Chirally imbalanced medium in a four-fermion interaction model beyond the large-N_c approximation

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- Prof. Drª. Dyana Duarte (UFSM)
- André Gonçalves (undergrad student at UFSM)

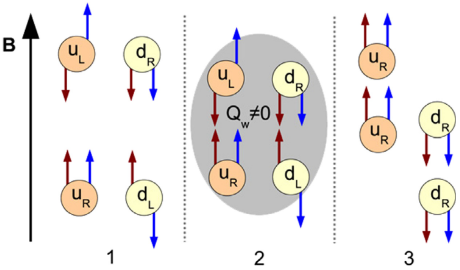
Where do we expect to see chiral imbalanced medium?

Different environments of chiral imbalanced medium:

- **Magnetic Fields-Chiral Magnetic Effect** [1];

$$\vec{J} = \frac{e\mu_5}{2\pi^2} \int d^3x\vec{B}. \quad (1)$$

- **High temperature QGP** - (Adler-Jackiw anomaly $\rightarrow$ chiral imbalanced medium);
- **Compact objects? (Li-Kang Yang et al. Symmetry 2020, 12(12), 2095)**



Ref: D. Kharzeev, L. Mclerran and H. Warringa. Nucl.Phys.A 803 (2008) 227-253

[1] K. Fukushima, D. Kharzeev, H. Warringa. Phys.Rev.D 78 (2008) 074033

What is the phase structure of a chiral imbalanced medium?

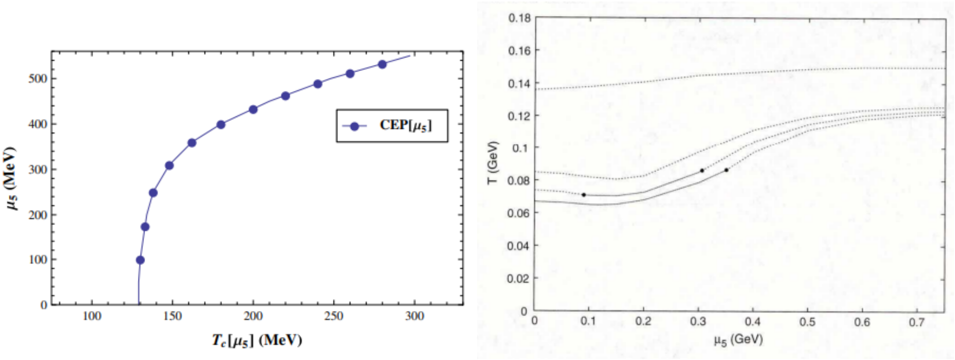


Figure: Left: Pseudocritical temperature of chiral transition[2]. Right: Pseudocritical temperatures of chiral transition with different values of baryon chemical potential[3].

[2] Bin Wang et al. Phys. Rev. D 91, 034017 (2015)
[3] Shu-Sheng Xu et al.Phys. Rev.D 91, 056003 (2015)

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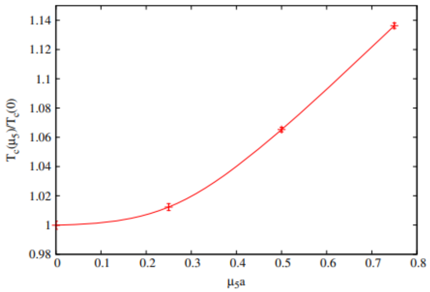


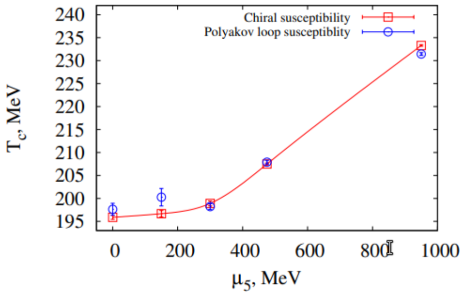


Figure: Left: Pseudocritical temperature of chiral transition[2]. Right: Pseudocritical temperatures of chiral and deconfinement transitions[3].

[2] V. V. Braguta, E.-M. Ilgenfritz, A. Yu. Kotov, B. Petersson and S. A. Skinderev. D 93, 034509 (2016)

[3] V. V. Braguta, et al. JHEP06(2015)094

What is the phase structure of a chiral imbalanced medium?

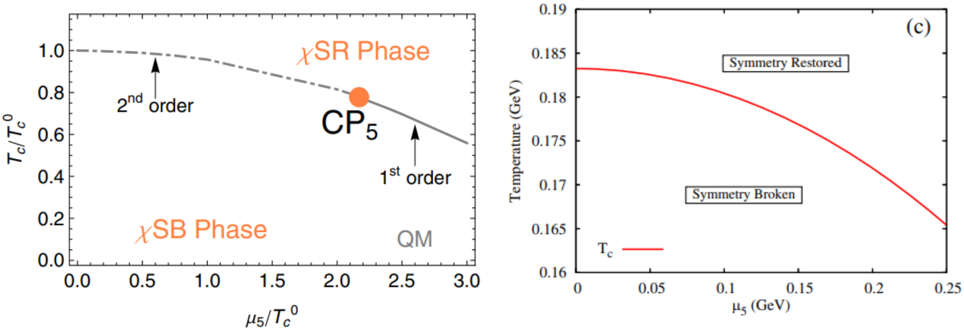


Figure: Left: Pseudocritical temperatures of chiral transition in the quark-meson model[4] and Nambu–Jona-Lasinio model [5].

[4] M. Ruggieri. Phys. Rev. D 84, 014011 (2011).

[5] Snigdha Ghosh et al. Phys. Rev. D 109, 016021 (2024)

Nambu–Jona-Lasinio model SU(2)

The lagrangian of the two-flavor NJL model within chiral imbalanced medium is given by

$$\mathcal{L} = \overline{\psi}\left[i\gamma^{\mu}\partial_{\mu} - \tilde{m} + \gamma^{0}\mu + \gamma_0\gamma_5\mu_5\right]\psi + G\left[(\overline{\psi}\psi)^2 + (\overline{\psi}i\gamma_5\vec{\tau}\psi)^2\right], \tag{2}$$

- the current quark masses matrix $\tilde{m} = \mathrm{diag}(m_u, m_d)$ in the isospin symmetry approximation, $m_u = m_d = m$; τ are the Pauli matrices; ψ is the spinor representing the quark fields $\psi = (\psi_u \quad \psi_d)^T$; G is the coupling constant and μ is the quark chemical potential.
- The new term $\mathcal{L}_{\mu 5} = \gamma_0\gamma_5\mu_5$ introduces effectively the difference between left- and right-handed particles.

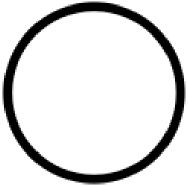
Recommended Reference: K. Fukushima, D. Kharzeev, H. Warringa. Phys.Rev.D 78 (2008) 074033

Thermodynamical potential at $\mu_5 = 0$

At $\mu_5 = 0$ the thermodynamical potential in the Large N_c approximation is given by

$$\mathcal{F}(T,\mu_q) = \frac{(M-m_0)}{4G} - 2N_cN_f\int_\Lambda \frac{d^3k}{(2\pi)^3}\omega(k)$$
$$-4N_cN_fT\left[\int_{\Lambda_T}\frac{d^3k}{(2\pi)^3}\log\left(1+e^{-(\omega(k)+\mu_q)/T}\right)\right.$$
$$\left.+\log\left(1+e^{-(\omega(k)-\mu_q)/T}\right)\right]$$

$$-\mathcal{F}^{(0)} =$$



The Feynman Diagram representation for the Free-energy at the Large N_c approximation.

- $\omega(k) = \sqrt{k^2+M^2} \Rightarrow$ quark energy dispersion relation;
- Sharp cutoff 3D, Λ, as a regularization procedure;
- thermal cutoff $\Lambda_T = \Lambda$ or $\Lambda_T = \infty$;

Thermodynamical potential at $\mu_5 \neq 0$

At finite μ_5, after solving the Functional Generator of the model, we obtain heuristically the change on the integrations

$$\begin{cases} 2\int \frac{d^3p}{(2\pi)^3} \Rightarrow \sum_{s=\pm 1} \int \frac{d^3p}{(2\pi)^3} \\ \omega(k) \Rightarrow \omega_s(k) \end{cases}$$

- where $\omega_s(k) = \sqrt{(|\vec{p}| + s\mu_5)^2 + M^2}$ is the new quark dispersion relation and the dispersion relation is given by

$$\mathcal{F}(T,\mu_5) = \frac{(M-m_0)}{4G} - N_c N_f \sum_{s=\pm 1} \int_\Lambda \frac{d^3k}{(2\pi)^3} \omega_s(k) - 2N_c N_f T \sum_{s=\pm 1} \int_{\Lambda_T} \frac{d^3k}{(2\pi)^3} \log\left(1 + e^{-\omega_s(k)/T}\right)$$

Thermodynamical potential - Regularization

Traditional regularization scheme (TRS)

This is just the application of a sharp 3D cutoff, Λ, on the vacuum divergent integrations.

$$\int \frac{d^3k}{(2\pi)^3}\omega_s(k) \Rightarrow \int_0^\Lambda \frac{dk}{2\pi^2}k^2\omega_s(k)$$

However, the vacuum contribution now has a dependence on μ_5!
Then Λ and μ_5 are **entangled**!

Medium separation Scheme (MSS)

It is a mathematical *trick* to **disentangle** vacuum contributions from the μ_5.

$$\int \frac{dk}{2\pi^2}k^2\omega_s(k)$$

$$\Rightarrow M^2 I_{\log}\left(\frac{M_0^2}{2}+\mu_5^2-\frac{M^2}{4}\right)+I_{\rm quad}\frac{M^2}{2}$$

$$-\frac{2M^2}{64\pi^2}+\frac{M^2(M^2-4\mu_5^2)}{32\pi^2}\log\left(\frac{M^2}{M_0^2}\right)$$

Recommended Reference: R. Farias, D. Duarte, G. Krein and R. Ramos. Phys.Rev.D 94, 074011 (2016)

Medium separation scheme

First, we can work with a more treatable integral by evaluating (euclidean space)

$$\frac{\partial}{\partial M^2}\left[\int \frac{d^3k}{(2\pi)^3}\omega_s(k)\right] = \int_{-\infty}^{+\infty}\frac{dk_4}{2\pi}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k_4^2+\omega_s^2(k)}, \tag{3}$$

then, we rewrite the integrand as

$$\frac{1}{k_4^2+\omega_s^2(k)} = \frac{1}{k_4^2+k^2+M_0^2} + \frac{k^2+M_0^2-\omega_s^2(k)}{(k_4^2+k^2+M_0^2)\left[k_4^2+\omega_s^2(k)\right]}, \tag{4}$$

which, when applied three times we obtain

$$\frac{1}{k_4^2+\omega_s^2(k)} = \frac{1}{k_4^2+\omega_0^2(k)} - \frac{A_s(k)}{(k_4^2+\omega_0^2(k))^2} + \frac{A_s^2(k)}{(k_4^2+\omega_0^2(k))^3} - \frac{A_s^3(k)}{(k_4^2+\omega_0^2(k))^3\left[k_4^2+\omega_s^2(k)\right]}, \tag{5}$$

where we have defined $\omega_0(k) = \sqrt{k^2+M_0^2}$; and $A_s(k) = \mu_5^2 + 2sk\mu_5 + M^2 - M_0^2$ and M_0 is the quark mass in the vacuum (i.e., computed at $T=0,\ \mu_5=0$).

Medium separation scheme

Let's look to each term separately

First term $\Rightarrow \int_{-\infty}^{+\infty} \frac{dk_4}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k_4^2 + \omega_0^2(k)} \Rightarrow$ Quadratically divergent integral

Second term $\Rightarrow \int_{-\infty}^{+\infty} \frac{dk_4}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{A_s(k)}{(k_4^2 + \omega_0^2(k))^2} \Rightarrow$ Logarithm divergent integral

Third term $\Rightarrow \int_{-\infty}^{+\infty} \frac{dk_4}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{A_s(k)^2}{(k_4^2 + \omega_0^2(k))^3} \Rightarrow$ Logarithm divergent integral

Fourth term $\Rightarrow \int_{-\infty}^{+\infty} \frac{dk_4}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{A_s(k)^3}{(k_4^2 + \omega_0^2(k))^3 (k_4^2 + \omega_s^2(k))} \Rightarrow$ finite integral

Mismatch between LQCD and Effective Models

Effective models using TRS regularization mismatch the $T_{pc} \times \mu_5$ relation with LQCD.

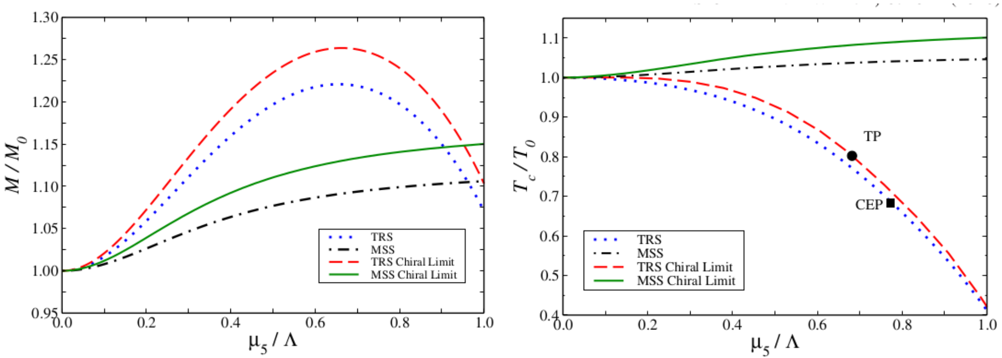


Figure: Left: Effective quark masses as a function of the chiral chemical potential at $T = 0$. Right: Pseudocritical temperature of chiral phase transition for TRS and MSS regularizations.

Recommended Reference: R. Farias, D. Duarte, G. Krein and R. Ramos. Phys.Rev.D 94, 074011 (2016)

Optimized perturbation theory

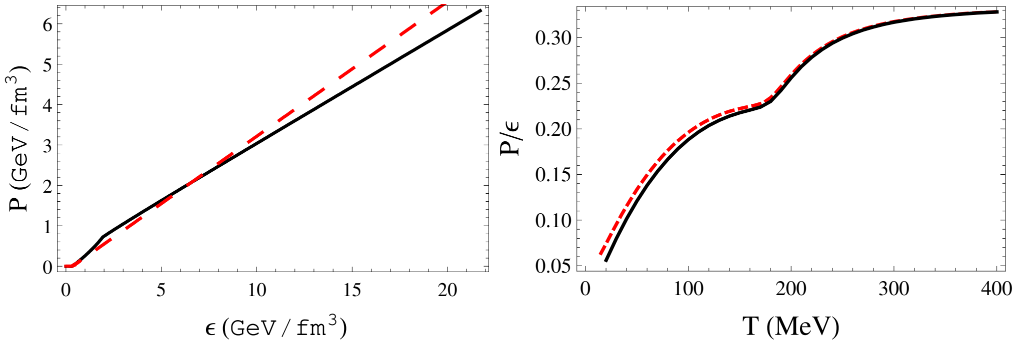


Figure: Left: EOS for NJL SU(2). Right: EOS as a function of the temperature in the NJL model. In both plots, the continuous line is for the OPT computation and the dashed for LN.

Jean-Loïc Kneur, Marcus B. Pinto and R. Ramos. Phys.Rev.C 81, 065205 (2010)

Optimized perturbation theory

The OPT approximation is based on the modification of the original Lagrangian by a fictitious new expansion parameter δ

$$\mathcal{L}(\delta) = (1-\delta)\mathcal{L}_0 + \delta\mathcal{L} = \mathcal{L}_0 + \delta(\mathcal{L} - \mathcal{L}_0) \quad (6)$$

- $\mathcal{L}_0$ is the free Lagrangian;
- $\mathcal{L}$ is the original Lagrangian;
- $\mathcal{L}(\delta)$ interpolates between the original Lagrangian $\delta = 1$ and the free theory $\delta = 0$;
- The relevant physical quantities are evaluated in series expansion in δ, which is treated as a small number.

For dimensional balance, $\mathcal{L}_0$ must have at least one arbitrary mass parameter η. Then, any physical quantity must be at least locally $\eta-$independent, i.e.,

$$\frac{\partial P(\eta)}{\partial \eta}\Big|_{\bar{\eta}} = 0. \quad \text{Principle of minimal sensitivity - PMS} \quad (7)$$

Optimized perturbation theory applied to NJL model

By using the Large N_c Nambu–Jona-Lasinio SU(2),

$$\mathcal{L}_{\mathrm{LN}} = \bar{\psi}\,(i\partial\!\!\!/ - m_c)\,\psi - \bar{\psi}\,(\sigma + i\gamma_5\tau\cdot\vec{\pi})\,\psi - G(\sigma^2 + \vec{\pi}^2) \tag{8}$$

we can rewrite the OPT Lagrangian as

$$\mathcal{L}_{\mathrm{OPT}} = \bar{\psi}\,[i\partial\!\!\!/ - m_c - \delta(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi}) - \eta(1-\delta)]\,\psi - G(\sigma^2 + \vec{\pi}^2)\,. \tag{9}$$

in which we will consider $\vec{\pi} = 0$.

$$\hat{\eta} = \eta + m_c - \delta(\eta - \sigma) \tag{10}$$

Optimized perturbation theory applied to NJL model

The Feynman diagrams at order δ in OPT to the NJL model

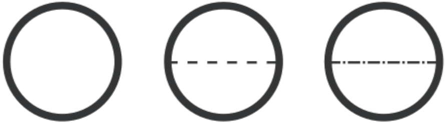


Figure: Feynman diagrams of OPT at order δ.

The free energy is given by

$$\mathcal{F}_{\rm OPT} = \frac{(M-m_c)^2}{4G} - N_f N_c I_1(T,\tilde{\mu}) + \delta N_f N_c (\eta + m_c)(\eta - M + m_c) I_2(T,\tilde{\mu})$$
$$+\delta N_f N_c \eta_0 I_3(T,\tilde{\mu}) + \delta G N_f N_c I_3^2(T,\tilde{\mu}) - \frac{1}{2}\delta G N_f N_c (\eta + m_c)^2 I_2^2(T,\tilde{\mu})$$

Jean-Loïc Kneur, Marcus B. Pinto and and R. Ramos. Phys.Rev.C 81, 065205 (2010)

Optimized perturbation theory applied to NJL model at $\mu_5 \neq 0$

We will have to solve the following integrations

$$I_1(T,\tilde{\mu}) = \sum_{s=\pm 1} \int \frac{d^3p}{(2\pi)^3} \left(\omega_s(p) + T \ln(1 + e^{-(\omega_s(p)-\tilde{\mu})/T}) + T \ln(1 + e^{-(\omega_s(p)+\tilde{\mu})/T} \right) \tag{11}$$

$$I_2(T,\tilde{\mu}) = \sum_{s=\pm 1} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_s(p)} \left(1 - \frac{1}{e^{(\omega_s(p)-\tilde{\mu})/T}+1} - \frac{1}{e^{(\omega_s(p)+\tilde{\mu})/T}+1} \right) \tag{12}$$

$$I_3(T,\tilde{\mu}) = \sum_{s=\pm 1} \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{e^{(\omega_s(p)-\tilde{\mu})/T}+1} - \frac{1}{e^{(\omega_s(p)+\tilde{\mu})/T}+1} \right) \tag{13}$$

where $\tilde{\mu} = \mu + \eta_0$[1].

[1] η_0 is the second mass parameter fixed by PMS condition.

Optimized perturbation theory applied to NJL model

OPT conditions

The principle of minimal sensitivity (PMS) applied to both variables η and η_0

$$\left.\frac{d\mathcal{F}_{\rm OPT}}{d\eta}\right|_{\bar{\eta}_0,\sigma,\delta=1} = \left.\frac{d\mathcal{F}_{\rm OPT}}{d\eta_0}\right|_{\bar{\eta},\sigma,\delta=1} = 0$$

Also the minimization of the chiral condensate

$$\left.\frac{d\mathcal{F}}{d\sigma}\right|_{\bar{\eta},\eta_0,\delta=1} = 0$$

Thermodynamics

The quantities we will explore in this work are

$$P = -\mathcal{F}(\bar{\sigma},\bar{\eta},\bar{\eta}_0)$$

$$\Rightarrow P_N(T,\mu) = P(T,\mu) - P(0,0)$$

$$\epsilon = -P + Ts + \mu\rho + \mu_5\rho_5$$

$$\rho = \left.\left(\frac{\partial P}{\partial \mu}\right)\right|_{T,\mu_5}, \quad \rho_5 = \left.\frac{\partial P}{\partial \mu_5}\right|_{T,\mu}$$

$$s = \frac{\partial P}{\partial T}, \quad c_s^2 = \frac{dP}{d\epsilon}, \quad \chi = m_c|\langle\bar{\psi}\psi\rangle|$$

Parametrizations

We adopt the following parameters (3D cutoff) of the Large N_c approximation

Λ [MeV]	$G\Lambda^2$	m_c [MeV]	$\langle \bar{u}u \rangle^{1/3}$ [MeV]	f_π [MeV]	m_π [MeV]
640	2.14	5.2	-246.986	92.4	136.6

The set of parameters with 3D sharp cutoff in the OPT approximation are given by:

Λ [MeV]	$G\Lambda^2$	m_c [MeV]	$\langle \bar{u}u \rangle^{1/3}$ [MeV]	f_π [MeV]	m_π [MeV]
635	1.98	5.14	-246.24	92.4	135

Jean-Loïc Kneur, Marcus B. Pinto and and R. Ramos. Phys.Rev.C 81, 065205 (2010)

Numerical Results

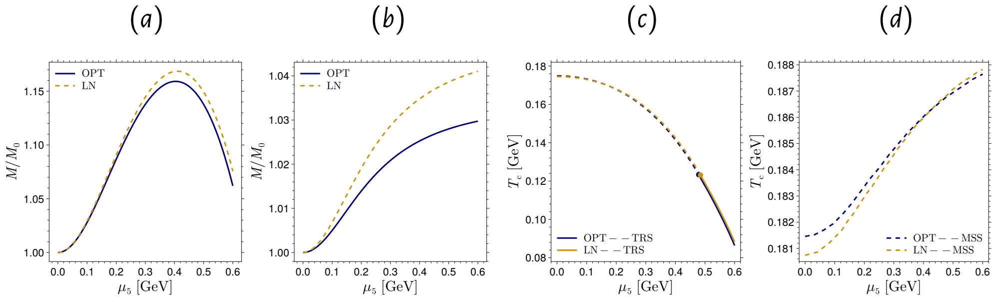


Figure: Effective quark masses as a function of the chiral chemical potential with TRS (a) and MSS (b). Right: Pseudocritical temperature as a function of the chiral chemical potential with TRS (c) and MSS (d).

A. Gonçalves, D. Duarte, R. Farias, M. Pinto, R. Ramos, **W.R. Tavares**. **To appear on Arxiv.**

Numerical Results

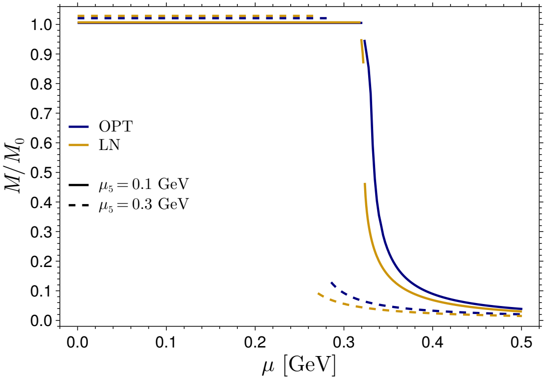


Figure: Effective quark masses as a function of the quark chemical potential at $T = 0$ in LN and BLN approximation, for different values of chiral chemical potential.

A. Gonçalves, D. Duarte, R. Farias, M. Pinto, R. Ramos, **W.R. Tavares**. **To appear on Arxiv.**

Numerical Results

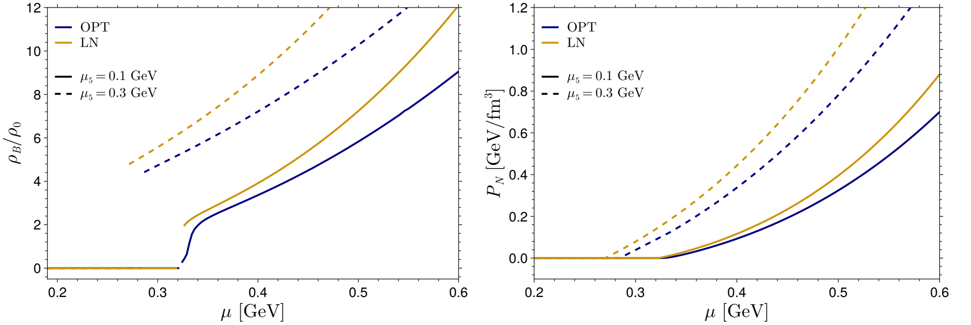


Figure: Left: Baryonic density as a function of quark chemical potential in LN and BLN approximations. Right: Normalized pressure as a function of quark chemical potential for different values of μ_5 in LN and BLN approximations.

A. Gonçalves, D. Duarte, R. Farias, M. Pinto, R. Ramos, **W.R. Tavares**. **To appear on Arxiv.**

Numerical Results

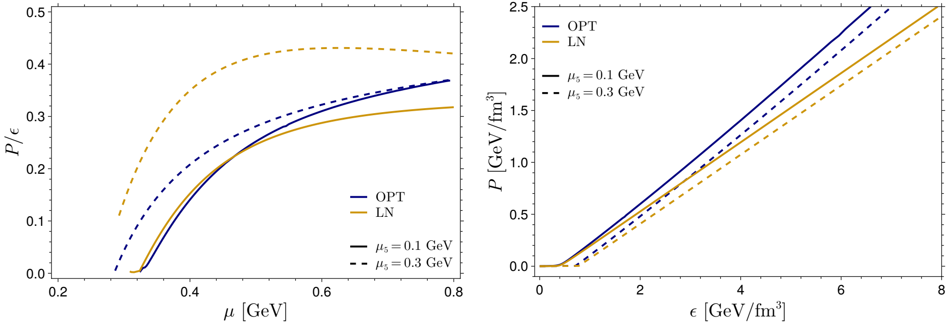


Figure: Left: Ratio P/ϵ as a function of the quark chemical potential for different values of μ_5 in LN and BLN approximations. Right: EOS ($P \times \epsilon$) for different values of μ_5 in LN and BLN approximations.

A. Gonçalves, D. Duarte, R. Farias, M. Pinto, R. Ramos, **W.R. Tavares**. **To appear on Arxiv.**

Numerical Results

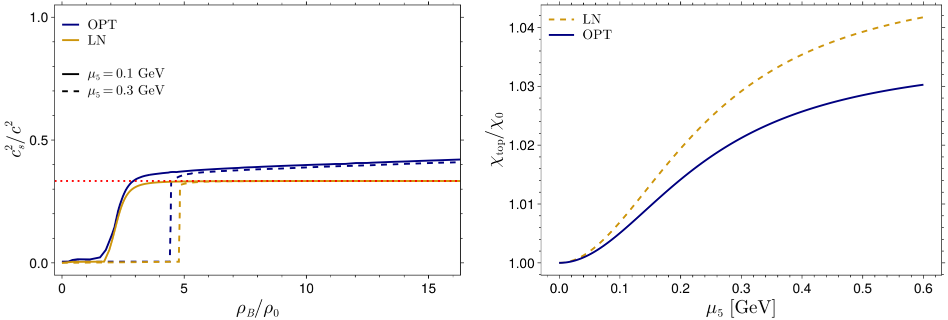


Figure: Left: Sound velocity as a function of the quark chemical potential for different values of μ_5 in LN and BLN approximations. Right: Topological susceptibility as a function of quark chemical potential for different values of μ_5 in LN and BLN approximations.

A. Gonçalves, D. Duarte, R. Farias, M. Pinto, R. Ramos, **W.R. Tavares**. **To appear on Arxiv.**

Conclusions

- **In the MSS scheme combined with the BLN approximation**, we observe an increase in the pseudocritical temperature as a function of the chiral chemical potential, in agreement with LQCD results.
- Non-trivial first-order phase transitions with μ and μ_5, observed in the LN approximation, are avoided in the BLN approximation.
- The BLN approximation underestimates both the pressure and the baryonic density compared to the LN results.
- Topological susceptibility and effective masses are underestimated in the BLN approximation compared to the LN results.

Thanks for your attention!

Acknowledgements

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