Aspects of Topology From Dirac Equation and Supersymmetry

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Collaborators

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01

Charged particles in Magnetic Fields

From Classical to Quantum Mechanics

Why electromagnetic interactions?

- Foundational to understanding from cyclotron motion to quantum spin dynamics
- Gauge invariance and the cornerstone of fundamental interactions
- Classical trajectories, QHE, Superconductivity, Landau levels, etc

Classical mechanics

Lorentz force

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

Magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Gauge invariance

$$\mathbf{A} \to \mathbf{A} + \nabla\chi$$

Free particle lagrangian

$$L_0 = \frac{1}{2}m\mathbf{v}^2$$

Coupling to e.m. potentials

$$L = \frac{1}{2}m\mathbf{v}^2 + q\mathbf{v}\cdot\mathbf{A} - q\phi$$

Quantum mechanics

Free particle Hamiltonian

$$H_0 = \frac{\mathbf{p}^2}{2m}$$

Coupling to e.m. potentials

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi$$

Schrödinger equation

$$\hat{H}\psi = \left[\frac{1}{2m}(-i\hbar\nabla - q\mathbf{A})^2 + q\phi\right]\psi$$

Gauge transformations

$$\mathbf{A} \to \mathbf{A} + \nabla\chi \qquad \psi \to \psi e^{iq\chi/\hbar}$$

Landau levels

Aspect	Non-Relativistic	Relativistic
Gauge Used	Landau Gauge ($\mathbf{A} = -By\hat{x}$)	Same
Hamiltonian	$\frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2$	$H = \gamma^0 \left[\boldsymbol{\gamma} \cdot (\mathbf{p} - q\mathbf{A}) + mc\right]$
Energy Levels	$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right)$	$E_n^{\pm} = \pm\sqrt{m^2c^4 + 2n\hbar\omega_c mc^2 + \hbar\omega_c s}$
Physical Insight	Quantization in the xy-plane; free motion in z-direction.	Spin and relativistic effects included.

02

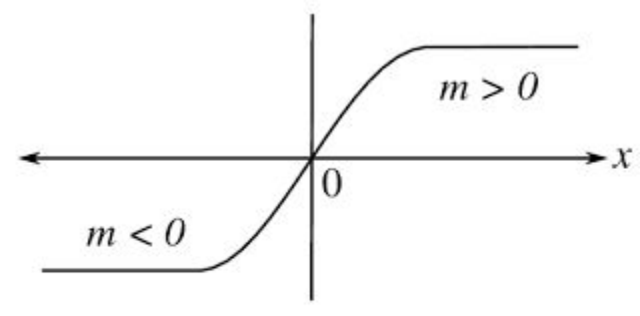
Jackiw-Rebbi model

SSH, Topoelectrical circuits and all that

Jackiw-Rebbi model

- Fermion field interacting with a spatially varying scalar field.
- Zero modes
- Topological system: Solitons or domain walls
- Hamiltonian

$$H = \gamma^0 \left[\boldsymbol{\gamma} \cdot \mathbf{p} + m(x) \right]$$



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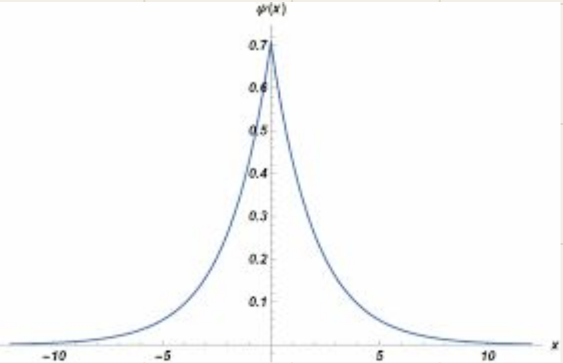
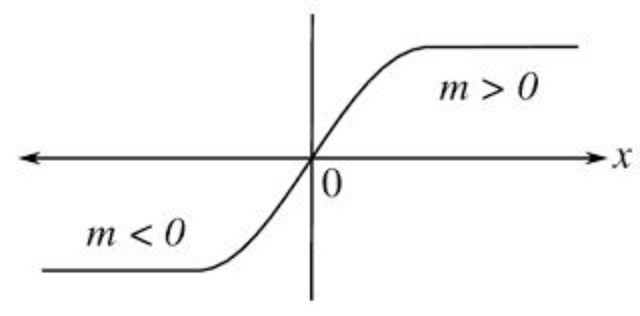
Jackiw-Rebbi model

- Topologically protected zero modes at domain walls
- Fractionalization of charges





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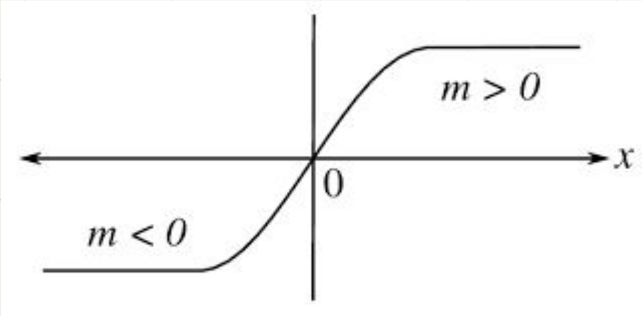
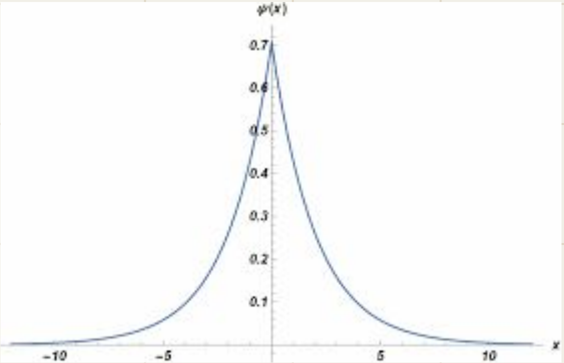
Jackiw-Rebbi model

- Topological insulators
- Gapless modes at interfaces where the bulk topological invariants change.





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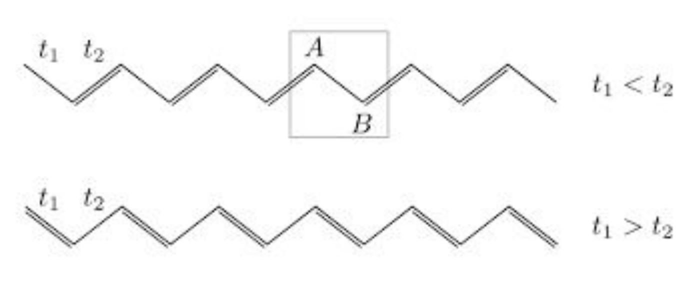
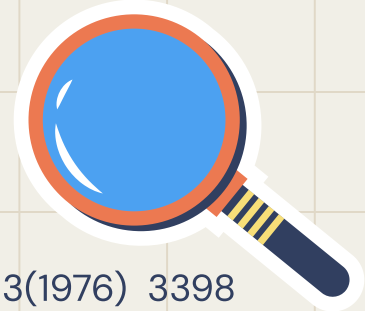
Jackiw-Rebbi model

- SSH model for polyacetylene chains
- 1D-TB model

$$H = \sum_n \left[t_1 c_n^\dagger c_{n+1} + t_2 c_{n+1}^\dagger c_n \right]$$

- Hosts topological edge states when the hopping amplitudes are reversed.
- The zero-energy edge states correspond to the JR zero modes.





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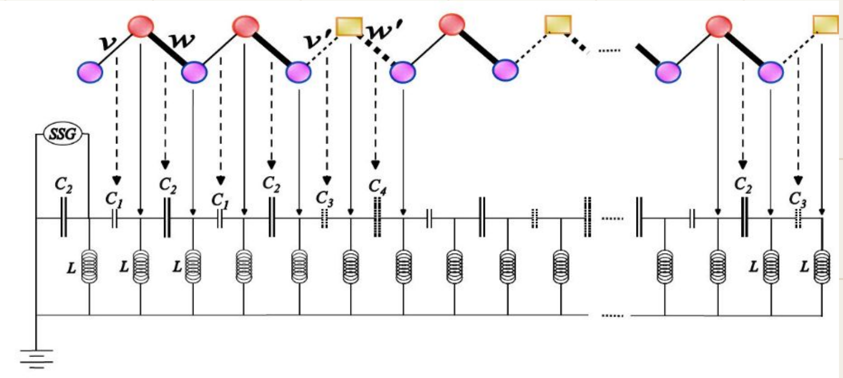
Jackiw-Rebbi model



- Topoelectric circuits
- Topological phases using classical electrical components
- Domain walls correspond to impedance mismatches
- Edge modes manifest as localized resonances in circuit voltage/current distributions

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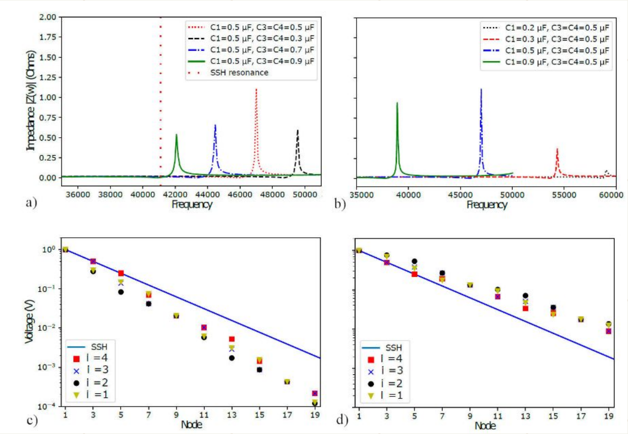

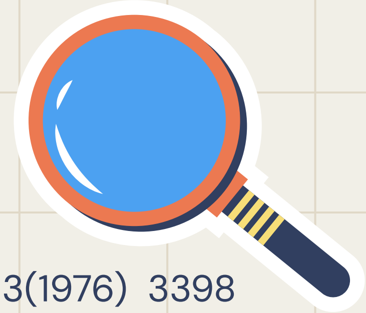

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Jackiw-Rebbi model

The **Jackiw-Rebbi model** has a compelling analogy with electrostatic systems. Consider the following comparison:

Feature	Jackiw-Rebbi Model	Electrostatic System
Domain Wall	Spatially varying mass term	Charged surface or potential interface
Zero-Energy Modes	Localized solutions at	Bound charges or fields localized at boundaries
Fractionalization	Fractional charges induced at solitons	Partial charges induced on conductors
Topological Nature	Protected states from topology	Surface integrals of electric flux (Gauss's law)

03

DMS oscillator

Non-minimal coupling in the Dirac equation

Dirac Oscillator

- Dirac Oscillator

$$H = \vec{\alpha} \cdot \vec{p} + \beta m + \frac{1}{2} m \omega^2 x^2$$

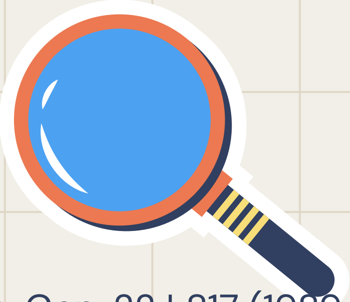
$$E_n = \pm \sqrt{m^2 + 2\hbar\omega \left(n + \frac{1}{2} \right)}$$

Dirac-Moshinsky-Szczepaniac Oscillator

- DMS Oscillator

$$\vec{p} \rightarrow \vec{p} - i\gamma^5 m\omega x$$

$$H = (\vec{\alpha} \cdot (\vec{p} - i\gamma^5 m\omega x)) + \beta m + \frac{1}{2} m\omega^2 x^2$$



J. Phys. A: Math. Gen. 22 L817 (1989)

Dirac-Moshinsky-Szczepaniac Oscillator

- Spin-orbit coupling
- Energy spectrum modification
- Position-spin
- Symmetry breaking
- Systems (graphene) with position dependent mass

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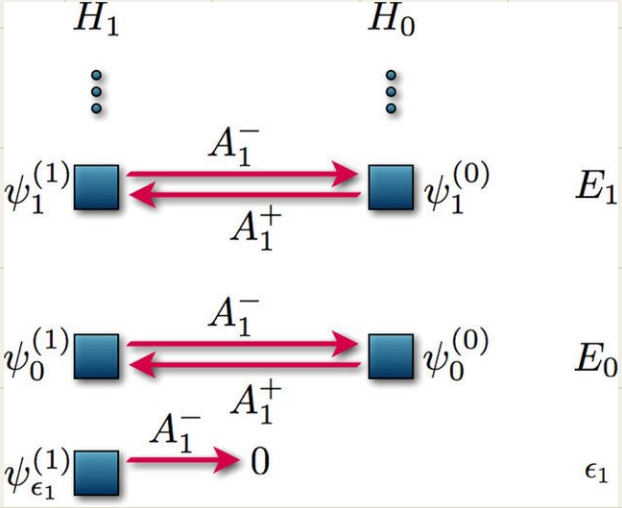
04

Topology and topological index

Where do we see topology?

SUSY-QM

$$\begin{aligned} H_0 &= A_1^+ A_1^- \\ H_1 &= A_1^- A_1^+ \end{aligned}$$



- Unbroken SUSY
- Witten index W=F-B=1

SUSY-DMS oscillator

- 1D DMS oscillator

$$\hat{H}_D\Psi = \left[\sigma_y \hat{p} + \sigma_x \left(\frac{E_x}{V_0}\right)\right]\Psi = \mathcal{E}\Psi.$$

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Wavefunction

$$\hat{H}_i\psi_i = \left(\frac{\partial^2}{\partial x^2} + U_i(x)\right)\psi_i(x) = 0$$

$$U_{1,2}(x) = \left[-\left(\frac{E_x}{V_0}\right)^2 + \mathcal{E}^2 \pm \frac{1}{V_0}\frac{dE_x}{dx}\right]$$

Supersymmetric pair of Schrödinger-like hamiltonians

$$\psi_i = C_{\mp}\exp\left[\mp\int\left(\frac{E_x}{V_0}\right)dx\right]$$

Zero modes

- Not normalizable simultaneously
- Spectra differ by a zero mode

G. GONZÁLEZ, *Rev. Mex. Fis.* E **65** (2019) 30–33

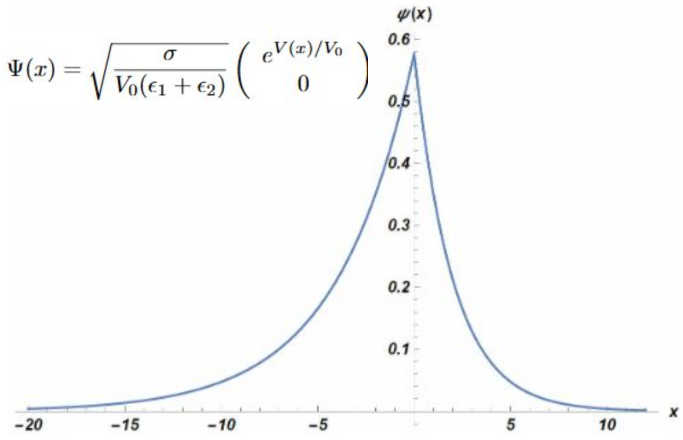
SUSY-DMS oscillator

- Infinitely charged sheet $\rho(x) = \sigma\delta(x)$

$$E_x(x) = \begin{cases} \dfrac{\sigma}{2\epsilon_1}, & \text{for} \quad x > 0 \\ -\dfrac{\sigma}{2\epsilon_2}, & \text{for} \quad x < 0. \end{cases}$$

$$V(x) = -\int_0^x E_x(x)dx = \begin{cases} -\dfrac{\sigma}{2\epsilon_1}x, & \text{for} \quad x > 0 \\ \dfrac{\sigma}{2\epsilon_2}x, & \text{for} \quad x < 0. \end{cases}$$

$$\Psi(x) = \sqrt{\frac{\sigma}{V_0(\epsilon_1+\epsilon_2)}} \begin{pmatrix} e^{V(x)/V_0} \\ 0 \end{pmatrix}$$



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SUSY-DMS oscillator

- 1D non-minimal oscillator

$$H = \sigma_x \left(\hat{p} + i\sigma_z W(x)\right)$$

Hamiltonian

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

Wavefunction

$$\left(\hat{p}^2 \mp W'(x) + W^2(x)\right)\psi_\pm = 0$$

Supersymmetric pair of Schrödinger-like hamiltonians

$$V_\pm(x) = \pm W'(x) + W^2(x)$$

SUSY-DMS oscillator

$$\psi_0^{\pm} = C_{\pm} e^{\mp \int dx W(x)}$$

Zero modes

- Not normalizable simultaneously
  - One additional zero mode in the spectrum of H_0
- Witten index preserved
  - Unbroken SUSY
- All topological indexes of the model can be expressed in terms of this one
  - Sak phase, winding numbers, and so on

SUSY-DMS oscillator

Zero modes in different potential with kink behavior

Spin-orbit interpretation

Connection with topological phases of the SSH model

Connection with topoelectric circuits

Thank you!