(\mathcal{E}_l-\mathcal{E}_{n_2})[2(\omega_{n_2}+\omega_l)^2\omega_{n_2}^2 + m_f^2((\omega_{n_2}+\omega_l)^2+\omega_{n_2}^2)]}{\mathcal{E}_l\mathcal{E}_{n_1}^3\mathcal{E}_{n_2}^3|\mathcal{E}_l-\mathcal{E}_{n_2}|(|\mathcal{E}_l-\mathcal{E}_{n_2}|+\mathcal{E}_{n_1})}\right\}$$

• On the lattice, one computes the f-flavor renormalized condensate (to eliminate additive and multiplicative divergences):

$$\Sigma_f^r(B,T) = \frac{m_f}{m_\pi^2 f_\pi^2}\left[\langle\bar{\psi}_f\psi_f\rangle_{B,T} - \langle\bar{\psi}_f\psi_f\rangle_{0,0}\right]$$

• NB: To obtain the vacuum condensate, one cannot simply take the zero-field limit since we assumed very large fields from the outset.

Strange quark number susceptibility

$$\chi^s = \frac{1}{T^2}\frac{\partial^2 P}{\partial\mu_s^2},$$

• Given the presence of a derivative with respect to the chemical potential, pure vacuum terms are excluded. —> advantage when comparing lattice results to pQCD, even if the T range in the simulations is still far from optimal for this purpose.

Running

$$\alpha_s(\bar\Lambda) = \frac{4\pi}{\beta_0 L}\left(1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln L}{L}\right)$$

$$m_s(\bar\Lambda) = \hat m_s \left(\frac{\alpha_s}{\pi}\right)^{4/9}\left[1 + 0.895062\left(\frac{\alpha_s}{\pi}\right)\right]$$

$$\beta_0 = 11 - 2N_f/3,\ \beta_1 = 51 - 19N_f/3,\ L = 2\ln\left(\bar\Lambda/\Lambda_{\overline{\rm MS}}\right)$$

$$\Lambda^{2+1}_{\overline{\rm MS}} = 343^{+18}_{-12}\ \mathrm{MeV}$$

$$\hat m_s^{2+1} \approx 248.7\ \mathrm{MeV}$$

Arbitrariness in choosing the renormalization scale:

- Thermal QCD: besides quark masses, the only scale is $T \gg m_f$ —> usual choice: Matsubara frequency $2\pi T$ with a band around it ($\pi T < \Lambda < 4\pi T$).

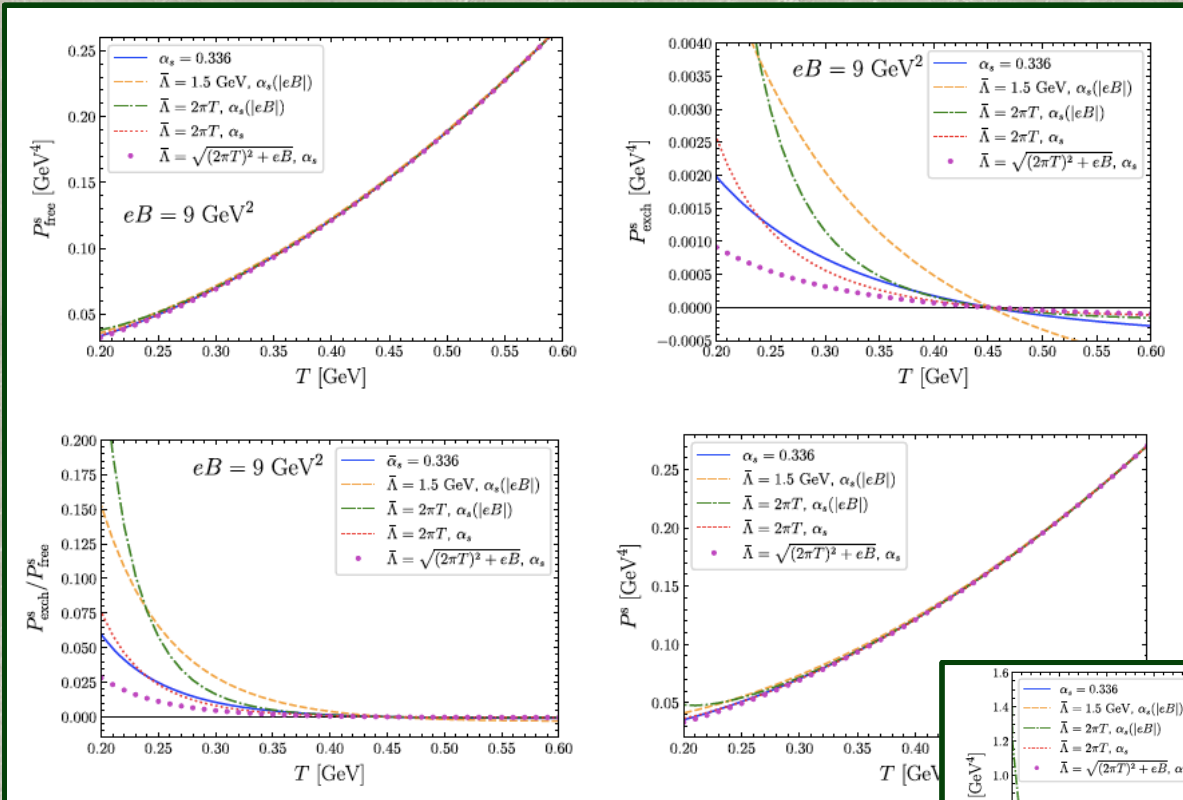
- In the present case, we have 3 mass scales: $(eB)^{1/2}$, T & m_f

In the literature, one can find a few different assumptions for the form of the running coupling.

We choose: QCD running with $\bar\Lambda = \sqrt{(2\pi T)^2 + eB}$ [extension of what is done in in-medium field theory]

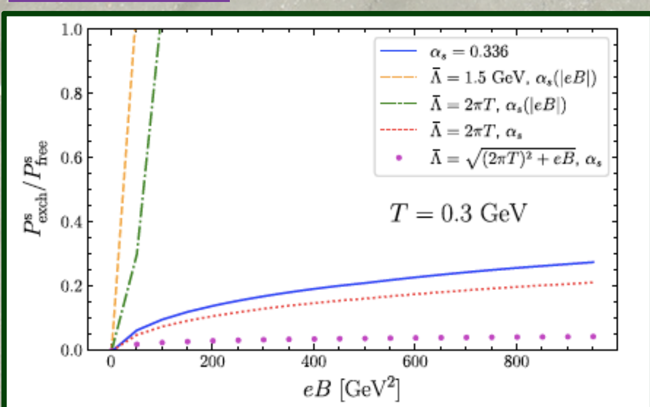
Results for the pressure

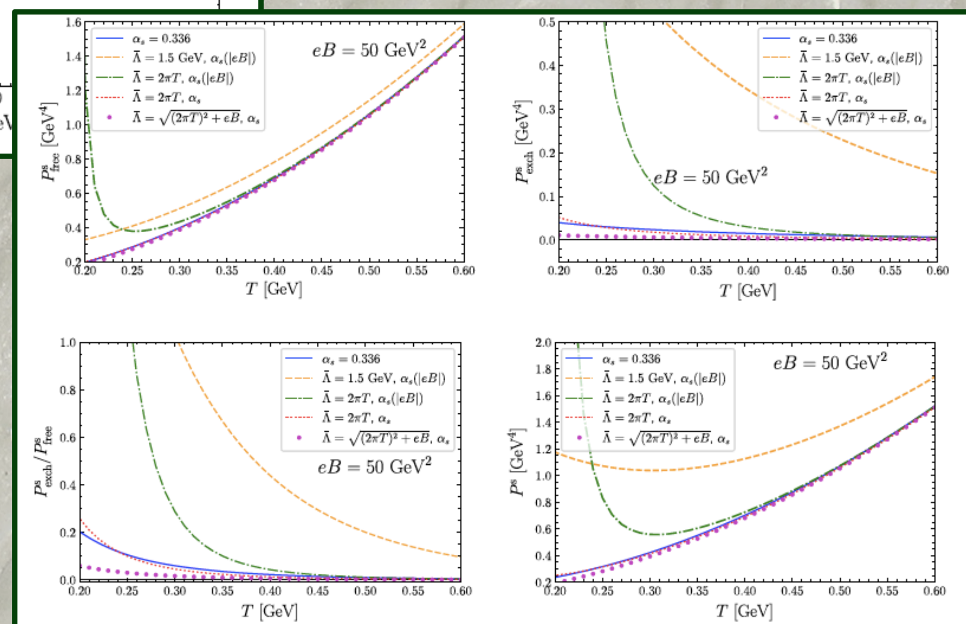
[ESF, Palhares & Restrepo (2023)]



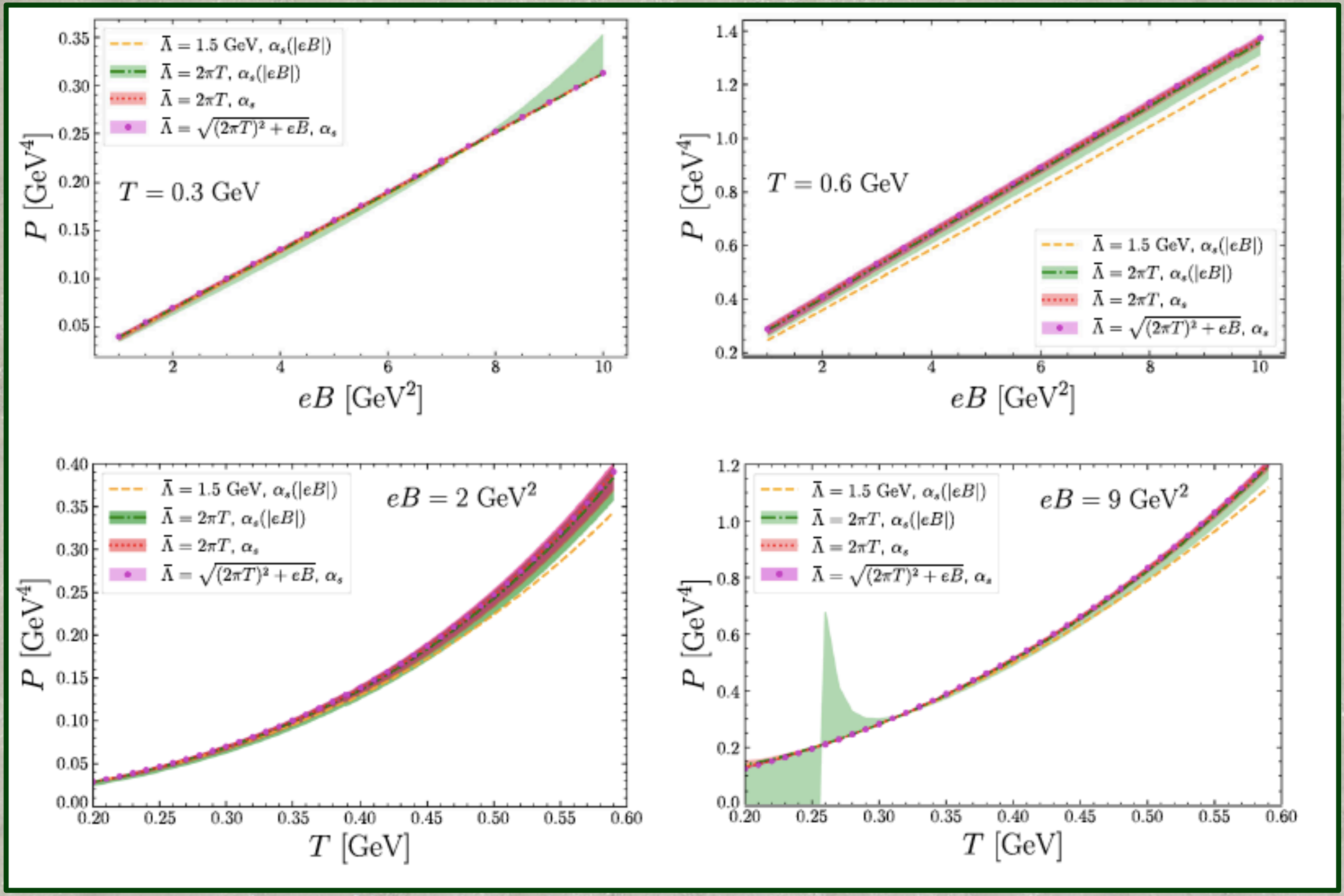
- Free pressure, exchange contribution, ratio, full pressure.
- Results for strange quark because mass effects are more relevant.
- P_{exch}/P_{free}: measure of the reliability of perturbation theory (more well behaved than the case in the absence of a large magnetic field. [Blaizot, ESF & Palhares (2013)]

Huge B fields





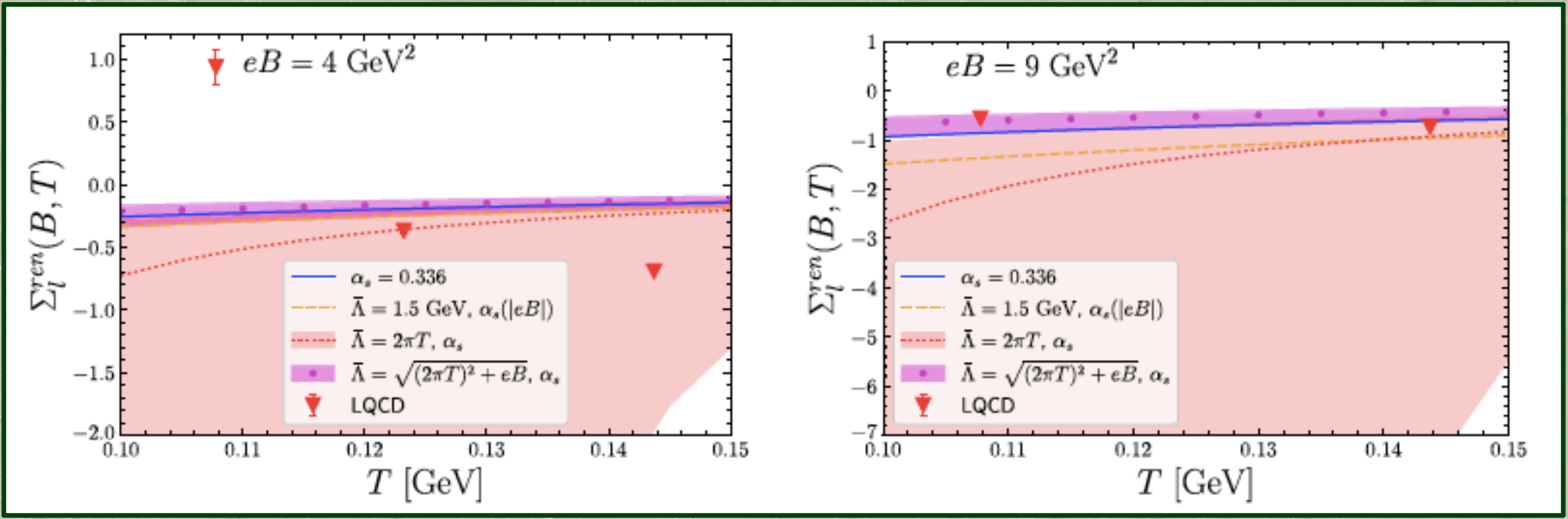
Full pressure with bands



- Bands: increasing/decreasing the central renormalization scale by a factor of 2.
- Bands ~ measure of the theoretical uncertainty of the perturbative series.
- Case (ii) has no band by construction, since Λ is fixed.

Results for the renormalized light chiral condensate

[ESF, Palhares & Restrepo (2023)]

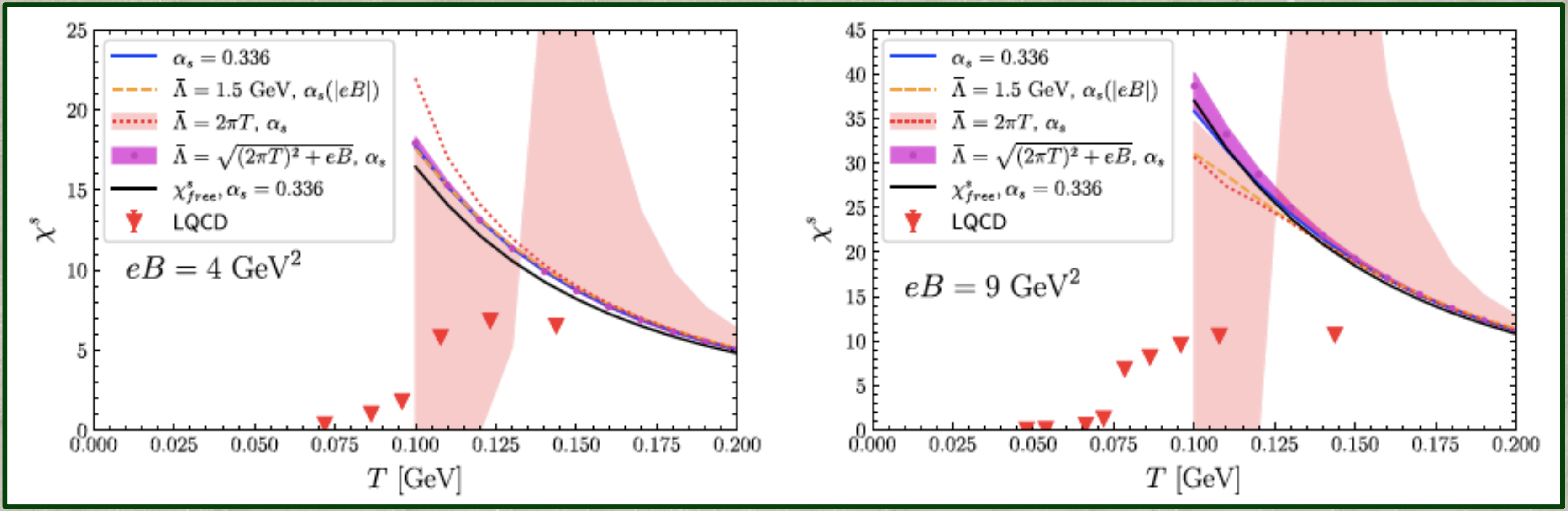


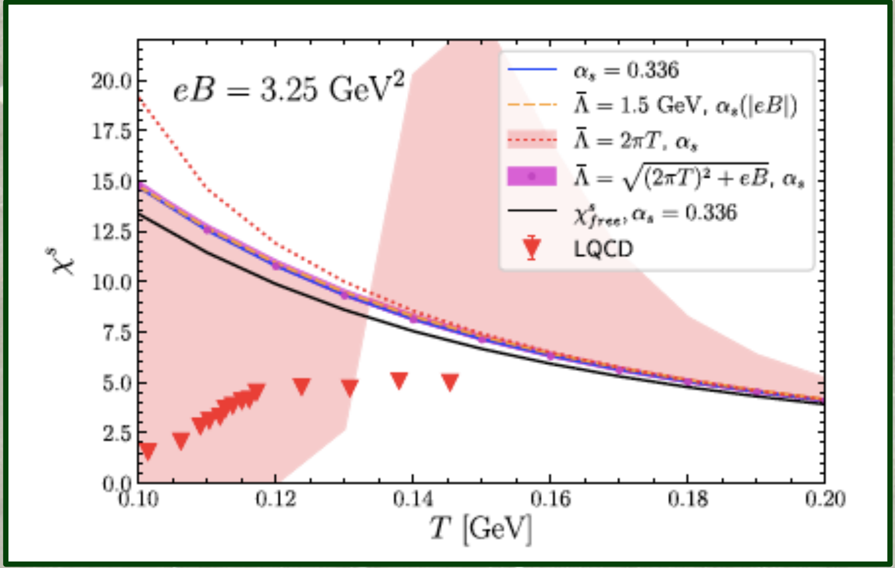
- Lattice data from D'Elia et al (2022).

- The width of the band for case (iii) basically diverges (not shown in the figures), case (iv) has a wide band that also diverges at some point for the susceptibility, and case (v) is always well behaved.

Results for the strange quark number susceptibility

[ESF, Palhares & Restrepo (2023)]





- Lattice data from D'Elia et al (2022) & Endrodi (2014).
- Same qualitative behavior for the different running choices.
- Lattice data still far from

$$m_s \ll T \ll \sqrt{eB}$$

Cold & dense QCD case

[ESF, Palhares & Restrepo (2024)]

• In the complementary scenario (T=0, nonzero density), the sum-integrals and distributions become simpler. At the end, we find:

$$\frac{P^{\rm LLL}_{\rm free}}{N_c} = -\sum_f \frac{(q_f B)^2}{2\pi^2}[x_f \ln \sqrt{x_f}] + \sum_f \frac{q_f B}{2\pi}\int \frac{dp_z}{2\pi}(\mu_f - E)\Theta(\mu_f - E)$$
$$= -\sum_f \frac{(q_f B)^2}{2\pi^2}[x_f \ln \sqrt{x_f}] + \sum_f \frac{q_f B}{2\pi}\int_0^{P_F} \frac{dp_z}{2\pi}(\mu_f - E)$$
$$= -\sum_f \frac{(q_f B)^2}{2\pi^2}[x_f \ln \sqrt{x_f}] + \sum_f \frac{(q_f B)}{4\pi^2}\left[\mu_f P_F - m_f^2 \log\left(\frac{\mu_f + P_F}{m_f}\right)\right]$$

$$E = \sqrt{p_z^2 + m_f^2}, x_f \equiv m_f^2/2q_f B$$

$$P_F = \sqrt{\mu_f^2 - m_f^2}$$

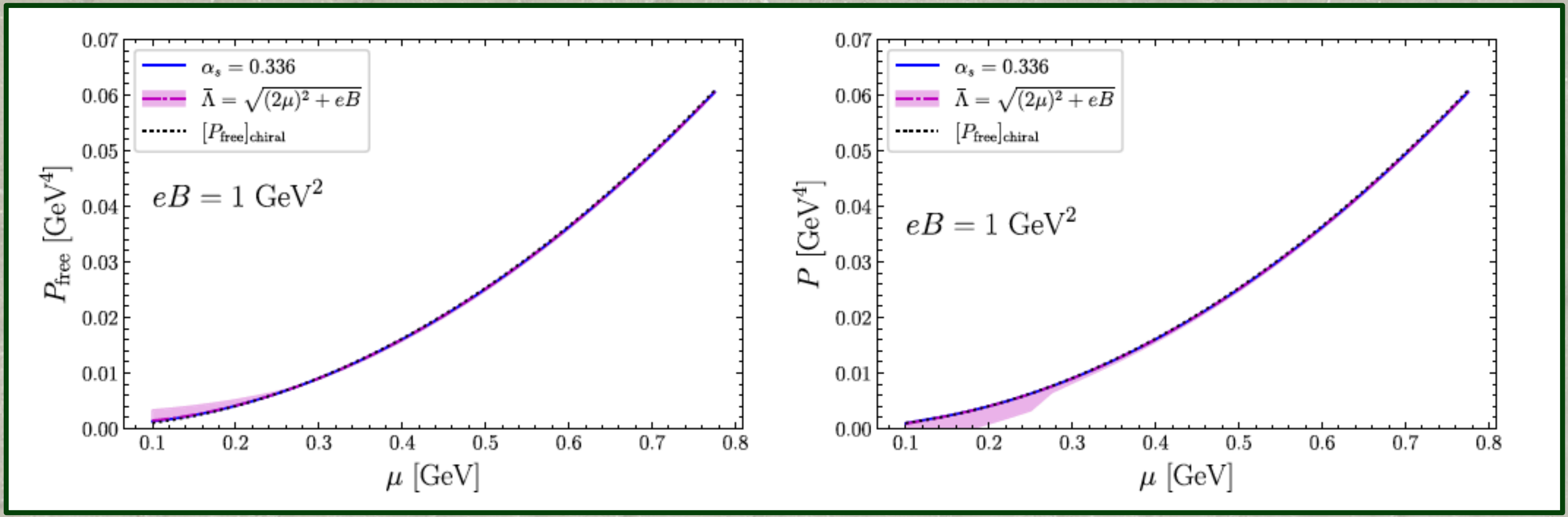
$$\frac{P^{\rm LLL}_{\rm exch}}{N_c} = -\frac{1}{2}g^2\left(\frac{N_c^2-1}{2N_c}\right)m_f^2\left(\frac{q_f B}{2\pi}\right)\int \frac{dm_k}{2\pi} m_k e^{-\frac{m_k^2}{2q_f B}}\int \frac{dp_z dq_z dk_z}{(2\pi)^3}(2\pi)\delta(k_z - p_z + q_z)$$
$$\times \frac{1}{\omega E_p E_q}\left\{\frac{\omega}{E_-^2 - \omega^2}\Theta(\mu_f - E_{\mathbf{p}})\Theta(\mu_f - E_{\mathbf{q}}) - \left[\frac{2(E_{\mathbf{q}} + \omega)}{(E_- - \omega)(E_+ + \omega)}\right]\Theta(\mu_f - E_{\mathbf{p}}) - \frac{1}{E_+ + \omega}\right\}$$

$$m_s \ll \mu_q \ll \sqrt{eB}$$

Here, we choose: QCD running with $\bar{\Lambda} = \sqrt{(2\mu_f)^2 + eB}$ [extension of what is done in in-medium field theory]

Results for the pressure

[ESF, Palhares & Restrepo (2024)]



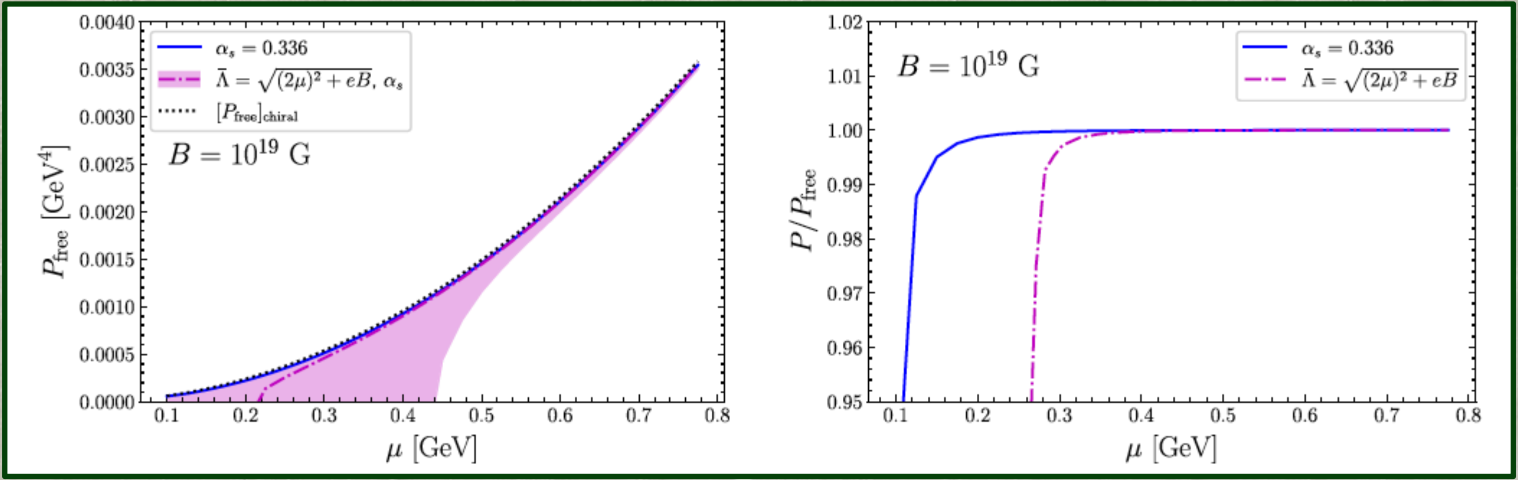
• For high B: two-loop contribution represents a correction of only a few percent.

• Even for much smaller fields (B ~ 10^{18} Gauss), possibly attainable in the core of magnetars, the exchange contribution remains quite small.

• On the other hand, RG running effects are always relevant and also affect the leading term via the running of the mass.

• For high B, one can build a pQCD-based simple analytic model for the EoS of cold and dense QM that could be used, phenomenologically, to describe magnetars.

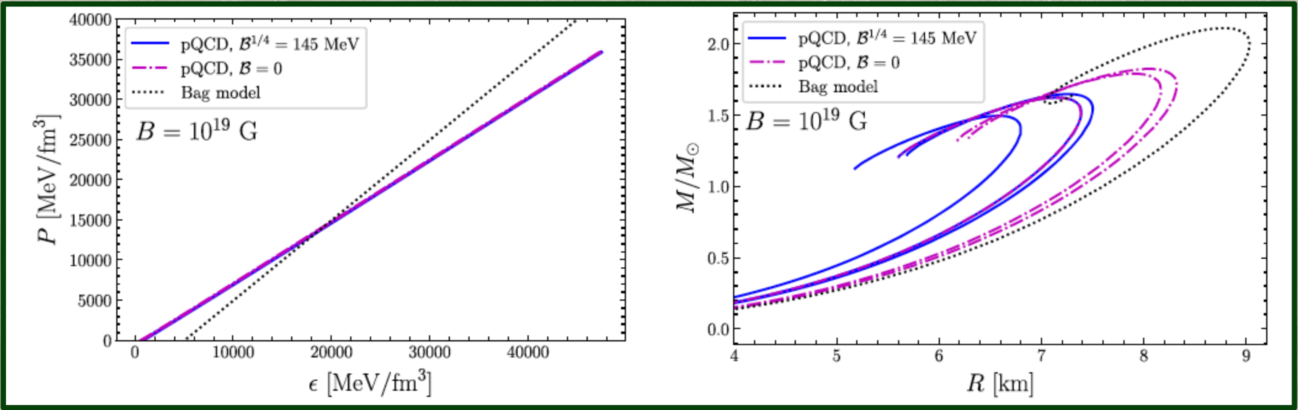
$$\frac{P_{\rm eff}}{N_c} = -\sum_f \frac{(q_f B)^2}{2\pi^2}\left[x_f \ln\sqrt{x_f}\right] + \sum_f \frac{(q_f B)}{4\pi^2}\left[\mu_f P_F - m_f^2 \log\left(\frac{\mu_f + P_F}{m_f}\right)\right]$$



[ESF, Palhares & Restrepo (2024)]

• NB: of course, we are pushing magnetic pQCD away from its region of validity. This pressure should be matched onto a low-density pressure for the description of a hybrid magnetar.

Constraints on quark magnetars from pQCD at very large B



- Charge-neutral and beta-equilibrated matter:

$$\boxed{\mu_d = \mu_s = \mu_u + \mu_e}$$

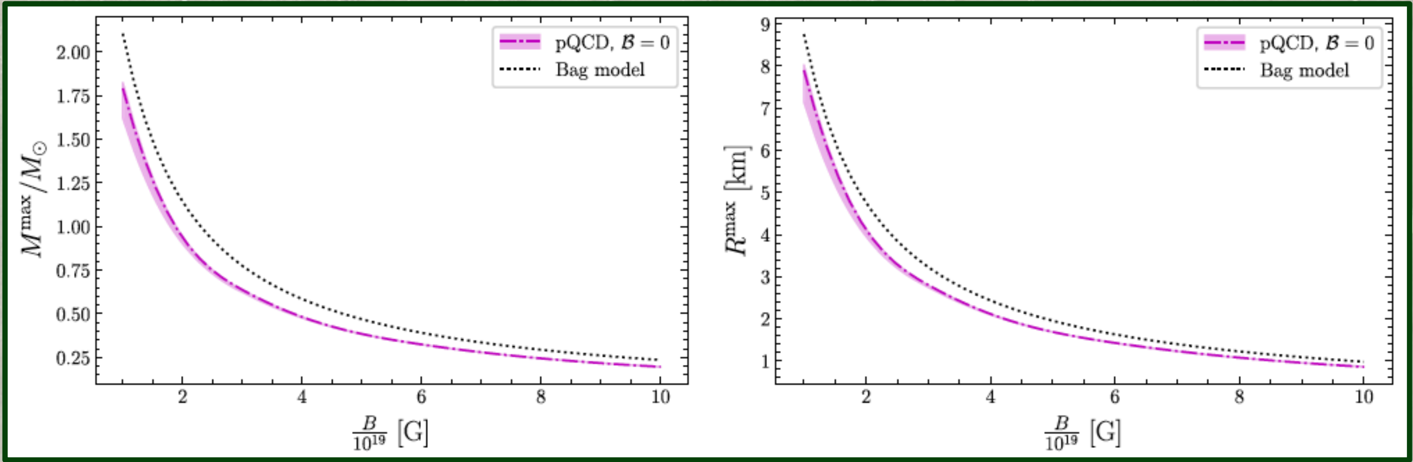
$$\boxed{\frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0}$$

- Densities:

$$\boxed{n_e = \frac{eB\mu_e}{2\pi^2}} \quad \boxed{n_f = N_c\frac{q_f BP_F}{2\pi^2}}$$

- Magnetic pQCD results valid for very high B.

- Effective models should ideally approach the perturbative results in this limit (constraints).



Final remarks

- Magnetic thermal/dense pQCD with physical quark masses indicates that loop corrections are subdominant in the pressure for large B. The NLO term is comparatively very small.

- Window of applicability is still narrow, but results obtained from a clean first principle calculation that can be systematically improved & controlled.

- Results for the chiral condensate & strange quark number susceptibility compared to recent lattice QCD data away from the chiral transition. Even if still out of the region of validity, pQCD results seem to be in the same ballpark, which is encouraging.

- It would be great to have lattice results at even higher B (and T)!

- For huge values of B, one can build a pQCD-inspired analytic model in the case of cold magnetic pQCD. Might be relevant for magnetar microphysics and can be used to constrain models.

Back up slides

Summary of a very long story:

➢ Usual chiral models (PQM, PNJL, …): do not capture the behavior of the critical line.

➢ Introducing a B dependence in their parameters does not help. [ESF, Mintz & Schaffner-Bielich (2013)] (other tentatives in the same direction alleged success, but either failed to satisfy physical conditions or had T_c turning up after some point)

➢ Including the actual running of the couplings of such chiral models with a scale that is B- and T-dependent didn't help either [Endrodi & ESF (unpublished tests in 2014)].

➢ Implementing (by hand) a "QCD-inspired running" that is B- and T-dependent [see e.g. Farias et al. (2014, 2016)] or including some B- and T-dependent dressing [see e.g. Ayala et al. (2015)] seems to do a good job for describing the behavior of $T_c(B)$ (even though, with some slightly different features as compared to lattice results) and condensates as one optimizes the model parameters. More on this later.

➢ Although artificially included, this points to a possible relevant role of asymptotic freedom in the phenomenon of inverse magnetic catalysis. And makes this description at least a good fit for several lattice results.

➢ On the other hand, MUCH simpler descriptions seemed to capture essential features of lattice results for the behavior of $T_c(B)$ soon after they appeared…

Slide from 2016…

From my perspective:

failure of effective chiral models...

Let's try something else

Bag model

[ESF & Palhares (2012)]

surprisingly good qualitative results - puzzling, hard to justify by itself.

Large N

[ESF, Noronha & Palhares (2013)]

good qualitative results - QCD in that limit! But hard to get more quantitative or improve...

Magnetic pQCD

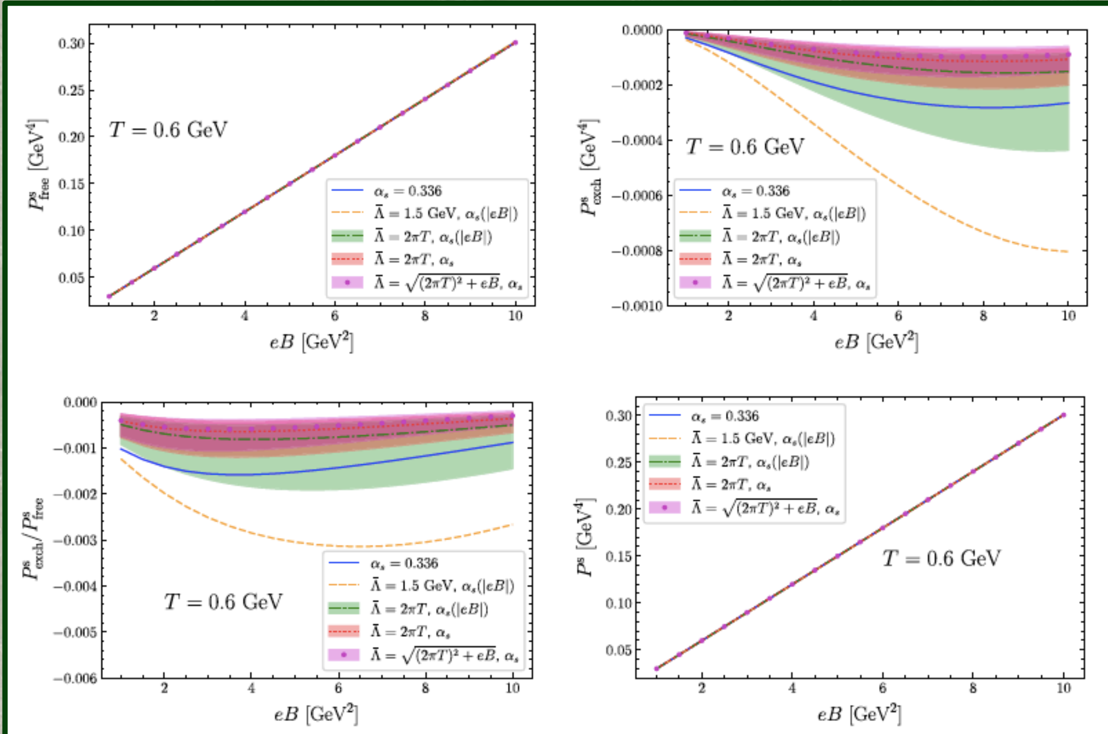
[Palhares (2012); Blaizot, ESF & Palhares (2013)]

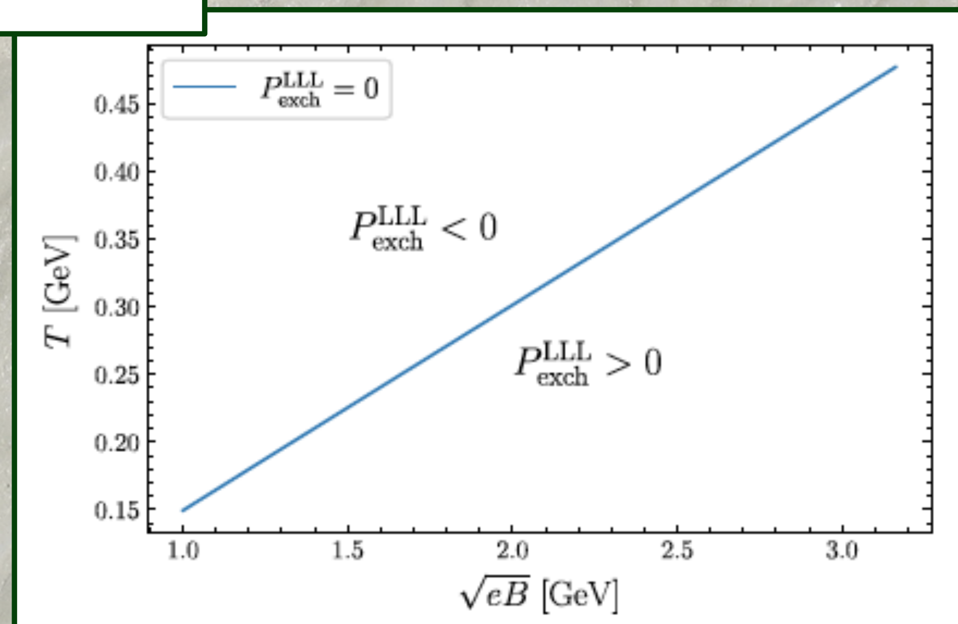
QCD in that limit! Systematic & controllable approximations, but...

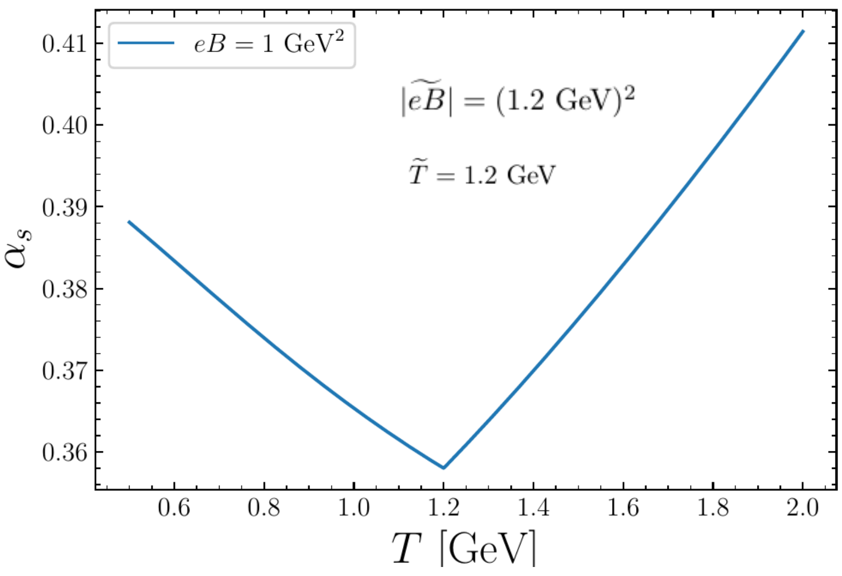
...needs really high B fields! Actually, needs

$$\boxed{m_s \ll T \ll \sqrt{eB}}$$

Already at the time of my old slide...


$T = 0.6$ GeV
$P^s_{\rm free}$ [GeV^4]
$P^s_{\rm exch}$ [GeV^4]
$P^s_{\rm exch}/P^s_{\rm free}$
P^s [GeV^4]
eB [GeV^2]
$\alpha_s = 0.336$
$\bar{\Lambda} = 1.5$ GeV, $\alpha_s(|eB|)$
$\bar{\Lambda} = 2\pi T$, $\alpha_s(|eB|)$
$\bar{\Lambda} = 2\pi T$, α_s
$\bar{\Lambda} = \sqrt{(2\pi T)^2 + eB}$, α_s


$P^{\rm LLL}_{\rm exch} = 0$
$P^{\rm LLL}_{\rm exch} < 0$
$P^{\rm LLL}_{\rm exch} > 0$
T [GeV]
$\sqrt{eB}$ [GeV]

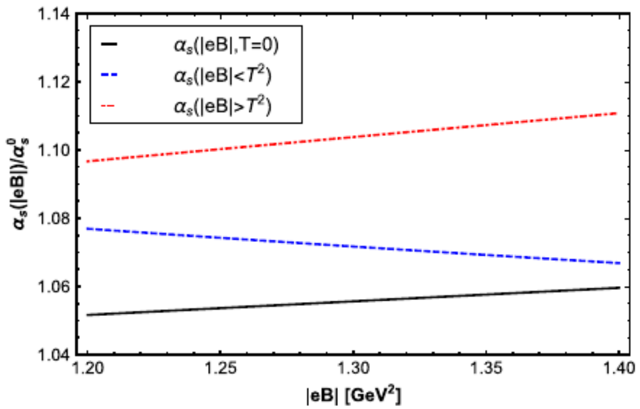


$|eB| < T^2$

$$\alpha_s(|eB|) = \frac{\alpha_s(Q^2 + \widetilde{|eB|})}{1 + b_1 \alpha_s(Q^2 + \widetilde{|eB|}) \ln\left(\frac{Q^2+|eB|}{Q^2+|eB|+T^2}\right)}$$

$|eB| > T^2$

$$\alpha_s(|eB|) = \frac{\alpha_s(Q^2 + \tilde{T}^2)}{1 + b_1 \alpha_s(Q^2 + \tilde{T}^2) \ln\left(\frac{Q^2+T^2}{Q^2+T^2+|eB|}\right)}$$



Usamos $\alpha_s(|eB|) = \dfrac{\alpha_s(\Lambda^2)}{1 + b_1 \ln(\frac{\Lambda^2}{\Lambda^2+|eB|})}$ com $\Lambda = 2\pi T$.

B. Karmakar et all PhysRevD.99.094002(2019)

Ayala et all PhysRevD.98.031501(2018)

[From talk by T. Restrepo (2022)]

• We show results for a few representative choices and discuss their implications for our observables.

• Although we have our preference for the most physical choice, we believe that, ultimately, this must be settled by direct comparison to lattice QCD simulations.

• We consider the following cases:

(i) A fixed value of $\alpha_s = 0.336$

[ignore running]

(ii) The running [Ayala et al (2018)]

$$\alpha_s(|eB|) = \frac{\overline{\alpha}_s(\bar{\Lambda}^2)}{1 + (\beta_0/4\pi)\overline{\alpha}_s(\bar{\Lambda}^2)\ln(\frac{\bar{\Lambda}^2}{\bar{\Lambda}^2+|eB|})}$$

$\overline{\alpha}_s(\bar{\Lambda}^2)$ (usual MSbar one-loop running)

$\bar{\Lambda} = 1.5 \text{ GeV}$

[Motivation: provide an understanding of inverse magnetic catalysis]

(iii) Same as previous, but with $\Lambda = 2\pi T$

[Karmakar et al (2019)]

(iv) QCD running with usual thermal choice ($\Lambda = 2\pi T$)

[ignore effect from B on the running scale]

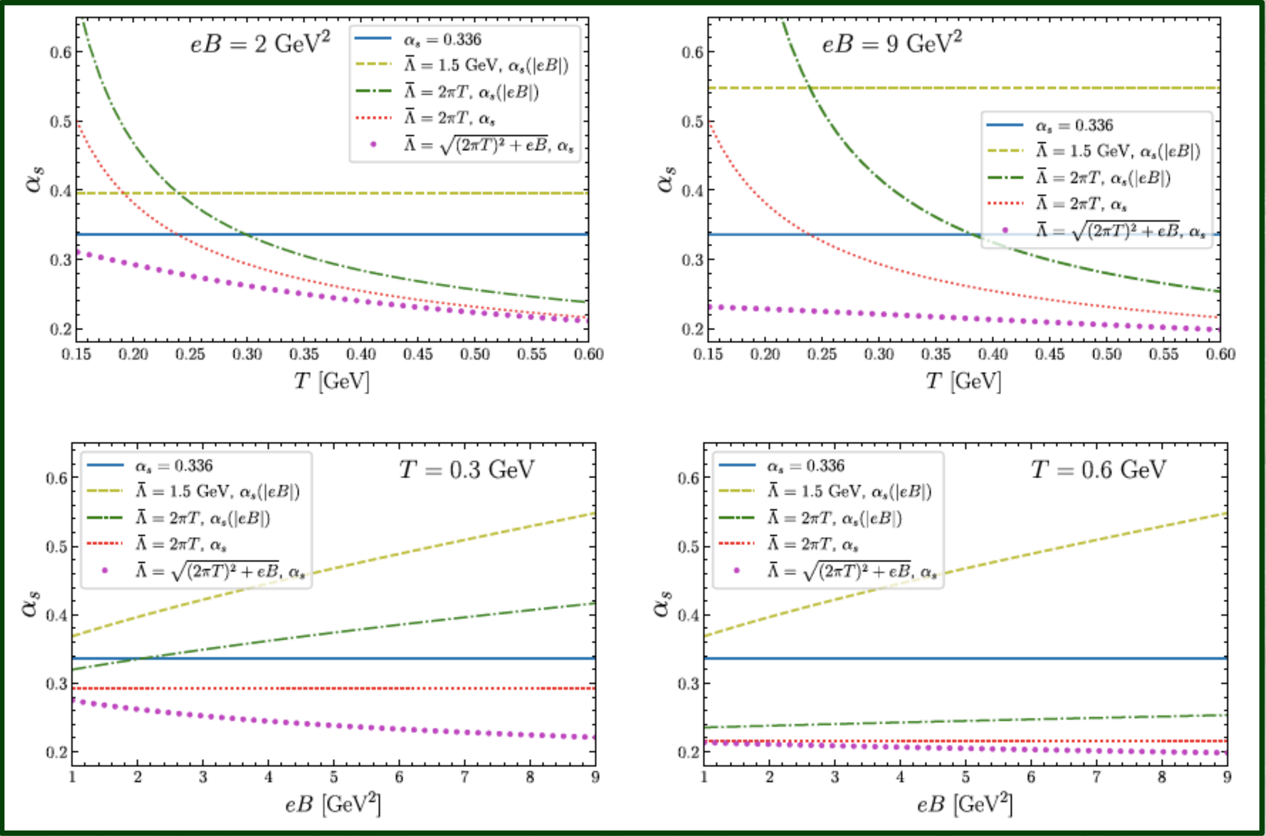
(v) QCD running with

$$\bar{\Lambda} = \sqrt{(2\pi T)^2 + eB}$$

[extension of what is done in in-medium field theory]

Results for the running coupling

[ESF, Palhares & Restrepo (2023)]



•Cases (ii) & (iii) display an unphysical behavior with increasing B: α_s simply grows while the energy density is also increasing. Then:

⋆ Perturbative calculations meaningless for high B.

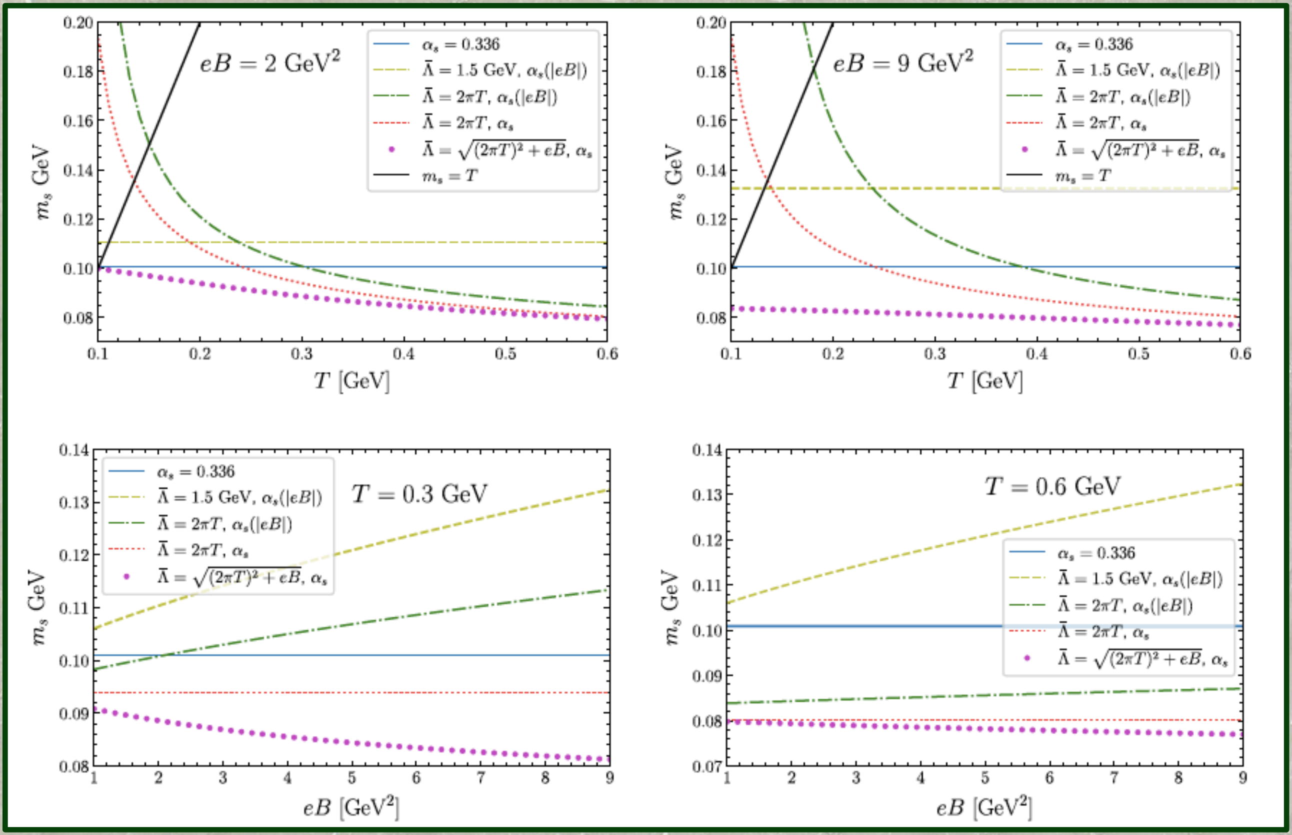
⋆ Incompatible with asymptotic freedom.

• In cases (iv) & (v), α_s exhibits the same qualitative (usual for QCD) behavior.

⋆ Quantitative difference because in case (v) B contributes to the running scale on an equal footing with respect to the temperature.

Results for the running strange quark mass

[ESF, Palhares & Restrepo (2023)]



- Black continuous line for $m_s = T$ (reminder of the constraint $m_s \ll T$).

- Behavior of different running cases analogous to what has been discussed for α_s.

- Quark mass increase with B probably related to the original motivation of running choices like cases (ii) & (iii) — trying to encode magnetic catalysis & inverse magnetic catalysis in the properties of the running of the strong coupling.

- We believe that only cases (iv) & (v) provide a physical description of the running coupling & running quark mass.

- Since it can also be tested by direct comparison to lattice data, we keep all cases in our results for the pressure, chiral condensate & strange quark number susceptibility.

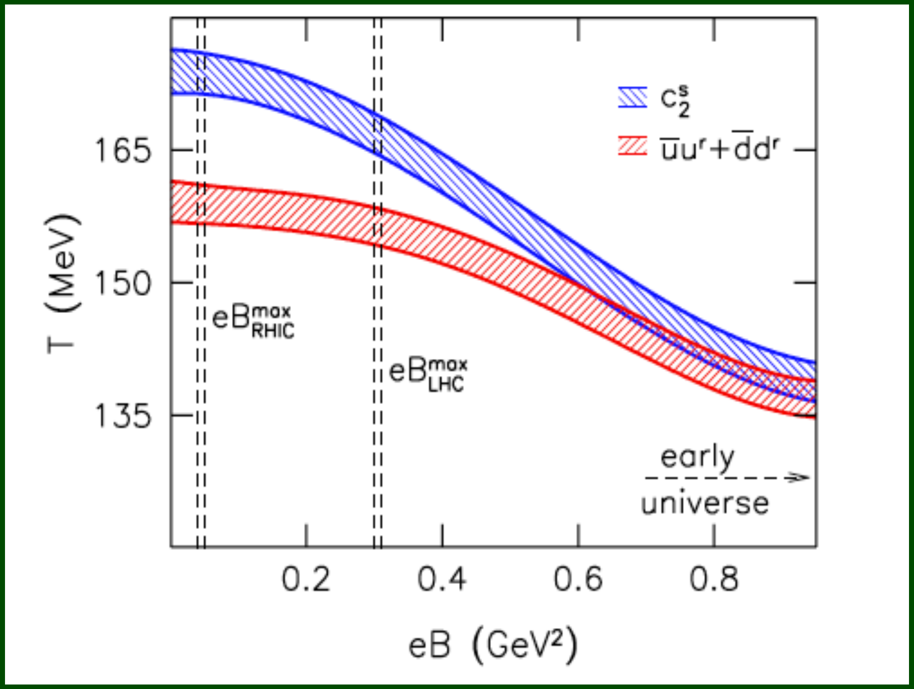
- Cases (ii) & (iii): much poorer convergence; becomes worse as one increases B.

- Compatible with the somewhat unphysical behavior observed in their running α_s & m_s.

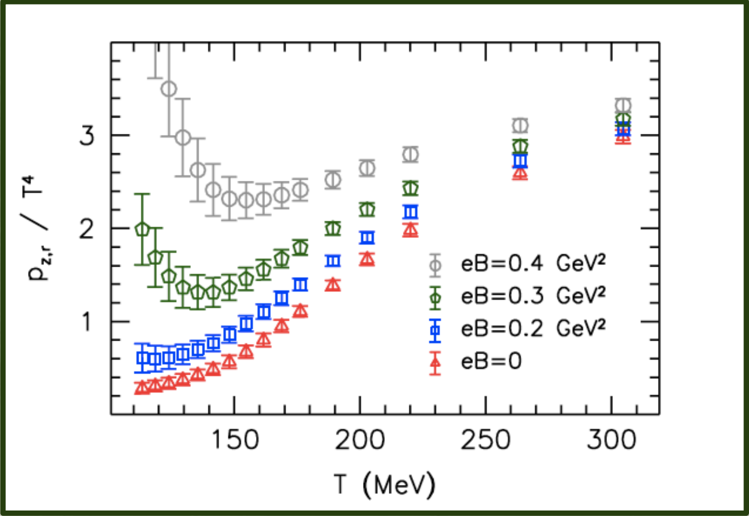
- Cases (iv) & (v): well behaved.

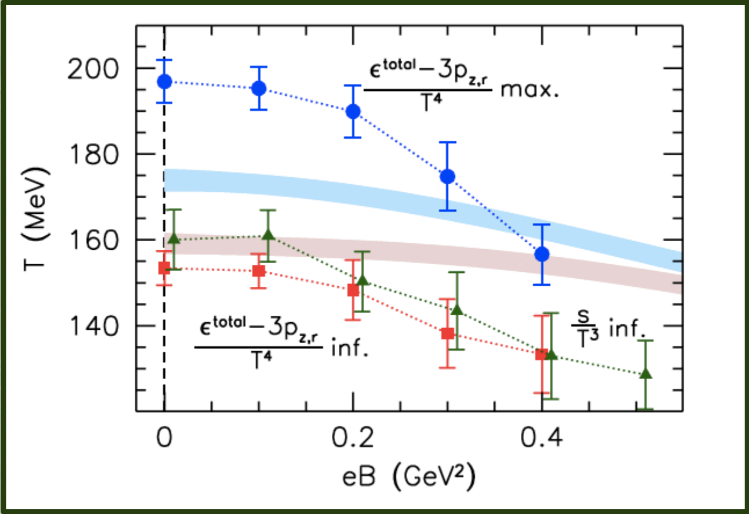
- Contribution from the exchange very small for the physical cases, even for huge B fields.

Lattice has already provided a great deal of info on the phase diagram (and essentially on all thermodynamic observables):



[Bali et al (2012)]





[Bali et al (2014)]

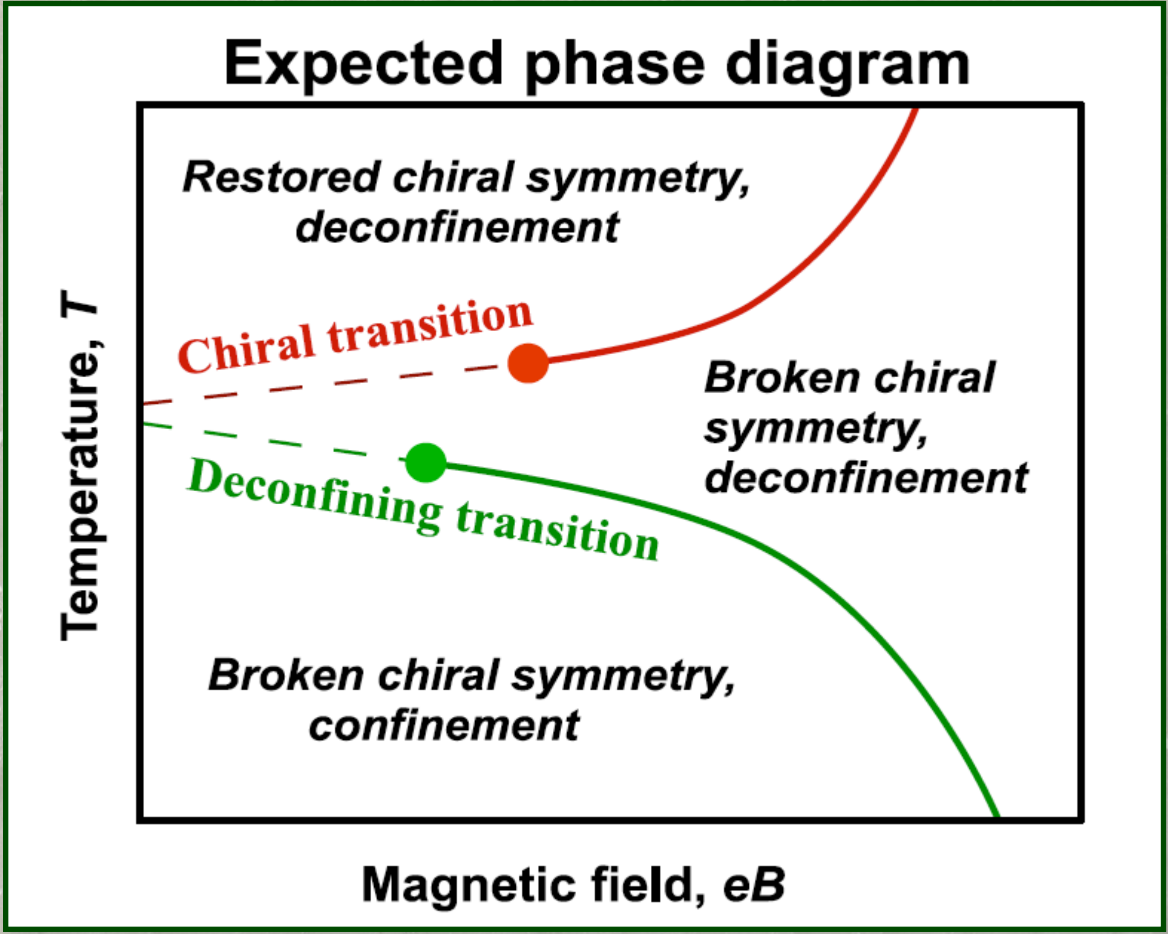
So, are we done??

Well, we would like to have some analytical understanding, too, which could provide new insights. Actually, it all started with effective models (before lattice QCD) but...

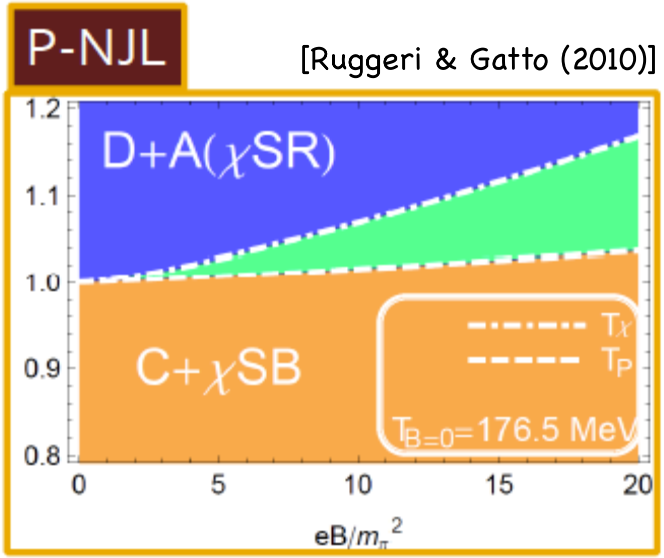
From the first papers:

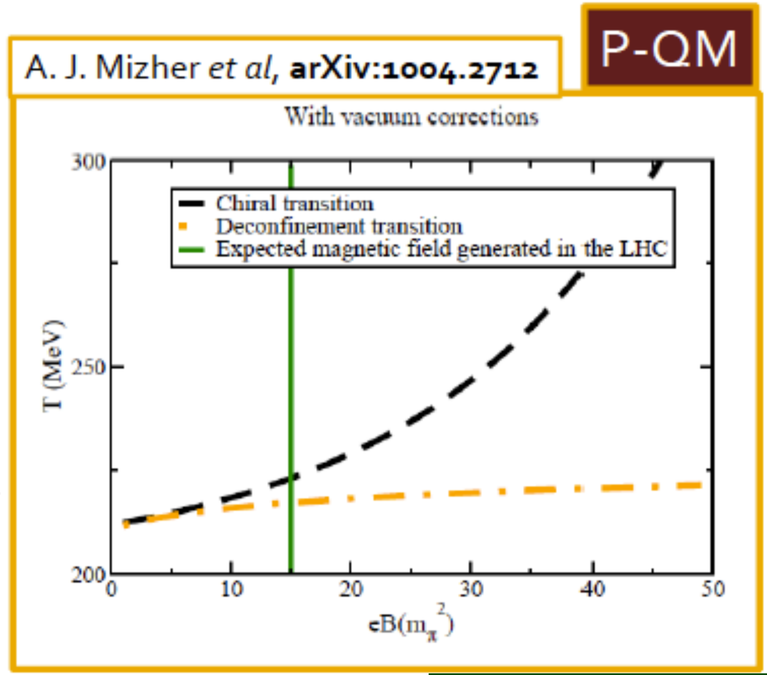
- Deconfining:
  Agasian & Fedorov (2008)

- Chiral:
  ESF & Mizher (2008)

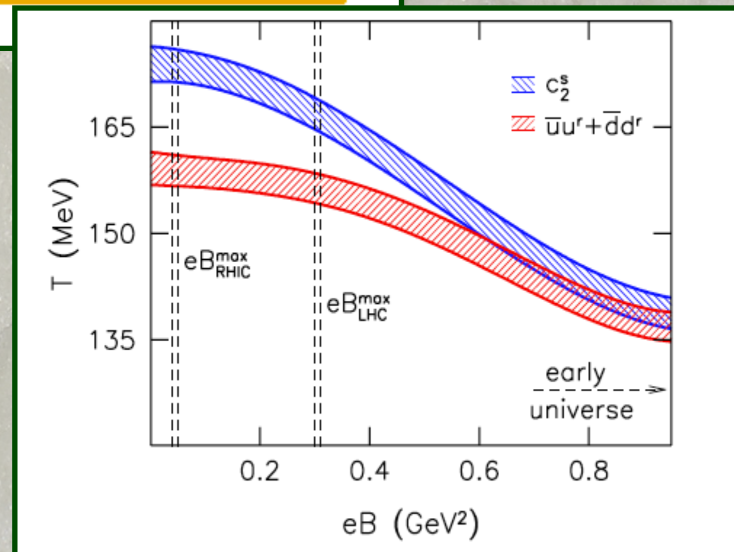


[Mizher, Chernodub & ESF (2010)]





Excellent agreement



So, all model predictions were basically wrong, except for T=0… (several came afterwards giving similar results, long list).

Clearly, effective chiral models were (are!) missing some crucial ingredient(s)!

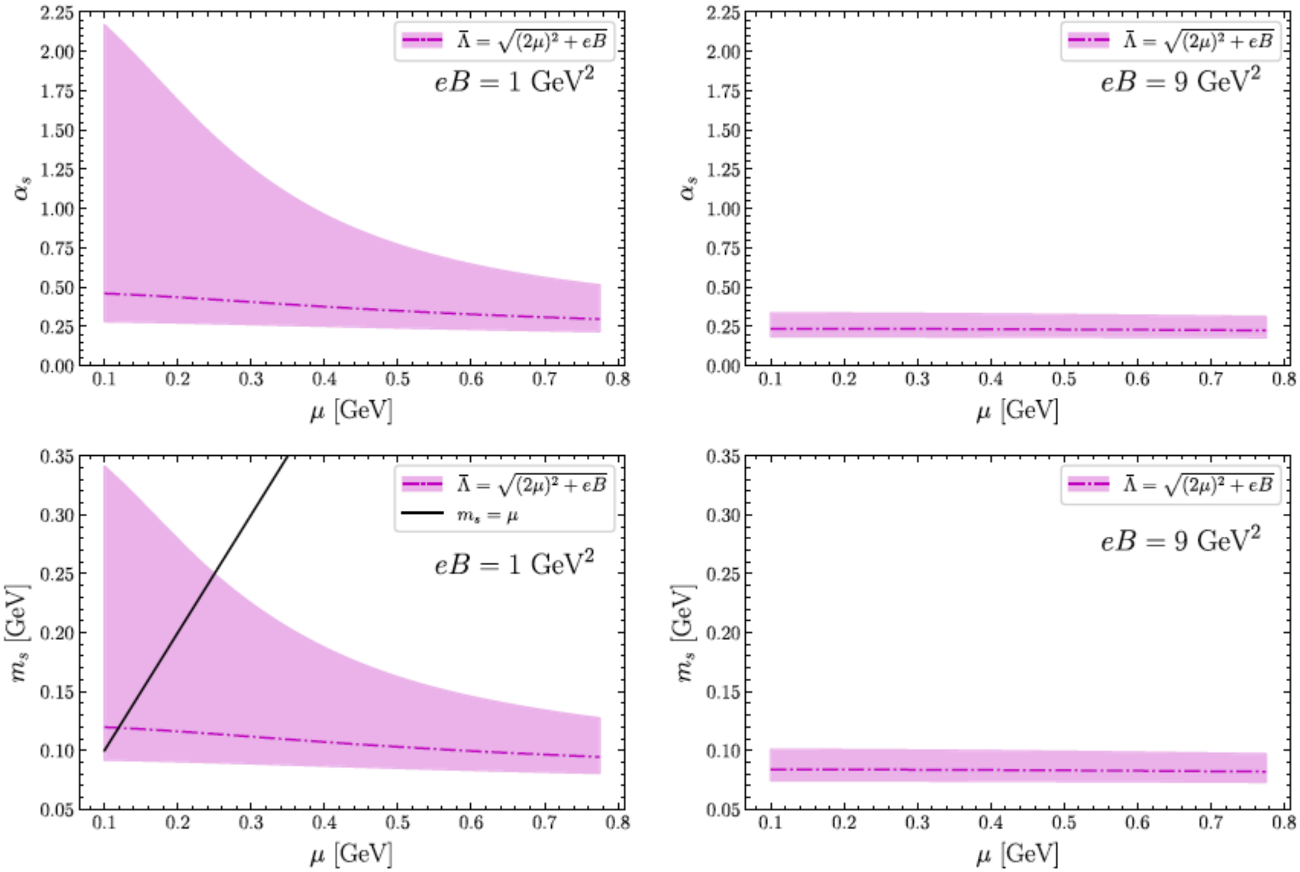


FIG. 1. Running coupling (top) and strange quark mass (bottom) as functions of the chemical potential for $eB = 1$ GeV^2 (left) and $eB = 9$ GeV^2 (right). The bands correspond to changes in the central scale by a factor of 2. For these plots $\mu = \mu_u = \mu_d = \mu_s$, assuming symmetric matter.