Fractional vorticity and Bogomol'nyi-Prasad-Sommerfield systems for the Gross-Pitaevskii equations

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Relevant references

Detailed version of this work

- F. Canfora and P. Pais,
  "Fractional vorticity, Bogomol'nyi-Prasad-Sommerfield systems and complex structures for the (generalized) spinor Gross-Pitaevskii equations", *Nucl. Phys.* **B 1017** (2025) 116955,
  arXiv:2502.00578 [cond-mat.quant-gas].
- F. Canfora and P. Pais,
  "A Bogomol'nyi-Prasad-Sommerfield bound with a first-order system in the $2D$ Gross-Pitaevskii equation", *Eur. J. Phys.* **C** (2025) to appear soon,
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The Gross-Pitaevskii Equation (GPE)

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- It describes many non-trivial experimental features of superfluids and supersolids.
- However, unlike what happens in the Ginzburg-Landau theory for superconductors, there are very few effective analytic tools to study GPE in two or more dimensions.

The Gross-Pitaevskii Equation (GPE)

We will consider stationary GPE in two spatial dimensions, without external potential or chemical potential[1,2]:

$$-\frac{\hbar^2}{2M}\triangle\Psi + g\,|\Psi|^2\,\Psi = 0\ .$$

where

$$\Psi = \rho\, e^{\mathrm{i}\,S} = \Phi_1 + \mathrm{i}\,\Phi_2\ ,$$

being g the coupling constant, Ψ the condensate wave function, ρ is the corresponding amplitude, and S is the phase.

[1] *L. Pitaevskii and S. Stringari, Bose-Einstein Condensation and Superfluidity. International Series of Monographs on Physics.*

[2] *C. Barenghi and N. Parker, A Primer on Quantum Fluids. SpringerBriefs in Physics.*

The Gross-Pitaevskii Equation (GPE)

It can be derived from the following energy-functional

$$E = \int_\Gamma d^2x \left[\frac{\hbar^2}{2M} \left| \overrightarrow{\nabla} \Psi \right|^2 + \frac{g}{2} \left| \Psi \right|^4 \right] = \frac{\hbar^2}{M} \int_\Gamma d^2x \left[\frac{1}{2} \left| \overrightarrow{\nabla} \Psi \right|^2 + \frac{g_{eff}}{4} \left| \Psi \right|^4 \right]$$

$$g_{eff} = \frac{2Mg}{\hbar^2} \ .$$

GPE, complex structure and fractional vorticity

Let us consider the following first-order elliptic system in two spatial dimensions

$$\partial_x \Phi_1 + \partial_y \Phi_2 = A\big(\Phi_1, \Phi_2\big) \ , \qquad (1)$$

$$\partial_y \Phi_1 - \partial_x \Phi_2 = B\big(\Phi_1, \Phi_2\big) \ , \qquad (2)$$

Let us consider the following first-order elliptic system in two spatial dimensions

$$\begin{aligned} \partial_x\Phi_1 + \partial_y\Phi_2 &= A\big(\Phi_1, \Phi_2\big) \ , \qquad (1) \\ \partial_y\Phi_1 - \partial_x\Phi_2 &= B\big(\Phi_1, \Phi_2\big) \ , \qquad (2) \end{aligned}$$

Take

$$\partial_x \,(1) + \partial_y (2) \ ,$$

and

$$\partial_y \,(1) - \partial_x (2) \ ,$$

From the above first-order system, one can deduce that $\Phi_1 = \Phi_1\left(\overrightarrow{x}\right)$, $\Phi_2 = \Phi_2\left(\overrightarrow{x}\right)$ satisfy the following second-order semilinear system of equations:

$$\begin{aligned}
\triangle\Phi_1 &= C_1\partial_x\Phi_2 + C_2\partial_y\Phi_2 + \frac{1}{2}\frac{\partial}{\partial\Phi_1}\left(A^2+B^2\right) \ , \\
\triangle\Phi_2 &= -C_1\,\partial_x\Phi_1 - C_2\,\partial_y\Phi_1 + \frac{1}{2}\frac{\partial}{\partial\Phi_2}\left(A^2+B^2\right) \ ,
\end{aligned}$$

where

$$C_1 = \frac{\partial A}{\partial\Phi_2} + \frac{\partial B}{\partial\Phi_1}\ , \quad C_2 = \frac{\partial B}{\partial\Phi_2} - \frac{\partial A}{\partial\Phi_1}\ .$$

GPE, complex structure and fractional vorticity

Therefore, when $C_1 = C_2 = 0$, one gets

$$\triangle\Phi_1 = \frac{1}{2}\frac{\partial}{\partial\Phi_1}\left(A^2 + B^2\right) ,$$
$$\triangle\Phi_2 = \frac{1}{2}\frac{\partial}{\partial\Phi_2}\left(A^2 + B^2\right) .$$

In other words, one can define a “superpotential” W, which is an analytic function of $Z = \Phi_1 + \mathrm{i}\,\Phi_2$ and then A and B are the real and imaginary parts of W:

$$W = W(Z), \quad Z = \Phi_1 + \mathrm{i}\,\Phi_2 ,$$
$$A = \mathrm{Re}\, W , \quad B = \mathrm{Im}\, W .$$

$|\Psi|^4$ interactions

If one considers the quadratic superpotential

$$W = \frac{\kappa}{\sqrt{2}} Z^2 \ ,$$

one arrives at the following BPS system

$$\begin{aligned} \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} &= \frac{\kappa}{\sqrt{2}} (\Phi_1^2 - \Phi_2^2) \ , \\ \frac{\partial \Phi_1}{\partial y} - \frac{\partial \Phi_2}{\partial x} &= \sqrt{2}\, \kappa \, \Phi_1 \, \Phi_2 \ , \end{aligned}$$

$|\Psi|^4$ interactions

Whose solutions are automatically solutions of the GPE

$$\begin{aligned} -\triangle\Phi_j + g_{eff}\left(\overrightarrow{\Phi}\cdot\overrightarrow{\Phi}\right)\Phi_j &= 0\ , \quad j=1,2 \\ (\Phi_1)^2+(\Phi_2)^2 &= \overrightarrow{\Phi}\cdot\overrightarrow{\Phi}\ , \\ g_{eff} &= \kappa^2\ , \end{aligned}$$

The energy can be rewritten as follows:

$$E=\frac{\hbar^2}{2M}\int_\Gamma d^2x\left[(\partial_x\Phi_1+\partial_y\Phi_2-A)^2+(\partial_y\Phi_1-\partial_x\Phi_2-B)^2\right]+Q_1+Q_2\ .$$

$|\Psi|^4$ interactions

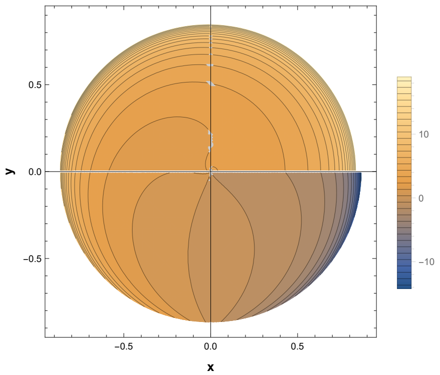
One kind of solution of the above BPS system

$$\rho(r) = \frac{2\sqrt{2}}{2\sqrt{2}\,A\,r^{\frac{1}{3}} - 3\kappa\,r} \ ,$$

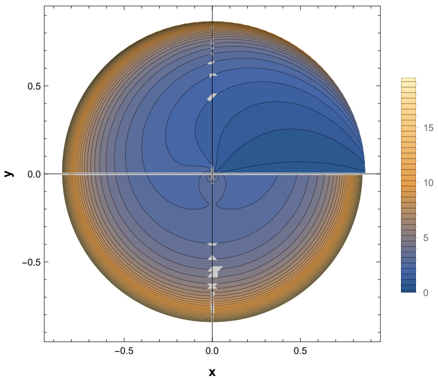
and

$$\begin{aligned} \Phi_1(r,\theta) &= \frac{2\sqrt{2}\,\cos\left(\frac{\theta}{3}+\theta_0\right)}{2\sqrt{2}\,A\,r^{\frac{1}{3}} - 3\kappa\,r} \ , \\ \Phi_2(r,\theta) &= \frac{2\sqrt{2}\,\sin\left(\frac{\theta}{3}+\theta_0\right)}{2\sqrt{2}\,A\,r^{\frac{1}{3}} - 3\kappa\,r} \ . \end{aligned}$$

The $\frac{1}{3}$-vorticity Solutions for $|\Psi|^4$ interactions



(a) Level curves for ϕ_1.



(b) Level curves for ϕ_2.

Figure: Level curves for solution the $\frac{1}{3}$-vorticity solution. We took $\kappa = 1$ and $A = 1$, and internal radius $a = 0.3$ while the external one is $R \approx 0.91$ in arbitrary units.

$|\Psi|^5$ interactions

Let us consider the following superpotential

$$W = \frac{\kappa}{\sqrt{5/2}} Z^{\frac{5}{2}} = \frac{\kappa}{\sqrt{5/2}} \left[\Phi_1 + \mathrm{i}\,\Phi_2\right]^{\frac{5}{2}} \ ,$$

$$Z = \rho\, e^{\mathrm{i}\, S} \to Z^{\frac{5}{2}} = \rho^{\frac{5}{2}}\, e^{\mathrm{i}\, \frac{5}{2}\, S} \ ,$$

One arrives at the following BPS system

$$\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} = \frac{\kappa}{\sqrt{5/2}} \rho^{\frac{5}{2}} \cos(\frac{5}{2} S) \ ,$$

$$\frac{\partial \Phi_1}{\partial y} - \frac{\partial \Phi_2}{\partial x} = \frac{\kappa}{\sqrt{5/2}} \rho^{\frac{5}{2}} \sin(\frac{5}{2} S) \ ,$$

$|\Psi|^5$ interactions

The solutions of the above system are automatically solutions of the GPE

$$
\begin{aligned}
-\triangle\Phi_j + g_{eff}\left(\overrightarrow{\Phi}\cdot\overrightarrow{\Phi}\right)^{\frac{3}{2}}\Phi_j &= 0\,, \quad j=1,2 \\
(\Phi_1)^2+(\Phi_2)^2 &= \overrightarrow{\Phi}\cdot\overrightarrow{\Phi}\,.
\end{aligned}
$$

$|\Psi|^5$ interactions

The simplest topologically non-trivial solution of the BPS system takes the form

$$\rho(r) = (A\, r^{3/7} - \frac{21}{4\sqrt{10}}\,\kappa\, r)^{-\frac{2}{3}} \,,$$

$$S(\theta) = \frac{2}{7}\,\theta + S_0 \,.$$

If one considers the quadratic superpotential

$$W = \frac{\kappa}{\sqrt{3}} Z^3 \ ,$$

then one arrives at the following BPS system

$$\begin{aligned} \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} &= \frac{\kappa}{\sqrt{3}} (\Phi_1^3 - 3\Phi_1 \Phi_2^2) \ , \\ \frac{\partial \Phi_1}{\partial y} - \frac{\partial \Phi_2}{\partial x} &= \frac{\kappa}{\sqrt{3}} \left(3\Phi_1^2 \Phi_2 - \Phi_2^3 \right) \ , \end{aligned}$$

whose solutions are automatically solutions of the GPE

$$
\begin{aligned}
-\triangle \Phi_j + g_{eff} \left(\overrightarrow{\Phi} \cdot \overrightarrow{\Phi} \right)^2 \Phi_j &= 0 \ , \quad j = 1, 2 \\
(\Phi_1)^2 + (\Phi_2)^2 &= \overrightarrow{\Phi} \cdot \overrightarrow{\Phi} \ , \quad g_{eff} = \kappa^2 \ ,
\end{aligned}
$$

while the energy becomes

$$
E = \frac{\hbar^2}{2M} \int_\Omega d^2x \left[\sum_{j=1}^{2} \left(\overrightarrow{\nabla} \Phi_j \right)^2 + \frac{g_{eff}}{3} \left(\overrightarrow{\Phi} \cdot \overrightarrow{\Phi} \right)^3 \right] \ ,
$$

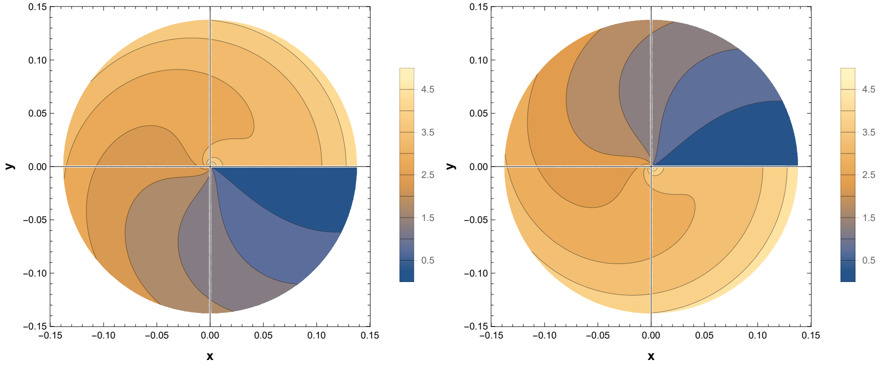
$|\Psi|^6$ interactions

The simplest topologically non-trivial solution of the above BPS system for the $|\Psi|^6$ self-interaction potential takes the form

$$
\begin{aligned}
\rho(r) &= (A\,r^{1/2} - \frac{4}{\sqrt{3}}\,\kappa\,r)^{-\frac{1}{2}} \,, \\
S(\theta) &= \frac{\theta}{4} + S_0 \,,
\end{aligned}
$$

where A and S_0 are two integration constants.

The $\frac{1}{4}$-vorticity Solutions for $|\Psi|^6$ interactions



(a) Level curves for ϕ_1. (b) Level curves for ϕ_2.

Figure: Level curves for solution the $\frac{1}{4}$-vorticity solution. We took $\kappa = 1$ and $A = 1$, and internal radius $a = 0.3$ while the external one is $R \approx 0.91$ in arbitrary units.

Final remarks

- In the present work, it has been shown that a large class of generalized GPE equations (with, for instance, the non-linear interaction terms $|\Psi|^5$ and $|\Psi|^6$)

Final remarks

- In the present work, it has been shown that a large class of generalized GPE equations (with, for instance, the non-linear interaction terms $|\Psi|^5$ and $|\Psi|^6$)
- This formalism allows the powerful analytic tools of BPS solitons to be applied to the topologically non-trivial configurations of GPE (usually analyzed only numerically).

Future work

Regarding the possible experimental settings where the predictions of these techniques could be tested, there are (at least) two natural candidates:

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- to connect the fractional statistics with the fractional vorticity

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Regarding the possible experimental settings where the predictions of these techniques could be tested, there are (at least) two natural candidates:

- to connect the fractional statistics with the fractional vorticity
- the (fractional) quantum Hall effect and the supersolids, as the experimental techniques will soon become precise enough to test the appearance of configurations with fractional vorticity

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Thank you